Around the Plancherel measure on integer partitions

(an introduction to Schur processes without Schur functions)

Jérémie Bouttier

A subject which I learned with Dan Betea, Cédric Boutillier, Guillaume Chapuy, Sylvie Corteel, Sanjay Ramassamy and Mirjana Vuletić

Institut de Physique Théorique, CEA Saclay
Laboratoire de Physique, ENS de Lyon

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Part I

20 March 2019

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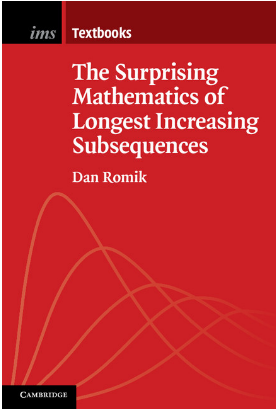

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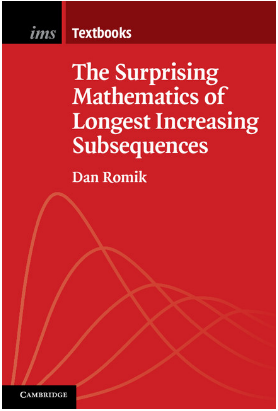

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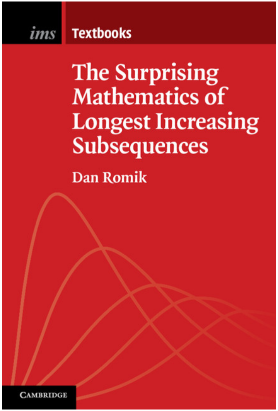

This is the material I would like to present here: fermions because of physics, saddle point computations because, well, we are in Aléa!

Integer partitions and Young diagrams/tableaux

An (integer) partition λ is a finite nonincreasing sequence of positive integers called parts:

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell > 0.$$

Its size is $|\lambda| := \sum \lambda_i$ and its length is $\ell(\lambda) := \ell$ (by convention $\lambda_n = 0$ for $n > \ell$).

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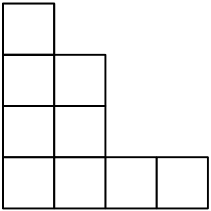

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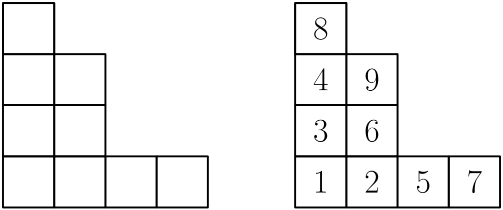

A standard Young tableau (SYT) of shape λ is a filling of the Young diagram of λ by the integers $1, \ldots, |\lambda|$ that is increasing along rows and columns. We denote by d_λ the number of SYTs of shape λ.

Plancherel measure

The Plancherel measure on partitions of size n is the probability measure such that

$$\mathrm{Prob}(\lambda) = \begin{cases} \frac{d_\lambda^2}{n!} & \text{if } \lambda \vdash n, \\ 0 & \text{otherwise.} \end{cases}$$

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which has (at least) two classical proofs:

- representation theory: $n!$ is the dimension of the regular representation of the symmetric group S_n, and d_λ is the dimension of its irreducible representation indexed by λ,
- bijection: the Robinson-Schensted correspondence is a bijection between S_n and the set of triples (λ, P, Q), where $\lambda \vdash n$ and P, Q are two SYTs of shape λ.

Connection with Longest Increasing Subsequences

A property of the Robinson-Schensted correspondence is that if $\sigma \mapsto (\lambda, P, Q)$, then the first part of λ satisfies

$$\lambda_1 = L(\sigma)$$

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The Longest Increasing Subsequence problem consists in understanding the asymptotic behaviour as $n \to \infty$ of $L_n := L(\sigma_n) = \lambda_1^{(n)}$, where σ_n denotes a uniform random permutation in S_n, and $\lambda^{(n)}$ the random partition to which it maps via the RS correspondence, and whose law is the Plancherel measure.

