Counting rooted planar quadrangulations

2. The functional equation in polynomial form

In[44]:= $P[Q\_, q\_, t\_, u\_] = \text{Factor}\left[u\left(Q - 1 - u\,Q^2 - t\,\frac{Q - 1 - u\,q}{u}\right)\right]$
factorise

Out[44]= $t - Q\,t - u + Q\,u + q\,t\,u - Q^2\,u^2$

3. Differentiating with respect to Q

$P^{(1,0,0,0)}[Q[U], Q_1, t, U] == 0$

$-t + U - 2\,U^2\,Q[U] == 0$

Existence of a solution : this equation amounts to U=t+2 U^2 Q[U] which has a unique solution by a fixed-point argument

4. The two other equations

$P^{(0,0,0,1)}[Q[U], Q_1, t, U] == 0$

$P[Q[U], Q_1, t, U] == 0$

$-1 + Q[U] - 2\,U\,Q[U]^2 + t\,Q_1 == 0$

$t - U - t\,Q[U] + U\,Q[U] - U^2\,Q[U]^2 + t\,U\,Q_1 == 0$

5. Manipulation of the polynomial system

We first express Q[U] in terms of U by the first equation

$\text{Solve}\left[P^{(1,0,0,0)}[Q[U], Q_1, t, U] == 0, Q[U]\right][[1]]$
résous

$\left\{Q[U] \to \frac{-t + U}{2\,U^2}\right\}$

We then substitute in the second equation and solve for Q_1

$\text{Solve}\left[\left(P^{(0,0,0,1)}[Q[U], Q_1, t, U] == 0\right) /. \left\{Q[U] \to \frac{-t + U}{2\,U^2}\right\}, Q_1\right][[1]]$
résous

$\left\{Q_1 \to \frac{t^2 - t\,U + 2\,U^3}{2\,t\,U^3}\right\}$

Finally we substitute all this in the third equation and solve for U (and pick the correct determination)

```
Factor[P[Q[U], Q1, t, U] == 0 /. {Q[U] → (-t + U)/(2 U^2), Q1 → (t^2 - t U + 2 U^3)/(2 t U^3)}]
factorise
```

```
Simplify[Solve[%, U], Assumptions → t > 0][[1]]
simplifie   résous    suppositions
```

$$\frac{3\,t^2 - 4\,t\,U + U^2 + 4\,t\,U^2}{4\,U^2} == 0$$

$$\left\{U \to -\frac{\left(-2 + \sqrt{1 - 12\,t}\right)t}{1 + 4\,t}\right\}$$

And now we plug this in Q_1, and compute the first few terms

```
FullSimplify[{Q1 → (t^2 - t U + 2 U^3)/(2 t U^3)} /. {U → -((-2 + Sqrt[1 - 12 t]) t)/(1 + 4 t)}, Assumptions → t > 0]
simplifie complètement                                                                           suppositions
```

```
Series[Q1 /. %, {t, 0, 10}]
développement en série entière
```

$$\left\{Q_1 \to \frac{-1 + \sqrt{1 - 12\,t} + 6\left(3 - 2\sqrt{1 - 12\,t}\right)t}{54\,t^2}\right\}$$

$$1 + 2\,t + 9\,t^2 + 54\,t^3 + 378\,t^4 + 2916\,t^5 + 24\,057\,t^6 + 208\,494\,t^7 + 1\,876\,446\,t^8 + 17\,399\,772\,t^9 + 165\,297\,834\,t^{10} + O[t]^{11}$$

Compare with the explicit formula

```
Table[(2 × 3^n (2 n)!)/(n! (n + 2)!), {n, 0, 10}]
table
```

