The Andersen–Kashaev Volume Conjecture for Twist Knots (and more !)

Fathi Ben Aribi

Sorbonne-Université - IMJ-PRG - INSPE Paris

Westlake Mathematics Seminar "Recent Progress on the Volume Conjecture"

20th October 2024

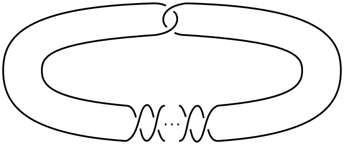
(joint work with François Guéritaud and Eiichi Piguet-Nakazawa)
Quantum Topology, 14 (2023), no. 2, pp. 285-406, arXiv:1903.09480

(joint work with Ka Ho Wong)
arXiv:2410.10776

http://www.normalesup.org/~benaribi/Beamer_Westlake.pdf

Our Goal

Proving the **Andersen–Kashaev volume conjecture** *for* **twist knots**.



0. **Context**: **quantum topology**, **volume conjectures**.
1. **Topology**: **triangulating** the **twist knot** complements
2. **Geometry**: the triangulations contain the **hyperbolicity**
3. **Algebra**: computing the **Teichmüller TQFT**
4. **Analysis**: the **hyperbolic volume** appears **asymptotically**

(Optional: parts/sketches of proofs, at the audience's preference)

Andersen–Kashaev '11: **Teichmüller TQFT** of a **triangulated** 3-manifold M, an "**infinite-dimensional** TQFT".
Its **partition function** $\{Z_{\mathbf{b}}(M) \in \mathbb{C}\}_{\mathbf{b}>0}$ yields an **invariant**.

Volume Conjecture (Andersen–Kashaev '11)

The **hyperbolic volume** $\mathrm{Vol}(S^3 \setminus K)$ *appears as an* **exponential decay rate** *in* $Z_{\mathbf{b}}(S^3 \setminus K)$ *for the limit* $\mathbf{b} \to 0^+$.

Andersen–Kashaev '11: Proof for 4_1 and 5_2.
Andersen–Nissen '17: Proof for 6_1.

Theorem (B.A.–Guéritaud–Piguet-Nakazawa '20)

The Conjecture holds for all **twist knot complements**.

Piguet-Nakazawa '21: Proof for integral DF of the Whitehead link.
Uemura '23: Proof for 7_3.
B.A.-Guéritaud '24+: Proof for $\Sigma_{1,1}$-bundles over S^1.

Theorem (B.A.-Wong '24)

Stronger Conjectures hold for **FAMED geometric triangulations**.

TOPOLOGY

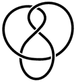
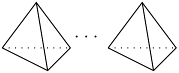


Diagram of a knot K
Triangulation of $S^3 \setminus K$



TOPOLOGY

TWIST KNOTS K_n

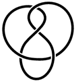
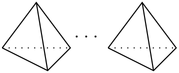


Diagram of a knot K

Triangulation of $S^3 \setminus K$

TH 1



TOPOLOGY

HYPERBOLIC GEOMETRY

TWIST KNOTS K_n

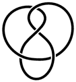
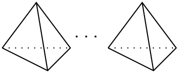


Diagram of a knot K

Triangulation of $S^3 \setminus K$

TH 1 **TH 2**

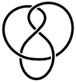


Hyperbolic volume

$\mathrm{vol}\left(S^3 \setminus K\right)$

TOPOLOGY

TWIST KNOTS K_n



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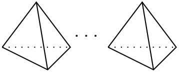
TH 1 TH 2



QUANTUM INVARIANTS

$J_K(N, q)$

Colored Jones polynomials

Teichmüller TQFT

$Z_{\mathbf{b}}(S^3 \setminus K)$

TH 3

HYPERBOLIC GEOMETRY

Hyperbolic volume

$\mathrm{vol}\left(S^3 \setminus K\right)$

TOPOLOGY

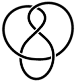
QUANTUM INVARIANTS

SEMI-CLASSICAL LIMIT

HYPERBOLIC GEOMETRY

TWIST KNOTS K_n

VOLUME CONJECTURES



$J_K(N,q) \xrightarrow{q = e^{2i\pi/N},\ N \to \infty} \sim e^{N \frac{\mathrm{vol}\left(S^3 \setminus K\right)}{2\pi}}$

