MSRI Summer school — Tropical geometry
August 2022

Exercises

1 Tropical algebra and geometry

Exercise 1
Prove the beginner's dream formula $(x \oplus y)^n = x^n \oplus y^n$.

Exercise 2
1. Let $P = \sum_0^d a_i x^i$ be a tropical polynomial of degree d. Prove that the graph has at most d corners.

2. What conditions on the coefficients ai must hold in order for the graph of P to have exactly d corners ? In this case, call $x_1, \ldots, x_d$ the x-coordinate of the corners.

3. Prove that P factors in linear factors : $P(x) = \prod(x \oplus x_i)$.

4. How to make the statement true without the assumption ? (*i.e.* what should be the multiplicity of a root, and what is the multiplicity of the root $-\infty$?)

Exercise 3
In analysis we define the exponential function as :

$$e^x = \sum_0^\infty \frac{x^n}{n!}.$$

Let us define the tropical exponential function by replacing all the operations by their tropical counterparts (be careful : what do division, and factorials correspond to tropically ?).

1. Describe the graph of the tropical exponential and its roots.

2. What about the formulas $e^{x+y} = e^x e^y$? And $e^x e^{-x} = 1$?

3. (after the valued fields and Kapranov's theorem) How is this compatible with the fact that the exponential has no zeros ?

Exercise 4
1. Is the graph of a tropical polynomial a tropical curve ?

2. What should we do to make it into one ?

3. Show that as in the classical case, where the graph of a polynomial P is defined by the equation $y - P(x) = 0$, the tropical graph is defined by the tropical polynomial $y \oplus P(x)$.

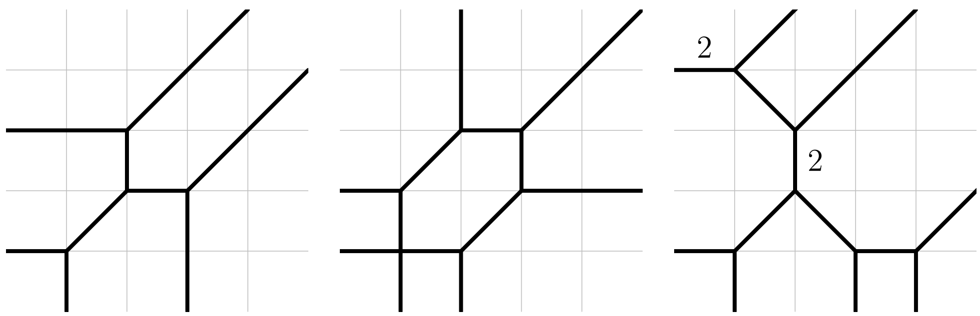
Exercise 5

Draw the tropical curves associated to the following polynomials and the dual subdivisions of their Newton polygon.

1. $f(x, y) = \max(0, x, y, x + y - 1)$
2. $f(x, y) = \max(0, -x, -y, x - 1, y - 2)$
3. $f(x, y) = \max(0, x, y, 2x - 3, 2y - 4, x + y - 1)$
4. $f(x, y) = \max(0, x, x + y - 1, x + 2y, 2x + y - 2)$
5. Draw a tropical curve of degree 3.
6. Draw a tropical curve of degree 3 with an edge of weight 2.

Exercise 6

Give an equation and the dual subdivision to the following tropical curves. (the bottom left corner is 0.) Is the equation unique ? Up to which choice ?



(a) (b) (c)

Exercise 7

1. If Γ is a degree d tropical curve, show that the sum of the areas of the polygons in the dual subdivision is equal to $\frac{d^2}{2}$.
2. Show that a degree d tropical curve has at most d^2 vertices.

Exercise 8

Going in higher dimension.

1. What does a tropical plane look like ? (Defining a tropical plane as the corner locus of a function $\max(a, x + b, y + c, z + d)$ in $\mathbb{R}^3$.)
2. What should be the definition of a tropical curve in dimension 3 ? Try to draw a tropical line in $\mathbb{R}^3$.

2 Geometry over a valued field

Exercise 9

1. Which of these expressions is a Puiseaux series ?
   - $c(t) = \frac{1}{t} + t$

- $c(t) = t^{1/2} + t^{1/3}$
- $c(t) = t^{\sqrt{2}}$
- $c(t) = \sum_0^\infty t^i$
- $c(t) = \sum_0^\infty t^{-i}$
- $c(t) = \sum_{-5}^\infty t^i$
- $c(t) = \sum_1^\infty t^{1/i}$

2. Let K be a field endowed with a valuation.
a) What is $\mathrm{val}(1)$?
b) If $x \neq 0$, how does $\mathrm{val}(1/x)$ compare to $\mathrm{val}(x)$?
c) What can you say about $\mathrm{val}(-x)$?

3. When they are, write down the valuations for all the Puiseaux series from first question.

4. Write an example of two Puiseaux series $c_1(t)$ and $c_2(t)$ such that

$$\mathrm{val}(c_1(t) + c_2(t)) > \min(\mathrm{val}(c_1(t)), \mathrm{val}(c_2(t))).$$

Exercise 10

Let D be the line of equation $z = w + 1$ in K^2. Check Kapranov's theorem for D : the image of D by $\mathrm{val} : K^2 \to (\mathbb{R} \cup \{-\infty\})^2$ is the tropical line of equation $\max(0, x, y)$.

Exercise 11

1. Let u and v be two vectors inside $\mathbb{Z}^2$. Prove that the index of the lattice that they span is equal to their determinant $|\det(u, v)|$.

2. If $u_1, \dots, u_n$ are n independent vectors inside $\mathbb{Z}^n$, show that the index of the lattice that they span is equal to $|\det(u_1, \dots, u_n)|$.