{1, 2, 9, 54, 378, 2916, 24 057, 208 494, 1 876 446, 17 399 772, 165 297 834}

6. Rational parametrization

We return to our quadratic equation relation U and perform the substitution

```
Solve[(3 t^2 - 4 t U + U^2 + 4 t U^2)/(4 U^2) == 0 /. U -> (R - 1)/(R (2 + R)), t][[1]]
```
résous

```
Simplify[{Q1 -> (t^2 - t U + 2 U^3)/(2 t U^3)} /. {U -> (R - 1)/(R (2 + R))} /. %]
```
simplifie

$$\left\{t \to \frac{-1+R}{3 R^2}\right\}$$

$$\left\{Q_1 \to -\frac{1}{3} (-4+R) R\right\}$$

7. Returning to the initial functional equation and computing the discriminant

In[45]:=
```
funQRu = Factor[P[Q, Q1, t, u] /. {t -> (-1 + R)/(3 R^2), Q1 -> -1/3 (-4 + R) R}]
```
factorise

```
Factor[Discriminant[%, Q]]
```
factorise discriminant

Out[45]= $-\frac{3-3 Q-3 R+3 Q R+4 R u+4 R^2 u-9 Q R^2 u+R^3 u+9 Q^2 R^2 u^2}{9 R^2}$

Out[46]= $-\frac{(-1+4 R u)\left(1-R+2 R u+R^2 u\right)^2}{9 R^4}$

Notice the square factor

8. The one-cut form

We simply check the announced form now

In[47]:=
```
Qonecut := ((1 - R + 3 R^2 u) - (1 - R + 2 R u + R^2 u) Sqrt[1 - 4 R u])/(6 R^2 u^2)

Simplify[funQRu /. Q -> Qonecut]
```
simplifie

Out[48]= 0

9. Expansion in u

In[49]:=
```
Series[Qonecut, {u, 0, 10}]
```
développement en série entière

$$\% - \text{Sum}\left[\frac{(2+R)\ \text{CatalanNumber}[k] + (1-R)\ \text{CatalanNumber}[k+1]}{3} R^k u^k, \{k, 0, 10\}\right]$$
somme

Out[49]= $1 + \left(\frac{4R}{3} - \frac{R^2}{3}\right) u + \left(3R^2 - R^3\right) u^2 + \left(8R^3 - 3R^4\right) u^3 + \left(\frac{70R^4}{3} - \frac{28R^5}{3}\right) u^4 + \left(72R^5 - 30R^6\right) u^5 + \left(231R^6 - 99R^7\right) u^6 + \left(\frac{2288R^7}{3} - \frac{1001R^8}{3}\right) u^7 + \left(2574R^8 - 1144R^9\right) u^8 + \left(8840R^9 - 3978R^{10}\right) u^9 + \left(\frac{92378R^{10}}{3} - \frac{41990R^{11}}{3}\right) u^{10} + O[u]^{11}$

Out[50]= $O[u]^{11}$

Basketball walks

Kernel and right-hand side

In[1]:=
$$K[t\_, u\_] := 1 - t\left(u^{-2} + u^{-1} + u + u^2\right)$$
$$R[t\_, u\_] := t\,(1+u) - \frac{t}{u}\left(G_1 + G_2 + \frac{G_1}{u}\right)$$

The initial terms as functions of the roots

In[7]:= $\text{Solve}[\{R[t, U_1] == 0, R[t, U_2] == 0\}, \{G_1, G_2\}]$
résous

Out[7]= $\{\{G_1 \to -U_1 U_2 (1 + U_1 + U_2),\ G_2 \to U_1 + U_1^2 + U_2 + 2 U_1 U_2 + U_1^2 U_2 + U_2^2 + U_1 U_2^2\}\}$

Actually we may write G_1 as a polynomial in the symmetric functions $e_1 = U_1 + U_2,\ e_2 = U_1 U_2$

In[8]:= $\text{SymmetricReduction}[G_1\ /.\ \%[[1]], \{U_1, U_2\}, \{e_1, e_2\}][[1]]$
réduction symétrique