Diagram of a knot K	Colored Jones polynomials	$\longrightarrow$ Sums $\downarrow$ Integrals $\longrightarrow$	
Triangulation of $S^3 \setminus K$	**Teichmüller TQFT**	$\longrightarrow$ **Integrals** $\longrightarrow$	**Hyperbolic volume**

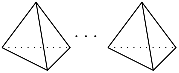
Saddle point method

$Z_{\mathbf{b}}(S^3 \setminus K) \xrightarrow[\mathbf{b} \to 0^+]{} \sim e^{\frac{1}{\mathbf{b}^2} \frac{-\mathrm{vol}\left(S^3 \setminus K\right)}{2\pi}}$

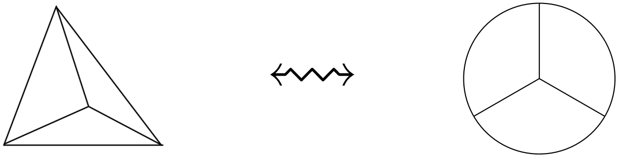
TH 1 TH 2

TH 3

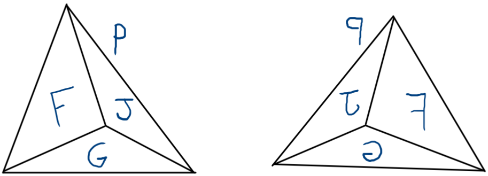
TH 4



A **tetrahedron** = compact, truncated or **ideal** (without vertices).



A **triangulation** $X = (T_1, \dots, T_N, \sim)$ of a 3-manifold M = N **tetrahedra** and a **gluing relation** $\sim$ of faces pairwise.

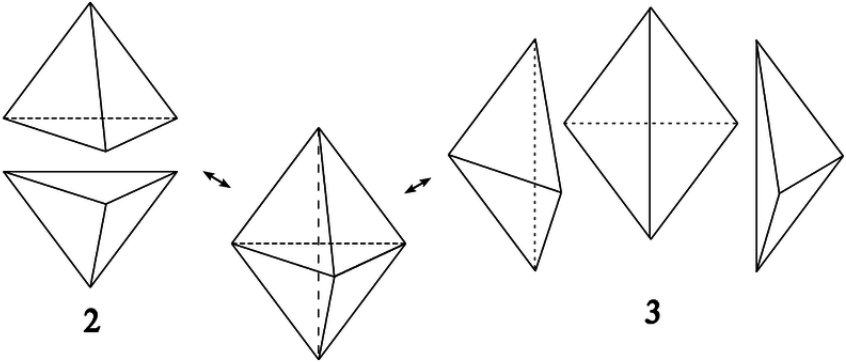


Example : $(T_1, T_2, \sim)$ triangulates either S^3 (compact T_i), ($S^3 \setminus 4$ points) (**ideal** T_i) or ($S^3 \setminus 4$ balls) (truncated T_i).

$(2,3)$-**Pachner moves** are moves between **ideal triangulations**.

Matveev-Piergallini: X and X' triangulate the **same** M if and only if they are related by a **finite sequence** of Pachner moves.

$\Rightarrow$ Useful for constructing **topological invariants** for M.

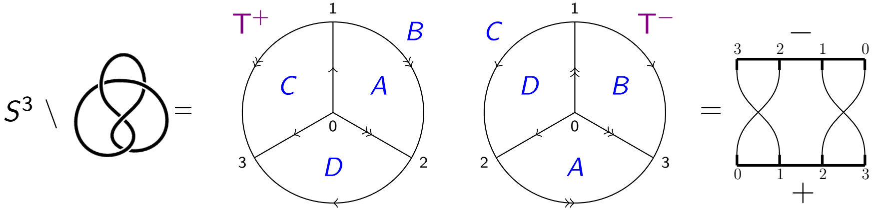


source of the picture: Wikipedia

Our **tetrahedra** have **ordered vertices** ($\Rightarrow$ **oriented edges** too).
$\rightsquigarrow$ two possible **signs** $\epsilon(T) \in \{\pm\}$.

A **triangulation** $X = (T_1, \dots, T_N, \sim)$ of a 3-manifold M is the datum of N **tetrahedra** and a **gluing relation** $\sim$ pairing their faces while **respecting the vertex order**.