3. ( ) If $u_1, \dots, u_r$ are r independent vectors inside $\mathbb{Z}^n$. Let V be their $\mathbb{R}$-span, and let $L = V \cap \mathbb{Z}^n$. Show that the index of the lattice spanned by $u_1, \dots, u_r$ inside L is equal to the integral length of the wedge product $u_1 \wedge \dots \wedge u_n \in \Lambda^r \mathbb{Z}^n$.

Exercise 12 : around tropical Bezout's theorem

We say that two tropical curves are transverse if they have a finite number of intersection points and those intersection points are disjoint from vertices of the curves. If Γ_1 and Γ_2 are two transverse tropical curves, and $p \in \Gamma_1 \cap \Gamma_2$, let $(\Gamma_1 \cdot \Gamma_2)_p = |\det(u_1, u_2)|$, where u_1 and u_2 are the vectors dual to the edges of Γ_1 and Γ_2 that contain p. We define the intersection number of Γ_1 and Γ_2 to be

$$\Gamma_1 \cdot \Gamma_2 = \sum_{p \in \Gamma_1 \cap \Gamma_2} (\Gamma_1 \cdot \Gamma_2)_p.$$

1. Show that $\Gamma_1 \cup \Gamma_2$ is a tropical curve. Assuming Γ_1 and Γ_2 are of respective degrees d_1 and d_2, what is its degree ?

2. Assuming Γ_1 and Γ_2 are of respective degrees d_1 and d_2, prove that $\Gamma_1 \cdot \Gamma_2 = d_1 d_2$. (*Ind. :* compute the area of the Newton polygon in different ways)

Exercise 13 ☂☂ : tropical Bezout's theorem in $(\mathbb{P}^1)^2$
We say that a tropical curve is of bidegree (a, b) if it has a ends going in each of the directions $(0, 1)$ and $(0, -1)$, and b ends in each of the directions $(1, 0)$ and $(-1, 0)$.

1. What is the Newton polygon of such a tropical curve ?

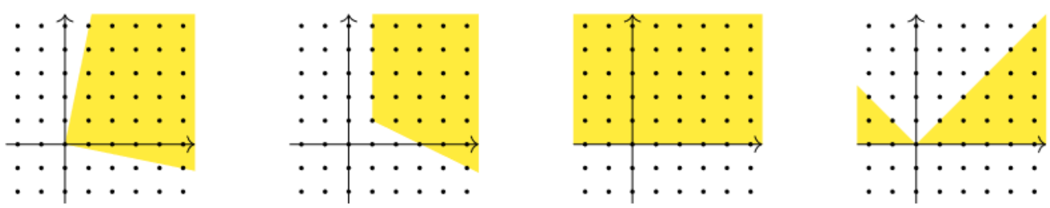
2. State and prove an analog to Bezout's theorem for these curves : "two curves of bidegrees (a_1, b_1) and (a_2, b_2) have $\cdot$ intersection points."

Exercise 14 ☂☂☂ : tropical Bernstein-Kouchnirenko's theorem
Let Γ_1 and Γ_2 be two tropical curves with respective Newton polygons Δ_1 and Δ_2. Show that if the curves are transverse, they have $\mathcal{A}(\Delta_1 + \Delta_2) - \mathcal{A}(\Delta_1) - \mathcal{A}(\Delta_2)$ intersection points.

3 Cone and fans

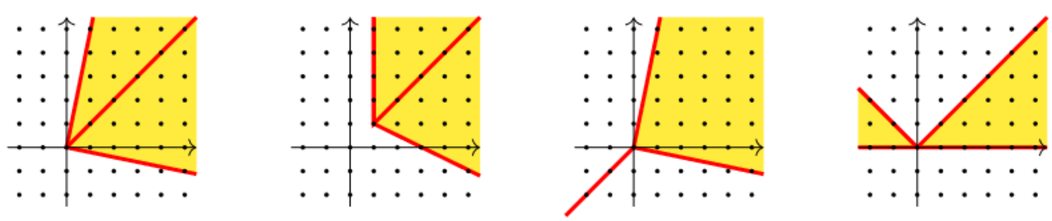
Exercise 15 ☂

1. Which of the following pictures represent a rational polyhedral cone ?



2. We did not give precise definitions of the notions of dimension of a cone, and face of a cone. Given the intuitive discussions we have had about them, try and formulate precise definitions for these concepts.

3. Which of the following pictures represent a rational polyhedral fan ? Which ones are pure dimensional ?



4. Given a rational polyhedral fan Σ, we define the support of Σ, denoted $|\Sigma|$, to be the set of points in $\mathbb{R}^n$ that belong to some cone of Σ. Decide which of the following statements are true :
a) The support of Σ is a linear subspace of $\mathbb{R}^n$.
b) The support of Σ is a convex subset of $\mathbb{R}^n$.
c) If $x \in |\Sigma|$ and λ is a non-negative number, then $\lambda x \in |\Sigma|$.
d) If $|\Sigma_1| = |\Sigma_2|$, then $\Sigma_1 = \Sigma_2$.

