# The Teichmüller TQFT volume conjecture for twist knots

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(joint work with François Guéritaud and Eiichi Piguet-Nakazawa)
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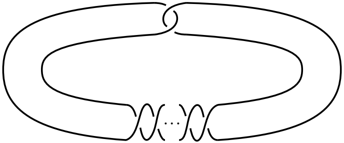
*"Topology and Geometry of Low-dimensional Manifolds 2022"*

**These slides are available on my webpage.**

## Our Goal

*Proving the* **Teichmüller TQFT volume conjecture** *for* **twist knots**.



0. **Context**: **quantum topology**, **volume conjectures**.
1. **Topology**: **triangulating** the **twist knot** complements
2. **Geometry**: the triangulations contain the **hyperbolicity**
3. **Algebra**: computing the **Teichmüller TQFT**
4. **Analysis**: the **hyperbolic volume** appears **asymptotically**

'84: **Jones polynomial**, new knot invariant.

'90: **Witten** retrieves the Jones polynomial via **quantum physics**.

90s: **New topological invariants** (TQFTs of **Reshitikin-Turaev**, **Turaev-Viro**, ...) are discovered via the intuition from physics.

Andersen-Kashaev '11: **Teichmüller TQFT** of a **triangulated** 3-manifold $M$, an "**infinite-dimensional** TQFT".

Its **partition function** $\{Z_{\mathbf{b}}(M) \in \mathbb{C}\}_{\mathbf{b}>0}$ yields an **invariant** of $M$.

**Volume Conjecture (Andersen-Kashaev '11)**

*If $M$ is a triangulated hyperbolic knot complement, then its* **hyperbolic volume** $\mathrm{Vol}(M)$ *appears as an* **exponential decrease rate** *in $Z_{\mathbf{b}}(M)$ for the limit* $\mathbf{b} \to 0^+$.

# TOPOLOGY

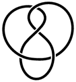
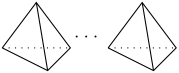


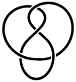
Diagram of a knot $K$

**Triangulation of $S^3 \setminus K$**



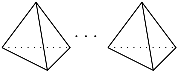
TOPOLOGY

TWIST KNOTS $K_n$



| Diagram of a knot $K$ |
| --- |
| **Triangulation of $S^3 \setminus K$** |

**TH 1**



TOPOLOGY

TWIST KNOTS $K_n$

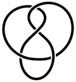
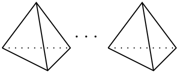


Diagram of a knot $K$

**Triangulation of $S^3 \setminus K$**

**TH 1** **TH 2**



HYPERBOLIC GEOMETRY

**Hyperbolic volume**

$$\mathrm{vol}\left(S^3 \setminus K\right)$$

TOPOLOGY

TWIST KNOTS $K_n$

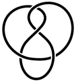
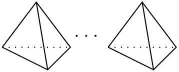


Diagram of a knot $K$

**Triangulation of $S^3 \setminus K$**

**TH 1** **TH 2**



QUANTUM INVARIANTS

$J_K(N, q)$

Colored Jones polynomials

**Teichmüller TQFT**

$Z_{\mathbf{b}}(S^3 \setminus K)$

**TH 3**

HYPERBOLIC GEOMETRY

**Hyperbolic volume**

$\mathrm{vol}\left(S^3 \setminus K\right)$

TOPOLOGY

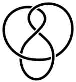
QUANTUM INVARIANTS

SEMI-CLASSICAL LIMIT

HYPERBOLIC GEOMETRY

TWIST KNOTS $K_n$

VOLUME CONJECTURES



$J_K(N,q) \xrightarrow{q=e^{2i\pi/N},\ N\to\infty} \sim e^{N\frac{\mathrm{vol}(S^3\setminus K)}{2\pi}}$

| Diagram of a knot $K$ | Colored Jones polynomials | $\longrightarrow$ Sums $\downarrow$ Integrals $\longrightarrow$ | **Hyperbolic volume** |
|---|---|---|---|
| **Triangulation of $S^3\setminus K$** | **Teichmüller TQFT** | $\longrightarrow$ **Integrals** $\longrightarrow$ | |