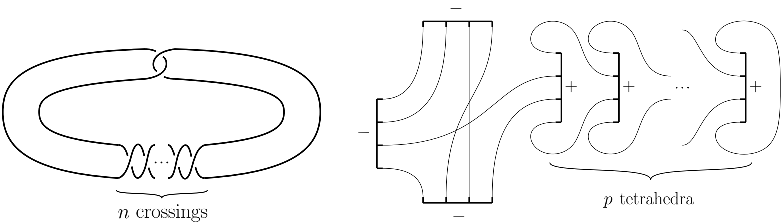
We consider **ideal triangulations** of **open** 3-manifolds, i.e. where the tetrahedra have their **vertices removed**.



$$X^3 = \{T^+, T^-\}, \quad X^2 = \{A, B, C, D\}, \quad X^1 = \{\rightarrow, \twoheadrightarrow\}, \quad X^0 = \{\cdot\}$$

face maps $x_0, \dots, x_3 \colon X^3 \to X^2$, for example $x_0(T^+) = B$.

Thurston: from a **diagram** of a knot K, one can construct an **ideal triangulation** X of the knot complement $M = S^3 \setminus K$.



The n-th twist knot K_n and the triangulation X_n (n odd, $p = \frac{n-3}{2}$)

Theorem (TH 1, B.A.-P.N. '18)

For all $n \geqslant 2$, we construct an **ideal triangulation** *X_n of the complement of the* **twist knot** *K_n, with $\left\lfloor \frac{n+4}{2} \right\rfloor$ tetrahedra.*

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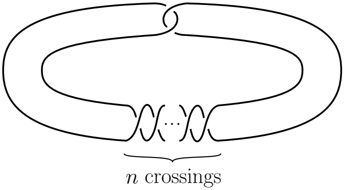

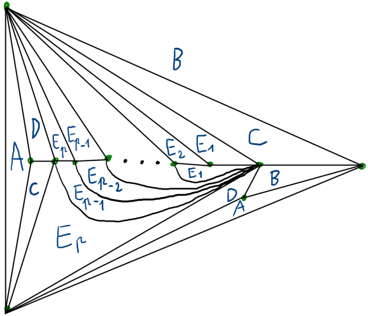

The n-th twist knot K_n and the triangulation X_n (n odd, $p = \frac{n-3}{2}$)

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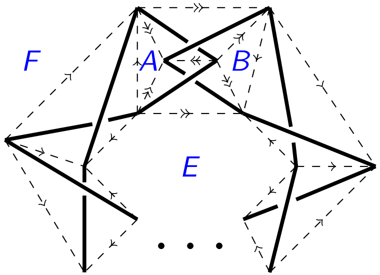
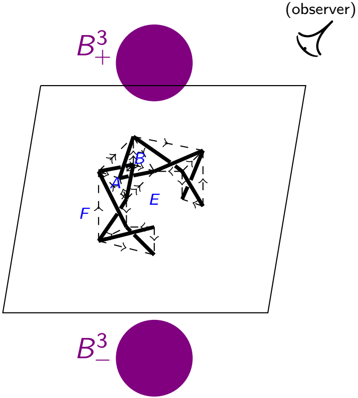
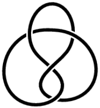
Sketch of proof of TH1: First draw a tetrahedron around each **crossing** of K, whose diagram lives in the **equatorial plane** of S^3.

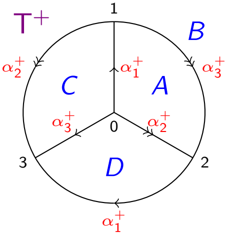


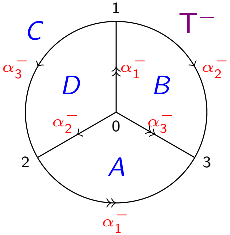


Then **collapse** the tetrahedra into segments ($K \rightsquigarrow \cdot$).
Hence the collapsed S^3 decomposes into **two polyhedra**.
Finally, **triangulate** the two polyhedra (several possible ways).

$\mathcal{A}_X$ is the space of **angle structures** on $X = (T_1, \dots, T_N, \sim)$, i.e. of $3N$-tuples $\alpha \in (0,\pi)^{3N}$ of **dihedral angles** on edges, such that the angle sum is π at each vertex and 2π around each edge.

