

Bridging the Rossby number gap in rapidly rotating thermal convection

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Geophysical and astrophysical fluid flows are typically driven by buoyancy and strongly constrained at large scales by planetary rotation. Rapidly rotating Rayleigh–Bénard convection (RRRBC) provides a paradigm for experiments and direct numerical simulations (DNS) of such flows, but the accessible parameter space remains restricted to moderately fast rotation rates (Ekman numbers $Ek \gtrsim 10^{-8}$), while realistic Ek for geo- and astrophysical applications are orders of magnitude smaller. On the other hand, previously derived reduced equations of motion describing the leading-order behaviour in the limit of very rapid rotation ($Ek \rightarrow 0$) cannot capture finite rotation effects, and the physically most relevant part of parameter space with small but finite Ek has remained elusive. Here, we employ the rescaled rapidly rotating incompressible Navier–Stokes equations (RRRiNSE) – a reformulation of the Navier–Stokes–Boussinesq equations informed by the scalings valid for $Ek \rightarrow 0$, recently introduced by Julien *et al.* (2024) – to provide full DNS of RRRBC at unprecedented rotation strengths down to $Ek = 10^{-15}$ and below, revealing the disappearance of cyclone–anticyclone asymmetry at previously unattainable Ekman numbers ($Ek \approx 10^{-9}$). We also identify an overshoot in the heat transport as Ek is varied at fixed $\tilde{Ra} \equiv RaEk^{4/3}$, where Ra is the Rayleigh number, associated with dissipation due to ageostrophic motions in the boundary layers. The simulations validate theoretical predictions based on thermal boundary layer theory for RRRBC and show that the solutions of RRRiNSE agree with the reduced equations at very small Ek .

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These results represent a first foray into the vast, largely unexplored parameter space of very rapidly rotating convection rendered accessible by RRRiNSE.

Key words: computational methods, rotating flows, turbulent convection

1. Introduction

The universe abounds with examples of turbulent flows that are driven by buoyancy and constrained by rotation. These highly complex flows shape our environment across scales: in protoplanetary disks, they mediate the formation of planetesimals (Lesur & Ogilvie 2010); in planetary atmospheres, including on Earth, they carry heat, momentum and moisture, crucially impacting the climate (Emanuel 1994; Dauxois *et al.* 2021; Siegelman *et al.* 2022); in the Earth's oceans (Marshall & Schott 1999; Cheon & Gordon 2019), they transport heat, carbon dioxide and nutrients, thereby crucially influencing biotopes (Severin *et al.* 2014); in planetary and stellar cores, they are responsible for the generation of large-scale magnetic fields via the dynamo instability (King *et al.* 2010; Jones 2011; Schaeffer *et al.* 2017); in the subsurface oceans of the gaseous giants' moons, they shape the icy crusts of Europa, Enceladus, Titan, etc. (Pappalardo *et al.* 1998; Mitri & Showman 2008; Nimmo & Pappalardo 2016; Soderlund *et al.* 2024); in stars like the Sun (Miesch 2000; Fan 2021) they are responsible for large-scale magnetic fields as well as rich dynamics such as the 22-year cycle in solar activity.

Such geo- and astrophysical flows, in spite of their differences, typically share two important properties. Owing to their turbulent nature, their dynamics results from a broad range of interacting spatial and temporal scales. Accordingly, advection by the flow dominates over viscous forces when the Reynolds number is large, $Re_H = UH/\nu \gg 1$, based here on the fluid layer depth H , a typical velocity of the flow U and kinematic viscosity ν . Simultaneously, advection by the flow occurs on time scales much longer than the planetary rotation period, as measured by small Rossby numbers $Ro_H = U/(2\Omega H) \ll 1$, where Ω is the planetary rotation rate.

Also of particular importance is the ratio of the viscous and Coriolis forces providing an *a priori* external parameter referred to as the Ekman number $Ek = Ro_H/Re_H = \nu/(2\Omega H^2)$, as well as the Prandtl number $\sigma = \nu/\kappa$ characterising the fluid under consideration, where κ is the thermal diffusivity. The strength of the thermal forcing is controlled by the Rayleigh number Ra , which is proportional to the temperature drop ΔT across the fluid layer. This number can be used to define an alternate *a priori* parameter, the convective Rossby number $Ro_{conv} = \sqrt{Ra/\sigma} Ek$, measuring the strength of the rotation relative to thermal forcing.

Celestial bodies typically feature extreme values for the non-dimensional parameters defined above, as summarised in table 1: rapid planetary rotation is reflected in very low Ekman and Rossby numbers, whereas large Reynolds numbers indicate the highly turbulent character of the flow. Such rapidly rotating, highly turbulent flows are challenging to analyse and specifically to simulate numerically owing to the sheer number of spatial and temporal degrees of freedom which they involve. One common approach for obtaining theoretical predictions at geophysically relevant parameters (specifically, $Ek \rightarrow 0$) relies upon identifying transitions between different flow regimes in a region of more moderate parameters ($Ek \gtrsim 10^{-8}$) that is amenable to direct numerical simulations (DNS) or laboratory experiments, together with the scaling laws for transport coefficients valid within each regime. These results are then extrapolated to extreme parameters on the assumption that the regimes and scaling laws observed for accessible parameter values

Celestial body	Ek	σ	Ro_H	Re_H
Earth's outer core	10^{-15}	0.1	10^{-7}	10^8
Mercury (core)	10^{-12}	0.1	10^{-4}	10^8
Jupiter (core)	10^{-19}	0.1	10^{-10}	10^9
Europa (ocean)	10^{-12}	11.0	$10^{-2.5}-10^{-1.5}$	$10^{9.5}-10^{10.5}$
Ganymede (ocean)	$10^{-10}-10^{-13}$	10.0	$10^{-3.5}-10^{1.5}$	$10^{9.5}-10^{11.5}$
Saturn (core)	10^{-18}	0.1	10^{-9}	10^9
Enceladus (ocean)	$10^{-10}-10^{-11}$	13.0	$10^{-3.5}-10^{-1}$	$10^{7.5}-10^9$
Titan (ocean)	$10^{-11}-10^{-12}$	10.0	$10^{-3}-1$	10^9-10^{11}
Neptune (core)	10^{-16}	10.0	10^{-6}	10^{10}
Uranus (core)	10^{-16}	10.0	10^{-6}	10^{10}
Sun (convection zone)	10^{-15}	10^{-6}	10^{-3}	10^{12}

Table 1. Non-dimensional parameter estimates for planetary (Schubert & Soderlund 2011), satellite (Soderlund 2019) and stellar interiors (Miesch 2000; Garaud & Garaud 2008). Estimates of the Rossby number are derived from the relation $Ro_H = Re_H Ek$.

extend to geo-/astrophysically relevant parameter values. Assuming that no non-trivial transition occurs outside the observed moderate parameter interval is a particularly strong assumption, akin to a leap of faith due to the current paucity of data at the most extreme parameter values.

The dynamics of the fluid flows found in the celestial bodies listed in table 1 is further rendered complex by additional ingredients such as spherical geometry, multiple contributors to the density, compressibility and the presence of magnetic fields. Absent such complexities, the quintessential paradigm for investigating rotationally influenced buoyant flows is provided by rapidly rotating Rayleigh–Bénard convection (RRRBC). A large number of studies have been published on this model system, which is very well suited for detailed experimental, numerical and theoretical studies, see for example Chandrasekhar (1953), Nakagawa & Frenzen (1955), Veronis (1959), Rossby (1969), Boubnov & Golitsyn (1986), Zhong *et al.* (1991), Julien *et al.* (1996), Knobloch (1998), Hart *et al.* (2002), Vorobieff & Ecke (2002), Boubnov & Golitsyn (2012), King *et al.* (2012), Stevens *et al.* (2013) and the recent review by Ecke & Shishkina (2023). In RRRBC, gravity is often taken to be antiparallel to the rotation axis on the assumption that the Froude number $Fr_\Omega \equiv \Omega^2 H/g$, where g is the gravitational acceleration, is small. The resulting configuration is translation-invariant and believed to be relevant to the polar regions of planets, moons and stars as well as to their interiors, although the case of misaligned gravity and rotation axis has also been studied (Hathaway & Somerville 1983; Julien & Knobloch 1998; Julien *et al.* 2006; Miquel *et al.* 2018; Novi *et al.* 2019; Cai 2020; Currie *et al.* 2020; Julien *et al.* 2025).

A major obstacle in the study of rapidly rotating convection, even in the highly idealised setting of RRRBC, is the difficulty, to date, of attaining the extremely low Ekman numbers, typically 10^{-12} or smaller (cf. table 1), that characterise geo- and astrophysical flows. By contrast, current state-of-the-art DNS of the full Navier–Stokes–Boussinesq equations that describe RRRBC have been restricted to $Ek \gtrsim 10^{-8}$ (Kunnen *et al.* 2016; Guervilly *et al.* 2019; Song *et al.* 2024a,b), despite enormous computational effort. Inclusion of magnetic fields in the simulations to capture the capability for dynamo action, as well as spherical shell geometry, further limits the reported range of nearly all studies to $Ek \gtrsim 10^{-7}$, $Re_H \lesssim 10^3$ (Schaeffer *et al.* 2017; Cooper *et al.* 2020; Mason *et al.* 2022; Kolhey *et al.* 2022; Gastine & Aurnou 2023; Majumder *et al.* 2024); simulations down to $Ek = 10^{-10}$

are available for linearised dynamics only (He *et al.* 2022). State-of-the-art laboratory experiments are likewise restricted to $Ek \gtrsim 10^{-8}$ (Shew & Lathrop 2005; Aurnou *et al.* 2015; Rajaei *et al.* 2017; Cheng *et al.* 2020; Lu *et al.* 2021; Hawkins *et al.* 2023; Pothérat & Horn 2024) although the Eindhoven TROCONVEX experiment is, in principle, capable of achieving yet lower $Ek \sim 5 \times 10^{-9}$, cf. Cheng *et al.* (2018). These accessible parameter values in numerical or laboratory experiments are orders of magnitude larger than the estimates for realistic geo- and astrophysical applications. On the other hand, existing reduced equations (Julien *et al.* 1998; Sprague *et al.* 2006), known as the non-hydrostatic quasi-geostrophic (NHQG) equations, describe the dynamics in the limit $Ek \rightarrow 0$, but their applicability is, by construction, confined to asymptotically small values of Ek . This leaves an unexplored gap corresponding to small but finite Ekman numbers, precisely the regime that is most relevant for geo- and astrophysical applications.

It is extremely rare to have a numerical or laboratory approach that allows one to span the parameter range from prototypical laboratory and DNS settings all the way to realistic geophysical and astrophysical settings. Remarkably, taking advantage of the characteristics of rapidly rotating thermal convection leads to a new numerical approach that relies on appropriately rescaled equations, and enables exploration of this very wide parameter range.

In the following section we describe the rescaled rapidly rotating Navier–Stokes equations (RRRiNSE) and sketch the numerical method used to solve them, followed in § 3 by a description of the results obtained, focusing both on bulk transport properties and the structure of the boundary layers at the top and bottom. The paper concludes in § 4 with a discussion of the significance of the RRRiNSE reformulation and prospects for future work. An Appendix provides additional details on the flow statistics and structure.

2. Methodology

In this section, we succinctly present the RRRiNSE, an equivalent reformulation of the full Navier–Stokes–Boussinesq equations, informed by the scaling behaviour valid in the limit of vanishing Ek (Julien *et al.* 1998; Julien *et al.* 2012*b*). The RRRiNSE formulation is described in detail in a recent paper (Julien *et al.* 2024), where it was first introduced and validated against existing results in the literature, and where the full details of the numerical implementation are given.

