Markov moves, L^2-Burau maps and link invariants

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Introduction

Nine decades of invariants

BRAIDS

KNOTS

3-MANIFOLDS

Burau Representation (Burau, 36)

Burau, 36

Alexander Polynomial Δ_K (Alexander, 28)

Milnor, 62

Reidemeister Torsions (Reidemeister, 35)

L^2-Burau maps of braids (B.A.-Conway, 18)

(B.A.-C., 18)

L^2-Alexander Invariant $\Delta_K^{(2)}$ (Li-Zhang, 06)

Li-Zhang, 06

L^2-Torsion (Carey-Mathai, Lück-Rothenberg, '90s)

Lück-Schick, 99

New Link Invariants ? (B.A., 21)

Hyperbolic Volume (Thurston, . . .)

Context and main result

<u>B.A., Conway 2018 :</u> Construction of the L^2-**Burau maps** $\mathcal{B}^{(2)}_{t,\gamma}$ of the braid groups B_n, with $t > 0$ and $\gamma \colon \mathbb{F}_n \twoheadrightarrow G$.

For $\gamma \colon \mathbb{F}_n \twoheadrightarrow G_K$, link with $T^{(2)}_K(t)$ **the** L^2-**Alexander torsion**.

Question

Which other **link invariants** *can be built via* $\det\left(\mathcal{B}^{(2)}_{t,\gamma} - \mathrm{Id}\right)$*?*

Theorem (B.A. 2021)

1. *For γ lower than G_K :* **twisted** L^2-**Alexander invariants**.
2. *For some other γ:* **no** *link invariance.*

<u>**Method :**</u> Apply **Markov moves** of braids to $\mathcal{B}^{(2)}_{t,\gamma}$.

Plan of the talk

1. Burau representation and generalizations
2. L^2-Burau maps
3. L^2-Burau maps and L^2-Alexander torsions
4. Finding new link invariants

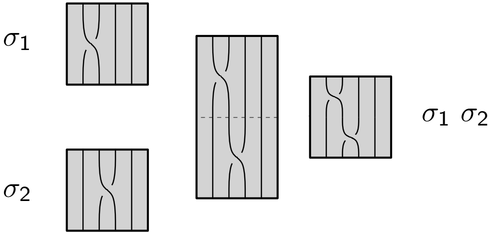
Burau representation and generalizations

Braids

The **braid group** B_n is defined by the presentation

$$\langle \sigma_1, \ldots, \sigma_{n-1} \mid \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1},\ \sigma_i\sigma_j = \sigma_j\sigma_i \text{ for } |i-j| \geqslant 2 \rangle$$

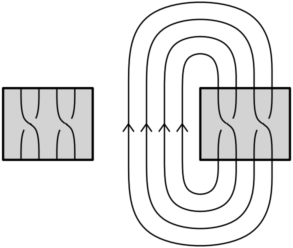
and a braid $\beta \in B_n$ also has a **topological description**, as a 1-submanifold of $\mathbb{D}^2 \times [0,1]$ with no horizontal tangent.



Braids and links

The **closure** $\hat{\beta}$ of a braid $\beta \in B_n$ is a **link** in $\mathbb{S}^3$.

$\beta = \sigma_1 \ \sigma_3$



$\hat{\beta} =$ 2-component unlink

Markov moves on braids $\sqcup_{n \geqslant 1} B_n$ are:

- Markov 1, the **conjugation**: $\beta \mapsto \alpha^{-1}\beta\alpha$, where $\alpha, \beta \in B_n$.
- Markov 2, the **stabilisations**: $B_n \ni \beta \mapsto \sigma_n^{\pm 1}\beta \in B_{n+1}$.

Theorem (Markov 1935)

$\hat{\beta}$ and $\hat{\beta}'$ are **isotopic links** *iff β and β' are related by a finite number of* **Markov moves**.