$S^3 \setminus$  $=$





$$\mathcal{A}_X = \left\{ \alpha = \begin{pmatrix} \alpha_1^+ \\ \alpha_2^+ \\ \alpha_3^+ \\ \alpha_1^- \\ \alpha_2^- \\ \alpha_3^- \end{pmatrix} \in (0,\pi)^6 \;\middle|\; \begin{array}{c} \alpha_1^+ + \alpha_2^+ + \alpha_3^+ = \pi \\ \alpha_1^- + \alpha_2^- + \alpha_3^- = \pi \\ \\ (\rightarrow)\ 2\alpha_1^+ + \alpha_3^+ + 2\alpha_2^- + \alpha_3^- = 2\pi \\ \\ (\twoheadrightarrow)\ 2\alpha_2^+ + \alpha_3^+ + 2\alpha_1^- + \alpha_3^- = 2\pi \end{array} \right\} \ni \begin{pmatrix} \frac{\pi}{3} \\ \vdots \\ \frac{\pi}{3} \end{pmatrix}$$

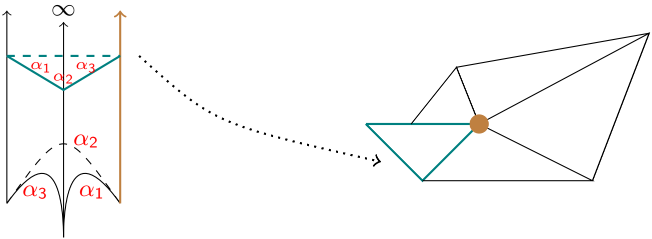
α fixed $\rightsquigarrow$ angle maps $\alpha_1, \alpha_2, \alpha_3 \colon X^3 \to \mathbb{R}$, for example $\alpha_2(T^+) = \alpha_2^+$.

The 3-dimensional **hyperbolic space** is $\mathbb{H}^3 = \mathbb{R}^2 \times \mathbb{R}_{>0}$ with

$$(ds)^2 = \frac{(dx)^2 + (dy)^2 + (dz)^2}{z^2},$$

a metric which has constant curvature -1.

A knot is **hyperbolic** if its complement M can be endowed with a complete hyperbolic metric of finite **volume** $\mathrm{Vol}(M)$.
$\rightsquigarrow$ a specific $\alpha \in \mathcal{A}_X$ on $X = (T_1, \ldots, T_N, \sim)$ triangulation of M.



$\alpha_1 + \alpha_2 + \alpha_3 = \pi$
$T \hookrightarrow \mathbb{H}^3$

$\sum_{\text{edge}} \alpha_j = 2\pi$ (+ others)
gluing gives a **manifold**

For all $n \geqslant 2$, the twist knot K_n is **hyperbolic**.

Theorem (TH2, B.A.-G.-P.N. '20)

For all $n \geqslant 2$, the triangulation X_n of $S^3 \setminus K_n$ is **geometric**, *i.e. it admits an angle structure $\alpha^0 \in \mathcal{A}_{X_n}$ corresponding to the* **complete** *hyperbolic structure on the complement of K_n.*

X **geometric** $\Leftrightarrow \exists$ solution to the **nonlinear** gluing equations of X (difficult!)

Casson-Rivin, Futer-Guéritaud: approach via $\mathcal{A}_X$, the solutions to the **linear** part: maximising the **volume fonctional**.

Dilogarithm function: $\mathrm{Li}_2(z) = -\int_0^z \log(1-u)\frac{du}{u}$ for $z \in \mathbb{C} \setminus [1,\infty)$.

Volume functional $\mathrm{Vol}\colon \mathcal{A}_X \to \mathbb{R}_{\geqslant 0}$ (**strictly concave**) is:

$$\mathrm{Vol}(\alpha) := \sum_{T \in X^3} \Im \mathrm{Li}_2(z(T)) + \arg(1 - z(T)) \log|z(T)|,$$

where $z(T) = \left(\frac{\sin \alpha_3(T)}{\sin \alpha_2(T)}\right)^{\epsilon(T)} e^{i\alpha_1(T)} \in \mathbb{R} + i\mathbb{R}_{>0}$ encodes the angles of T.