Some partial history of the LIS problem

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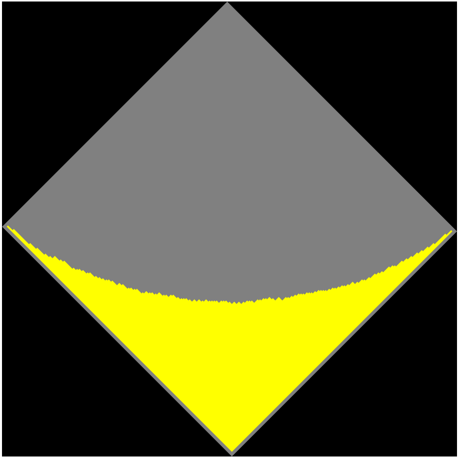
Limit shape



A Plancherel random partition of size 10000 (courtesy of D. Betea)

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- Baik-Deift-Johansson (1999) proved the most precise result

$$\mathbb{P}\left(\frac{L_n - 2\sqrt{n}}{n^{1/6}} \leq s\right) = F_{GUE}(s), \qquad n \to \infty$$

where F_{GUE} is the Tracy-Widom GUE distribution. (See Chapter 2.) The unusual exponent $n^{1/6}$ was previously conjectured by Odlyzko-Rains and Kim based on numerical evidence and bounds.

Topics of the lectures

We will discuss some properties of the Plancherel measure.

1. We will show that the poissonized Plancherel measure (to be defined) is closely related with a determinantal point process (DPP) called the discrete Bessel process. Plan:
   - Some general theory of DPPs
   - Connection with Plancherel measure via fermions
2. We will then investigate asymptotics, in the following regimes:
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These results were obtained indepently in two papers by Borodin, Okounkov and Olshanski (2000) and by Johansson (2001). But we use a different approach developed later by Okounkov *et al.*, which may be generalized to Schur measures and Schur processes. We concentrate on the Plancherel measure for simplicity.

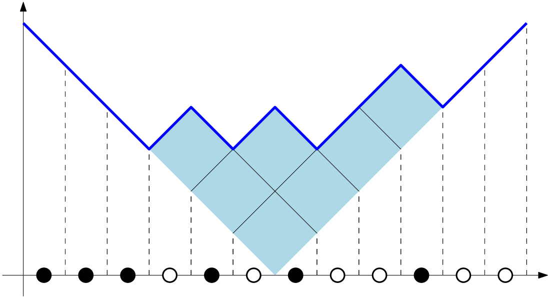
Poissonized Plancherel measure

The poissonized Plancherel measure of parameter θ is the measure

$$\mathrm{Prob}(\lambda) = \frac{d_\lambda^2}{(|\lambda|!)^2} \theta^{|\lambda|} e^{-\theta}.$$

It is a mixture of the Plancherel measures of fixed size, where the size is a Poisson random variable of parameter θ.
We denote by $\lambda^{\langle\theta\rangle}$ a random partition distributed according to the poissonized Plancherel measure, $\lambda^{(n)}$ denoting a Plancherel random partition of size n.

Partitions and particle configurations



To a partition λ, here $(4, 2, 1)$, we associate a set $S(\lambda) \subset \mathbb{Z}' := Z + \frac{1}{2}$ by

$$S(\lambda) = \{\lambda_1 - \frac{1}{2}, \lambda_2 - \frac{3}{2}, \lambda_3 - \frac{5}{2}, \ldots\}$$

Here $S(\lambda) = \{\frac{7}{2}, \frac{1}{2}, \frac{-3}{2}, \frac{-7}{2}, \frac{-9}{2}, \ldots\}$. Elements of $S(\lambda)$ ("particles" $\bullet$) correspond to the down-steps of the blue curve.

Main result of today

Theorem [Borodin-Okounkov-Olshanski 2000, Johansson 2001]

The particle configuration $S(\lambda^{\langle\theta\rangle})$ associated with the poissonized Plancherel measure is a determinantal point process in the sense that, for any distinct points $\{u_1, \ldots, u_n\} \subset \mathbb{Z}'$, we have

$$\mathbb{P}\left(\{u_1, \ldots, u_n\} \subset S(\lambda^{\langle\theta\rangle})\right) = \det_{1 \le i,j \le n} \mathbf{J}_\theta(u_i, u_j).$$

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The correlation kernel $\mathbf{J}_\theta$ is the discrete Bessel kernel

$$\mathbf{J}_\theta(s,t) = \sum_{\ell \in \mathbb{Z}'_{>0}} J_{s+\ell}(2\sqrt{\theta}) J_{t+\ell}(2\sqrt{\theta}), \qquad s, t \in \mathbb{Z}'$$

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By the general theory of DPPs, knowing $\mathbf{J}_\theta$ gives all the information on the point process.