The Burau representations

Burau representation:

$$\mathcal{B}\colon B_n \ni \sigma_i \mapsto \mathrm{Id}_{i-1} \oplus \begin{pmatrix} 1-T & 1 \\ T & 0 \end{pmatrix} \oplus \mathrm{Id}_{n-i-1} \in GL_n(\mathbb{Z}[T, T^{-1}])$$

Reduced Burau representation:

$$\overline{\mathcal{B}}\colon B_n \ni \sigma_i \mapsto \mathrm{Id}_{i-2} \oplus \begin{pmatrix} 1 & T & 0 \\ 0 & -T & 0 \\ 0 & 1 & 1 \end{pmatrix} \oplus \mathrm{Id}_{n-i-2} \in GL_{n-1}(\mathbb{Z}[T, T^{-1}])$$

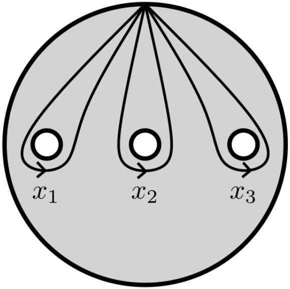
Theorem (Burau 1936)

Let $\beta \in B_n$. Then the **Alexander polynomial** $\Delta_{\hat{\beta}}$ *of the link $\hat{\beta}$ satisfies (up to product by a unit of $\mathbb{Z}[T, T^{-1}]$):*

$$\Delta_{\hat{\beta}}(T) = \frac{1-T}{1-T^n} \det(\overline{\mathcal{B}}(\beta) - \mathrm{Id}_{n-1}).$$

Braids and free groups

Free group $\mathbb{F}_n = \langle x_1, \dots, x_n \mid \rangle \cong \pi_1(D^2 \setminus \{p_1, \dots, p_n\})$.



Artin: A braid $\beta \in B_n$ induces an automorphism $h_\beta \in Aut(\mathbb{F}_n)$.

Example: For $\sigma_1 \in B_2$, h_{σ_1}: $x_1 \mapsto x_1 x_2 x_1^{-1}$, $x_2 \mapsto x_1$.

$$B_n \hookrightarrow Aut(\mathbb{F}_n)$$

Fox calculus

The **Fox derivatives** on $\mathbb{F}_n$ are linear maps $\dfrac{\partial}{\partial x_i} : \mathbb{Z}\mathbb{F}_n \to \mathbb{Z}\mathbb{F}_n$ (where $i = 1, \ldots, n$), inductively defined by:

$$\frac{\partial}{\partial x_i}(x_j) = \delta_{i,j}, \qquad \frac{\partial}{\partial x_i}\left(x_j^{-1}\right) = -\delta_{i,j} x_j^{-1},$$
$$\frac{\partial}{\partial x_i}(uv) = \frac{\partial}{\partial x_i}(u) + u\frac{\partial}{\partial x_i}(v).$$

$\left(\dfrac{\partial f(x_j)}{\partial x_i}\right)_{i,j} \in GL_n(\mathbb{Z}\mathbb{F}_n)$ is the **Fox jacobian** of $f \in Aut(\mathbb{F}_n)$.

Example: For $\sigma_1 \in B_2$, $h_{\sigma_1} :\ x_1 \mapsto x_1 x_2 x_1^{-1},\ x_2 \mapsto x_1$, and

$$\left(\frac{\partial h_{\sigma_1}(x_j)}{\partial x_i}\right) = \begin{pmatrix} 1 - x_1 x_2 x_1^{-1} & 1 \\ x_1 & 0 \end{pmatrix}.$$

$$Aut(\mathbb{F}_n) \hookrightarrow GL_n(\mathbb{Z}\mathbb{F}_n)$$

The change of coefficients $\kappa(\Phi_n, \gamma, t)$

We construct a **ring morphism** $\kappa(\Phi_n, \gamma, t)$ for

- $\Phi_n \colon \mathbb{F}_n \to \mathbb{Z}, x_i \mapsto 1$, the **augmentation** epimorphism,
- $\gamma \colon \mathbb{F}_n \to G$ such that Φ_n factors through γ,
- $t > 0$.

We define $\kappa(\Phi_n, \gamma, t) \colon \mathbb{Z}\mathbb{F}_n \to \mathbb{R}G$ by $g \mapsto t^{\phi(g)}\gamma(g)$.