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Sketch of proof of TH2:

- Check that the open polyhedron $\mathcal{A}_X$ is **non-empty**.
- General fact: the **complete** structure α^0 exists $\Leftrightarrow$ $\max_{\mathcal{A}_X} \mathrm{Vol}$ is reached in $\mathcal{A}_X$.
- Prove that $\max_{\mathcal{A}_X} \mathrm{Vol}$ cannot be on $\partial\mathcal{A}_X$ (case-by-case).

$S(\mathbb{R}^n)$ = rapidly decreasing functions $f\colon \mathbb{R}^n \to \mathbb{C}$.

$S'(\mathbb{R}^n)$ = dual of $S(\mathbb{R}^n)$, **tempered distributions**.

Example: $X^2 = \{A, B\}$, **Dirac delta function**
$\delta(A) \in S'(\mathbb{R}^{X^2}) \cong S'(\mathbb{R}^2)$ acts by: $\forall f \in S(\mathbb{R}^2)$,

$$\delta(A) \cdot f = \iint_{(A,B)\in\mathbb{R}^2} dA dB \, \delta(A) \, f(A,B) = \int_{B\in\mathbb{R}} dB \, f(0,B) \quad \in \mathbb{C}.$$

⚠ **Product** of Dirac deltas is sometimes but not always **defined**.

$\delta(A)\delta(A)$ is not defined (because of **linear dependance**).

$\delta(A+B)\delta(A-B) = \frac{1}{2}\delta(A)\delta(B) = (f \mapsto \frac{1}{2} f(0,0))$ is well-defined.

Partition function for the triangulation X (and $\alpha \in \mathcal{A}_X$, $\mathbf{b} > 0$):

$$Z_{\mathbf{b}}(X, \alpha) = \int_{\overline{x} \in \mathbb{R}^{X^2}} d\overline{x} \prod_{T_1, \ldots, T_N} p_{\mathbf{b}}(T)(\alpha)(\overline{x}) \qquad \in \mathbb{C}.$$

Tetrahedral operator: $p_{\mathbf{b}}(T)(\alpha)(\overline{x}) \in S'(\mathbb{R}^{X_2})$ is equal to

$$\frac{\delta\left(x_0(T) - x_1(T) + x_2(T)\right) e^{\left(2\pi i \epsilon(T) x_0(T) + (\mathbf{b}+\mathbf{b}^{-1})\alpha_3(T)\right)(x_3(T) - x_2(T))}}{\Phi_{\mathbf{b}}\left((x_3(T) - x_2(T)) - \frac{i(\mathbf{b}+\mathbf{b}^{-1})}{2\pi}\epsilon(T)(\alpha_2(T) + \alpha_3(T))\right)^{\epsilon(T)}}.$$

Faddeev's quantum dilogarithm:

$$\Phi_{\mathbf{b}}(x) := \exp\left(\int_{z \in \mathbb{R} + i0^+} \frac{e^{-2izx}\, dz}{4 \sinh(z\mathbf{b}) \sinh(z\mathbf{b}^{-1}) z}\right).$$

Proposition (Andersen-Kashaev '11)

$|Z_{\mathbf{b}}(X, \alpha)|$ *is* **invariant** *under angled Pachner moves on* (X, α).

Partition function for the triangulation X (and $\alpha \in \mathcal{A}_X$, $\mathbf{b} > 0$):

$$Z_{\mathbf{b}}(X, \alpha) = \int_{\overline{x} \in \mathbb{R}^{X^2}} d\overline{x} \prod_{T_1, \dots, T_N} p_{\mathbf{b}}(T)(\alpha)(\overline{x}) \qquad \in \mathbb{C}.$$

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Volume Conjecture (Andersen-Kashaev '11)

Let X be a triangulation of a **hyperbolic knot complement** M.