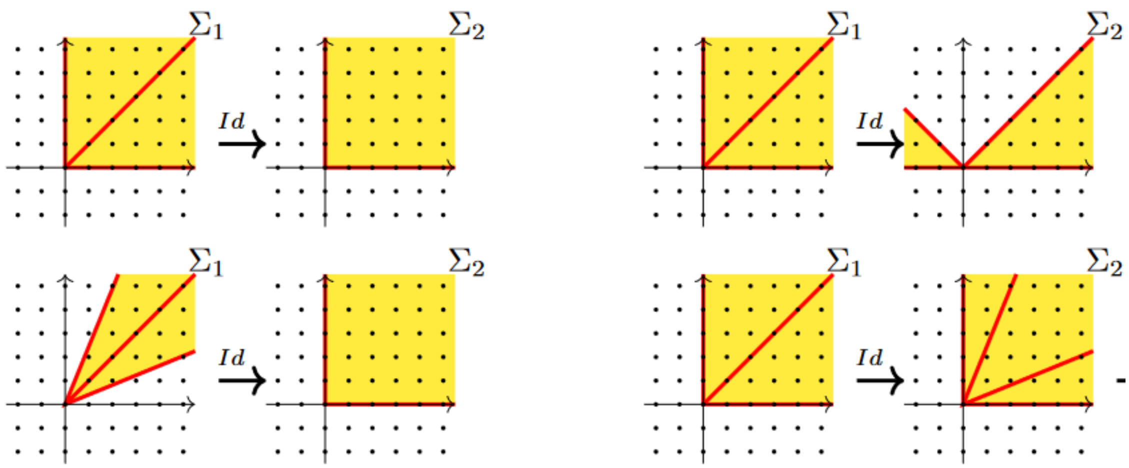
Exercise 16

1. What does the balancing condition state if Σ is a one-dimensional fan? What are the normal vectors to a codimension one face of Σ?

2. Let $v_\tau = (1,0,0), v_1 = (1,1,0), v_2 = (2,2,2)$ and $v_3 = (1,-3,-2)$. Consider the two dimensional marked fan $\Sigma \subset \mathbb{R}^3$ where rays are directed by v_1, v_2, v_3 and v_τ, and we have the 2-dimensional cones $\sigma_1 = \langle v_\tau, v_1 \rangle$, $\sigma_2 = \langle v_\tau, v_2 \rangle$ and $\sigma_3 = \langle v_\tau, v_3 \rangle$.
a) Compute the weights $\omega_\Sigma(\sigma_i)$ for the three top dimensional cones
b) Compute the normal vectors and check that Σ is balanced at the face τ.

Exercise 17

1. For each of the pictures below, consider the identity function $\mathbb{R}^2 \to \mathbb{R}^2$. Decide if it induces a map of fans. If it does, show how the cones of Σ_1 and Σ_2 should be subdivided in order for the map of fans to send cones to cones.

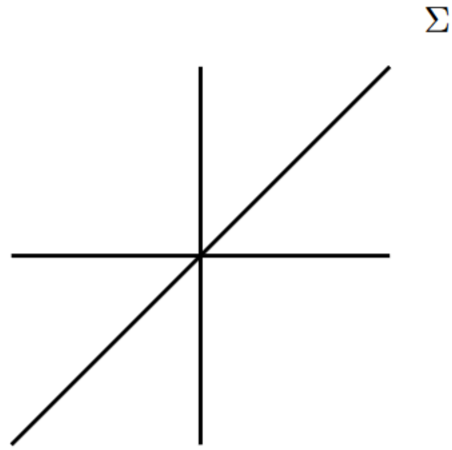


2. If Σ is a pure dimensional fan of dimension k, what is the dimension of $f_*(\Sigma)$?

3. Consider the fan $\Sigma \subset \mathbb{R}^2$ consisting of four rays generated by $\pm e_1, \pm e_2$.
a) What are the conditions on the weights on the four rays for Σ to be a balanced fan?
b) Now consider the map $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x,y) = 2x + 3y$. Describe the fan $f_*(\Sigma)$ and check it is balanced.

Exercise 18

Show that the fan in the picture below is not irreducible, and further that it may be decomposed as a sum of irreducible balanced fans in more than one way.



Exercise 19

Suppose Σ_1, Σ_2 are two balanced fans of the same dimension, Σ_1 is irreducible, and $|\Sigma_2| \subset |\Sigma_1|$. Then, up to subdivision, there exists a positive rational number λ such that

$$\Sigma_1 = \lambda \Sigma_2.$$

(*Ind. :* Weights on Σ_2 extended by 0 also make Σ_1 balanced)

Exercise 20

Let $\Sigma \subset \mathbb{R}^3$ be a pure two dimensional fan whose maximal cones consist of the twelve coordinate orthants of $\mathbb{R}^3$.

1. Choose a marking on this fan that makes it into a balanced fan.

2. What are the weights of the maximal cones with your choice of marking ?

3. Consider the linear function $f : \mathbb{R}^3 \to \mathbb{R}^2$ given by $f(x, y, z) = (x + y, 2y + 3z)$. Describe the fan $f_*(\Sigma)$.

4 Abstract tropical curves and their moduli

Exercise 21

1. What are the topological types of abstract, rational, stable, n-pointed tropical curves, for $n = 3, 4, 5$?

2. What are the minimum and maximum number of compact edges that an abstract rational stable n-pointed tropical curve can have ?

3. ( ) We say that a tropical curve is of genus g if it is connected and the first Betti number of the graph as topological space is g. What are the minimum and maximum number of compact edges that an abstract genus g stable n-pointed tropical curve can have ?

Exercise 22

1. Describe the space $M_{0,3}^{\mathrm{trop}}$.

2. Understand that the space $M_{0,5}^{\mathrm{trop}}$ is two dimensional, and it is (combinatorially) represented as the cone over the Petersen graph.

3. What is the dimension of $M_{0,n}^{\mathrm{trop}}$? Give a combinatorial description of the tropical curves parameterized by the top dimensional cones, and by the codimension one cones.

Exercise 23

1. Understand the forgetful morphism $\pi_5 : M_{0,5}^{\mathrm{trop}} \to M_{0,4}^{\mathrm{trop}}$.

2. Show that the tropical forgetful morphisms map cones to cones. Characterize which cones it is bijective on, and which cones it con- tracts to lower dimensional cones.