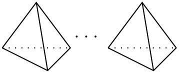
**Saddle point method**

$Z_{\mathbf{b}}(S^3\setminus K) \xrightarrow[\mathbf{b}\to 0^+]{} \sim e^{\frac{1}{\mathbf{b}^2}\frac{-\mathrm{vol}(S^3\setminus K)}{2\pi}}$

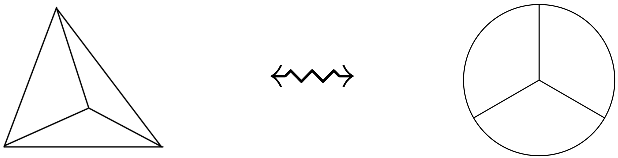
TH 1 TH 2

TH 3

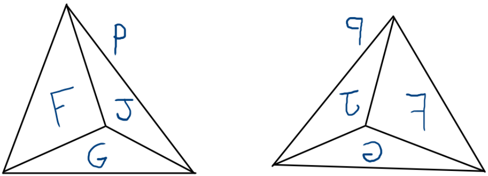
TH 4



A **tetrahedron** = compact, truncated or **ideal** (without vertices).



A **triangulation** $X = (T_1, \ldots, T_N, \sim)$ of a 3-manifold $M$ = $N$ **tetrahedra** and a **gluing relation** $\sim$ of faces pairwise.

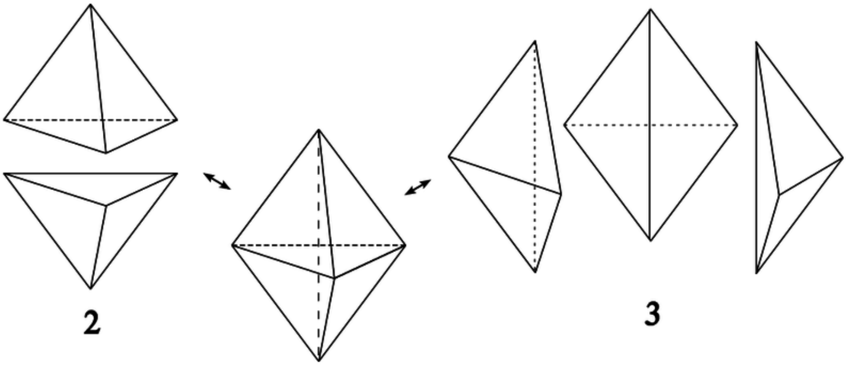



Example : $(T_1, T_2, \sim)$ triangulates either $S^3$ (compact $T_i$), $(S^3 \setminus 4 \text{ points})$ (**ideal** $T_i$) or $(S^3 \setminus 4 \text{ balls})$ (truncated $T_i$).

$(2,3)$-**Pachner moves** are moves between **ideal triangulations**.

Matveev-Piergallini: $X$ and $X'$ triangulate the **same** $M$ if and only if they are related by a **finite sequence** of Pachner moves.

$\Rightarrow$ Useful for constructing **topological invariants** for $M$.

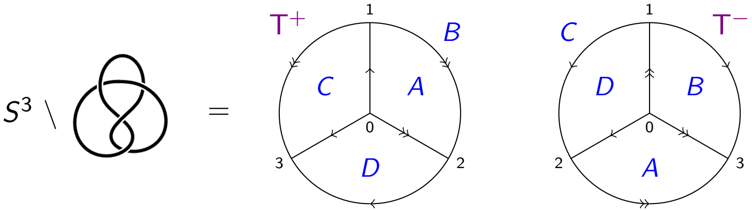



source of the picture: Wikipedia

Our **tetrahedra** have **ordered vertices** ($\Rightarrow$ **oriented edges** too).
$\rightsquigarrow$ two possible **signs** $\epsilon(T) \in \{\pm\}$.