(1) $\exists\, \lambda_X$ *linear combination of dihedral angles,* $\exists$ *smooth function* $J_X : \mathbb{R}_{>0} \times \mathbb{R} \to \mathbb{C}$ *such that* $\forall$ *angle structures* α, $\forall\, \mathbf{b} > 0$,

$$|Z_{\mathbf{b}}(X, \alpha)| = \left| \int_{x \in \mathbb{R}} J_X(\mathbf{b}, x) e^{-(\mathbf{b}+\mathbf{b}^{-1}) x \lambda_X(\alpha)} dx \right|.$$

Volume Conjecture (Andersen-Kashaev '11)

Let X be a triangulation of a **hyperbolic knot complement** *M.*

(1) *$\exists\ \lambda_X$ linear combination of dihedral angles, $\exists$ smooth function $J_X \colon \mathbb{R}_{>0} \times \mathbb{R} \to \mathbb{C}$ such that $\forall$ angle structures α, $\forall\ \mathbf{b} > 0$,*

$$|Z_{\mathbf{b}}(X,\alpha)| = \left| \int_{x\in\mathbb{R}} J_X(\mathbf{b},x) e^{-(\mathbf{b}+\mathbf{b}^{-1})x\lambda_X(\alpha)} dx \right| .$$

(2) *The* **hyperbolic volume** $\mathrm{Vol}(M)$ *is obtained as the following* **semi-classical limit***:*

$$\lim_{\mathbf{b}\to 0^+} 2\pi\mathbf{b}^2 \ \log|J_X(\mathbf{b},0)| \ = \ -\mathrm{Vol}(M).$$

Theorem (TH3, B.A.-P.N. '18)

(1) is proven for **all twist knots***, via algebraic computations.*

Theorem (TH4, B.A.-G.-P.N. '20)

(2) is proven for **all twist knots***, via asymptotic analysis.*

Proof of TH3, easiest example: For $K = 4_1$, we find $Z_{\mathbf{b}}(X, \alpha) =$

$$\iiiint \frac{dAdBdCdD \quad \delta(B-D+C)\,\delta(C-A+B) \quad \Phi_{\mathbf{b}}\left(D-B+\frac{i(\mathbf{b}+\mathbf{b}^{-1})}{2\pi}(\alpha_2^- + \alpha_3^-)\right)}{e^{\left(2\pi i B+(\mathbf{b}+\mathbf{b}^{-1})\alpha_3^+\right)(C-A)}e^{\left(-2\pi i C+(\mathbf{b}+\mathbf{b}^{-1})\alpha_3^-\right)(B-D)}\Phi_{\mathbf{b}}\left(A-C-\frac{i(\mathbf{b}+\mathbf{b}^{-1})}{2\pi}(\alpha_2^+ + \alpha_3^+)\right)}.$$

Then we change the variables: $2x = B + C + \frac{i(\mathbf{b}+\mathbf{b}^{-1})}{2\pi}(\alpha_1^+ - \alpha_1^-)$, $2y = B - C + \frac{i(\mathbf{b}+\mathbf{b}^{-1})}{2\pi}(\alpha_1^+ + \alpha_1^- - 2\pi)$ and $A = D = B + C$.

Thus, by taking the module, $|Z_{\mathbf{b}}(X, \alpha)| =$

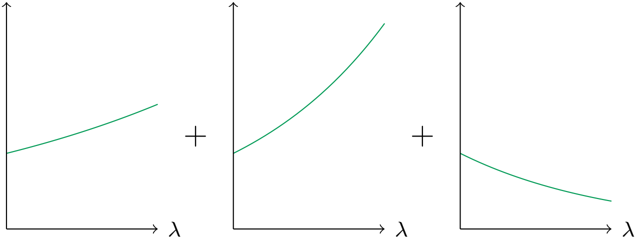
$$\left| \iint \frac{dxdy \quad \Phi_{\mathbf{b}}(x+y)}{e^{-8\pi i xy}\Phi_{\mathbf{b}}(x-y)} e^{-(\mathbf{b}+\mathbf{b}^{-1})\left((2\alpha_2^+ + \alpha_3^+)(x+y)+(2\alpha_2^- + \alpha_3^-)(x-y)\right)} \right|$$

Finally we obtain (1) via $(\rightarrow)$ $2\alpha_1^+ + \alpha_3^+ + 2\alpha_2^- + \alpha_3^- = 2\pi$, with

$$J_X(\mathbf{b}, x) = \int_{y \in \Gamma} dy\, e^{8\pi i xy} \frac{\Phi_{\mathbf{b}}(x+y)}{\Phi_{\mathbf{b}}(x-y)} \quad \text{and} \quad \lambda_X(\alpha) = 4\alpha_2^+ + 2\alpha_3^+.$$