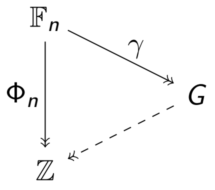
Example: For $n = 2$ and $\gamma = T^{\Phi_2} \colon \left(\mathbb{F}_2 \to T^{\mathbb{Z}} = \{T^m, m \in \mathbb{Z}\}\right)$,

$$\kappa(\Phi_2, T^{\Phi_2}, t) \colon \begin{pmatrix} 1 - x_1 x_2 x_1^{-1} & 1 \\ x_1 & 0 \end{pmatrix} \mapsto \begin{pmatrix} 1 - tT & 1 \\ tT & 0 \end{pmatrix}.$$

$\kappa(\Phi_n, \gamma, t)$ induces $GL_n(\mathbb{Z}\mathbb{F}_n) \to GL_n(\mathbb{R}G)$.

A family of maps between Artin and Burau

$$B_n \overset{Artin}{\hookrightarrow} Aut(\mathbb{F}_n) \overset{Fox}{\hookrightarrow} GL_n(\mathbb{Z}\mathbb{F}_n) \overset{\kappa(\Phi_n,\gamma,t)}{\longrightarrow} GL_n(\mathbb{R}G)$$



We consider the maps

$$\beta \mapsto \kappa(\Phi_n, \gamma, t)\left(\frac{\partial h_\beta(x_j)}{\partial x_i}\right)_{1\leqslant i,j\leqslant n}$$

for different $\gamma\colon \mathbb{F}_n \to G$.

- $\gamma = \mathrm{Id}_{\mathbb{F}_n}$: jacobian of the **Artin action**, always **injective**.

$$B_2 \ni \sigma_1 \mapsto \begin{pmatrix} 1 - t\ x_1x_2x_1^{-1} & 1 \\ t\ x_1 & 0 \end{pmatrix}.$$

- $\gamma = T^{\Phi_n}$: **Burau representation**, **non injective** for $n \geqslant 5$.

$$B_2 \ni \sigma_1 \mapsto \begin{pmatrix} 1 - tT & 1 \\ tT & 0 \end{pmatrix}.$$

L^2-Burau maps

The general idea of L^2-invariants

X topological space, cellular structure $\longrightarrow$ $C(X)$ $\mathbb{C}$-vector spaces, operators over $\mathbb{C}^n$ $\longrightarrow$ $I(X)$ Numerical invariant

$X \downarrow \widetilde{X}$; $C(\widetilde{X}) \uparrow C(X)$: $\underset{\mathbb{C}[\pi_1(X)]}{\otimes} \mathbb{C}$

$\widetilde{X}$ Universal covering $\longrightarrow$ $C(\widetilde{X})$ $\mathbb{C}[\pi_1(X)]$-modules, operators over $\mathbb{C}[\pi_1(X)]$

$C(\widetilde{X}) \downarrow C^{(2)}(X)$: $\underset{\mathbb{C}[\pi_1(X)]}{\otimes} \ell^2(\pi_1(X))$

$C^{(2)}(X)$ $\pi_1(X)$-Hilbert modules, operators over $\ell^2(\pi_1(X))$ $\longrightarrow$ $I^{(2)}(X)$ L^2-invariant

An L^2 point of view

$$B_n \overset{Artin}{\hookrightarrow} Aut(\mathbb{F}_n) \overset{Fox}{\hookrightarrow} GL_n(\mathbb{Z}\mathbb{F}_n) \overset{\kappa(\Phi_n,\gamma,t)}{\longrightarrow} GL_n(\mathbb{R}G)$$

Problem : Extracting info from $M_n(\mathbb{R}G)$ if G is not abelian.
$\rightsquigarrow$ On $\ell^2(G)$ there is the **Fuglede-Kadison determinant**.

