L^2-torsion and hyperbolic volume (HQI Workshop, CIRM)

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Introduction

Plan of the talk

1. Introduction
2. Reidemeister torsions
3. Topological L^2-torsion
4. Analytic L^2-torsion
5. Connexion with hyperbolic volume

Theorem (Lück-Schick 99, others)

Let M^3 be a complete hyperbolic manifold (closed or cusped) with finite volume $\mathrm{vol}(M)$. Then:

- *its L^2-Betti numbers $b_*^{(2)}(M)$ are all 0,*
- *its topological and analytic L^2-torsions ($T^{(2)}$ and $\rho^{(2)}$) satisfy*

$$T^{(2)}(M) = \exp\left(\rho^{(2)}(M)\right) = \exp\left(\frac{\mathrm{vol}(M)}{6\pi}\right).$$

Nine decades of invariants

Alexander Polynomial Δ_K (Alexander, 28)

Milnor, 62 ⟷

Reidemeister Torsions (Reidemeister, 35)

Cheeger-Muller, 78 ⟷

Analytic Torsion (Ray-Singer, 73)

L^2-Torsion (Carey-Mathai, Lück-Rothenberg, '90s)

Burghelea-Friedlander-Kappeler-McDonald, 96 ⟷

L^2-Analytic Torsion (Lott, Mathai, Novikov-Shubin, '90s)

Lück-Schick, 99 ↕

Hyperbolic Volume (Thurston, . . .)

The principle of (topological) L^2-invariants

X topological space, cellular structure $\longrightarrow$ $C(X)$ $\mathbb{C}$-vector spaces, operators over $\mathbb{C}^n$ $\longrightarrow$ $I(X)$ Numerical invariant

$X \downarrow \widetilde{X}$

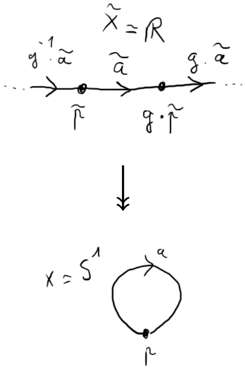
$C(\widetilde{X}) \uparrow C(X)$: $\underset{\mathbb{C}[\pi_1(X)]}{\otimes} \mathbb{C}^n$

$\widetilde{X}$ Universal covering $\longrightarrow$ $C(\widetilde{X})$ $\mathbb{C}[\pi_1(X)]$-modules, operators over $\mathbb{C}[\pi_1(X)]$

$C(\widetilde{X}) \downarrow C^{(2)}(X)$: $\underset{\mathbb{C}[\pi_1(X)]}{\otimes} \ell^2(\pi_1(X))$

$C^{(2)}(X)$ $\mathcal{N}(\pi_1(X))$-Hilbert modules operators over $\ell^2(\pi_1(X))$ $\longrightarrow$ $I^{(2)}(X)$ L^2-invariant

Reidemeister torsions



$$C_*(\widetilde{X}) = 0 \to \bigoplus_{n\in\mathbb{Z}} \mathbb{Z}(g^n\widetilde{a}) \xrightarrow[(g-1)]{\partial} \bigoplus_{n\in\mathbb{Z}} \mathbb{Z}(g^n\widetilde{p}) \to 0$$

where $C_1(\widetilde{X}) = \bigoplus_{n\in\mathbb{Z}} \mathbb{Z}(g^n\widetilde{a}) = \mathbb{Z}[G]\widetilde{a}$ and $C_0(\widetilde{X}) = \bigoplus_{n\in\mathbb{Z}} \mathbb{Z}(g^n\widetilde{p}) = \mathbb{Z}[G]\widetilde{p}$.

$G = \pi_1(X) = \langle g | \rangle \cong \mathbb{Z}$

$$C_*(X) = 0 \to \underset{\mathbb{Z}a}{C_1(X)} \xrightarrow[(0)]{\partial} \underset{\mathbb{Z}p}{C_0(X)} \to 0$$

where $C_1(X) = \mathbb{Z}a$ and $C_0(X) = \mathbb{Z}p$.

$\left(C_*(\widetilde{X}), \partial\right)$ contains the **topological info** of the CW-complex X.

Reidemeister torsion is an invariant obtained from $\left(C_*(\widetilde{X}), \partial\right)$.

Reidemeister torsion : the example of S^1

From a group representation $\phi : G \to GL_d(K)$, you can build the **twisted cellular chain complex** $C_*(X, \phi) := K^d \otimes_{\mathbb{Z}G} C_*(\widetilde{X})$.

$$C_*(\widetilde{X}) = \ 0 \to \mathbb{Z}[G]\widetilde{a} \xrightarrow[(g-1)]{\partial} \mathbb{Z}[G]\widetilde{p} \to 0$$

$$C_*(X,\phi) = \ 0 \to K^d(1 \otimes \widetilde{a}) \xrightarrow[(\phi(g)-Id_{K^d})]{\partial^\phi} K^d(1 \otimes \widetilde{p}) \to 0$$

$b_*(X, \phi) \in \mathbb{N}$ are the **twisted Betti numbers**. If they are zero, we can define the **Reidemeister torsion** $T(X, \phi) \in K^*/\det(\phi(K))$ as an alternated product of determinants of submatrices of ∂^ϕ.

