Equations with one catalytic variable in enumerative combinatorics: exercise session

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1 Rooted planar quadrangulations

A (rooted planar) *quadrangulation* is a rooted planar map where every face has degree 4. A quadrangulation cannot have loops (why?) but multiple edges are allowed. We denote by q_n the number of quadrangulations with n *faces*. The number of edges is then $2n$, here it is better to count by the number of faces to avoid parity issues. Our purpose is to compute q_n using the method studied in the lecture.

For this we must consider a more general counting problem. A *quadrangulation with a boundary* is a rooted planar map where every face, except possible the outer face, has degree 4. We do not assume the boundary to be simple. The outer degree is necessarily even, and we denote by $q_{n,k}$ the number of quadrangulations with n *inner faces* and outer degree $2k$. It is elementary to check that $q_n = q_{n,1}$ for $n \geq 2$ (think about gluing together the two boundary edges).

Short formulation of the problem (for those who want no indications)

> Find a nice explicit formula for q_n (and possibly $q_{n,k}$ as a bonus) using the method studied in the lecture.

Detailed questions (for those who want some indications)

1. Using Tutte's recursive decomposition (see Figure 1 for a reminder), check that $q_{n,k}$ is given by the recurrence relation

$$q_{n,k} = \sum_{n'=0}^{n} \sum_{k'=0}^{k-1} q_{n',k'} q_{n-n',k-1-k'} + q_{n-1,k+1} \tag{1}$$

   with the initial data $q_{0,0} = 1$, $q_{0,k} = 0$ if $k > 0$.

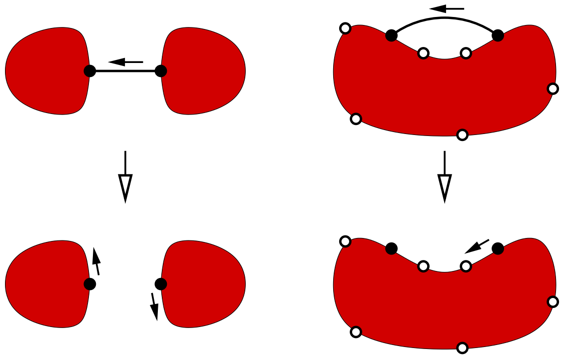

Figure 1: Reminder of Tutte's recursive decomposition.

2. Deduce that the bivariate generating function $Q(t,u) := \sum_{n\geq 0}\sum_{k\geq 0} q_{n,k}t^n u^k$ satisfies
$$Q(t,u) = 1 + uQ(t,u)^2 + t\frac{Q(t,u) - 1 - uQ_1(t)}{u} \tag{2}$$
where $Q_1(t) := \sum_{n\geq 0} q_{n,1}t^n$. Rewrite this equation in the form $P(Q(t,u), Q_1(t), t, u) = 0$ with $P(Q,q,t,u)$ a polynomial in 4 variables.

3. Show that there exists a unique series $U(t) \in \mathbb{Z}[[t]]$ such that
$$\frac{\partial P}{\partial Q}\left(Q(t,U(t)), Q(t,1), t, u\right) = 0. \tag{3}$$

4. ("Question de cours") Deduce two other polynomial relations between $U(t)$, $Q(t,U(t))$, $Q_1(t)$.

5. By manipulating the three polynomial equations (a computer might help), show that
$$Q_1(t) = \frac{(1-12t)^{3/2} - 1 + 18t}{54t^2} \tag{4}$$
and deduce an expression for q_n.

   (Hint: first eliminate $Q(t,U(t))$, then $Q_1(t)$, to deduce a quadratic polynomial relation between $U(t)$ and t. Wait, haven't we seen the series $Q_1(t)$ before?)

6. A useful alternative form: show that Q_1 admits the *rational parametrization*
$$Q_1 = \frac{4R - R^2}{3}, \qquad t = \frac{R-1}{3R^2}. \tag{5}$$
(Hint: substitute $U = \frac{R-1}{R(R+2)}$ in the previous quadratic relation between U and t.)

7. Replace t and Q_1 by their rational parametrization in the initial functional equation (2), which is a quadratic polynomial in $Q(t,u)$, and compute the discriminant: what do you notice?

8. Deduce the *one-cut form*

$$Q(t,u) = \frac{(1 - R + 3R^2u) - (1 - R + 2Ru + R^2u)\sqrt{1 - 4Ru}}{6R^2u^2}. \tag{6}$$

9. Deduce that quadrangulations with fixed outer degree $2k$ are counted by

$$[u^k]Q(t,u) = \frac{(2 + R)C_k + (1 - R)C_{k+1}}{3} R^k \tag{7}$$

where $C_k = \frac{1}{k+1}\binom{2k}{k}$ is the k-th Catalan number.