A **triangulation** $X = (T_1, \ldots, T_N, \sim)$ of a 3-manifold $M$ is the datum of $N$ **tetrahedra** and a **gluing relation** $\sim$ pairing their faces while **respecting the vertex order**.

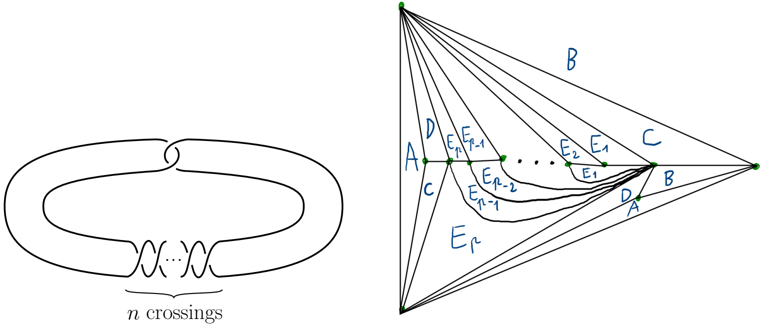
We consider **ideal triangulations** of **open** 3-manifolds, i.e. where the tetrahedra have their **vertices removed**.




$$X^3 = \{T^+, T^-\}, \quad X^2 = \{A, B, C, D\}, \quad X^1 = \{\rightarrow, \twoheadrightarrow\}, \quad X^0 = \{\cdot\}$$

face maps $x_0, \ldots, x_3 \colon X^3 \to X^2$, for example $x_0(T^+) = B$.

Thurston: from a **diagram** of a knot $K$, one can construct an **ideal triangulation** $X$ of the knot complement $M = S^3 \setminus K$.

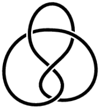


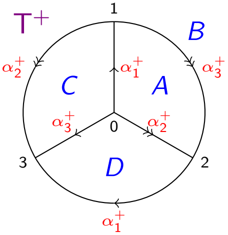

The $n$-th twist knot $K_n$ and the triangulation $X_n$ ($n$ odd, $p = \frac{n-3}{2}$)

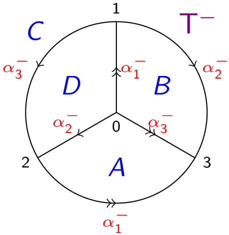
**Theorem (TH 1, B.A.-P.N. '18)**

*For all $n \geqslant 2$, we construct an* **ideal triangulation** *$X_n$ of the complement of the* **twist knot** *$K_n$, with $\left\lfloor \frac{n+4}{2} \right\rfloor$ tetrahedra.*

$\mathcal{A}_X$ is the space of **angle structures** on $X = (T_1, \ldots, T_N, \sim)$, i.e. of $3N$-tuples $\alpha \in (0,\pi)^{3N}$ of **dihedral angles** on edges, such that the angle sum is $\pi$ at each vertex and $2\pi$ around each edge.

$S^3 \setminus$  $=$




$$\mathcal{A}_X = \left\{ \alpha = \begin{pmatrix} \alpha_1^+ \\ \alpha_2^+ \\ \alpha_3^+ \\ \alpha_1^- \\ \alpha_2^- \\ \alpha_3^- \end{pmatrix} \in (0,\pi)^6 \;\middle|\; \begin{array}{c} \alpha_1^+ + \alpha_2^+ + \alpha_3^+ = \pi \\ \alpha_1^- + \alpha_2^- + \alpha_3^- = \pi \\ \\ (\rightarrow)\ 2\alpha_1^+ + \alpha_3^+ + 2\alpha_2^- + \alpha_3^- = 2\pi \\ \\ (\twoheadrightarrow)\ 2\alpha_2^+ + \alpha_3^+ + 2\alpha_1^- + \alpha_3^- = 2\pi \end{array} \right\} \ni \begin{pmatrix} \frac{\pi}{3} \\ \vdots \\ \frac{\pi}{3} \end{pmatrix}$$