In short, the RRRiNSE retain all terms in the Navier–Stokes–Boussinesq equations while rescaling each variable according to its leading-order asymptotic scaling behaviour valid for $Ek \rightarrow 0$ in terms of the natural small parameter of rapidly rotating convection $\varepsilon \equiv Ek^{1/3}$ (Chandrasekhar 1953; Sprague *et al.* 2006). In particular, the rapid rotation generates a high degree of anisotropy, which is explicitly incorporated in the RRRiNSE formulation: vertical length scales are non-dimensionalised by the layer height H , while horizontal length scales are significantly smaller, given by εH . Consequently, time is measured in units of the viscous diffusion time based upon εH . As shown by Julien *et al.* (2024), this procedure ensures an adequate preconditioning of the equations of motion at small but finite Ek , thus facilitating DNS in this highly relevant but unexplored regime.

2.1. Rescaled Navier–Stokes equations

The convective fluid layer confined by horizontal surfaces separated by a distance H is subjected to rotation around the vertical axis $\boldsymbol{\Omega} = \Omega \hat{\mathbf{e}}_z$ and uniform gravity $-g \hat{\mathbf{e}}_z$. The temperature difference between the bottom and the top plate is $\Delta_T > 0$. In the

Oberbeck–Boussinesq limit, the fluid density varies linearly with temperature $\rho = \rho_0[1 - \alpha(T - T_0)]$ near a reference state (ρ_0, T_0) , with both the thermal expansion coefficient α , the fluid kinematic viscosity ν , and the temperature diffusivity κ considered constant. The dimensional governing equations for the solenoidal velocity field \mathbf{U} , the temperature T and the pressure P are

$$\partial_t \mathbf{U} + \mathbf{U} \cdot \nabla \mathbf{U} + 2\boldsymbol{\Omega} \times \mathbf{U} = -\frac{1}{\rho_0} \nabla P + \alpha g T \hat{\mathbf{e}}_z + \nu \Delta \mathbf{U}, \quad \nabla \cdot \mathbf{U} = 0, \quad (2.1)$$

$$\partial_t T + \mathbf{U} \cdot \nabla T = \kappa \Delta T. \quad (2.2)$$

As described in detail by Julien *et al.* (2024), the power behind the RRRiNSE formulation lies in their use of rotation-influenced characteristic scales for non-dimensionalisation. For convenience, we introduce the small parameter $\varepsilon \equiv Ek^{1/3} = (\nu/2\Omega H^2)^{1/3}$ and measure lengths in units of the layer depth H in the vertical, but in units of $\ell_* = \varepsilon H$ in the horizontal, so that $(X, Y, Z) = (\varepsilon H \tilde{x}, \varepsilon H \tilde{y}, H \tilde{z})$. In these coordinates, the rescaled anisotropic gradient $\nabla = \varepsilon^{-1} H^{-1} \tilde{\nabla}_\varepsilon \equiv \varepsilon^{-1} H^{-1} (\partial_{\tilde{x}}, \partial_{\tilde{y}}, \varepsilon \partial_{\tilde{z}})$ and the Laplace operator becomes $\Delta = \varepsilon^{-2} H^{-2} \tilde{\nabla}_\varepsilon^2$, where we have introduced the anisotropic diffusion operator $\tilde{\nabla}_\varepsilon^2 = \partial_{\tilde{x}\tilde{x}} + \partial_{\tilde{y}\tilde{y}} + \varepsilon^2 \partial_{\tilde{z}\tilde{z}}$. All tildes will be dropped in the following. Time is non-dimensionalised using a diffusive time scale based on the horizontal scale rather than the layer depth alone: $t_* = \ell_*^2/\nu = \varepsilon^2 H^2/\nu$. Accordingly, the velocity scale is $U_* = \ell_*/t_* = \nu/(\varepsilon H)$, giving the non-dimensional velocity $\mathbf{u} = \mathbf{U}/U_* \equiv (u, v, w)$; the pressure scale $P_* = \rho_0 U_*^2/\varepsilon = \rho_0 \nu^2/(\varepsilon \ell_*^2)$ is used to define the non-dimensional pressure $p \equiv P/P_*$. This isotropic velocity scale, combined with our anisotropic coordinate system, results in horizontally dominated material derivatives

$$D_t^{\perp, \varepsilon} = \partial_t + u \partial_x + v \partial_y + \varepsilon w \partial_z = \partial_t + \mathbf{u}_\perp \cdot \nabla_\perp + \varepsilon w \partial_z. \quad (2.3)$$

Here, the perpendicular symbol \perp in the subscript denotes the projection upon the horizontal plane: specifically, $\nabla_\perp = (\partial_x, \partial_y, 0)^\top$ and $\mathbf{u}_\perp = \mathbf{u} - w \hat{\mathbf{e}}_z$. The incompressibility condition becomes

$$\nabla_\perp \cdot \mathbf{u}_\perp + \varepsilon \partial_z w = 0. \quad (2.4)$$

The (dimensional) temperature T is decomposed as the sum of three terms: an unstable linear stratification, a vertical profile $\Theta(z)$, and horizontal fluctuations $\theta(x, y, z)$ with zero horizontal mean

$$T - T_0 = \Delta_T [1 - z + \Theta(z) + \varepsilon \theta(x, y, z)], \quad (2.5)$$

where $0 \leq z \leq 1$. We thus formally embed in our decomposition one hallmark of rapidly rotating convection: horizontal temperature fluctuations are dominated by vertical variations. With these elements, the dimensionless governing equations for the horizontal fluctuations $(\partial_x, \partial_y) \neq (0, 0)$ of momentum and temperature are found to be

$$D_t^{\perp, \varepsilon} \mathbf{u} + \varepsilon^{-1} \hat{\mathbf{e}}_z \times \mathbf{u} = -\varepsilon^{-1} \nabla_\varepsilon p + \frac{\tilde{Ra}}{\sigma} \theta \hat{\mathbf{e}}_z + \nabla_\varepsilon^2 \mathbf{u}, \quad (2.6a)$$

$$D_t^{\perp, \varepsilon} \theta + w (\partial_z \Theta - 1) = \frac{1}{\sigma} \nabla_\varepsilon^2 \theta, \quad (2.6b)$$

where $\tilde{Ra} = Ek^{4/3} Ra$, and the Rayleigh number Ra is given by $Ra = \alpha g H^3 \Delta_T / (\nu \kappa)$ in terms of the thermal expansion coefficient α of the fluid and gravitational acceleration g .

By denoting the horizontal average of any quantity using an overline

$$\overline{q}^{xy} \equiv \frac{1}{L_x L_y} \int_0^{L_x} dx \int_0^{L_y} dy q(x, y, z, t), \quad (2.7)$$

we write separately the equations governing the vertical profiles of horizontal velocity (\overline{u}^{xy} , \overline{v}^{xy}) and temperature Θ

$$\partial_t \overline{u}^{xy} + \varepsilon \partial_z \overline{w} \overline{u}^{xy} - \varepsilon^{-1} \overline{v}^{xy} = \varepsilon^2 \partial_{zz} \overline{u}^{xy}, \quad (2.8a)$$

$$\partial_t \overline{v}^{xy} + \varepsilon \partial_z \overline{w} \overline{v}^{xy} + \varepsilon^{-1} \overline{u}^{xy} = \varepsilon^2 \partial_{zz} \overline{v}^{xy}, \quad (2.8b)$$

$$\varepsilon^{-2} \partial_t \Theta + \overline{\partial_z w \theta}^{xy} = \frac{1}{\sigma} \partial_{zz} \Theta. \quad (2.8c)$$

Equations (2.4), (2.6) and (2.8) form the RRRiNSE system, which is analytically equivalent to the dimensional equations (2.1), (2.2) as we have only rescaled, but not discarded any terms in the equations. Numerically, however, these equations differ drastically in the sense that (2.4), (2.6) and (2.8) can be solved accurately for very small values of Ekman number, whereas the poor conditioning of the unscaled equations (2.1), (2.2) renders their solution impossible in practice whenever $Ek \lesssim 10^{-9}$ for reasons explained by Julien *et al.* (2024).

2.2. Asymptotically reduced non-hydrostatic quasi-geostrophic equations

In the limit of rapid rotation $Ek, Ro \rightarrow 0$, one can derive a reduced set of governing equations for the flow (presented below) in a systematic and controlled way using asymptotic methods (Julien *et al.* 1998; Sprague *et al.* 2006; Julien & Knobloch 2007; Julien *et al.* 2012b). Here, we emphasise, in a quick sketch, how the rescaled equations (2.6) naturally lead to the reduced equations when the formal limit $\varepsilon \rightarrow 0$ is considered. Upon inspection, if the scaling employed for deriving the rescaled equations holds and all dimensionless variables remain $O(1)$, the leading-order terms at order $O(\varepsilon^{-1})$ in (2.6a) indicate geostrophic balance

$$\widehat{\mathbf{e}}_z \times \mathbf{u}_\perp = -\nabla_\perp p, \quad (2.9)$$

so that the pressure serves as the geostrophic streamfunction

$$\mathbf{u} = -\nabla \times (p \widehat{\mathbf{e}}_z) + w \widehat{\mathbf{e}}_z + O(\varepsilon) \quad (2.10a)$$

$$= -\partial_y p \widehat{\mathbf{e}}_x + \partial_x p \widehat{\mathbf{e}}_y + w \widehat{\mathbf{e}}_z + O(\varepsilon). \quad (2.10b)$$

To obtain prognostic equations for p , w , θ , Θ we substitute expression (2.10) into the rescaled momentum equation and project the result upon $\widehat{\mathbf{e}}_z$, obtaining at leading order in ε

$$\partial_t w + J[p, w] + \partial_z p = \frac{\widetilde{Ra}}{\sigma} \theta + \nabla_\perp^2 w, \quad (2.11a)$$

where the Jacobian $J[p, w] \equiv \partial_x p \partial_y w - \partial_y p \partial_x w$ indicates horizontal advection of w and $\nabla_\perp^2 \equiv \partial_{xx} + \partial_{yy}$ emphasises that horizontal diffusion dominates. We complement the vertical velocity equation with a governing equation for the streamfunction p obtained by projecting the curl of the momentum equation on the vertical. Using incompressibility, one proves $\widehat{\mathbf{e}}_z \cdot \nabla \times (\widehat{\mathbf{e}}_z \times \mathbf{u}) = -\varepsilon \partial_z w$, so that the leading-order contribution of $\widehat{\mathbf{e}}_z \cdot \nabla \times$ (2.6a) yields

$$\partial_t \nabla_\perp^2 p + J[p, \nabla_\perp^2 p] - \partial_z w = \nabla_\perp^4 p. \quad (2.11b)$$

Finally, one readily obtains the leading-order contribution to the governing equation for the temperature fluctuation θ and the mean temperature Θ

$$\partial_t \theta + J[p, \theta] + w(\partial_z \Theta - 1) = \frac{1}{\sigma} \nabla_{\perp}^2 \theta, \quad (2.11c)$$

$$\varepsilon^{-2} \partial_t \Theta + \overline{\partial_z w \theta}^{xy} = \frac{1}{\sigma} \partial_{zz} \Theta. \quad (2.11d)$$

Equations (2.11) constitute the reduced NHQG equations previously derived using a multiple-scale asymptotic expansion (Julien *et al.* 1998; Sprague *et al.* 2006). These equations filter out fast inertial waves and thin Ekman layers and govern the leading-order geostrophically balanced flow.