Out[8]= $-e_2 - e_1 e_2$

Series expansion of $U_{1,2}$, here $\tau = t^{1/2}$ or $-t^{1/2}$

```
InverseSeries[Series[u / Sqrt[1 + u + u^3 + u^4], {u, 0, 20}] /. u → τ]
série inverse       développe…   racine carrée
```

$$\tau + \frac{\tau^2}{2} + \frac{\tau^3}{8} + \frac{\tau^4}{2} + \frac{159\,\tau^5}{128} + \frac{3\,\tau^6}{2} + \frac{1761\,\tau^7}{1024} + \frac{7\,\tau^8}{2} + \frac{229\,819\,\tau^9}{32\,768} + 11\,\tau^{10} + \frac{4\,551\,367\,\tau^{11}}{262\,144} + \frac{65\,\tau^{12}}{2} + \frac{256\,435\,147\,\tau^{13}}{4\,194\,304} + \frac{213\,\tau^{14}}{2} + \frac{6\,269\,791\,041\,\tau^{15}}{33\,554\,432} + \frac{693\,\tau^{16}}{2} + \frac{1\,386\,188\,792\,787\,\tau^{17}}{2\,147\,483\,648} + 1176\,\tau^{18} + \frac{36\,980\,416\,515\,147\,\tau^{19}}{17\,179\,869\,184} + 4017\,\tau^{20} + O[\tau]^{21}$$

An algebraic equation for G_1

Notice that the equation K[t,U]=0 can be rewritten as the system

X=U+1/U

$1 = t(X^2 + X - 2)$

The second equation admits two roots X_1 and X_2, and it is not difficult to check that each of them fixes one of the U_i

```
X₁ := U₁ + 1 / U₁
X₂ := U₂ + 1 / U₂
(* since X₁ and X₂ are the two roots of X²+X-2-1/t,
we have X₁X₂=-2-1/t and X₁+X₂=-1 *)
(* but X₁X₂ and X₁+X₂ are also symmetric functions in U₁ and U₂,
   hence polynomials in e₁=U₁+U₂, e₂=U₁U₂ *)
(* so we deduce a system of two polynomial equations for e₁ and e₂ *)
Numerator[Factor[X₁ X₂ + 2 + 1 / t]];
numérateur  factorise
SymmetricReduction[%, {U₁, U₂}, {e₁ , e₂}][[1]] == 0
réduction symétrique
Numerator[Factor[X₁ + X₂ + 1]];
numérateur  factorise
SymmetricReduction[%, {U₁, U₂}, {e₁ , e₂}][[1]] == 0
réduction symétrique
```

Out[32]= $t + t\,e_1^2 + e_2 + t\,e_2^2 == 0$

Out[34]= $e_1 + e_2 + e_1\,e_2 == 0$

Since $G_1 = -e_2 - e_1 e_2$, we obtain by elimination an algebraic equation for G_1 which admits a unique “good” solution, whose expansion yields the wanted numbers

```
In[40]:= Eliminate[{t + t e₁² + e₂ + t e₂² == 0, e₁ + e₂ + e₁ e₂ == 0, G₁ == -e₂ - e₁ e₂}, {e₁, e₂}]
élimine
```

Out[40]= $(-1 + 2\,t)\,G_1 + (-1 + 3\,t)\,G_1^2 + 2\,t\,G_1^3 + t\,G_1^4 == -t$

In[41]:=
```
Solve[%, G1][[2]]
FullSimplify[Series[G1 /. %, {t, 0, 10}], Assumptions → t > 0]
```
résous / simplifie complè··· / développement en série entière / suppositions

Out[41]= $\left\{G_1 \to \frac{1}{2}\left(-1 + \sqrt{-3 + \frac{2}{t} - \frac{2\sqrt{1-4t}}{t}}\right)\right\}$

Out[42]= $t + t^2 + 3t^3 + 7t^4 + 22t^5 + 65t^6 + 213t^7 + 693t^8 + 2352t^9 + 8034t^{10} + O[t]^{11}$

And Catalan numbers at the end

In[43]:= $\text{FullSimplify}\left[\left(1 + G_1 + G_1^2\right) \text{ /. } G_1 \to \frac{1}{2}\left(-1 + \sqrt{-3 + \frac{2}{t} - \frac{2\sqrt{1-4t}}{t}}\right)\right]$

simplifie complètement

Out[43]= $\frac{2}{1 + \sqrt{1 - 4t}}$