(Hint: recall that $\sqrt{1 - 4x} = 1 - 2\sum_{k\geq 0} C_k x^{k+1}$, substitute into the one-cut form, and observe that we do not need to care about the “leading terms” since they only serve to kill negative powers of u.)

10. (For the brave) Apply the Lagrange inversion formula to deduce an expression for $q_{n,k}$.

2 Basketball walks

A *basketball walk* with n steps is a sequence $(x_0, \ldots, x_n) \in \mathbb{Z}^{n+1}$ such that $x_t - x_{t-1} \in \{-2, -1, 1, 2\}$ for all t. The name was coined by Ayyer and Zeilberger because the walk record the score difference between two teams playing a basketball game at a time where three-pointers did not exist. See [1] and references therein.

Let g_n denote the number of basketball walks where the initial value is $x_0 = 0$, the final value is $x_n = 1$, and we have $x_t > 0$ for any $t > 0$. The purpose of the exercise is to show that

$$G(t) := \sum_{n=1}^{\infty} g_n t^n = -\frac{1}{2} + \frac{1}{2}\sqrt{\frac{2 - 3t - 2\sqrt{1 - 4t}}{t}} \tag{8}$$

and that

$$1 + G(z) + G(z)^2 = \mathrm{Cat}(z) \tag{9}$$

where $\mathrm{Cat}(z)$ is the generating function of Catalan numbers (this latter identity was the motivation of [1]).

1. Let $g_{n,k}$ denote the number of basketball walks satisfying the same property as above, except that the final value is $x_n = k \geq 1$. Show that the bivariate generating function $G(t,u) := \sum_{n\geq 1}\sum_{k\geq 1} g_{n,k} t^n u^{k-1}$ satisfies the functional equation

$$K(t,u)G(t,u) = t(1 + u) - \frac{t}{u}\left(G(t,0) + \frac{\partial G}{\partial u}(t,0) + \frac{G(t,0)}{u}\right) \tag{10}$$

with the kernel $K(t,u) := 1 - t(u^{-2} + u^{-1} + u + u^2)$.

2. Show that the kernel equation $K(t, U(t)) = 0$ admits two roots $U_{1,2}(t)$ in $\mathbb{C}[[t^{1/2}]]$.

3. Explain how $G(t,0)$ is given in terms of these roots.

4. Do the algebra!

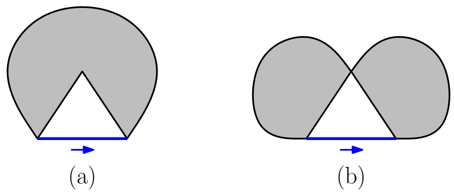

Figure 2: Recursive decomposition for triangulations with a simple boundary.

3 Triangulations with simple boundaries

A (rooted planar) *triangulation* is a rooted planar map whose all faces have degree 3. We may enumerate them using the recursive decomposition we have seen before, by a straightforward adaptation of the first exercise. But, following the original approach of Tutte and Brown, we may also proceed slightly differently by writing down a functional equation for triangulations with a *simple* boundary, i.e. we assume that there is no separating vertex incident to the outer face (a vertex is said separating if its removal disconnects the map).

To add even more fun, we may actually distinguish different classes of triangulations, depending on their possible "singularities": general triangulations may have loops and multiple edges, and we refer to them as *type I* triangulations, *type II* triangulations may have multiple edges but no loops, finally *type III* triangulations have neither loops nor multiple edges. In fact, a triangulation is of type II if and only if it is 2-connected, and of type III if and only if it is 3-connected. (A map or a graph is said *k-connected* if it remains connected whenever one removes at most $(k-1)$ of its vertices.)

In this exercise we concentrate on the type II case. We denote by $T_{n,k}$ the number of triangulations of type II with n *inner* vertices (vertices not on the boundary) and a boundary of length $m+2$. We may see that the number of triangulations of type II without boundary is $T_{n-2,0} = T_{n-3,1}$. Show that the bivariate generating function satisfies the functional equation

$$T(t,u) = 1 + t\frac{T(t,u) - T(t,0)}{u} + uT(t,u)^2. \tag{11}$$

and solve it! (i.e. determine at least $T(t,0)$ and possible even $T(t,u)$ – there is still a one-cut form, and we may obtain an explicit formula for $T_{n,k}$).

If you want more, redo the same exercise for types I and III. How are these different series related?

You may also try counting quadrangulations with a simple boundary (in which case one gets a *cubic* equation, which may still be solved by the same method). How is the series related to the one considered in the first exercise? (Buzzword: free cumulants.)

References

[1] Jérémie Bettinelli, Éric Fusy, Cécile Mailler, and Lucas Randazzo. A bijective study of basketball walks. *Sém. Lothar. Combin.*, 77:Art. B77a, 24, [2016-2018].