Tomorrow

Asymptotics of $\mathbf{J}_\theta$, using saddle point computations. Again this is different from the original techniques of BOO/J, our approach follows Okounkov and Reshetikhin and are robust ("universality").

Part II

21 March 2019

A refresher on fermions

A fermionic configuration is a subset $S \subset \mathbb{Z}' := Z + \frac{1}{2}$ that contains finitely many positive elements, and whose complement contains finitely many negative elements. We denote by $\mathcal{S}$ the (countable) set of fermionic configurations.

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Partitions are embedded into fermionic configurations by the mapping

$$\lambda \mapsto S(\lambda) := \{\lambda_1 - \frac{1}{2}, \lambda_2 - \frac{3}{2}, \lambda_3 - \frac{5}{2}, \dots\}$$

It is not a bijection but the mapping $(\lambda, c) \mapsto S(\lambda) + c$, with $c \in \mathbb{Z}$, is.

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The fermionic Fock space $\mathcal{F}$ consists of columns vectors indexed by $\mathcal{S}$. The standard basis is denoted by $(v_S)_{S \in \mathcal{S}}$ and the dual basis (of row vectors) by $(v_S^*)_{S \in \mathcal{S}}$. Operators on $\mathcal{F}$ are naively viewed as matrices with rows and columns indexed by $\mathcal{S}$.

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We use the shorthand notation $v_\lambda := v_{S(\lambda)}$ for partitions and $v_\emptyset := v_{\mathbb{Z}'_{<0}}$ corresponds to the (nonzero!) vacuum vector.

A refresher on fermions

We defined the fermionic creation/annihilation operators through their action on the standard basis:

$$\psi_k v_S := \begin{cases} 0 & \text{if } k \in S, \\ (-1)^{\#(S \cap \mathbb{Z}'_{>k})} v_{S \cup \{k\}} & \text{if } k \notin S, \end{cases}$$

$$\psi_k^* v_S := \begin{cases} 0 & \text{if } k \notin S, \\ (-1)^{\#(S \cap \mathbb{Z}'_{>k})} v_{S \setminus \{k\}} & \text{if } k \in S. \end{cases}$$

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These operators satisfy the canonical anticommutation relations (CAR)

$$\begin{aligned} \psi_k \psi_\ell + \psi_\ell \psi_k &= 0 \\ \psi_k^* \psi_\ell^* + \psi_\ell^* \psi_k^* &= 0 \\ \psi_k \psi_\ell^* + \psi_\ell^* \psi_k &= \delta_{k,\ell}. \end{aligned}$$

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The diagonal operator $N_k := \psi_k \psi_k^*$ "measures" whether there is a particle at position k.

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We defined the "box" creation/annihilation operators by

$$\alpha^* := \sum_{k\in\mathbb{Z}'} \psi_k \psi^*_{k+1}, \qquad \alpha := \sum_{k\in\mathbb{Z}'} \psi_{k+1}\psi^*_k.$$

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In terms of partitions their action read

$$v^*_\lambda \alpha^* = \sum_{\mu:\lambda\nearrow\mu} v^*_\mu, \qquad \alpha v_\lambda = \sum_{\mu:\lambda\nearrow\mu} v_\mu$$

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By iterating we get

$$v^*_\emptyset (\alpha^*)^n = \sum_{\lambda\vdash n} d_\lambda v^*_\lambda, \qquad \alpha^n v_\emptyset = \sum_{\lambda\vdash n} d_\lambda v_\lambda$$

or, equivalently,

$$v^*_\emptyset e^{x\alpha^*} = \sum_\lambda \frac{d_\lambda x^{|\lambda|}}{|\lambda|!} v^*_\lambda, \qquad e^{x\alpha} v_\emptyset = \sum_\lambda \frac{d_\lambda x^{|\lambda|}}{|\lambda|!} v_\lambda.$$

A refresher on fermions

Final result of yesterday

The correlation function $\rho(U)$ of the poissonized Plancherel measure admit the fermionic expression (with $\theta = x^2$)

$$\rho(U) := \mathbb{P}\left(\{u_1, \ldots, u_n\} \subset S(\lambda^{\langle\theta\rangle})\right) = \frac{v_\emptyset^* e^{x\alpha^*} N_{u_1} \cdots N_{u_n} e^{x\alpha} v_\emptyset}{e^{x^2}}$$