If $\phi(g) - Id$ is invertible, then $T(S^1, \phi) = \dfrac{1}{\det(\phi(g) - Id)}$.

Topological L^2-torsions

L^2-invariants

Base (Hilbert) space: $\ell^2(G)$, completion of $\mathbb{C}G$.

Right-multiplication operator $R_g : \ell^2(G) \to \ell^2(G), h \mapsto hg$.

von Neumann trace $\mathrm{tr}_G : (\lambda_1 Id + \ldots + \lambda_g R_g) \mapsto \lambda_1 \in \mathbb{C}$.

von Neumann dimension $\dim_G : \ell^2(G)^{\oplus r} \mapsto r \geqslant 0$.

Fuglede-Kadison determinant $\det_G(A) \geqslant 0$ of the G-equivariant operator A over $\ell^2(G)$, defined by:

$$\det{}_G(A) = \lim_{\epsilon \to 0^+} \left(\exp \circ \left(\frac{1}{2} \mathrm{tr}_G \right) \circ \ln \right) (A^* A + \epsilon Id).$$

Example : $G = \mathbb{Z}/3\mathbb{Z}$. $\ell^2(G) = \mathbb{C}^3$, R_g is a permutation matrix, $\mathrm{tr}_G = \frac{1}{3}\mathrm{tr}_{\mathbb{C}}$, $\dim_G = \frac{1}{3}\dim_{\mathbb{C}}$, $\det_G = |\det|^{1/3}$.

Topological L^2 torsion

$$C_*^{(2)}(X) = \ell^2(G) \underset{\mathbb{Z}G}{\otimes} C_*(\widetilde{X})$$

$$= \ 0 \to \ell^2(G)(1 \otimes \widetilde{a}) \xrightarrow[(R_g - Id)]{\partial^{(2)}} \ell^2(G)(1 \otimes \widetilde{p}) \to 0$$

$$C_*(\widetilde{X}) = \ 0 \to \mathbb{Z}[G]\widetilde{a} \xrightarrow[(g-1)]{\partial} \mathbb{Z}[G]\widetilde{p} \to 0$$

L^2-**torsion** is a variant of Reidemeister torsion, with a twist by the infinite-dimensional **right regular representation** $G \overset{R}{\hookrightarrow} B(\ell^2(G))$.

$$b_*^{(2)}(X) = \dim_G \left(Ker(\partial^{(2)}) \ / \ \overline{Im(\partial^{(2)})} \right) \in \mathbb{R}_{\geqslant 0}$$

are the L^2 **Betti numbers** of X. If they **vanish**, we define the L^2-**torsion** $T^{(2)}(X) \in \mathbb{R}_{>0}$ as an alternated product of $\det_G\left(\partial^{(2)}\right)$.

For S^1, we have $T^{(2)}(S^1) = \dfrac{1}{\det_G(R_g - Id)} = 1.$

Computing the L^2-torsion

$M = S^3 \setminus 4_1$ is hyperbolic, with volume $\mathrm{vol}(M) = 2.029...$

Lück-Schick, etc. : $T^{(2)}(M) = \exp\left(\dfrac{\mathrm{vol}(M)}{6\pi}\right) = 1.113...$

From $G = \pi_1(M) = \langle x, y | xyx^{-1}yx = yxy^{-1}xy \rangle$, we compute

$T^{(2)}(M) = \det_G(A)$ with $A = Id - R_y - R_{xyx^{-1}} - R_{yxy^{-1}} + R_{xyx^{-1}y}$.

(From a braid closure presentation for G, A is a L^2-**Burau map**).

Via developing $\det_G = \exp \mathrm{tr}_G \ln$, we can compute $T^{(2)}(M)$ from the sequence of $\mathrm{tr}_G\left((A^*A)^n\right), n \in \mathbb{N}$ **(not known in general)**.
$\rightsquigarrow$ counting closed paths on the Cayley graph of G.

Analytic L^2-torsions

Laplacian on differential forms

M^d riemannian, $\Omega^p(M) := C^\infty(M, (T_x M)^*)$ its space of p-forms.

L^2 metric on $\Omega^*(M) \rightsquigarrow$ adjoint d^*, **Laplacian** $\Delta_p := dd^* + d^*d$.

Hodge-de Rham : Betti number $b_p(M) = \dim \ker \Delta_p$.

Example : $M = S^1 = \mathbb{R}/\mathbb{Z}$.