Exercise 24

1. a) How many top-dimensional cones are there in $M_{0,n}^{\mathrm{trop}}$ for $n = 3, 4, 5$?

b) What about $M_{0,6}^{\mathrm{trop}}$?

c) ( ) More generally, how many top-dimensional cones are there in $M_{0,n}^{\mathrm{trop}}$? (Proceed by induction using the forgetful morphism)

2. a) How many codimension 1 cones are there in $M_{0,n}^{\mathrm{trop}}$ for $n = 3, 4, 5$?

b) What about $M_{0,6}^{\mathrm{trop}}$?

c) ( ) More generally, how many codimension 1 cones are there in $M_{0,n}^{\mathrm{trop}}$? (Use the fact that a codimension 1 cone is adjacent to three faces)

Exercise 25

Recall the map $\Phi : (a_i) \in \mathbb{R}^n \longmapsto (a_i + a_j) \in \mathbb{R}^{\binom{n}{2}}$ and Q the quotient $\mathbb{R}^{\binom{n}{2}}/\mathrm{Im}\Phi$. We have a map

$$\mathrm{dist} : M_{0,n}^{\mathrm{trop}} \to Q,$$

that associates to a tropical curve the family of distances between its ends.

1. a) Study the function $\mathrm{dist} : M_{0,4}^{\mathrm{trop}} \to \mathbb{R}^6$.

b) Show that it is injective, and linear on each cone of $M_{0,4}^{\mathrm{trop}}$.

c) Show the image cannot be made into a balanced fan in $\mathbb{R}^6$.

d) Show that $v_{12} + v_{13} + v_{14}$ lies in the image of Φ and conclude that $M_{0,4}^{\mathrm{trop}}$ can be made into a balanced fan in Q.

2. Let Γ be an abstract, rational, tropical, n-pointed curve. Each edge e of Γ defines a two-part partition $I_e \sqcup I_e^c$ of the set of indices by considering the indices that lie on either side of the edge. If we denote by $l(e)$ the length of the edge e, show that we have :

$$\mathrm{dist}(\Gamma) = \sum_e l(e) v_{I_e} \in Q.$$

3. Given a topological type T, consider the cone $C_T \simeq R^m_{\geqslant 0}$ and denote by e_i the standard basis vector corresponding to the i-th edge e_i. Show that

$$\mathrm{dist}(e_i) = v_{I_{e_i}}.$$

Exercise 26

1. Show that for $n \geqslant 4$, $\Phi(1, n-3, \dots, n-3) = \sum_{|I|=2, 1\notin I} v_I$.

2. Prove that $\mathrm{dist} : M_{0,n}^{\mathrm{trop}} \to Q$ is injective.

3. Show that the forgetful map $\mathrm{ft}_{n+1} : M_{0,n+1}^{\mathrm{trop}} \to M_{0,n}^{\mathrm{trop}}$ is universal in the sense that $\mathrm{ft}_{n+1}^{-1}(\Gamma) \simeq \Gamma$.

Exercise 27
Prove that $M_{0,n}^{\mathrm{trop}}$ is connected through codimension 1.

Exercise 28
1. Given 4 distinct points p_1, p_2, p_3 and p_4 on $\mathbb{P}^1$, prove we can choose a coordinate on $\mathbb{P}^1$ (making it into $K \cup \{\infty\}$) such that $p_1 = 0$, $p_1 = 1$, $p_3 = \infty$.
The coordinate λ of the remaining p_4 is called the *cross-ratio* of the numbers. It is invariant by an automorphism of $\mathbb{P}^1$ (a map of the form $[z : w] \mapsto [az+bw : cz+dw]$, where $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in GL_2(K)$.
2. Taking a coordinate on $\mathbb{P}^1$, map $(\mathbb{P}^1_K, p_1, p_2, p_3, p_4)$ to $(K^*)^2$ by the following map :

$$t \in \mathbb{P}^1 \backslash \{p_1, p_2, p_3, p_4\} \longmapsto \left(\frac{t-p_1}{t-p_3}, \frac{t-p_2}{t-p_4} \right).$$

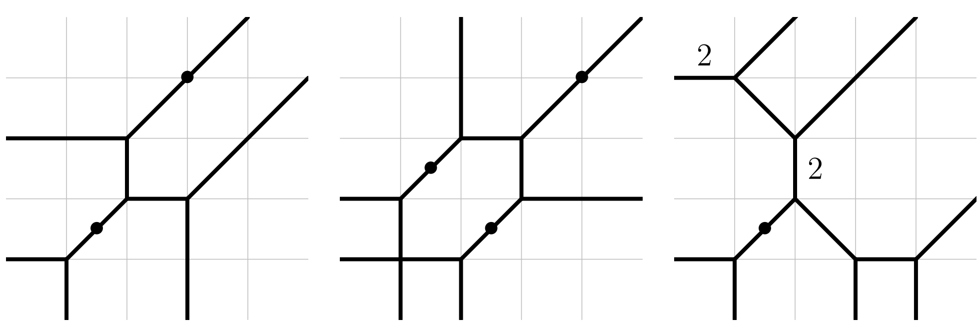
a) Check that this map is well-defined does not depend on the choice of a coordinate up to multiplication by an element in $(K^*)^2$. (One can use the fact that given two distinct points, there is a unique rational function on $\mathbb{P}^1$ having zero and pole at these points up to multiplication)
b) The image is a curve of bidegree $(1,1)$.

3. Assuming that K is a valued field, the curve has a tropicalization, which is its image by the valuation. What can it look like ?

4. If the tropical curve inside $\mathbb{R}^2$ has a bounded edge, prove that the valuation of the cross-ratio of the points is equal to the length of this unique bounded edge of the curve.

5 Moduli spaces of tropical stable maps

Exercise 29
What is the degree and to which moduli space do these tropical curves belong to ?