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- apply Wick's lemma to get

$$\rho(U) = \det_{1 \le i,j \le n} v_\emptyset^* \widehat{\psi}_{u_i} \widehat{\psi}^*_{u_j} v_\emptyset = \det_{1 \le i,j \le n} \mathbf{J}_\theta(u_i, u_j).$$

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and combining everything

$$\rho(U) = \frac{v_\emptyset^* e^{x\alpha^*} \psi_{u_1} \psi_{u_1}^* \cdots \psi_{u_n} \psi_{u_n}^* e^{x\alpha} v_\emptyset}{e^{x^2}} = v_\emptyset^* \widehat{\psi}_{u_1} \widehat{\psi}_{u_1}^* \cdots \widehat{\psi}_{u_n} \widehat{\psi}_{u_n}^* v_\emptyset.$$

Wick's lemma (fermionic version)

Let $\langle \mathcal{O} \rangle := v_\emptyset^* \mathcal{O} v_\emptyset$ denote the vacuum expectation value of an operator $\mathcal{O}$.

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Let $\varphi_1, \varphi_3, \ldots, \varphi_{2n-1}$ denote linear combinations of the ψ_k's and $\varphi_2^*, \varphi_4^*, \ldots, \varphi_{2n}^*$ denote linear combinations of the ψ_k^*'s. Then we have

$$\langle \varphi_1 \varphi_2^* \varphi_3 \varphi_4^* \cdots \varphi_{2n-1} \varphi_{2n}^* \rangle = \det_{1 \leq i,j \leq n} C_{i,j}$$

where $C_{i,j} = \begin{cases} \langle \varphi_{2i-1} \varphi_{2j}^* \rangle & \text{if } i \leq j \\ -\langle \varphi_{2j}^* \varphi_{2i-1} \rangle & \text{if } i > j \end{cases}$ ("time-ordered correlator").

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Example

For $n = 2$ we have

$$\langle \varphi_1 \varphi_2^* \varphi_3 \varphi_4^* \rangle = \begin{vmatrix} \langle \varphi_1 \varphi_2^* \rangle & \langle \varphi_1 \varphi_4^* \rangle \\ -\langle \varphi_2^* \varphi_3 \rangle & \langle \varphi_3 \varphi_4^* \rangle \end{vmatrix} = \langle \varphi_1 \varphi_2^* \rangle \cdot \langle \varphi_3 \varphi_4^* \rangle + \langle \varphi_1 \varphi_4^* \rangle \cdot \langle \varphi_2^* \varphi_3 \rangle.$$

Applying Wick's lemma

We deduce that

$$\rho(U) = \langle \widehat{\psi}_{u_1} \widehat{\psi}^*_{u_1} \cdots \widehat{\psi}_{u_n} \widehat{\psi}^*_{u_n} \rangle = \det_{1 \le i,j \le n} \langle \widehat{\psi}_{u_i} \widehat{\psi}^*_{u_j} \rangle$$

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The final step is to observe that

$$\langle \psi_k \psi^*_\ell \rangle = \delta_{k,\ell} \mathbb{1}_{k<0}$$

and therefore

$$\langle \widehat{\psi}_s \widehat{\psi}^*_t \rangle = \sum_{u \in \mathbb{Z}'_{>0}} J_{s+u}(2x) J_{t+u}(2x) =: \mathbf{J}_\theta(s,t).$$

Main result of "yesterday"

Theorem [Borodin-Okounkov-Olshanski 2000, Johansson 2001]

The particle configuration $S(\lambda^{\langle\theta\rangle})$ associated with the poissonized Plancherel measure is a determinantal point process in the sense that, for any distinct points $\{u_1, \ldots, u_n\} \subset \mathbb{Z}'$, we have

$$\mathbb{P}\left(\{u_1, \ldots, u_n\} \subset S(\lambda^{\langle\theta\rangle})\right) = \det_{1 \le i,j \le n} \mathbf{J}_\theta(u_i, u_j).$$

The correlation kernel $\mathbf{J}_\theta$ is the discrete Bessel kernel

$$\mathbf{J}_\theta(s,t) = \sum_{\ell \in \mathbb{Z}'_{>0}} J_{s+\ell}(2\sqrt{\theta}) J_{t+\ell}(2\sqrt{\theta}), \qquad s,t \in \mathbb{Z}'$$

where J_n is the Bessel function of order n.