We replace the algebra $\mathbb{R}G$ by the Hilbert space

$$\ell^2(G) := \left\{ \sum_{g\in G} \lambda_g g \;\middle|\; \lambda_g \in \mathbb{C}, \sum_{g\in G} |\lambda_g|^2 < \infty \right\}.$$

Typical **bounded** G-equivariant operators:

$$R_h : \ell^2(G) \to \ell^2(G), g \mapsto gh,$$

the **right multiplication** operator by $h \in G$.

The operation $R.$ induces $\quad GL_n(\mathbb{R}G) \hookrightarrow B\left(\ell^2(G)^n\right).$

L^2-Burau maps of braids

$$B_n \overset{Artin}{\hookrightarrow} Aut(\mathbb{F}_n) \overset{Fox}{\hookrightarrow} GL_n(\mathbb{Z}\mathbb{F}_n) \overset{\kappa(\Phi_n,\gamma,t)}{\longrightarrow} GL_n(\mathbb{R}G) \overset{R.}{\hookrightarrow} B\left(\ell^2(G)^n\right),$$

The L^2-**Burau map** associated to $t > 0$ and $\gamma \colon \mathbb{F}_n \to G$ is:

$$\mathcal{B}^{(2)}_{t,\gamma} \colon \begin{pmatrix} B_n & \to & B\left(\ell^2(G)^n\right) \\ \beta & \mapsto & R_A \end{pmatrix}, \text{with } A = \kappa(\Phi_n, \gamma, t)\left(\frac{\partial h_\beta(x_j)}{\partial x_i}\right)_{1 \leqslant i,j \leqslant n}.$$

The **reduced** L^2-**Burau map** associated to the same t, γ is:

$$\overline{\mathcal{B}}^{(2)}_{t,\gamma} \colon \begin{pmatrix} B_n \to B\left(\ell^2(G)^{n-1}\right) \\ \beta \mapsto R_{A'} \end{pmatrix}, \text{with } A' = \kappa(\Phi_n, \gamma, t)\left(\frac{\partial h_\beta(g_j)}{\partial g_i}\right)_{1 \leqslant i,j \leqslant n-1},$$

where $g_1 = x_1,\ g_2 = x_1x_2,\ \ldots,\ g_n = x_1 \ldots x_n$ also generate $\mathbb{F}_n$.

L^2-Burau maps of braids

$$B_n \overset{Artin}{\hookrightarrow} Aut(\mathbb{F}_n) \overset{Fox}{\hookrightarrow} GL_n(\mathbb{Z}\mathbb{F}_n) \xrightarrow{\kappa(\Phi_n,\gamma,t)} GL_n(\mathbb{R}G) \overset{R.}{\hookrightarrow} B\left(\ell^2(G)^n\right),$$

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The **reduced** L^2-**Burau map** associated to the same t, γ is:

$$\overline{\mathcal{B}}^{(2)}_{t,\gamma} \colon \begin{pmatrix} B_n \to B\left(\ell^2(G)^{n-1}\right) \\ \beta \mapsto \quad R_{A'} \end{pmatrix}, \text{with } A' = \kappa(\Phi_n, \gamma, t)\left(\frac{\partial h_\beta(g_j)}{\partial g_i}\right)_{1 \leqslant i,j \leqslant n-1},$$

where $g_1 = x_1,\ g_2 = x_1 x_2,\ \dots,\ g_n = x_1 \dots x_n$ also generate $\mathbb{F}_n$.

Multiplicativity and computations

(Anti-)**multiplication formula**: $\forall \alpha, \beta \in B_n$,

$$\mathcal{B}^{(2)}_{t,\gamma}(\alpha\beta) = \mathcal{B}^{(2)}_{t,\gamma}(\beta) \circ \mathcal{B}^{(2)}_{t,\gamma\circ h_\beta}(\alpha).$$

Consequence : The more **injective** $\gamma \colon \mathbb{F}_n \twoheadrightarrow G$ is, the less $\mathcal{B}^{(2)}_{t,\gamma}$ looks like an (anti-)**group representation**.

$\mathcal{B}^{(2)}_{t,\gamma}$ is an (anti-)rep. of the **subgroup** $\{\beta \in B_n, \gamma \circ h_\beta = \gamma\}$.