(a) (b) (c)

Exercise 30

We consider degree Δ curves with n marked points, and the i-th evaluation map

$$\mathrm{ev}_i : M_{0,n}^{\mathrm{trop}}(\mathbb{R}^2, \Delta) \to \mathbb{R}^2.$$

1. Given a combinatorial type, write down an expression for ev_i in a suitable basis of the orthant.

2. We now choose as a basis the position of P_1 and the natural coordinates (z_{ij}) inside $\mathbb{R}^{\binom{n}{2}}$. Show that $\mathrm{ev}_1((z_{ij}), x, y) = (x, y)$.

3. Using the same coordinates, show that for any $i \neq 1$,

$$\mathrm{ev}_i((z_{ij}), x, y) = (x, y) + \frac{1}{2} \sum_{k \neq 1, i} (z_{1k} - z_{ik}) v_k,$$

where k is any of the $|\Delta| + n$ marked points and v_k is the slope for the k-th marked point.

Exercise 31

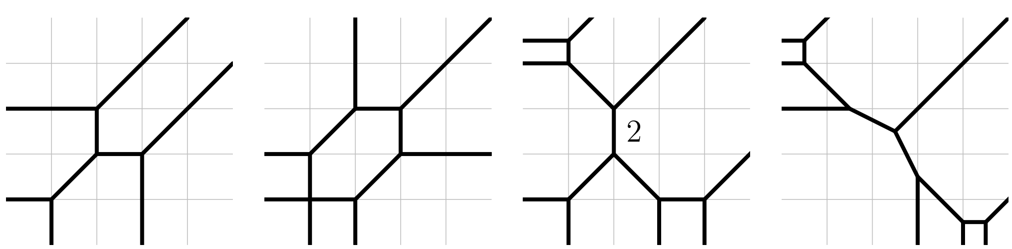
Let Δ be a collection of primitive vectors inside $\mathbb{Z}^2$, and let $M_{0,n}^{\mathrm{trop}}(\mathbb{R}^2, \Delta)$ be the moduli space of rational tropical curves of degree Δ. We have the evaluation map

$$\mathrm{ev} : M_{0,n}^{\mathrm{trop}}(\mathbb{R}^2, \Delta) \to (\mathbb{R}^2)^n.$$

1. What are the dimensions of domain and codomain ? For which value of n do they agree ?

2. If the dimension of both spaces agree and Γ belongs to a top-dimensional cone of $M_{0,n}^{\mathrm{trop}}(\mathbb{R}^2, \Delta)$, we define the multiplicity m_Γ of Γ to be absolute value of the determinant of ev on the cone.
a) Picking a basis comprised of the image of vertex and the edge lengths of the curve, write down the matrix of the evaluation map.
b) If we pick a different vertex, we get two different matrices. How are they related ?
c) Show that if Γ has a flat vertex or if the complement of marked points in Γ contains a string, the multiplicity m_Γ is 0.
d) Prove that $m_\Gamma = \prod_V |\det(a_V, b_V)|$, where the product is made over the trivalent vertices of Γ, and $a_V, b_V, -a_V - b_V$ are the directing vectors of the edges adjacent to V.

3. Compute the multiplicity of the following tropical curves.



(a) (b) (c) (d)

Exercise 32

We consider stable rational tropical curves of degree d inside $\mathbb{R}^2$.

1. Prove that the space of deformation of these curves is of dimension $3d-1$.

2. We consider a generic family $\mathcal{P}$ of $3d-1$ points inside $\mathbb{R}^2$.
a) Prove that only a finite number of degree d rational tropical curves can pass through $\mathcal{P}$. (Use the evaluation map)
b) Prove that each of these curve has only trivalent vertices and no flat vertices.
c) Let Γ be a curve passing through $\mathcal{P}$. Prove that each component of the complement of marked points in Γ contains a unique unbounded end.

Exercise 33 : higher genus curve
We consider genus g tropical curves of degree d inside $\mathbb{R}^2$.

1. Assume Γ is a *simple* tropical curve : only trivalent vertices, and no flat vertices.
a) Prove that Γ has $3g-3+3d$ bounded edges.
b) Prove that each cycle not mapped to a line imposes 2 conditions on the edge lengths of the curves.
c) Prove that the space of deformation of Γ is of dimension $3d-1+g$.
We now consider the moduli space $M_{g,n}^{\text{trop}}(\mathbb{R}^2,d)$, with $n=3d-1+g$ marked points, and the evaluation map.

2. a) If Γ is not simple, prove that its image under the evaluation map is of dimension strictly smaller than $2n$.
b) Deduce that if $\mathcal{P}$ is generic, there is a finite number of degree d genus g tropical curves passing through $\mathcal{P}$, and that they are simple.
c) Let Γ be a curve passing through $\mathcal{P}$. Prove that each component of the complement of marked points in Γ contains a unique unbounded end.

Exercise 34
We define N_Δ^{trop} to be the degree of the evaluation map

$$\mathrm{ev} : M_{0,n}^{\text{trop}}(\mathbb{R}^2,\Delta) \to (\mathbb{R}^2)^n,$$

where $n=|\Delta|-1$. Try to compute N_Δ^{trop} in the following cases :
1. $\Delta=\{(0,-1),(-1,0),(1,1)\}$
2. $\Delta=\{(0,-1)^2,(-1,0)^2,(1,1)^2\}$
3. $\Delta=\{(0,-1),(-1,0),(-1,2),(2,-1)\}$
4. $\Delta=\{(0,-1)^2,(-1,0)^2,(0,1)^2,(1,0)^2\}$

Exercise 35 : curves in $(\mathbb{P}^1)^2$
We now consider curves inside the quadric surface $\mathbb{P}^1\times\mathbb{P}^1$. We say that a tropical curve is of bidegree (a,b) if it has a ends in directions $(1,0)$ and $(-1,0)$, and b ends in directions $(0,1)$ and $(0,-1)$.