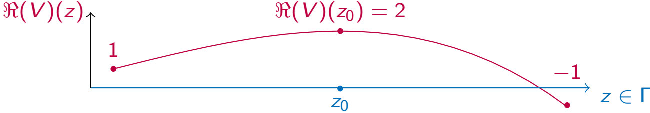
The **saddle point method** gives (under technical conditions) **asymptotics** of complex **integrals with parameters** of the form:

$$\left| \int_{\Gamma} \exp\left(\frac{1}{\mathbf{b}^2} V(z) \right) dz \right| \underset{\mathbf{b} \to 0^+}{\approx} \exp\left(\frac{1}{\mathbf{b}^2} \Re(V)(z_0) \right).$$



$\left(\lambda \leftrightarrow \frac{1}{\mathbf{b}^2}\right)$

$$\exp(\lambda \cdot 1) + \exp(\lambda \cdot 2) + \exp(\lambda \cdot (-1)) \underset{\lambda \to \infty}{\approx} \exp(\lambda \cdot 2)$$



z_0 = saddle point

Theorem (TH4, B.A.-G.-P.N. '20)

$$\lim_{\mathbf{b}\to 0^+} 2\pi\mathbf{b}^2 \, \log|J_{X_n}(\mathbf{b},0)| \; = \; -\mathrm{Vol}(S^3 \setminus K_n).$$

Sketch of proof: (a) Semi-classical approximation:

$$|J_{X_n}(\mathbf{b},0)| \underset{\mathbf{b}\to 0^+}{\approx} \left| \int_\Gamma \exp\left(\frac{1}{\mathbf{b}^2} V(z)\right) dz \right|.$$

comes from $\log\Phi_\mathbf{b} \underset{\mathbf{b}\to 0^+}{\approx} \mathrm{Li}_2$ + technical error bounds

(b) Saddle point method:

$$\left| \int_\Gamma \exp\left(\frac{1}{\mathbf{b}^2} V(z)\right) dz \right| \underset{\mathbf{b}\to 0^+}{\approx} \exp\left(\frac{1}{\mathbf{b}^2} \Re(V)(z_0)\right).$$

we check that z_0 exists thanks to TH2 (geometricity).

(c) Finally, $\Re(V)(z_0) = -\frac{1}{2\pi}\mathrm{Vol}(S^3 \setminus K_n)$, from $\mathrm{Li}_2 \leftrightarrow \mathrm{Vol}$.

B.A. - Wong '24: For general M, TH3 (Algebra) & TH4 (Analysis) follow from the triangulation being **geometric** and **FAMED**.

A triangulation is **FAMED** (*Face Adjaceny Matrices with Edge Duality*) if there is a specific matrix relation between

- the **face gluing** matrices associated with $\delta\left(x_0(T) - x_1(T) + x_2(T)\right)$ and $x_3(T) - x_2(T)$

  For 4_1: $\begin{pmatrix} -1 & 1 & 1 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & -1 \\ -1 & 1 & 0 & 0 \end{pmatrix}$

- the **Neumann-Zagier** matrices associated with **edge equations** on the angle/shape structures.

  For 4_1: $(\rightarrow)\ 2\alpha_1^+ + \alpha_3^+ + 2\alpha_2^- + \alpha_3^-$ & $\lambda_X(\alpha) = 4\alpha_2^+ + 2\alpha_3^+$

$\rightarrow$ Asymptotics of $Z_{\mathbf{b}}(X, \alpha)$, of $J_X(\mathbf{b}, x)$, weak AJ-conjecture
$\rightarrow$ The X_n for the twist knots are FAMED

Ongoing projects:

(with Guéritaud) Proof for **fibered** M^3 with fiber a punctured torus

(with Baseilhac) Vol Conj for **BB invariants** for **twist knots**

Future possible directions:

Algorithm Knot diagram $\longrightarrow$ Triangulation (many choices)

New formulations of Teichmüller TQFT (links, unordered X)

Apply **geometric triangulations** to **other volume conjectures**

Hope: $\exists$ geometric triangulation $\Rightarrow$ volume conjecture is true.
It suffices to prove that every triangulation is **FAMED**!