We can still **compute** $\mathcal{B}^{(2)}_{t,\gamma}(\beta)$ via the images of the σ_i.

$$\mathcal{B}^{(2)}_{t,\gamma} \colon B_2 \ni \sigma_1 \mapsto \begin{pmatrix} \mathrm{Id} - tR_{\gamma(x_1 x_2 x_1^{-1})} & \mathrm{Id} \\ tR_{\gamma(x_1)} & 0 \end{pmatrix} \in B\left(\ell^2(G)^{\oplus 2}\right).$$

L^2-Burau maps and L^2-Alexander torsions

Fuglede-Kadison determinants of G-equivariant operators

The **von Neumann trace** $\mathrm{tr}_G : (\lambda_1 Id + \ldots + \lambda_g R_g) \mapsto \lambda_1 \in \mathbb{C}$.

The **Fuglede-Kadison determinant** $\det_G$ is :

$$\det{}_G : R_{M_n(\mathbb{Z}G)} (\subset B(\ell^2(G)^n)) \to \mathbb{R}_{\geqslant 0}$$

$$\det{}_G(A) := \lim_{\epsilon \to 0^+} \left(\exp \circ \left(\frac{1}{2} \mathrm{tr}_G \right) \circ \ln \right) (A^* A + \epsilon Id).$$

$\leftrightsquigarrow$ counting loops on the **Cayley graph** of G. (hard to compute !)

Ex : If $g \in G$ has ∞ order, then $\det_G(\mathrm{Id} + tR_g) = \max\{1, |t|\}$.

Ex : For $G = \mathbb{Z}/3\mathbb{Z}$: $\ell^2(G) = \mathbb{C}^3$, R_g is a permutation matrix, $\mathrm{tr}_G = \frac{1}{3}\mathrm{tr}_{\mathbb{C}}$, $\det_G = |\det|^{1/3}$.

L^2-Alexander torsions of links

(Dubois-Friedl-Lück 14) L^2-**Alexander torsion** of the link L :

$$T_L^{(2)} : \begin{pmatrix} \mathbb{R}_{>0} & \longrightarrow & \mathbb{R}_{\geqslant 0} \\ t & \longmapsto & T_L^{(2)}(t) \end{pmatrix}$$

defined with $\det_{\pi_1(S^3 \setminus L)}$ and **Fox calculus**.

Some properties:

(Lück-Schick 99) $T_L^{(2)}(1)$ contains the **volume** of $S^3 \setminus L$.

(Friedl-Lück, Liu 15) $T_L^{(2)}(0^+)$, $T_L^{(2)}(+\infty)$ yield the **genus** $g(L)$.

(Miscellaneous) $T_L^{(2)}$ is the **zero map** iff L is **split**.

L^2-Burau map at the link group

For $\gamma = \gamma_\beta \colon \mathbb{F}_n \to G_\beta \cong \pi_1\left(S^3 \setminus \hat{\beta}\right)$ quotient by $h_\beta(x_i) = x_i$, the **reduced** L^2-Burau map $\overline{\mathcal{B}}^{(2)}_{t,\gamma_\beta}$ gives the L^2-Alexander torsion $T^{(2)}_{\hat{\beta}}$:

Theorem (B.A.-Conway 18)

Let $\beta \in B_n$, $L = \hat{\beta}$ its closure and $t > 0$. Then:

$$T^{(2)}_L(t) \cdot \max(1,t)^n \; = \; \det{}_{G_L}\left(\overline{\mathcal{B}}^{(2)}_{t,\gamma_\beta}(\beta) - \mathrm{Id}^{\oplus(n-1)}\right).$$

Finding new link invariants

Markov functions

Markov moves on braids $\sqcup_{n\geqslant 1} B_n$ are:

- Markov 1, the **conjugation**: $\beta \mapsto \alpha^{-1}\beta\alpha$, where $\alpha, \beta \in B_n$.
- Markov 2, the **stabilisations**: $B_n \ni \beta \mapsto \sigma_n^{\pm 1}\beta \in B_{n+1}$.