The Ekman number dependence that remains in the governing equation (2.11d) for the mean temperature reflects the fact that the time scale $\tau = \varepsilon^2 t$ associated with the evolution of Θ is much slower than the convective time scale. Previous studies (Sprague *et al.* 2006; Julien *et al.* 2012b) have established that, in the limit of horizontally extended domains where a large number of convective cells or plumes contribute to the horizontal spatial averaging, this term can be omitted for the computation of global diagnostics of a statistically stationary state $\partial_t \Theta \approx 0$. This assumption can be justified based on the following rationale: as the horizontal extent $L_{\perp} = L_x = L_y$ of the domain increases, so does the number of independent plumes N_p that feed back into the temperature profile. The central limit theorem then suggests that, in the ergodic limit (time and ensemble averages behave similarly), the amplitude of temporal fluctuations of the vertical temperature profile $\Theta(z)$ should decrease like $1/\sqrt{N_p} \approx 1/L_{\perp}$, justifying the neglect of the term $\partial_t \Theta$. In the context of the RRRiNSE formulation, this assumption has been successfully validated and has proved to shorten the transient that leads to the stationary state (Julien *et al.* 2024). Thus in all simulations of the RRRiNSE (2.4), (2.6) and (2.8) and the NHQG equations (2.11) described in this paper, $\partial_t \Theta$ is set to zero, as is expected to be the case for a sufficiently large horizontal extent L_{\perp} of the domain.

2.3. Details on the numerical method

We compute solutions to both the rescaled equations (2.4), (2.6) and (2.8) and the reduced equations (2.11) using the pseudo-spectral code Coral (Miquel 2021), a flexible platform for solving partial differential equations in a Cartesian domain. We assume periodicity in (x, y) and thus decompose the problem variables using a Fourier basis in the horizontal, whereas a Chebyshev expansion is used in the vertical in order to impose boundary conditions on the bounding surfaces $z=0, 1$ using a Galerkin basis recombination. Differential operators are discretised using the quasi-inverse method (Clenshaw 1957; Greengard 1991; Julien & Watson 2009) to optimise memory usage and accuracy. The resolution was varied from 192 grid points in each direction (corresponding to 128 alias-free modes) at the lowest values of Ra , up to 768 grid points (512 Chebyshev modes) in the vertical direction and 576 grid points (384 Fourier modes) in the horizontal directions at larger Ra . The time marching relies on the family of stable semi-implicit Runge–Kutta schemes introduced by Ascher *et al.* (1997), specifically their four-stage, third-order scheme, known as ARS443.

For some of the runs presented here, we have used low-amplitude noise as the initial condition, and waited for the convective turbulent flow to settle into a statistically stationary state after a transient period. However, for strongly supercritical flow, this transient may prove much more turbulent than the stationary state itself. Hence, in some cases, we resort to an alternative: instead of low-amplitude noise, the initial condition for

a given run was taken from the solution computed at a neighbouring position in parameter space. This procedure facilitates the computation of the transient.

Once the transient has subsided, we compute time averages of different diagnostic quantities reported here. For instance, the barotropic vertical vorticity displayed in [figure 2](#) is defined as

$$\overline{\omega_z}(x, y) = \int_0^1 \omega_z \, dz, \quad (2.12)$$

where the vertical vorticity is given by $\omega_z = \partial_x v - \partial_y u$. The Nusselt number is computed in terms of a volume average as $Nu(t) - 1 = \sigma \langle w\theta \rangle_{vol}$, where the volume average is given by

$$\langle \cdot \rangle_{vol} = \overline{(\cdot)}^{xy} = \frac{1}{L_x L_y} \int_0^{L_x} dx \int_0^{L_y} dy \int_0^1 (\cdot) dz, \quad (2.13)$$

and the small-scale Reynolds number based on the vertical velocity and the horizontal length scale is given, in the non-dimensional RRRiNSE formulation, by $Re(t) \equiv \sqrt{\langle w^2 \rangle_{vol}}$. In addition, we denote the combined space and time average by

$$\langle \cdot \rangle = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} dt \frac{1}{L_x L_y} \int_0^{L_x} dx \int_0^{L_y} dy \int_0^1 (\cdot) dz. \quad (2.14)$$

3. Results

Here, we present the first DNS of RRRBC at realistically small Ekman numbers for planetary, satellite and stellar interiors, using the RRRiNSE formulation, and analyse the physics of the flow in this extreme parameter range. Our extensive set of simulations was performed using the pseudospectral solver Coral (Miquel 2021) for a periodic domain of horizontal extent $10\ell_c$, where $\ell_c \approx 4.82Ek^{1/3}$ is the critical wavelength in the limit $Ek \rightarrow 0$ (Chandrasekhar 1953). Specifically, we consider a fluid with $\sigma = 1$ subject to stress-free boundaries (for simplicity) at top and bottom where constant temperatures are imposed, and periodic boundary conditions in the horizontal directions. We explore unprecedentedly small but finite Ekman numbers, as low as $Ek = 10^{-15}$, values that are more than six orders of magnitude smaller than those previously attainable in DNS or in the laboratory.

3.1. Overview of parameter space

[Figure 1\(a\)](#) shows the parameter space explored in our simulations. The top panel shows the estimates of these parameters for planetary interiors and oceans on icy satellites given in [table 1](#) in the plane spanned by the Ekman number Ek and the bulk Reynolds number Re_H , together with blue circles showing the values reached in our simulations, indicating that we attain realistically small Ek equal to the estimated values for the outer core of the Earth ($Ek \approx 10^{-15}$) and the convective zone of the Sun (not shown, cf. [table 1](#)). We further indicate the approximate range of parameters reached in previous numerical and laboratory studies, which are restricted to higher Ek . Dashed lines highlight important regime boundaries where the Rossby number Ro equals unity in different parts of the flow. The green dashed line is an estimate of where the bulk of the flow is expected to become rotationally supported as the rotation rate increases, namely $Ro_H = 1$, i.e. $Re_H = 1/Ek$. In contrast, the rotation rate required to generate rotational support in the thermal boundary layers at the top and bottom is much higher. This rotation rate is determined by the condition that the local Rossby number in the thermal boundary layer $Ro_{BL} = 1$ (red dashed line). As shown by Julien *et al.* (2012a) this condition is satisfied

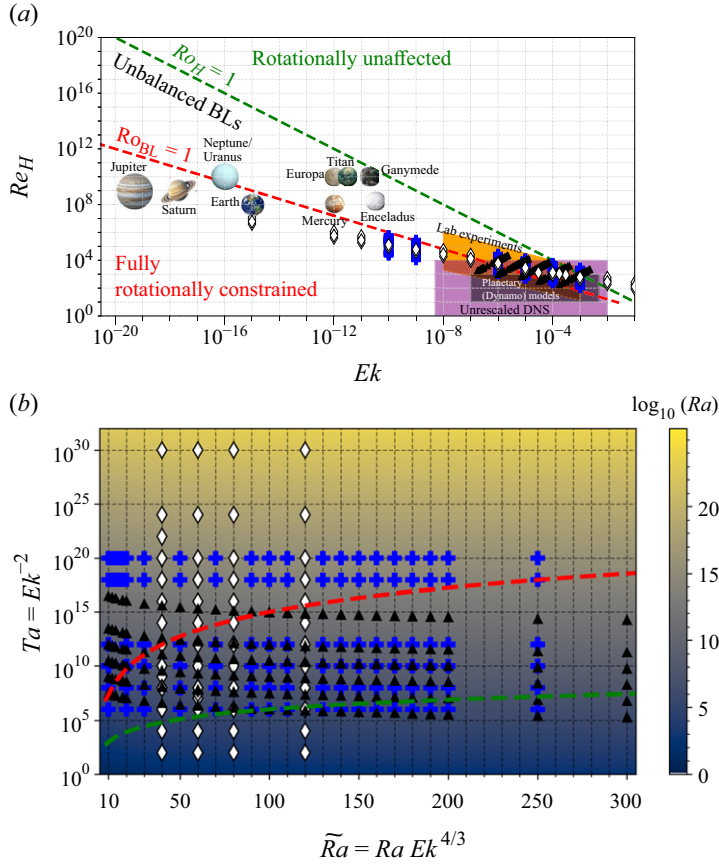


Figure 1. Overview of the parameter space. Panel *a*: estimates of the non-dimensional parameters for different celestial objects (see table 1) shown in the plane spanned by the Ekman and Reynolds numbers. The parameters reached in previous laboratory experiments and simulations are indicated by shaded regions. The dashed lines indicate regime boundaries discussed in the text: the green dashed line corresponds to the bulk Rossby number $Ro_H = 1$ while the red dashed line corresponds to the local Rossby number $Ro_{BL} = 1$ in the thermal boundary layers (BLs). These lines are given by $Re_H = 1/Ek$ and $Re_H = Ek^{-3/5}$, respectively. The regime bounded by these two dashed lines is characterised by unbalanced boundary layers and a balanced bulk flow. The symbols refer to the simulations summarised in panel *b*: $\tilde{Ra} = const.$ (white diamonds), $Ra = const.$ (black triangles) and $Ek = const.$ (blue plus signs). Panel *b*: overview of simulations (with symbols identical to panel *a*) in the plane spanned by the reduced Rayleigh number $\tilde{Ra} = Ra Ek^{4/3}$ and the Taylor number $Ta = Ek^{-2}$. Dashed lines represent regime boundaries corresponding to those in panel *a*.

when the convective Rossby number associated with the bulk flow $Ro_{conv} = Ek^{1/5}$. With velocity following the rotational free-fall scale, $U = \alpha g \Delta T H^2 / (2\Omega)$, we see that $Re_H = Ro_{conv}^2 / Ek$, cf. Julien *et al.* (2024), and hence that $Re_H = Ek^{-3/5}$ (red dashed line in figure 1a).

In rapidly rotating flows, higher values of Ra are required to trigger convection as a consequence of the Taylor–Proudman constraint (Proudman 1916). Specifically, the critical Rayleigh number for convection in the limit of small Ek is proportional to $Ek^{-4/3}$ as shown by Chandrasekhar (1953). Therefore, it is natural to introduce $\tilde{Ra} \equiv Ra Ek^{4/3}$, the reduced Rayleigh number, which measures the supercriticality of rapidly rotating convection. Panel *b* of figure 1 shows an alternative representation of the parameter space in terms of the reduced Rayleigh number \tilde{Ra} and the Taylor number $Ta = Ek^{-2}$.

We reach Taylor numbers up to 10^{30} and reduced Rayleigh numbers up to 300, corresponding to rapidly rotating and highly turbulent flows. The associated bare Rayleigh number Ra , indicated by colours in the background, ranges over a similarly wide interval from $O(10^3)$ to very large values in excess of 10^{20} .

This wide array of numerical simulations allows us to probe, for the first time in DNS, the physics of rotating convection from moderate Ekman numbers down to the fully rotationally constrained regime, previously only accessible in asymptotically reduced equations (Sprague *et al.* 2006; Julien *et al.* 2012a) within the $Ek \rightarrow 0$ limit.