1. What is Bezout's theorem for curves of bidegree (a,b) ?

2. What is the dimension of the moduli space of stable maps $M_{0,n}^{\text{trop}}(\mathbb{P}^1\times\mathbb{P}^1,(a,b))$ of genus 0 curves of bidegree (a,b) with n marked points.

3. If we choose $n = 2a + 2b - 1$ marked points in generic position, argue that there is a finite number of curves, whose count with multiplicity m_Γ does not depend on the choice of the points. We denote it by $N_{a,b}$.

4. What is the value of $N_{a,0}$?

5. What is the value of $N_{a,1}$?

6. ( ) What is the value of $N_{2,2}$?

Exercise 36 : on the invariance of the degree

Recall that the multiplicity of a tropical curve is defined by $m_\Gamma = \prod_V m_V$.

1. We wish to prove by hand that the degree of ev is well-defined : the weighted number of rational tropical curves passing through a point configuration $\mathcal{P}$ does not depend on the choice of the points. To do this, we choose a generic path $\mathcal{P}_t$ between two generic point configurations $\mathcal{P}_0$ and $\mathcal{P}_1$.

a) Prove that if $\mathcal{P}_t$ is generic, the count is locally invariant around $\mathcal{P}_t$ as each of the curves passing through $\mathcal{P}_t$ can be deformed.

b) Prove that the "worse" thing that can happen when moving $\mathcal{P}$ is that a curve with a quadrivalent vertex can appear. (Use a dimension argument)

c) Write down the possibilities of deformation around the quadrivalent vertex.

d) Prove that we have the local invariance around a curve with a quadrivalent vertex.

e) Conclude.

2. Prove similarly that the *refined multiplicity* $M_\Gamma = \prod_V (q^{m_V/2} - q^{-m_V/2}) \in \mathbb{Z}[q^{\pm 1/2}]$, which is a Laurent polynomial, also provides an invariant count.

Exercise 37

Given Δ, is the naive number (*i.e.* without multiplicity) of degree Δ rational tropical curves passing through a generic point configuration $\mathcal{P}$ constant ?

6 Kontsevich's formula and Mikhalkin's theorem

Exercise 38

We have the map $\pi : M_{0,n}^{\mathrm{trop}}(\mathbb{P}^2, d) \to \mathbb{R} \times \mathbb{R} \times (\mathbb{R}^2)^{n-2} \times M_{0,4}^{\mathrm{trop}}$ that is comprised of evaluation of the first coordinate of first marked point, second coordinate of second marked point, position of the other marked points, and the image by the forgetful morphism remembering the relative position of the first four marked points.

1. Taking $n = 3d$, prove that both spaces have dimension $6d - 1$.

2. Prove that the degree of π is constant.

3. Give a geometric description of the preimage of $(x_0, y_0, P_3, \dots, P_n, \gamma)$.

Exercise 39

Consider the evaluation map ev : $M_{0,n}^{\mathrm{trop}}(\mathbb{P}^2, d) \to \mathbb{R} \times \mathbb{R} \times (\mathbb{R}^2)^{n-2}$, which is π without the forgetful map to $M_{0,4}^{\mathrm{trop}}$, where $n = 3d$, and consider the preimage of $(x_0, y_0, P_3, \dots, P_n)$.

1. Let Γ be a tropical curve passing through $P_3, \ldots, P_n$. Prove that Γ contains a unique *string* : a path from one unbounded end to another not meeting any point $P_3, \ldots, P_n$.

2. Assuming there are no contracted edge on Γ, argue that the curves in the preimage of $(x_0, y_0, P_3, \ldots, P_n)$ having the same combinatorial type of Γ are obtained by deforming the string.

3. Prove that all the edges adjacent to the string either contain a marked point or are bounded.

4. Prove that unless the string contains no bounded edge, it is only possible to deform the string in a bounded set. If the string contains no bounded edge, then one direction is unbounded.

5. Using the previous question, prove that the image of $\mathrm{ev}^{-1}(x_0, y_0, P_3, \ldots, P_n)$ under ft : $M_{0,n}^{\mathrm{trop}}(\mathbb{P}^2, d) \to M_{0,4}^{\mathrm{trop}}$ is bounded.

6. Deduce that if γ is large enough, curves in $\pi^{-1}(x_0, y_0, P_3, \ldots, P_n, \gamma)$ have a contracted edge.

Exercise 40

1. Take a large coordinate γ on the cone $12//34$ of $M_{0,4}^{\mathrm{trop}}$ and Γ a curve in $\pi^{-1}(x_0, y_0, P_3, \ldots, P_n, \gamma)$. It contains a unique bounded edge. Assume this edge is adjacent to a vertex itself adjacent to the ends 1 and 2.
a) Argue that the image of Γ is in fact passing through $3d - 1$ points.
b) Prove that the multiplicity of π at Γ is in fact m_Γ.

2. We no longer assume that the contracted edge is adjacent to a vertex itself adjacent to ends 1 and 2.
a) Prove that the image of Γ is reducible. We write it $\Gamma_1 \cup \Gamma_2$, of respective degrees d_1 and d_2.
b) How do the marked point dispatch on the two components ?
c) Show that the multiplicity of π at Γ is equal to

$$(\Gamma_1 \cdot \mathbb{R} \times \{x_0\})_{p_1} (\Gamma_1 \cdot \{y_0\} \times \mathbb{R})_{p_2} (\Gamma_1 \cdot \Gamma_2)_p m_{\Gamma_1} m_{\Gamma_2},$$

where p is the point to which the contracted edge is mapped, and p_1 and p_2 are first and second marked points.
d) Conversely, compute the contribution of pairs of reducible solutions satisfying the assumptions found in the previous question.