Markov : $\hat{\beta}$ and $\hat{\beta}'$ are **isotopic links** iff β and β' are related by a finite number of **Markov moves**.

A function F on $\sqcup_{n\geqslant 1} B_n$ is a **Markov function** if it is invariant under all the Markov moves.
$\rightsquigarrow$ F provides a **invariant of knots and links**.

Question

For **which** $\gamma\colon \mathbb{F}_n \twoheadrightarrow G$ *is* $\beta \mapsto \det{}_G \left(\overline{\mathcal{B}}^{(2)}_{t,\gamma}(\beta) - \mathrm{Id}^{\oplus(n-1)}\right)$ *a* **Markov function***?*

Markov-admissibility of a family of epimorphisms

A family $\mathcal{Q}$ of **epimorphisms of free groups**

$$\mathcal{Q} = \left\{ Q_\beta \colon \mathbb{F}_{n(\beta)} \twoheadrightarrow G_{Q_\beta} \mid \beta \in \sqcup_{n \geqslant 1} B_n \right\}$$

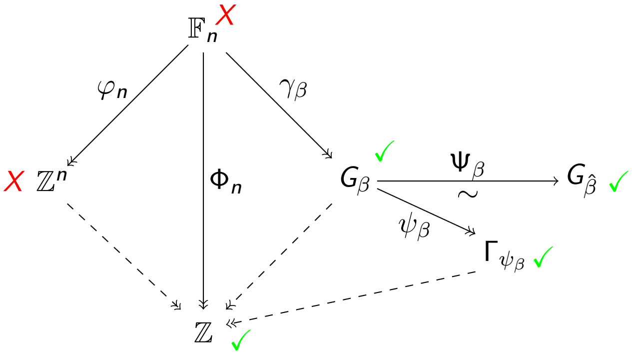
is **Markov-admissible** if for all pairs of braids β, β' related by a Markov move, Q_β and $Q_{\beta'}$ are "comparable".

$$\begin{array}{ccc} \mathbb{F}_n & \xrightarrow[\sim]{h_\alpha} & \mathbb{F}_n \\ \downarrow Q_\beta & & \downarrow Q_{\alpha^{-1}\beta\alpha} \\ G_{Q_\beta} & \xrightarrow[\sim]{\exists\ \chi^{\mathcal{Q}}_{\beta,\alpha}} & G_{Q_{\alpha^{-1}\beta\alpha}} \end{array} \qquad \begin{array}{ccc} \mathbb{F}_n & \xhookrightarrow{\iota_n} & \mathbb{F}_{n+1} \\ \downarrow Q_\beta & & \downarrow Q_{\sigma_n^\varepsilon \beta} \\ G_{Q_\beta} & \xhookrightarrow{\exists\ \sigma^{\mathcal{Q}}_{\beta,\varepsilon}} & G_{Q_{\sigma_n^\varepsilon \beta}} \end{array}$$

Goal : Constructing a **Markov function** on $\sqcup_{n \geqslant 1} B_n$ via $\mathcal{Q}$.

Several Markov-admissible families of epimorphisms

- $Id_{\mathbb{F}_n}$:= the **identity** on $\mathbb{F}_n$.
- φ_n := the **abelianisation** of $\mathbb{F}_n$, onto $\mathbb{Z}^n$.
- γ_β := the quotient by $h_\beta(x_i) = x_i$ ($\leftrightsquigarrow$ **closure** of β).
- Φ_n := the **augmentation** of $\mathbb{F}_n$, onto $\mathbb{Z}$.



Is $\det\left(\mathrm{Burau}^{(2)} - \mathrm{Id}\right)$ a **Markov function**? ✓: Yes. X: No.