3.2. Flow morphology – bulk and boundary layers

An important feature of rapidly rotating flows is the formation of large-scale flow structures (vortices or jets), associated with spectral condensation of kinetic energy at the domain scale as a result of an inverse energy cascade (Favier *et al.* 2014; Guervilly *et al.* 2014; Rubio *et al.* 2014). This behaviour, which is robust for rotating convection in different domain geometries (Guervilly & Hughes 2017; Julien *et al.* 2018; Lin & Jackson 2021), is also observed more broadly in anisotropic turbulent flows, such as stably stratified rotating turbulence, turbulence subject to strong magnetic fields or geometric confinement to a thin-layer geometry, among others (Alexakis & Biferale 2018; Alexakis 2023; van Kan 2024). In the case of a periodic, horizontally square Cartesian domain, the condensate in the limit of asymptotically large rotation $Ek \rightarrow 0$ takes the form of a large-scale vortex dipole, first observed in the reduced equations (Julien *et al.* 2012b; Rubio *et al.* 2014), resembling findings in two-dimensional turbulence (Boffetta & Ecke 2012). In contrast, DNS of rotating convection at previously attainable moderately small Ekman and Rossby numbers reveal an asymmetry between cyclones and anticyclones, namely a preference for strong cyclonic structures and diffuse, weaker anticyclones (Favier *et al.* 2014). While a dominance of cyclones over anticyclones is also observed in stably stratified flows at moderately small Rossby numbers (Roulet & Klein 2010; Gallet *et al.* 2014), the transition from the moderately small Rossby number regime to very small Ek (relevant to the celestial bodies listed in table 1), where the cyclone–anticyclone symmetry is expected to be restored, could not previously be reached in RRRBC.

To address this question, we illustrate and analyse in figure 2 how the vertically averaged (i.e. barotropic) vertical vorticity $\overline{\omega_z}(x, y)$ (see § 2) changes as Ek varies for a fixed supercriticality $Ra = 120$. At moderate rotation rates ($Ek = 10^{-6}$ and $Ro \approx 0.4$, panel *d*), a striking asymmetry exists between the strong, coherent cyclone and a weak, incoherent anticyclonic region. Here, the *a posteriori* Rossby number Ro is computed as the root mean square of the ratio $\|\mathbf{u} \cdot \nabla \mathbf{u}_\perp\| / (2\Omega \|\mathbf{u}_\perp\|)$, where $\mathbf{u}_\perp = (u, v, 0)$. As rotation increases (top row of figure 2, right to left), this asymmetry decreases but remains visible for $Ek = 10^{-8}$ ($Ro \approx 0.07$, panel *b*), a value close to the current state of the art for RRRBC studies. The anticipated cyclone–anticyclone symmetry present in the asymptotic rapidly rotating limit $Ek \rightarrow 0$ based on the NHQG equations is in fact observed for $Ek = 10^{-15}$ ($Ro \approx 8 \times 10^{-5}$, panel *a*). To quantify this qualitative observation, we display in figure 2(e) the skewness of the barotropic vertical vorticity

$$\text{Skew}(\overline{\omega_z}) = \left\langle (\overline{\omega_z} - \langle \overline{\omega_z} \rangle)^3 \right\rangle / \left\langle (\overline{\omega_z} - \langle \overline{\omega_z} \rangle)^2 \right\rangle^{3/2}. \quad (3.1)$$

A finite positive value, indicating the prevalence of strong cyclones, is obtained for $Ek \gtrsim 10^{-9}$, whereas an approximately vanishing skewness is observed at smaller Ekman numbers, indicating a restored (statistical) cyclone–anticyclone symmetry, with a relatively sharp transition between the two regimes.

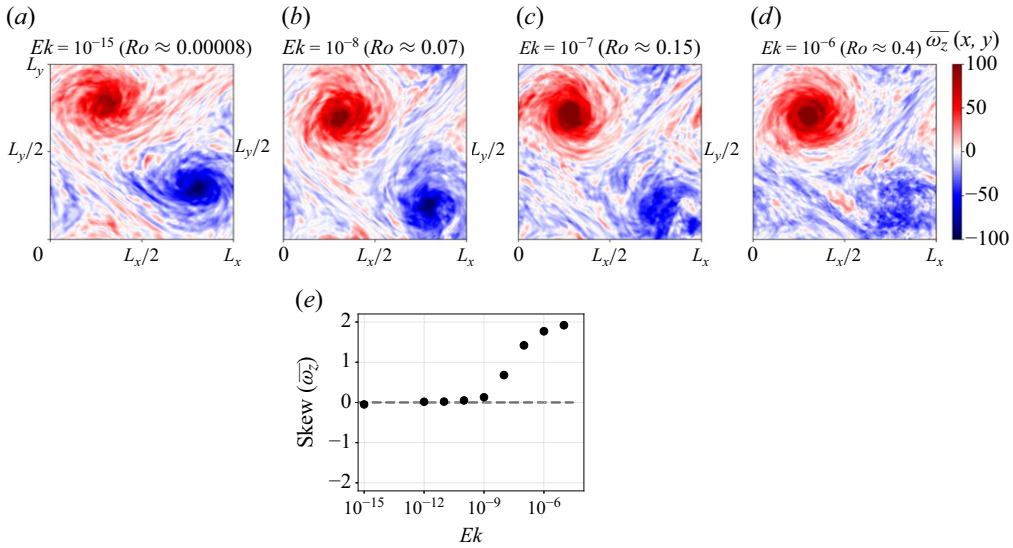


Figure 2. Top row: vertically averaged (i.e. barotropic) vertical vorticity field $\overline{\omega_z}(x, y)$ at $Ek = 10^{-15}$ (panel a), 10^{-8} (panel b), 10^{-7} (panel c), 10^{-6} (panel d) and $Ra = 120$ (all panels). Panel e: skewness of the barotropic vorticity field versus Ek indicating a strong cyclone–anticyclone asymmetry for $Ek \gtrsim 10^{-9}$, and an approximate (statistical) cyclone–anticyclone symmetry for $Ek \lesssim 10^{-9}$.

In addition to the barotropic component of the flow, which informs us about the bulk flow structure, the flow inside the thermal boundary layer near the stress-free, constant-temperature boundaries at the top and bottom is of great importance. Figure 3(a) shows visualisations of the flow inside the bottom thermal boundary layer (at height $z = \delta_{\omega_z}$, defined thereafter, cf. figure 5) at $Ra = 120$ for four distinct Ekman numbers $Ek = 10^{-15}$, 10^{-8} , 10^{-7} , 10^{-6} in terms of three fields: the vertical vorticity ω_z , the temperature perturbation θ away from the horizontally averaged temperature profile and the vertical velocity w at the top of the (bottom) boundary layer, where the flow transitions between qualitatively distinct regimes. Figure 3(b) shows histograms of each field at the same Ekman numbers, while figure 3(c) displays the associated statistical skewness versus Ek . At very low Ekman numbers, $Ek = 10^{-15}$, where the bulk flow features two large-scale vortices and displays an approximate statistical cyclone–anticyclone symmetry, the same symmetry is obeyed in the boundary layer as evidenced by the approximately vanishing skewness not only of ω_z , but also θ and w (see also Supplementary Movies 1–3). A near perfect correlation can be seen between the vertical vorticity and θ . We mention that the RRRINSE formulation remains numerically stable even when the Ekman number is reduced further, as far as $Ek = 10^{-24}$; the corresponding flow fields obtained from DNS are shown in Appendix A.

As the Ekman number increases, the approximate symmetry of the histograms persists up to $Ek \approx 10^{-9}$ (not shown). At $Ek \approx 10^{-8}$, where cyclone–anticyclone asymmetry and cyclone dominance emerges in the barotropic mode, the boundary layer flow morphology is also drastically altered: in addition to the trace of the strong large-scale cyclone, similarly strong, albeit short-lived, anticyclonic vortices emerge in the boundary layer, whose anticyclonic core is seen in some cases to be surrounded by a cyclonic shield (see also Supplementary Movie 4). These strong vortices with anticyclonic cores have a clear signature in the temperature fluctuation θ as strong, cold perturbations (in the

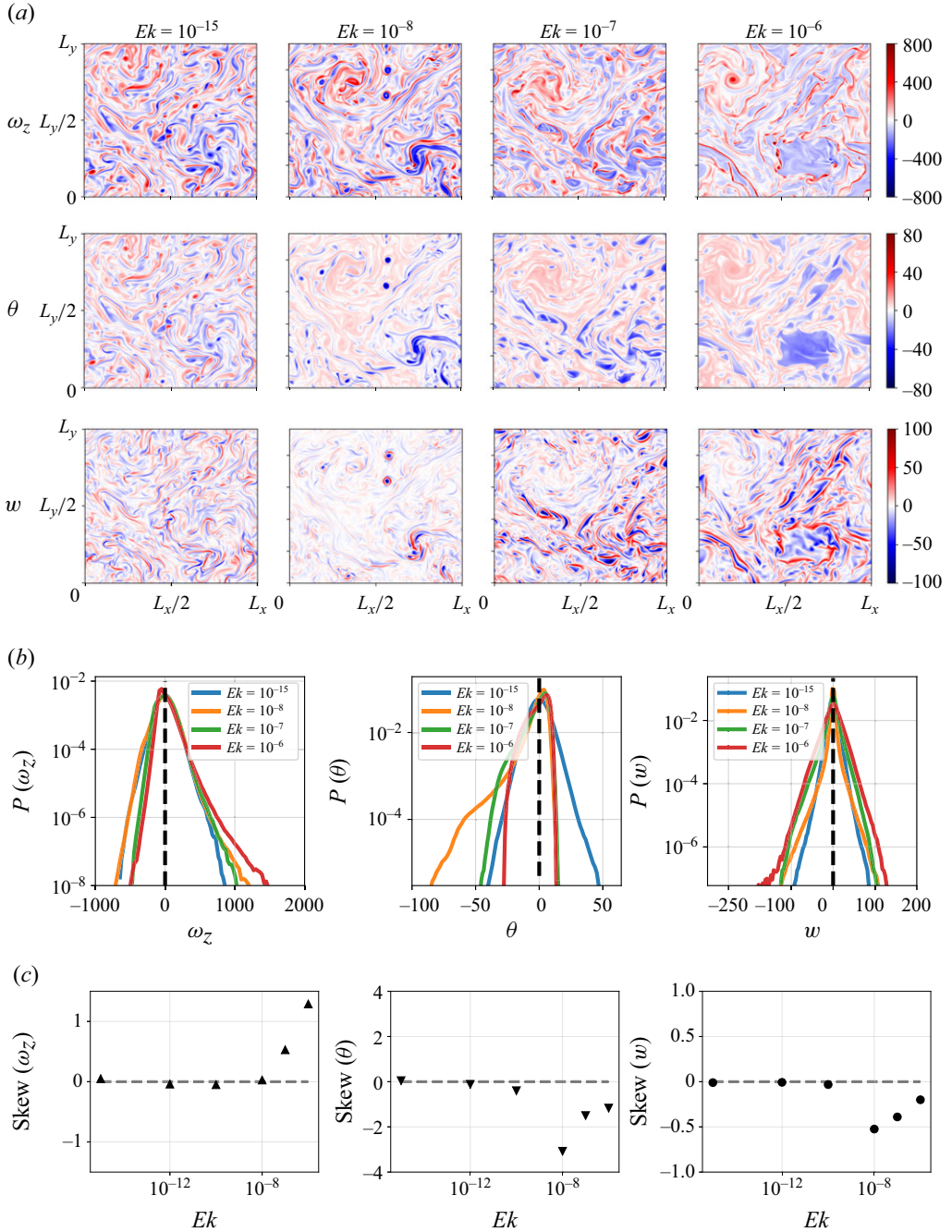


Figure 3. Panel *a*: overview of boundary layer flow morphology in terms of the vertical vorticity ω_z , temperature perturbation θ and vertical velocity w at different Ek and $Ra = 120$. Panel *b*: histograms of ω_z , θ , w at different Ek . Near-Gaussian statistics are observed at $Ek = 10^{-15}$ with larger skewness at larger Ek . Panel *c*: skewness of ω_z , θ , w versus Ek . All results shown are obtained at $z = \delta_{\omega_z}$ (top of the momentum boundary layer near the bottom boundary) defined in figure 5(a).

bottom boundary layer shown here, see Supplementary Movie 5), and also in the vertical velocity w which shows descending flow inside the vortex core, surrounded by rising fluid motion (see Supplementary Movie 6). As a result, the statistics of θ and w are skewed towards negative values at this Ek . It is important to stress that the structures displayed in figure 3 and discussed above are confined within the boundary layer, and do not extend into the bulk flow. To the best of our knowledge, the strong, partially shielded, short-lived anticyclonic vortices in the boundary layer with stress-free boundaries are described here for the first time, although shielded structures have been previously observed in various studies of rotating convection (Sprague *et al.* 2006; Julien *et al.* 2012b; Stellmach *et al.* 2014; Song *et al.* 2024b). We also note that similar structures were predicted based on Ekman layer theory for no-slip boundary conditions by Julien *et al.* (1996) in response to a Gaussian temperature perturbation. As Ek is increased to 10^{-7} and 10^{-6} , cyclones clearly dominate the flow in the boundary layer (as in the bulk), while the strong localised anticyclones become more diffuse, leading to a significant skewness in the statistics of ω_z . The responses in θ and w remain visible in the boundary layer, particularly outside the large-scale cyclone, although their skewness decreases with further increase in Ek .