3. We now take a large coordinate γ on the cone $13//24$ of $M_{0,4}^{\mathrm{trop}}$ and Γ a curve in $\pi^{-1}(x_0, y_0, P_3, \ldots, P_n, \gamma)$. It still contains a contracted edge.
a) Prove that the image of Γ is reducible. We write it $\Gamma_1 \cup \Gamma_2$, of respective degrees d_1 and d_2.
b) How do the marked point dispatch on the two components ?
c) Show that the multiplicity of π at Γ is equal to

$$(\Gamma_1 \cdot \mathbb{R} \times \{x_0\})_{p_1} (\Gamma_2 \cdot \{y_0\} \times \mathbb{R})_{p_2} (\Gamma_1 \cdot \Gamma_2)_p m_{\Gamma_1} m_{\Gamma_2},$$

where p is the point to which the contracted edge is mapped, and p_1 and p_2 are first and second marked points.

4. As the degree of π is constant, equate the degrees at two well-chosen points and finish proving Kontsevich's formula.

Exercise 41

The cohomology of $\mathbb{P}^2$ is generated by the neutral class $1 = h^0$, the class Poincaré dual to a line $h = h^1$, and its square h^2 which is Poincaré dual to a point :

$$H^*(\mathbb{P}^2, \mathbb{Q}) = \mathbb{Q}h^0 \oplus \mathbb{Q}h^1 \oplus \mathbb{Q}h^2.$$

Let $\langle (h^0)^{a_0}, (h^1)^{a_1}, (h^2)^{a_2} \rangle_{0,d}$ be the number of degree d maps $u : (\mathbb{P}_1, (p_i), (q_j), (r_l)) \to \mathbb{P}^2$ up to automorphisms such that :

- p_i are mapped to fixed points $P_i \in \mathbb{P}^2$ for $1 \leqslant i \leqslant a_2$,
- q_j are mapped to fixed lines $L_j \subset \mathbb{P}^2$ for $1 \leqslant j \leqslant a_1$,
- r_l are mapped to $\mathbb{P}^2$,

if this number is finite. Otherwise, we put it equal to 0.

1. a) What is the image of a degree 0 map ?
b) How many marked points can it have ?
c) Show that the only non-zero invariants for $d = 0$ are obtained for $(a_0, a_1, a_2) = (1, 2, 0)$ and $(2, 0, 1)$.
2. Argue that if $d \geqslant 1$ and $a_0 > 0$, the number is 0.
3. a) If $d \geqslant 1$, $a_0 = 0$ and $a_1 \geqslant 1$, prove that

$$\langle (h^1)^{a_1}, (h^2)^{a_2} \rangle_{0,d} = d \langle (h^1)^{a_1 - 1}, (h^2)^{a_2} \rangle_{0,d}.$$

Deduce that you can assume $a_1 = 0$ as well.
b) Show that $\langle (h^2)^{a_2} \rangle_{0,d} = 0$ unless $a_2 = 3d - 1$, where the value is N_d.
4. We set

$$\Phi(x_0, x_1, x_2) = \sum_{a_0, a_1, a_2, d} \frac{x_0^{a_0}}{a_0!} \frac{x_1^{a_1}}{a_1!} \frac{x_2^{a_2}}{a_2!} \langle (h^0)^{a_0}, (h^1)^{a_1}, (h^2)^{a_2} \rangle_{0,d}.$$

Show that we have

$$\Phi(x_0, x_1, x_2) = \frac{1}{2}(x_0 x_1^2 + x_0^2 x_2) + \sum_{d \geqslant 1} N_d \frac{x_2^{3d-1}}{(3d-1)!} e^{d x_1}.$$

This series is called the Gromov-Witten potential. We use it in the next exercise.

Exercise 42

We consider the Gromov-Witten potential Φ from the previous question and the cohomology of $\mathbb{P}^2$ with coefficients in the ring of power series $\mathbb{Q}[\![x_0, x_1, x_2]\!]$, whose base is given by h^0, h^1, h^2. Recall that as a (graded) algebra, we have

$$H^*(\mathbb{P}^2, K) \simeq K[h]/(h^3).$$

We introduce on $H^*(\mathbb{P}^2, \mathbb{Q}[\![x_0, x_1, x_2]\!])$ the following weird product

$$h^i * h^j = \sum_{k=0}^{2} \Phi_{ijk} h^{2-k},$$

for $0 \leqslant i, j \leqslant 2$, where Φ_{ijk} is the third partial derivative of Φ.

1. Prove that when $x_0 = x_1 = x_2 = 0$, we recover the usual cup-product on $H^*(\mathbb{P}^2, \mathbb{Q}) \simeq \mathbb{Q}[h]/(h^3)$.

2. Check that the only non-zero third partial derivatives involving a derivation with respect to x_0 are

$$\Phi_{011} = \Phi_{002} = 1.$$

3. Show that even for this weird product, h^0 is the neutral element.

4. Compute the three products $h^1 * h^1$, $h^2 * h^2$ and $h^1 * h^2$ in terms of the partial derivatives of Φ.

5. Prove that the above product is associative if and only if one has the following equation :

$$\Phi_{111}\Phi_{122} + \Phi_{222} = \Phi_{112}^2.$$

(Notice one has only to check the associativity for $h^1 * h^1 * h^2$ and $h^1 * h^2 * h^2$ products.)

6. Prove the associativity using Kontsevich's formula.

Exercise 43

We now consider curves inside the quadric surface $\mathbb{P}^1 \times \mathbb{P}^1$. Recall the definition of the numbers $N_{a,b}$. We consider the map

$$\pi : M_{0,n}^{\mathrm{trop}}(\mathbb{P}^1 \times \mathbb{P}^1, (a,b)) \to \mathbb{R} \times \mathbb{R} \times (\mathbb{R}^2)^{n-2} \times M_{0,4}^{\mathrm{trop}},$$

as in the proof of Kontsevich's formula : first coordinate of first marked point, second coordinate of second marked point, evaluation of remaining marked points, cross-ratio of first four marked points.