Obtaining a Markov function

For $\mathcal{Q} = \{Q_\beta\}_{\beta \in \sqcup_{n \geqslant 1} B_n}$ Markov-admissible and $t > 0$, we define:

$$F_{\mathcal{Q}} := \left(\begin{array}{ccc} \sqcup_{n\geqslant 1} B_n & \to & \mathcal{F}(\mathbb{R}_{>0}, \mathbb{R}_{>0}) / \{t \mapsto t^m, m \in \mathbb{Z}\} \\ \beta & \mapsto & \left[t \mapsto \dfrac{\det_{G_{Q_\beta}}\left(\overline{\mathcal{B}}^{(2)}_{t,Q_\beta}(\beta) - \mathrm{Id}^{\oplus(n-1)} \right)}{\max(1,t)^n} \right] \end{array} \right)$$

Theorem (B.A. 2021)

1. *If $\{Q_\beta\} = \{\psi_\beta \circ \gamma_\beta\}$, then $F_{\mathcal{Q}}$ is a Markov function, and the associated invariant is a L^2-**Alexander torsion** twisted by ψ_β.*
2. *If the Q_β are the **identity** maps $\mathrm{Id}_{\mathbb{F}_{n(\beta)}}$ or the **abelianisations** $\varphi_{n(\beta)}$, then $F_{\mathcal{Q}}$ is **not** a Markov function.*

Sketch of proof for Theorem 1

Conjugation : For $\gamma\colon \mathbb{F}_n \twoheadrightarrow G$, the product formula gives:

$$\overline{\mathcal{B}}^{(2)}_{t,\gamma}(\alpha^{-1}\beta\alpha) = \overline{\mathcal{B}}^{(2)}_{t,\gamma}(\alpha) \circ \overline{\mathcal{B}}^{(2)}_{t,\gamma\circ h_\alpha}(\beta) \circ \left(\overline{\mathcal{B}}^{(2)}_{t,\gamma\circ h_{\alpha^{-1}\beta\alpha}}(\alpha)\right)^{-1},$$

a **conjugation** if γ goes deep enough so that $\gamma \circ h_{\alpha^{-1}\beta\alpha} = \gamma$.

$\checkmark$ for $\gamma = \psi_\beta \circ \gamma_\beta$.

Stabilisations : Via elementary operations, we get

$$\overline{\mathcal{B}}^{(2)}_{t,\gamma}\left(\sigma_n^{\pm 1}\beta\right) \approx \begin{pmatrix} \overline{\mathcal{B}}^{(2)}_{t,\gamma}(\beta) & * \\ (\gamma \circ h_\beta - \gamma)(*) & * \end{pmatrix},$$

where $*$ has a known Fuglede-Kadison determinant.

$\checkmark$ for $\gamma = \psi_\beta \circ \gamma_\beta$.

Computing Fuglede-Kadison determinants for Theorem 2

In both cases we prove that $F_{\mathcal{Q}}(\sigma_1^{-1}) \neq F_{\mathcal{Q}}(\sigma_2\sigma_1^{-1})$.

Difficulty: In order to prove that two Fuglede-Kadison determinants are **different**, we must **compute** them.

- For the abelianisations $\varphi_n \colon \mathbb{F}_n \to \mathbb{Z}^n$,
  $F_{\mathcal{Q}}(\sigma_2\sigma_1^{-1}) = \det{}_{\mathbb{Z}^3}(\mathrm{Id} + R_x + R_y) = 1.38135... \neq 1 = F_{\mathcal{Q}}(\sigma_1^{-1})$.
  (Boyd, computations of **Mahler measures**)

- For the identity maps $\mathrm{Id}_{\mathbb{F}_n}$,
  $F_{\mathcal{Q}}(\sigma_2\sigma_1^{-1}) = \det{}_{\mathbb{F}_2}(\mathrm{Id} + R_x + R_y) = \frac{2}{\sqrt{3}} = 1.15... \neq 1 = F_{\mathcal{Q}}(\sigma_1^{-1})$
  (Bartholdi-Dasbach-Lalin, **counting paths** on **trees**)

Potential future leads

- Finding larger classes of $\mathcal{Q}$ such that $F_{\mathcal{Q}}$ is not a Markov function.
- Trying to obtain link invariants from L^2-Burau maps via **other formulas** than $\det(\cdot - \mathrm{Id})$.
- (with C. Anghel-Palmer) Adapting these techniques to future L^2 versions of **Lawrence representations** of the braid groups, which are known to yield **quantum invariants**.