"Today": asymptotics

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$$J_n(2x) = \frac{1}{2i\pi} \oint_{|z|=r} e^{x(z-z^{-1})} \frac{dz}{z^{n+1}}$$

$$\mathbf{J}_\theta(s,t) = \frac{1}{(2i\pi)^2} \oint\!\!\oint_{|z|>|w|>0} \frac{e^{x(z-z^{-1})}}{e^{x(w-w^{-1})}} \cdot \frac{dz \cdot dw}{(z-w) z^{s+\frac{1}{2}} w^{-t+\frac{1}{2}}}.$$

Bulk limit: discrete sine kernel

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Theorem 1 [BOO/J]

Fix $A \in \mathbb{R}$ and consider the asymptotic regime $\theta = x^2 \to \infty$ with $s, t \sim Ax$ and $s - t$ fixed. Then we have

$$\mathbf{J}_\theta(s,t) \to \mathbf{K}_{\sin}(s-t;\chi) := \begin{cases} \frac{\chi}{\pi} & \text{if } s = t, \\ \frac{\sin \chi(s-t)}{\pi(s-t)} & \text{if } s \neq t, \end{cases}$$

where

$$\chi := \begin{cases} \arccos(A/2) & \text{if } |A| \leq 2, \\ 0 & \text{if } A > 2, \\ \pi & \text{if } A < -2. \end{cases}$$

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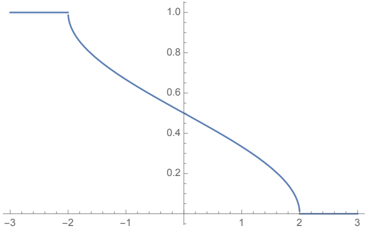
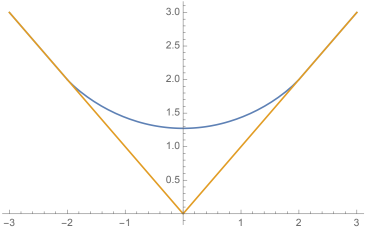
We deduce immediately that, if $u_1, \ldots, u_n$ are such that $u_i \sim Ax$ and $u_i - u_j$ remains fixed for all i, j, then

$$\mathbb{P}\left(\{u_1, \ldots, u_n\} \subset S(\lambda^{\langle\theta\rangle})\right) \to \det_{1 \leq i,j \leq n} \mathbf{K}_{\sin}(u_i - u_j; \chi).$$

Connection with Vershik-Kerov-Logan-Shepp

In particular, for $s = t$ we obtain the one-point function (particle density).





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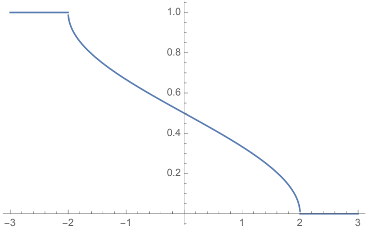

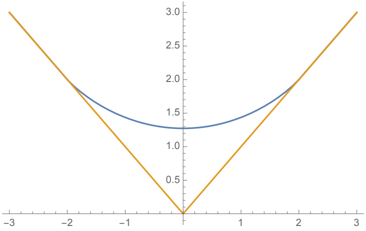

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We do not quite recover their theorem: here we do a first moment calculation, we should also do second moment to prove concentration, and depoissonize.

Edge limit ($A = 2$): Airy kernel

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Theorem 2 [BOO/J]

Fix $\sigma, \tau \in R$ and consider the asymptotic regime $\theta = x^2 \to \infty$ with

$$s = 2x + \sigma x^{1/3} + o(x^{1/3}), \qquad t = 2x + \tau x^{1/3} + o(x^{1/3}).$$

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where Ai is the Airy function given by $\mathrm{Ai}(y) = \frac{1}{2i\pi} \int_{\Re(\zeta)=1} e^{\frac{\zeta^3}{3} - y\zeta} d\zeta$.

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The Baik-Deift-Johansson theorem follows by a depoissonization argument!