1. We take $n = 2a + 2b$. Show that both spaces have the same dimension.

2. Recall that $N_{a,b}$ is the number of bidegree (a, b) rational curve passing through $2a + 2b - 1$ points. Adapt the proof from Kontsevich's formula to prove the following recursive formula

$$N_{a,b} + \sum_{\substack{a_1+a_2=a \\ b_1+b_2=b}} a_1 b_1 (a_1 b_2 + a_2 b_1) \binom{2a+2b-4}{2a_1+2b_1-1} N_{a_1,b_1} N_{a_2,b_2} = \sum_{\substack{a_1+a_2=a \\ b_1+b_2=b}} a_1 b_2 (a_1 b_2 + a_2 b_1) \binom{2a+2b-4}{2a_1+2b_1-2} N_{a_1,b_1} N_{a_2,b_2}.$$

Proceed as follows :

- Notice that the degree of π is well-defined.
- Prove that points with a large coordinate on $M_{0,4}^{\mathrm{trop}}$ in type $12//34$ and $13//24$ have a contracted edge.
- Compute the multiplicity of π at curves having a unique contracted bounded edge.
- Equate the degree of π at two points with large coordinate on $M_{0,4}^{\mathrm{trop}}$ in type $12//34$ and $13//24$.

Exercise 44

The cohomology ring of $(\mathbb{P}^1)^2$ is given by

$$H^*((\mathbb{P}^1)^2, \mathbb{Q}) = \mathbb{Q}1 \oplus (\mathbb{Q}\alpha \oplus \mathbb{Q}\beta) \oplus \mathbb{Q}\alpha\beta,$$

where $\alpha^2 = \beta^2 = 0$. As in the case of the projective plane, we consider the "Gromov-Witten invariants" of $(\mathbb{P}^1)^2$:

$$\langle 1^{a_0}, \alpha^{a_\alpha}, b^{a_\beta}, (\alpha\beta)^{a_2} \rangle_{0,(a,b)}$$

is the up to automorphisms number of bidegree (a, b) maps $u : \mathbb{P}^1 \to (\mathbb{P}^1)^2$ mapping the $a_0 + a_\alpha + a_\beta + a_2$ marked points to nothing/vertical lines/horizontal lines/points, or 0 if this number is infinite.

1. Assume $(a, b) = (0, 0)$. Prove that the only non-zero $\langle 1^{a_0}, \alpha^{a_\alpha}, b^{a_\beta}, (\alpha\beta)^{a_2} \rangle_{0,(0,0)}$ are obtained for tuples

$$(2, 0, 0, 1), (1, 1, 1, 0).$$

2. a) Prove that if $(a, b) \neq (0, 0)$ and $a_0 > 0$, we have $\langle 1^{a_0}, \alpha^{a_\alpha}, b^{a_\beta}, (\alpha\beta)^{a_2} \rangle_{0,(a,b)} = 0$.

b) Prove that if $a_\alpha \geqslant 1$, we have

$$\langle \alpha^{a_\alpha}, b^{a_\beta}, (\alpha\beta)^{a_2} \rangle_{0,(a,b)} = a \langle \alpha^{a_\alpha - 1}, b^{a_\beta}, (\alpha\beta)^{a_2} \rangle_{0,(a,b)},$$

and if $a_\beta \geqslant 1$,

$$\langle \alpha^{a_\alpha}, b^{a_\beta}, (\alpha\beta)^{a_2} \rangle_{0,(a,b)} = b \langle \alpha^{a_\alpha}, b^{a_\beta - 1}, (\alpha\beta)^{a_2} \rangle_{0,(a,b)}.$$

Thus, we can assume $a_\alpha = a_\beta = 0$.

c) Prove that $\langle (\alpha\beta)^{a_2} \rangle_{0,(a,b)} = 0$ unless $a_2 = 2a + 2b - 1$ where the value is $N_{a,b}$.

3. We consider the Gromov-Witten potential

$$\Phi(x_0, x_\alpha, x_\beta, x_2) = \sum_{a_0, a_\alpha, a_\beta, a_2, (a,b)} \frac{x_0^{a_0}}{a_0!} \frac{x_\alpha^{a_\alpha}}{a_\alpha!} \frac{x_\beta^{a_\beta}}{a_\beta!} \frac{x_2^{a_2}}{a_2!} \langle 1^{a_0}, \alpha^{a_\alpha}, \beta^{a_\beta}, (\alpha\beta)^{a_2} \rangle_{0,(a,b)}.$$

Show that the series admits the following form :

$$\Phi = \frac{1}{2} x_0^2 x_2 + x_a x_b + \sum_{(a,b) \neq (0,0)} N_{(a,b)} \frac{x_2^{2a+2b-1}}{(2a + 2b - 1)!} e^{a x_\alpha + b x_\beta}.$$

4. We define the following weird product on the cohomology ring of $(\mathbb{P}^1)^2$ with coefficients in the ring of power series $\mathbb{Q}[\![x_0, x_a, x_b, x_2]\!]$: let $PD(1) = \alpha\beta$, $PD(\alpha\beta) = 1$, $PD(\alpha) = \beta$ and $PD(\beta) = \alpha$. Then, for $h, h' \in \{1, \alpha, \beta, \alpha\beta\}$,

$$h * h' = \sum_{k \in \{1, \alpha, \beta, \alpha\beta\}} \Phi_{hh'k} PD(k).$$

a) Show that 1 is still a neutral element for $*$.

b) As in the case of $\mathbb{P}^2$, show that it is associative using Kontsevich's formula for $(\mathbb{P}^1)^2$.

Exercise 45

Can you figure out a three dimensional version of Kontsevich's formula ?