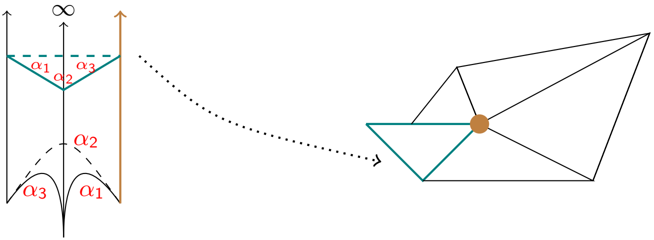
$\alpha$ fixed $\rightsquigarrow$ angle maps $\alpha_1, \alpha_2, \alpha_3 \colon X^3 \to \mathbb{R}$, for example $\alpha_2(T^+) = \alpha_2^+$.

The 3-dimensional **hyperbolic space** is $\mathbb{H}^3 = \mathbb{R}^2 \times \mathbb{R}_{>0}$ with

$$(ds)^2 = \frac{(dx)^2 + (dy)^2 + (dz)^2}{z^2},$$

a metric which has constant curvature $-1$.

A knot is **hyperbolic** if its complement $M$ can be endowed with a complete hyperbolic metric of finite **volume** $\mathrm{Vol}(M)$.
$\rightsquigarrow$ a specific $\alpha \in \mathcal{A}_X$ on $X = (T_1, \dots, T_N, \sim)$ triangulation of $M$.




$\alpha_1 + \alpha_2 + \alpha_3 = \pi$
$T \hookrightarrow \mathbb{H}^3$

$\sum_{\text{edge}} \alpha_j = 2\pi$ (+ others)
gluing gives a **manifold**

For all $n \geqslant 2$, the twist knot $K_n$ is **hyperbolic**.

**Theorem (TH2, B.A.-G.-P.N. '20)**

*For all $n \geqslant 2$, the triangulation $X_n$ of $S^3 \setminus K_n$ is* **geometric**, *i.e. it admits an angle structure $\alpha^0 \in \mathcal{A}_{X_n}$ corresponding to the* **complete** *hyperbolic structure on the complement of $K_n$.*

$X$ **geometric** $\Leftrightarrow \exists$ solution to the **nonlinear** gluing equations of $X$ (difficult!)

Existence of a **geometric** $X$ for any $M \rightsquigarrow$ **Open question** !

Casson-Rivin, Futer-Guéritaud: approach via $\mathcal{A}_X$, the solutions to the **linear** part: maximising the **volume fonctional**.

**Dilogarithm** function: $\mathrm{Li}_2(z) = -\int_0^z \log(1-u)\frac{du}{u}$ for $z \in \mathbb{C} \setminus [1,\infty)$.

**Volume functional** $\mathrm{Vol}\colon \mathcal{A}_X \to \mathbb{R}_{\geqslant 0}$ (**strictly concave**) is:

$$\mathrm{Vol}(\alpha) := \sum_{T \in X^3} \Im \mathrm{Li}_2(z(T)) + \arg(1 - z(T)) \log |z(T)|,$$

where $z(T) = \left(\frac{\sin \alpha_3(T)}{\sin \alpha_2(T)}\right)^{\epsilon(T)} e^{i\alpha_1(T)} \in \mathbb{R} + i\mathbb{R}_{>0}$ encodes the angles of $T$.

**Theorem (TH2, B.A.-G.-P.N. '20)**

*For all $n \geqslant 2$, the triangulation $X_n$ of $S^3 \setminus K_n$ is* **geometric**, *i.e. it admits an angle structure $\alpha^0 \in \mathcal{A}_{X_n}$ corresponding to the* **complete** *hyperbolic structure on the complement of $K_n$.*

Sketch of proof of TH2:

- Check that the open polyhedron $\mathcal{A}_X$ is **non-empty**.
- General fact: the **complete** structure $\alpha^0$ exists $\Leftrightarrow \max_{\mathcal{A}_X} \mathrm{Vol}$ is reached in $\mathcal{A}_X$.
- Prove that $\max_{\mathcal{A}_X} \mathrm{Vol}$ cannot be on $\partial \mathcal{A}_X$ (case-by-case).