3.3. Flow statistics

Going beyond the visualisations shown in figure 3, we display in figure 4 key quantitative information on the flow in terms of important non-dimensional numbers. We define the non-dimensional heat flux across the layer, measured by the Nusselt number $Nu - 1 \equiv \sigma \langle w\theta \rangle$ where $\langle \cdot \rangle$ is the combined volume and time average, and denote by $Nu(t)$ the instantaneous Nusselt number obtained by averaging only over the volume, not time. In addition, we measure the small-scale Reynolds number Re , based on the horizontal length scale and the root-mean-square vertical velocity, by taking the time average of the instantaneous Reynolds number $Re(t) = \sqrt{\langle w^2 \rangle_{vol}}$ (see § 2). Both Nu and Re are commonly used to quantify turbulent convection since their statistics are known to converge quickly in the nonlinear regime following the initial exponential growth phase of the convective instability, see e.g. Maffei *et al.* (2021). This remains true even in the presence of an inverse cascade leading to a slow growth in the horizontal kinetic energy. Figure 4(a) displays the time series of the instantaneous Nusselt number at $Ek = 10^{-10}$ for three different reduced Rayleigh numbers $\tilde{Ra} = 40, 60, 80$, showing increasing turbulent heat flux (mean and fluctuations) with increasing \tilde{Ra} . Rescaling the Nusselt numbers by the well-known turbulent scaling law (Julien *et al.* 2012b) for rapidly rotating convection $Nu - 1 \sim \tilde{Ra}^{3/2}$ (see panel b), the data collapse satisfactorily for $\tilde{Ra} = 60, 80$, while the least turbulent simulation at $\tilde{Ra} = 40$ shows a small mismatch. The corresponding theoretical scaling prediction for the Reynolds number is $Re \sim \tilde{Ra}$, which we use to rescale our Reynolds number data.

Averaging over time series like those shown in the top row of figure 4, we obtain the Nusselt and Reynolds number statistics in panels c and d of figure 4, shown versus the Taylor number $Ta = Ek^{-2}$. Specifically, four sets of simulations at constant $\tilde{Ra} = 40, 60, 80, 120$ are shown for a wide range of $10^2 \leq Ta \leq 10^{30}$. Triangles in different orientations for each \tilde{Ra} show the observed average Nu and Re , while error bars indicate the measured standard deviation about those mean values. Horizontal dashed lines indicate, for each \tilde{Ra} , the values corresponding to the reduced NHQG equations, with a shaded region indicating the corresponding standard deviation measured in the NHQG equations. At small Ta (large Ek), for the given \tilde{Ra} , the flow is only weakly

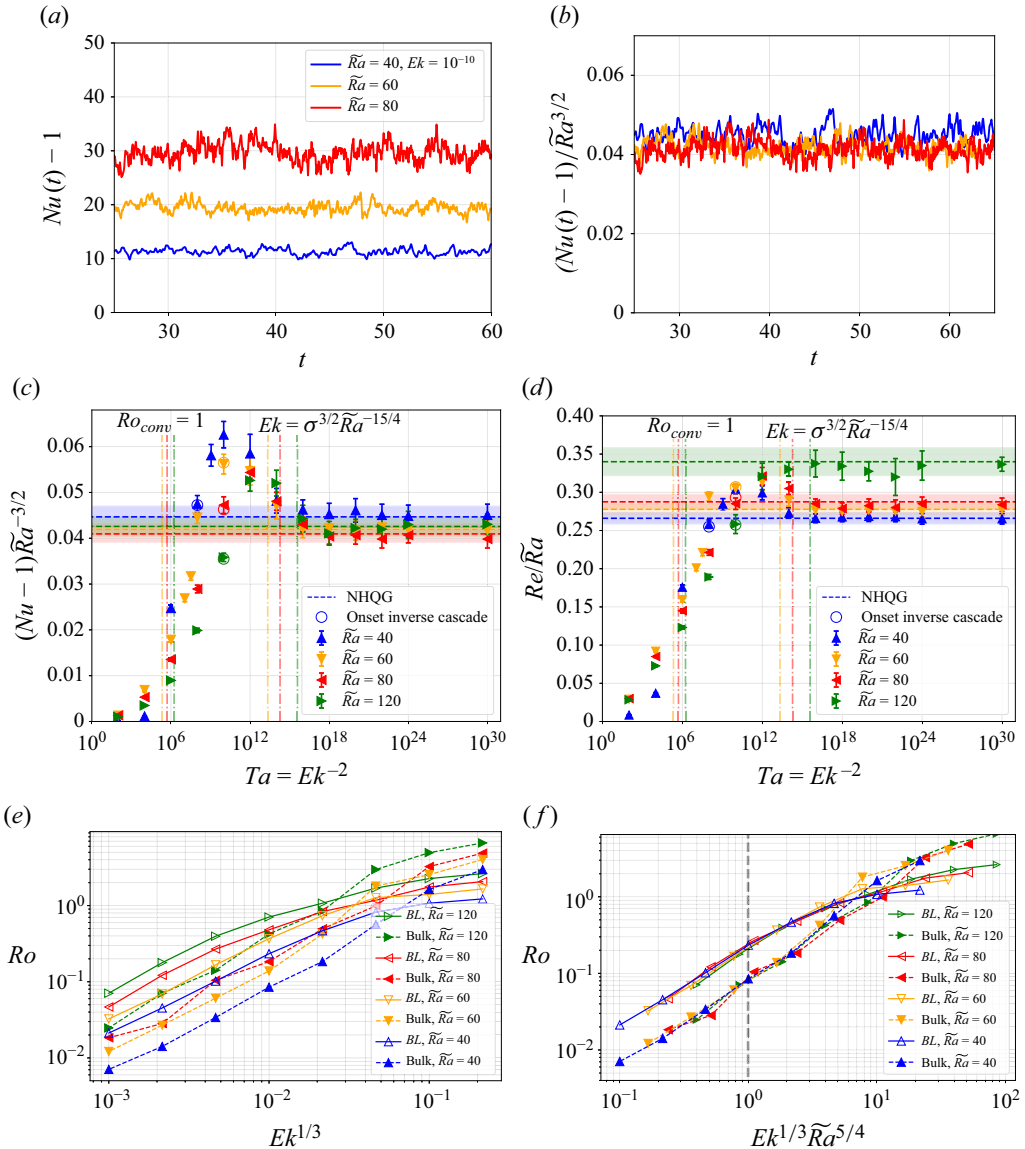


Figure 4. Panel *a*: sample time series of $Nu - 1$ at $Ek = 10^{-10}$ for $\tilde{Ra} = 40, 60, 80$ in the statistically stationary state, revealing turbulent fluctuations about a well-defined mean value. Both mean and amplitude of fluctuations increase with \tilde{Ra} . Panel *b*: same data, collapsed by rescaling the y axis by $\tilde{Ra}^{3/2}$. Panel *c*: temporally averaged $Nu - 1$ in the stationary state, compensated by $\tilde{Ra}^{3/2}$, with error bars indicating standard deviation, vs $Ta = Ek^{-2}$. Here, Nu increases with Ta at fixed \tilde{Ra} up to a maximum, beyond which it decreases to converge to a Ta -independent constant, which agrees with the value obtained in the NHQG equations at the same \tilde{Ra} (horizontal dashed line), within one standard deviation (shaded area). Vertical dashed lines (same colour scheme) correspond to $Ro_{conv} = 1$ and $Ro_{BL} = 1$. Panel *d*: corresponding data for the small-scale Reynolds number $Re \equiv \sqrt{\langle w^2 \rangle}$, compensated by \tilde{Ra} , show a similar structure with an overshoot and eventual convergence to the NHQG prediction at large Ta . Panel *e*: *a posteriori* Rossby numbers versus $Ek^{1/3}$ for the bulk and the boundary layer, both of which exhibit collapse in panel *f* when shown versus $Ek^{1/3} \tilde{Ra}^{5/4}$.

supercritical and therefore features only small Nu and Re . A break is seen in the curves around $Ta = 10^6$ ($Ek = 10^{-3}$), where the bulk convective Rossby number equals one, marked by faint vertical dashed lines, indicating that the bulk flow begins to be rotationally constrained. The inverse cascade sets in around $Ta = 10^8$ for $\widetilde{Ra} = 40$, and closer to $Ta = 10^{10}$ for larger $\widetilde{Ra} = 60, 80, 120$ (hollow circular markers). This is in agreement with the literature, see e.g. Favier *et al.* (2014). As Ta increases (Ek decreases) further from this point, an overshoot in Nu is observed leading to a maximum in the convective heat transport for the given control parameter \widetilde{Ra} , whose amplitude and location depend on \widetilde{Ra} . This is followed by convergence within one standard deviation to the value obtained in the NHQG equations at yet larger Ta , beyond a threshold indicated by a second set of vertical dashed lines, one for each \widetilde{Ra} , corresponding to convective Rossby number in the boundary layer of order unity, a condition equivalent to $Ek = \sigma^{3/2} \widetilde{Ra}^{-15/4}$ (Julien *et al.* 2012a), i.e. the rotation rate where the boundary layer also becomes rotationally constrained. These regime boundaries are equivalent to those shown in the top panel of figure 1. A similar structure is observed in the Reynolds number data, which collapse slightly less satisfactorily under the rescaling (this is known to be a consequence of the presence of the large-scale vortex (Maffei *et al.* 2021; Oliver *et al.* 2023)). At small Ta , the flow is only weakly supercritical and therefore the Reynolds number is small. It increases with Ta up to a maximum, beyond which it decreases to converge (within one standard deviation) to the value obtained from the NHQG equations. Since Taylor numbers in most laboratory experiments or unrescaled DNS have been limited to $Ta \lesssim 10^{14} - 10^{16}$, this Nu and Re vs Ta dependence has not previously been described. In particular, the presence of a maximum in the heat transport and turbulence intensity at a finite Taylor (or equivalently, Ekman) number as well as the statistical convergence to the NHQG equations are reported here for the first time.

In addition to the *a priori* (bulk/boundary layer) convective Rossby numbers shown by the dashed vertical lines in panels *c* and *d* of figure 4, panels *e* and *f* show the *a posteriori* Rossby number Ro computed as the root mean square of the ratio $\|\mathbf{u} \cdot \nabla \mathbf{u}_\perp\| / (2\Omega \|\mathbf{u}_\perp\|)$ in the bulk and in the boundary layer, where it takes distinct values (see Appendix A). When shown versus $\varepsilon = Ek^{1/3}$, the standard small parameter of rotating convection, the data, while scattered, clearly show that the bulk Rossby number increases monotonically with Ek as expected, crossing unity near the observed onset of the inverse cascade. It is interesting that this occurs at Ekman numbers which are two orders of magnitude smaller than what may naively be expected based on the convective Rossby number $Ro_c = 1$. When the same data are plotted against the rescaled parameter $Ek^{1/3} \widetilde{Ra}^{5/4}$, it collapses to a remarkable degree, both in the bulk and in the boundary layer. We stress that this is somewhat surprising, since this effective small parameter is obtained explicitly from the derivation of the reduced equations for the boundary layer (Julien *et al.* 2012a,b). However, we note that the bulk flow develops cyclone–anticyclone symmetry at similar Ekman numbers as the boundary layer, which is another empirical link between the bulk and boundary layer flows observed here. The collapse of the data in figure 4(e,f) implies in particular that when $Ek^{1/3} \widetilde{Ra}^{5/4} = 1$, below which the average Nusselt and Reynolds numbers converge to the NHQG data, there is a well-defined associated critical Rossby number in the boundary layer, which is found to be approximately 0.2. This is where the boundary layer loses rotational constraint and becomes unstable to strong ageostrophic motions with pronounced vertical gradients. These additional vertical gradients are associated with additional dissipation contributions. In steady state, the dissipation rates of kinetic energy and temperature variance are directly linked to the Nusselt number, via

the so-called power integrals. In the RRRiNSE formulation, these read

$$\langle |\nabla_{\perp} \theta|^2 \rangle + \varepsilon^2 \langle (\partial_z \theta)^2 \rangle + \langle (\partial_z \Theta)^2 \rangle = Nu - 1, \quad (3.2a)$$

$$\langle |\partial_x \mathbf{u}|^2 + |\partial_y \mathbf{u}|^2 \rangle + \varepsilon^2 \langle |\partial_z \mathbf{u}|^2 \rangle = \frac{\widetilde{Ra}}{\sigma^2} (Nu - 1), \quad (3.2b)$$

where $\langle \cdot \rangle$ is the combined time and volume average over the whole domain. Directly measuring all contributions in our simulations, we find that both balances are approximately satisfied (indicating that non-stationarity is weak, see [Appendix A](#)), with important contributions to the dissipation of temperature variance originating in the boundary layers near $Ek^{1/3} \widetilde{Ra}^{5/4} = 1$. The increase in the Nusselt number away from the small Ek NHQG limit can therefore be understood as a consequence of increased dissipation due to enhanced (vertical and horizontal) gradients, at least in part due to the strong anticyclonic vortical structures appearing in the boundary layer at $Ek = 10^{-8}$ (which is near $Ek^{1/3} \widetilde{Ra}^{5/4} = 1$ for $\widetilde{Ra} = 120$), cf. [figure 3](#).

3.4. Boundary layer structure

From the above analysis, it is apparent that the boundary layers play a non-trivial role in the transition from small to large Ek , controlling the departure from the NHQG branch. To further characterise the boundary layer structure, we measure and analyse vertical root-mean-square profiles of different quantities as a function of z . [Figure 5\(a\)](#) shows a log–log plot of the thermal boundary layer depth δ_{θ} computed as the location of the maximum in the root-mean-square vertical profile of θ (shown in the inset for different Ek at $\widetilde{Ra} = 120$). At small Ekman numbers, δ_{θ} converges to a constant value for a given reduced Rayleigh number, corresponding to the NHQG limit. [Figure 5\(c\)](#) confirms that the thermal boundary layer depth scales with $\widetilde{Ra}^{-15/8}$, which is the scaling obtained from reduced NHQG equations for the boundary layer (Julien *et al.* 2012a,b). As Ek is increased close to the point where the boundary layer loses rotational support ($Ek = \sigma^{3/2} \widetilde{Ra}^{-15/4}$, not shown), the boundary layer depth undershoots before increasing with a power law not far from $Ek^{1/2}$ (and steeper than $Ek^{1/3}$, not shown). [Figure 5\(b\)](#) shows an alternative measure of the boundary layer thickness, based on the root-mean-square vertical profile of $\partial_z \omega_z$. The inset shows sample profiles at $\widetilde{Ra} = 120$ for different Ekman numbers. A local maximum is seen in these profiles, which we use to define a momentum boundary layer thickness δ_{ω_z} . Since we are considering stress-free boundaries, where no linear Ekman layer is present, the existence of this momentum boundary layer is non-trivial. We interpret δ_{ω_z} as a measure of how far the effect of the stress-free boundary penetrates into the bulk of the flow.

We have furthermore verified that the momentum boundary layer seen here in the RRRiNSE formulation is also present in the reduced NHQG equations (not shown), although to our knowledge this has not been described previously. Like δ_{θ} , at sufficiently small Ek , δ_{ω_z} becomes constant, while at larger values of Ek , there is again an undershoot followed by a power-law scaling regime where δ_{ω_z} increases approximately as $Ek^{1/2}$. The approximate $Ek^{1/2}$ scaling, which is observed both in terms of δ_{θ} and in δ_{ω_z} and is also observed for no-slip boundary conditions (Song *et al.* 2024b), is surprising in the present case, given that we consider stress-free boundaries, which do not lead to the formation of linear Ekman layers. While a boundary layer thickness proportional to $Ek^{1/2}$ is theoretically predicted for the transitional thermal boundary layer (Julien *et al.* 2012b), the presence of such a scaling at significantly larger Ekman numbers, where the boundary

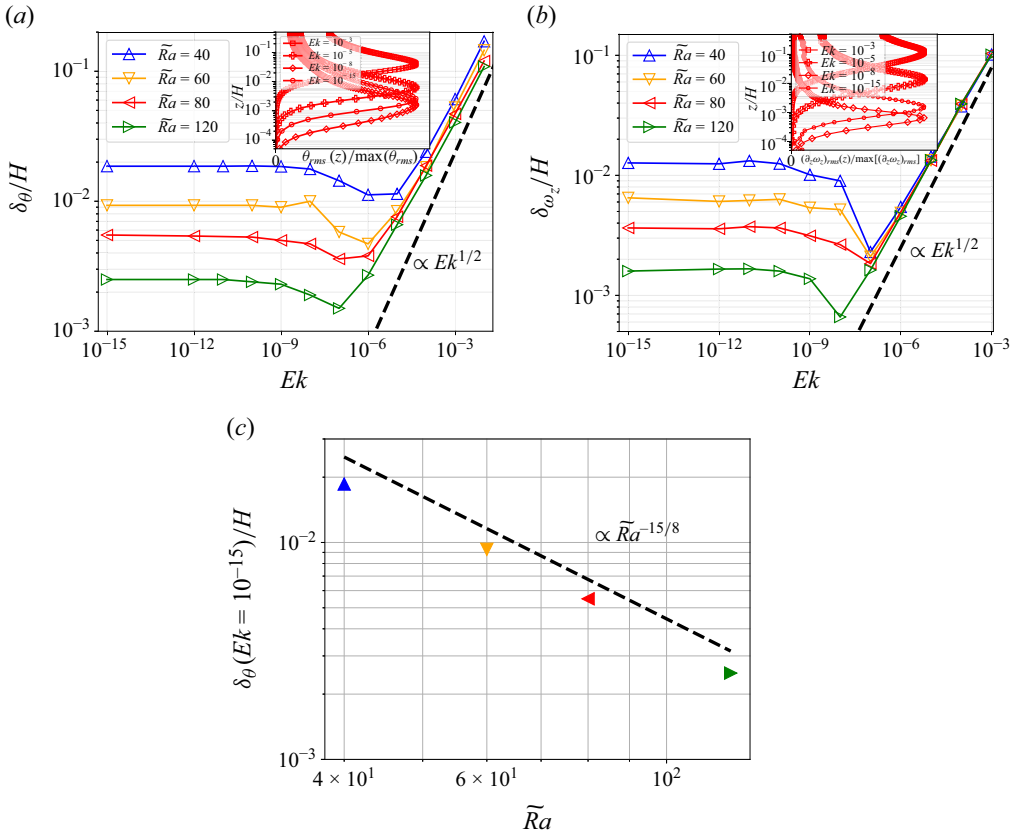


Figure 5. Boundary layer structure for a range of Ek and \tilde{Ra} . Panel *a*: log-log plot of the non-dimensional widths of the thermal boundary layer δ_θ/H , given by the location in z of the maxima in the vertical root-mean-square (r.m.s.) profiles of θ shown in the inset, versus Ek for different \tilde{Ra} . Dashed line indicates a $Ek^{1/2}$ power law. Panel *b*: log-log plot of the momentum boundary layer thickness δ_{ω_z}/H based on the location of the maxima of the r.m.s. profile of $\partial_z \omega_z$ shown in the inset. Panel *c*: log-log plot of the non-dimensional thermal boundary layer thickness δ_θ/H at $Ek = 10^{-15}$ compared with the power law $\tilde{Ra}^{-15/8}$ predicted by the NHQG equations (Julien *et al.* 2012*b*).

layer flow is no longer rotationally dominated, goes beyond this prediction and the precise origin of this scaling law remains to be elucidated.

3.5. Alternative cuts through parameter space

The results presented thus far were obtained along a particular cut through the physical control parameter space, namely varying Ek at fixed \tilde{Ra} . This is the natural approach in the framework of the RRRinSE. By contrast, keeping \tilde{Ra} fixed in an experimental setup or in an unrescaled DNS requires tuning two parameters. In such contexts, it is instead more natural to vary Ra at fixed Ek (equivalent to varying \tilde{Ra} at fixed Ek) or vary Ek at fixed Ra (equivalent to varying \tilde{Ra} at fixed Ra). To make contact with existing experimental and numerical studies, the corresponding cuts through parameter space are shown in figure 6. Panel *a* shows $Nu - 1$ versus Ra for different sets of RRRinSE simulations, each with Ra fixed to a different value between $Ra = 10^6$ and $Ra = 10^{10}$, as well as the corresponding values from the NHQG equations ($Ra \rightarrow \infty$) at the same \tilde{Ra} . The NHQG data approximately follow the turbulent $\tilde{Ra}^{3/2}$ scaling law, cf. Figure 4, while at finite