**Partition function** for the triangulation X (and $\alpha \in \mathcal{A}_X$, $\mathbf{b} > 0$):

$$Z_{\mathbf{b}}(X,\alpha) = \int_{\overline{x} \in \mathbb{R}^{X^2}} d\overline{x} \prod_{T_1,\dots,T_N} p_{\mathbf{b}}(T)(\alpha)(\overline{x}) \qquad \in \mathbb{C}.$$

**Tetrahedral operator**: $p_{\mathbf{b}}(T)(\alpha)(\overline{x}) \in S'(\mathbb{R}^{X_2})$ is equal to

$$\frac{\delta\left(x_0(T) - x_1(T) + x_2(T)\right) e^{\left(2\pi i \epsilon(T) x_0(T) + (\mathbf{b}+\mathbf{b}^{-1})\alpha_3(T)\right)(x_3(T) - x_2(T))}}{\Phi_{\mathbf{b}}\left((x_3(T) - x_2(T)) - \frac{i(\mathbf{b}+\mathbf{b}^{-1})}{2\pi}\epsilon(T)(\alpha_2(T) + \alpha_3(T))\right)^{\epsilon(T)}}.$$

**Faddeev's quantum dilogarithm**:

$$\Phi_{\mathbf{b}}(x) := \exp\left(\int_{z \in \mathbb{R}+i0^+} \frac{e^{-2izx}\, dz}{4\sinh(z\mathbf{b})\sinh(z\mathbf{b}^{-1})z}\right).$$

**Proposition (Andersen-Kashaev '11)**

$|Z_{\mathbf{b}}(X,\alpha)|$ *is* **invariant** *under angled Pachner moves on* $(X,\alpha)$.

**Partition function** for the triangulation X (and $\alpha \in \mathcal{A}_X$, $\mathbf{b} > 0$):

$$Z_{\mathbf{b}}(X, \alpha) = \int_{\overline{x} \in \mathbb{R}^{X^2}} d\overline{x} \prod_{T_1, \dots, T_N} p_{\mathbf{b}}(T)(\alpha)(\overline{x}) \qquad \in \mathbb{C}.$$

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**Volume Conjecture (Andersen-Kashaev '11)**

*Let $X$ be a triangulation of a* **hyperbolic knot complement** $M$.

(1) *$\exists\ \lambda_X$ linear combination of dihedral angles, $\exists$ smooth function $J_X : \mathbb{R}_{>0} \times \mathbb{R} \to \mathbb{C}$ such that $\forall$ angle structures $\alpha$, $\forall\ \mathbf{b} > 0$,*

$$|Z_{\mathbf{b}}(X, \alpha)| = \left| \int_{x \in \mathbb{R}} J_X(\mathbf{b}, x) e^{-(\mathbf{b}+\mathbf{b}^{-1}) x \lambda_X(\alpha)} dx \right|.$$

## Volume Conjecture (Andersen-Kashaev '11)

*Let $X$ be a triangulation of a* **hyperbolic knot complement** *$M$.*

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$$|Z_{\mathbf{b}}(X,\alpha)| = \left| \int_{x\in\mathbb{R}} J_X(\mathbf{b},x) e^{-(\mathbf{b}+\mathbf{b}^{-1})x\lambda_X(\alpha)} dx \right|.$$

(2) *The* **hyperbolic volume** *$\mathrm{Vol}(M)$ is obtained as the following* **semi-classical limit***:*

$$\lim_{\mathbf{b}\to 0^+} 2\pi\mathbf{b}^2 \ \log|J_X(\mathbf{b},0)| \ = \ -\mathrm{Vol}(M).$$