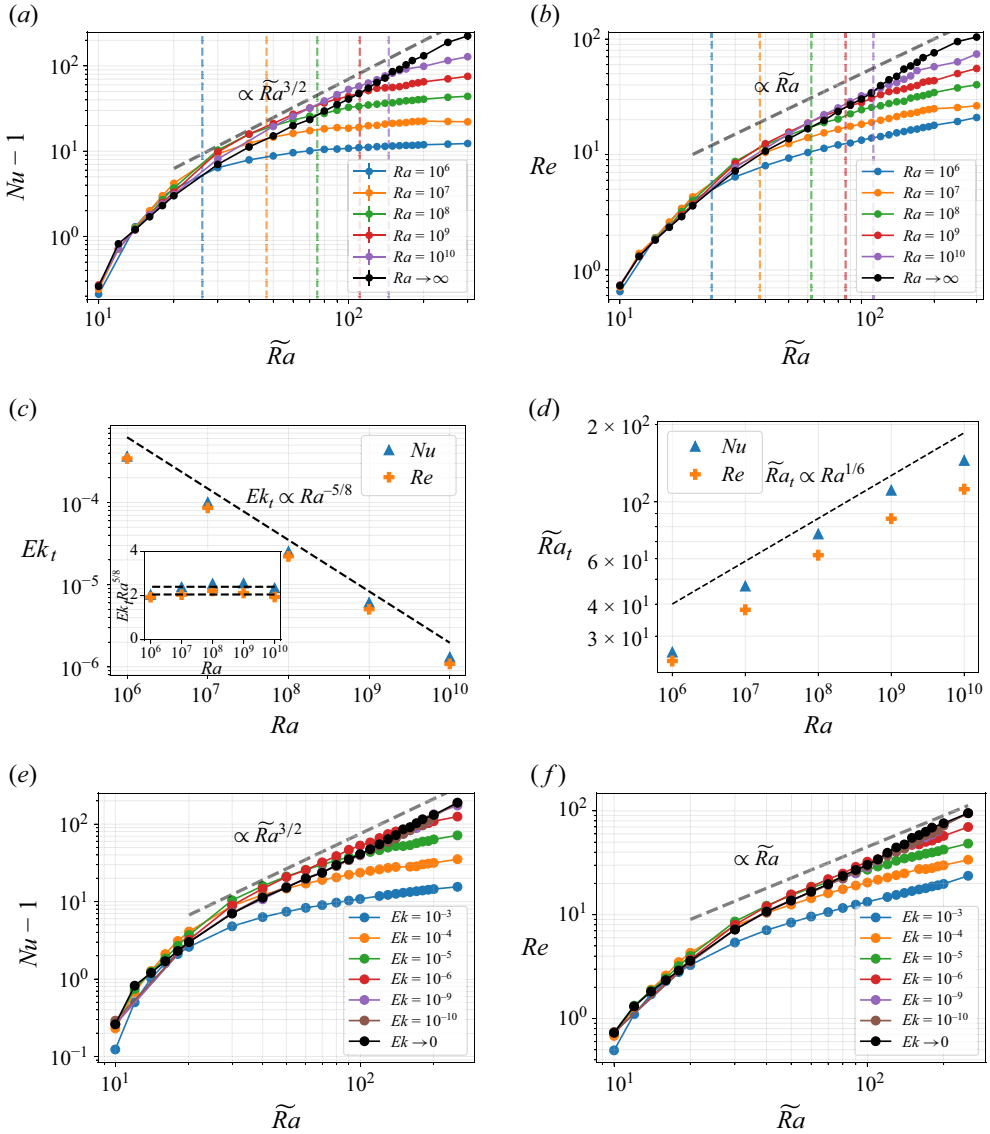


Figure 6. Nusselt number (panel *a*) and Reynolds number (panel *b*) versus the reduced Rayleigh number \tilde{Ra} at fixed bare Rayleigh number Ra . Dashed lines indicate intersection of a given $Ra = \text{const.}$ set with the asymptotic limit. Transitional Ekman number (panel *c*) and reduced Rayleigh number (panel *d*) versus the bare Rayleigh number Ra . $Nu - 1$ (panel *e*) and Re (panel *f*) versus \tilde{Ra} at fixed Ekman number.

Ra , there is a well-defined transitional value \tilde{Ra}_t of \tilde{Ra} where $Nu - 1$ intersects the $Ra \rightarrow \infty$ limit, and subsequently follows a shallower slope. This structure of the Nusselt number flattening out beyond a threshold $\tilde{Ra} = \tilde{Ra}_t$ is robust among the datasets of runs with different Ra . Qualitatively similar behaviour was recently reported for RRRBC with no-slip boundaries (Song *et al.* 2024b), although the eventual low Ek regime remains elusive in that case. Panel *b* shows the corresponding Reynolds number data for the same simulations. As discussed earlier, the corresponding turbulent scaling law $Re \propto \tilde{Ra}$ has proved to be more elusive owing to the impact of the inverse cascade, see Maffei *et al.* (2021), Oliver *et al.* (2023) and figure 4. Panel *b* shows a similar structure to panel *a*,

in that the Re curves intersect the curve corresponding to the NHQG equations at a (numerically slightly different) threshold value for each \tilde{Ra} . Since these runs are performed at constant Ra , each \tilde{Ra} is equivalent to an Ekman number Ek . Therefore Ra_t is equivalent to a transitional Ekman number Ek_t .

Panel *c* shows Ek_t , corresponding to the vertical dashed lines in panels *a* and *b*, indicating that the data are quantitatively compatible with the scaling law $Ek_t \propto Ra^{-5/8}$. This scaling law corresponds to the theoretical prediction (Julien *et al.* 2012*a*) for the Ekman number at which the boundary layer loses rotational support. Here, we recover this scaling from a simple measurement of the Nusselt and Reynolds numbers. An equivalent representation of the same data is shown in panel *d* in terms of the reduced Rayleigh number \tilde{Ra} versus the bare Rayleigh number Ra , which also reveals satisfactory agreement with the corresponding theoretical scaling prediction $\tilde{Ra}_t \propto Ra^{1/6}$.

Panels *e* and *f* of figure 6 show a third possible cut through the parameter space of rotating convection (qualitatively resembling results presented in Song *et al.* 2024*b*), holding Ek fixed and changing Ra (or equivalently changing \tilde{Ra}). The turbulent scaling laws with \tilde{Ra} compare similarly well with these datasets and exhibit similar changes in scaling in the Nusselt and Reynolds numbers with \tilde{Ra} at finite Ek , up to $Ek = 10^{-6}$, while for $Ek = 10^{-9}$ and $Ek = 10^{-10}$ the values of \tilde{Ra} attainable in our simulations were not sufficiently high to observe a departure from the asymptotic curve corresponding to the NHQG equations.

The flattening of the Nusselt number curves with increasing \tilde{Ra} at finite Ekman numbers is a robust signal that could potentially be reproduced in laboratory experiments. Our results in figure 6 qualitatively resemble similar findings recently obtained for RRRBC with no-slip boundaries (Song *et al.* 2024*b*).

4. Discussion

The results described above show that the RRRiNSE formulation, based on low Ekman number asymptotics, allows efficient DNS far beyond the current state of art in laboratory experiments or unrescaled DNS, down to $Ek = 10^{-15}$ and below, which is comparable to the estimated values of Ek in the outer core of the Earth and the convection zone of the Sun. Specifically, this indicates that the RRRiNSE enable DNS at Ekman numbers which are over six orders of magnitude smaller than the smallest value obtained previously, in an impressive effort and at great computational cost, by means of unrescaled DNS (Song *et al.* 2024*a,b*). Using the RRRiNSE formulation, we revealed rich new physics in this previously inaccessible parameter regime. First, we uncovered a transition at $Ek \approx \sigma^{3/2} \tilde{Ra}^{-15/4}$ ($Ek \approx 10^{-9}$ for $\tilde{Ra} = 120$) from a regime at larger Ek where the depth-averaged flow features a strong large-scale cyclone and a weak diffuse anticyclone towards a regime characterised by cyclone–anticyclone symmetry and a large-scale vortex dipole, in agreement with the predictions of the NHQG equations. Second, we identified a non-trivial transition in the boundary layer dynamics corresponding to the loss of rotational support in the boundary layer (confirming a previously untestable theoretical prediction of Julien *et al.* 2012*b*), a transition accompanied by the emergence of strong, albeit short-lived, anticyclonic structures near the Ek threshold that weaken as Ek increases, leading to dominance of strong cyclonic structures. It is interesting to note that the cyclone–anticyclone symmetry breaking in the bulk approximately coincides in Ek with the boundary layer transition, although a theoretical explanation of this observation remains unavailable. While flows in the Earth’s atmosphere are characterised by only moderately small Rossby numbers and are therefore dominated by strong cyclones, our results suggest

that in the interior of the Earth or other celestial bodies, cyclones and anticyclones may be statistically of equal strength.

We showed quantitatively that the time-averaged Nusselt and Reynolds numbers in steady state reflect these regime transitions, taking values consistent with the NHQG limit for $Ek \lesssim \sigma^{3/2} \widetilde{Ra}^{-15/4}$, while overshooting as Ek is increased at fixed \widetilde{Ra} , reaching a maximum at a bulk Rossby number much less than one, and decreasing to close to zero as Ek increases further at fixed \widetilde{Ra} due to reduced supercriticality. The increase in the Nusselt number was shown to be tied to increased dissipation associated with a change in the boundary layer flow morphology. The boundary depth was quantified in terms of the thermal fluctuations and vertical vorticity, revealing that, at $Ek \lesssim \sigma^{3/2} \widetilde{Ra}^{-15/4}$, the boundary layer depth becomes Ek -independent, taking the value associated with the NHQG equations, but undershooting for $Ek \gtrsim \sigma^{3/2} \widetilde{Ra}^{-15/4}$ before increasing approximately with an Ekman layer-like scaling $Ek^{1/2}$ as Ek increases. Owing to the presence of stress-free boundary conditions, this behaviour cannot be explained by a linear Ekman layer and is therefore a nonlinear effect. Finally, we have considered alternative cuts through the parameter space, with one set of runs varying Ek at fixed Ra and another set of runs varying Ra at fixed Ek . This procedure revealed that at finite Ekman numbers, the Nusselt and Reynolds numbers remain close to the values found in the NHQG equations as \widetilde{Ra} increases, but start to deviate from them at a value of \widetilde{Ra} that depends on Ra in a way consistent with boundary layer theory (Julien *et al.* 2012a). We mention that the NHQG equations apply to systems with no-slip boundaries since in the limit $Ek \rightarrow 0$ the no-slip boundaries become effectively stress-free (Julien *et al.* 2012b). At finite Ek , however, Ekman boundary layers and the associated Ekman pumping necessarily modify the boundary layer structure, an effect that can be included in the NHQG equations following Julien *et al.* (2016).

The results presented here are based on the RRRiNSE reformulation of the equations and explain why previous state-of-the-art DNS and laboratory experiments could not reach the parameter regime for observing the transition to fully rotationally constrained dynamics. The RRRiNSE formulation therefore opens the door to the further exploration of the parameter regime of very small but finite Ekman and Rossby numbers, highly relevant to planetary, satellite and stellar interiors. This is all the more timely since current and future observational missions prompted by the last Planetary Decadal Survey (National Academies of Sciences, Engineering, and Medicine 2023), including the recently launched ESA JUICE (JUper Icy Moons Explorer) mission (Grasset *et al.* 2013) and NASA's Europa Clipper mission (Howell & Pappalardo 2020) will provide new data requiring interpretation based on faithful model simulations at parameters as close as possible to realistic values. Our results are a first step in this direction.

The RRRBC as studied here, with antiparallel gravity and rotation axis is an appropriate, albeit idealised, model of the North Pole regions of the celestial bodies listed in table 1. Owing to the latitudinal non-uniformity of rotating convection in spherical shells (Gastine & Aurnou 2023), the RRRiNSE formulation cannot be straightforwardly generalised to that setting, but an investigation of the RRRiNSE with misaligned gravity and rotation axes could provide important insights in this regard. While RRRiNSE enables simulations at realistic values of Ek , many challenges remain: the increasing resolution constraint for increasing levels of turbulence cannot be circumvented, and for simplicity our simulations considered stress-free boundaries as well as Prandtl number $\sigma = 1$ while ignoring other important ingredients such as magnetic fields. To further test the applicability of our findings to planets, icy moons and stars, future work must also investigate how the regime transitions described in this study are influenced by additional

effects, including fluids with Prandtl numbers different from one, compressibility, internal heating (Barker *et al.* 2014; Bouillaut *et al.* 2021; Hadjerci *et al.* 2024) and strong magnetic fields, as well as alternative choices of boundary conditions. The doors to these explorations are now open thanks to the RRRiNSE formulation. Comparison of the RRRiNSE numerical results with future laboratory experiments would also be highly desirable.

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A. Additional details on flow statistics and structure

Supplementary Movies 1 to 3 show the evolution of the vertical vorticity ω_z , the temperature perturbation θ and the vertical velocity w , respectively, at the top of the momentum boundary layer $z = \delta_{\omega_z}$ for $Ek = 10^{-15}$ and $Ra = 120$. Supplementary Movies 4 to 6 provide the same information for $Ek = 10^{-8}$. At $Ek = 10^{-15}$, all fields display close to zero skewness and cyclones are of approximately equal strength and structure as anticyclones. In contrast, at $Ek = 10^{-8}$, the boundary layer is characterised by the presence of strong, albeit short-lived, anticyclonic vortical structures typically with a shielded structure that impose a clear signature on θ and w , associated with cold spots in the centre where the fluid descends while rising within a surrounding ring. Figure 7 shows an analysis of the dissipation budget predicted by the power-integral relations given in (3.2) in the main text, as a function of the Taylor number $Ta \equiv Ek^{-2}$. Panel *a* shows that in the statistically stationary state at $Ra = 120$ the sum of the dissipation contributions from the temperature variations due to gradients in the horizontal and vertical directions sums up approximately to $Nu - 1$ as predicted by (3.2a). The departure from the approximately constant Nusselt number at small Ek is seen to be associated with an increase in dissipation due to both horizontal and vertical gradients. Panel *b* shows the contributions from vertical gradients, which primarily stem from the boundary layers near the departure from the NHQG limit. Panel *c* shows the kinetic energy dissipation budget as a function of the Taylor number at $Ra = 120$. The sum of the dissipation of kinetic energy due to vertical and horizontal gradients shows a satisfactory overall agreement with $(Nu - 1)Ra/\sigma^2$, as predicted by (3.2b). However,

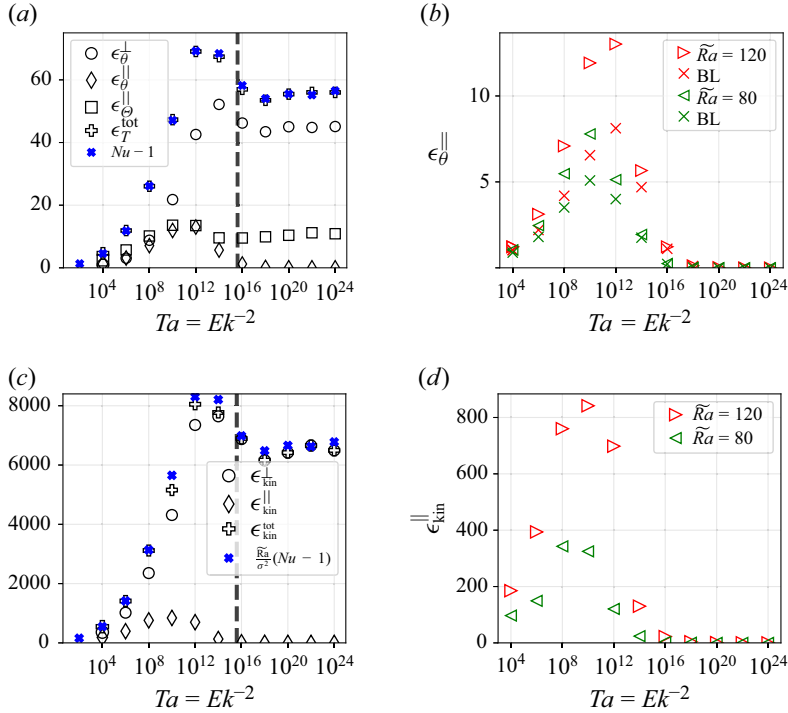


Figure 7. Panel *a* shows the dissipation budget of the temperature variance arising from vertical flow variations at $\tilde{Ra} = 120$ versus $Ta = Ek^{-2}$, with $\epsilon_{\theta}^{\perp} \equiv \langle |\nabla_{\perp} \theta|^2 \rangle$, $\epsilon_{\theta}^{\parallel} \equiv \epsilon^2 \langle (\partial_z \theta)^2 \rangle$ and $\epsilon_{\theta}^{\text{tot}} \equiv \langle (\partial_z \Theta)^2 \rangle$, showing that the power-integral equation (3.2a) of the main text is well satisfied and hence that a statistically steady state has been reached. The overshoot in $Nu - 1$ is primarily due to increased $\epsilon_{\theta}^{\perp}$ and $\epsilon_{\theta}^{\parallel}$. Panel *b* shows the dissipation due to vertical variations in θ at $\tilde{Ra} = 80, 120$ versus $Ta = Ek^{-2}$ together with the boundary layer contributions (marked by \times , integrated over a depth of $2\delta_{\theta}$). Panel *c* shows the kinetic energy budget showing that the contributions from the horizontal ($\epsilon_{kin}^{\perp} \equiv \langle |\nabla_{\perp} \mathbf{u}|^2 \rangle$) and vertical gradients ($\epsilon_{kin}^{\parallel} \equiv \epsilon^2 \langle |\partial_z \mathbf{u}|^2 \rangle$) approximately add up to $(Nu - 1)Ra/\sigma^2$, as predicted by (3.2b) of the main text. A small mismatch is seen at $Ek \lesssim 10^{-5}$ since the emerging large-scale vortex has not fully saturated in amplitude. Panel *d* shows the contribution from vertical variation to the kinetic energy dissipation versus Ta at $\tilde{Ra} = 80, 120$.

while for $Ta \lesssim 10^8$, where no inverse energy cascade is observed, the agreement is close to perfect, it is notable that for $Ta \gtrsim 10^{10}$, the measured kinetic energy is slightly below the value predicted by (3.2b). This is because the large-scale vortex is still slowly growing in amplitude via the condensation process. Panel *d* shows the contributions from vertical gradients, which show a similarly drastic transition from negligibly small to finite values, contributing to the observed overshoot in the Nusselt number.

The RRRiNSE formulation remains numerically stable at even lower Ekman numbers than those discussed in the main text. Figure 8 illustrates this in terms of the flow field obtained for $\tilde{Ra} = 120$ at $Ek = 10^{-24}$. The flow morphology is indistinguishable from that at $Ek = 10^{-15}$. Specifically, in the boundary layer, there is no cyclone–anticyclone asymmetry and the ω_z and θ fields are nearly perfectly correlated. Moreover, the barotropic vertical vorticity displays a counter-rotating vortex dipole at the domain scale, similar to what is seen in two-dimensional turbulence. While the observed flow structure and the associated flow statistics at $Ek = 10^{-24}$ are close to $Ek = 10^{-15}$ for the case considered here, more analysis is needed to determine the lowest Ekman numbers attainable with

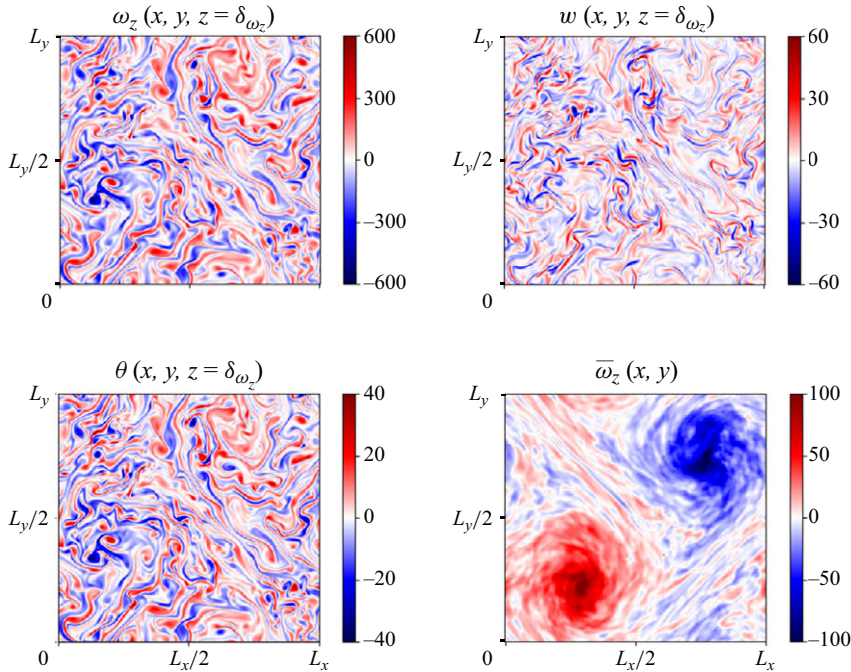


Figure 8. Snapshots of the flow state obtained via DNS of the RRRiNSE at $Ek = 10^{-24}$ and $\tilde{Ra} = 120$ in terms of ω_z , w , θ at the top of the boundary layer $z = \delta_{\omega_z}$, and the depth-averaged vorticity field $\bar{\omega}_z$ revealing a symmetric large-scale vortex dipole.

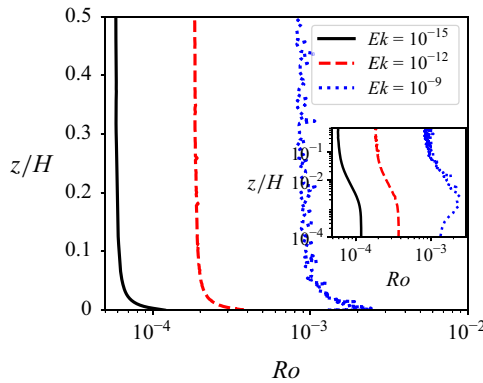


Figure 9. Vertical profiles of the r.m.s. local Rossby number $|\mathbf{u} \cdot \nabla \mathbf{u}| / (2\Omega |\mathbf{u}_\perp|)$ versus non-dimensional height at $\tilde{Ra} = 120$ for Ekman numbers $Ek = 10^{-15}$, 10^{-12} , 10^{-9} . Inset shows same data in log-log representation, showing that bulk and boundary layer have distinct values of the Rossby number. At $Ek = 10^{-9}$, there is a local maximum in the Rossby number observed near the top of the thermal boundary layer. The values shown in the main text are obtained by averaging over the thermal boundary layer volume.

RRRiNSE given finite machine precision, and to characterise the breakdown of the method.

At a given Ekman number, the importance of the Coriolis force compared with nonlinear acceleration is not uniform in the vertical direction. To illustrate this, figure 9 shows the vertical r.m.s. profiles of the Rossby number, computed based on the ratio

$|\mathbf{u} \cdot \nabla \mathbf{u}_\perp| / (2\Omega|\mathbf{u}_\perp|)$. This quantity has a well-defined value in the bulk of the flow, but assumes a distinct value in the boundary layer, as can be seen in the doubly logarithmic representation in the inset. These two distinct values characterising the bulk flow and the boundary layer are shown in panels *e* and *f* of figure 4 in the main text.

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