## Theorem (TH3, B.A.-P.N. '18)

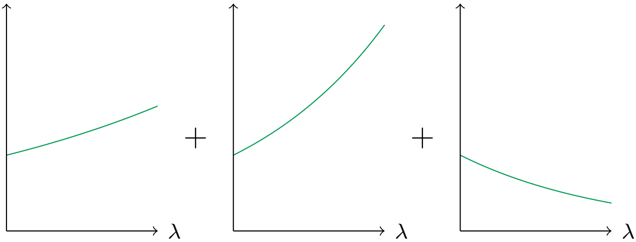
*(1) is proven for* **all twist knots***, via algebraic computations.*

## Theorem (TH4, B.A.-G.-P.N. '20)

*(2) is proven for* **all twist knots***, via asymptotic analysis.*

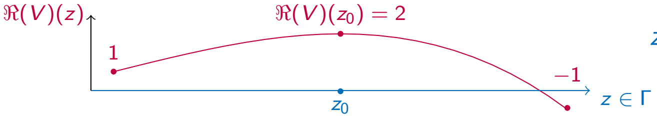
The **saddle point method** gives (under technical conditions) **asymptotics** of complex **integrals with parameters** of the form:

$$\left| \int_{\Gamma} \exp\left(\frac{1}{\mathbf{b}^2} V(z)\right) dz \right| \underset{\mathbf{b}\to 0^+}{\approx} \exp\left(\frac{1}{\mathbf{b}^2} \Re(V)(z_0)\right).$$




$$\left(\lambda \leftrightarrow \frac{1}{\mathbf{b}^2}\right)$$

$$\exp(\lambda \cdot 1) + \exp(\lambda \cdot 2) + \exp(\lambda \cdot (-1)) \underset{\lambda\to\infty}{\approx} \exp(\lambda \cdot 2)$$




$z_0$ = saddle point

**Theorem (TH4, B.A.-G.-P.N. '20)**

$$\lim_{\mathbf{b}\to 0^+} 2\pi\mathbf{b}^2 \ \log|J_{X_n}(\mathbf{b},0)| \ = \ -\mathrm{Vol}(S^3\setminus K_n).$$

Sketch of proof: (a) Semi-classical approximation:

$$|J_{X_n}(\mathbf{b},0)| \underset{\mathbf{b}\to 0^+}{\approx} \left|\int_\Gamma \exp\left(\frac{1}{\mathbf{b}^2}V(z)\right)dz\right|.$$

comes from $\log\Phi_{\mathbf{b}} \underset{\mathbf{b}\to 0^+}{\approx} \mathrm{Li}_2$ + technical error bounds

(b) Saddle point method:

$$\left|\int_\Gamma \exp\left(\frac{1}{\mathbf{b}^2}V(z)\right)dz\right| \underset{\mathbf{b}\to 0^+}{\approx} \exp\left(\frac{1}{\mathbf{b}^2}\Re(V)(z_0)\right).$$

we check that $z_0$ exists thanks to TH2 (geometricity).

(c) Finally, $\Re(V)(z_0) = -\frac{1}{2\pi}\mathrm{Vol}(S^3\setminus K_n)$, from $\mathrm{Li}_2 \leftrightarrow \mathrm{Vol}$.

**Future possible directions**:

Understand the **combinatorial simplifications** in $Z_{\mathbf{b}}(X, \alpha)$
($\leftrightarrow$ Neumann-Zagier matrices?)

**Hope**: $\exists$ **geometric triangulation** of *M*
$\Rightarrow$ the Teichmüller TQFT Volume Conjecture is **true**.

Apply **geometric triangulations** to **other volume conjectures**:

- Chen–Yang volume conjecture for Turaev–Viro invariants,
- Baseilhac–Benedetti volume conjecture for Quantum Hyperbolic Invariants.

**Thank you for your attention!**