$\Omega^0(S^1) = \{f : \mathbb{R} \to \mathbb{R} \mid f\ 1-\text{periodic}\}$
$\Omega^1(S^1) = \{f \cdot dx \mid f : \mathbb{R} \to \mathbb{R}\ 1-\text{periodic}\}$

$$0 \to \Omega^0(S^1) \xrightarrow{d} \Omega^1(S^1) \to 0$$

$d\colon f \mapsto (f' \cdot dx), \quad d^*\colon (f \cdot dx) \mapsto (-f'), \quad \Delta\colon f \mapsto -f''$

$H_0(S^1)$ and $H_1(S^1)$ are the **constants**, thus $b_0(S^1) = b_1(S^1) = 1$.

Lapalacian and analytic torsion (sketch)

Δ has eigenvalues $\lambda_n \geqslant 0$ for all $n \in \mathbb{N}$. We want to define

$$\det(\Delta) \text{ " } = \text{ " } \prod_{n \in \mathbb{N}} \lambda_n .$$

We can do this via (...) the operator $e^{-t\Delta}$ and its **trace**

$$\mathrm{tr}_{C^\infty(M)}(e^{-t\Delta}) = \sum_{n \in \mathbb{N}} e^{-t\lambda_n}.$$

Analytic torsion: $\rho(M) := \frac{-1}{2} \sum_{p \geqslant 0} (-1)^p p \cdot \ln\left(\det(\Delta_p)\right)$

Example : $M = S^1 = \mathbb{R}/\mathbb{Z}$, $\Delta \colon f \mapsto -f''$.
$\lambda_n = (2\pi n)^2$, $\det(\Delta) = 1$, $\rho(S^1) = 0$.

Theorem (Cheeger-Muller, without details)

Analytic torsion $\rho(M)$ equals Reidemeister torsion $\ln(T(M))$.

Interlude on the heat kernel

If $f : M \to \mathbb{R}$ is the **temperature** and t is the **time**, then $e^{-t\Delta}(f) \colon M \to \mathbb{R}$ is the **temperature at the time** t.

Indeed, $g(t,x) := \left(e^{-t\Delta}(f)\right)(x)$ satisfies the **heat equation**

$$\frac{\partial}{\partial t} g(t,x) + \Delta_x g(t,x) = 0.$$

The **heat kernel** $e^{-t\Delta}(x,y) \in \mathbb{R}$ is the ratio of temperature transferred from y to x during t. It satisfies

$$\forall f \in C^\infty(M), \forall x \in M, \ \left(e^{-t\Delta}(f)\right)(x) = \int_M e^{-t\Delta}(x,y)\, f(y)\, d\mathrm{vol}_M(y).$$

Example : $M = S^1 = \mathbb{R}/\mathbb{Z}$: $\quad e^{-t\Delta}(x,y) = \frac{1}{\sqrt{4\pi t}}\, e^{-\frac{|x-y|^2}{4t}}$.

Analytic L^2-Betti numbers

Let E complete Riemannian and $G \curvearrowright E$ cocompact by isometries.
Let $M = E/G$, $G = \pi_1(M)$, $\mathcal{F} \subset E$ **fundamental domain** for G.

Example : $E = \mathbb{H}^3$, M hyperbolic closed.

Example : $E = \mathbb{R}$, $M = S^1 = \mathbb{R}/\mathbb{Z}$, $\mathcal{F} = (0,1)$.

Analytic L^2-Betti number:
$b_p^{(2)}(M) = \lim_{t\to\infty} \int_{\mathcal{F}} \mathrm{tr}_{\mathbb{C}} \left(e^{-t\Delta_p}(x,x) \right) d\mathrm{vol}_M(x)$

Dodziuk : $b_p^{(2),ana}(M) = b_p^{(2),topo}(M)$.

Example : $M = S^1 = \mathbb{R}/\mathbb{Z}$, $e^{-t\Delta}(x,x) = \frac{1}{\sqrt{4\pi t}}$,
$b_0^{(2)}(S^1) = b_1^{(2)}(S^1) = \lim_{t\to\infty} \frac{1}{\sqrt{4\pi t}} \int_{\mathcal{F}} 1 = 0$.

L^2-analytic torsion of a closed Riemannian M

Δ_p Laplacian on p-forms of M. We want to define

$$\rho^{(2)}(M) \text{ " = " } \frac{1}{2}\sum_{p\geqslant 0}(-1)^p p \cdot \ln(\det(\Delta_p)).$$

The **heat kernel** gives $\theta_p(M)(t) := \int_{\mathcal{F}} \mathrm{tr}_{\mathbb{C}}(e^{-t\Delta_p}(x,x))d\mathrm{vol}_M(x)$, and the L^2-**Betti number** : $\theta_p(M)(\infty) = b_p^{(2)}(M)$.

The L^2-**analytic torsion** is defined by:

$$\rho^{(2)}(M) := \frac{1}{2}\sum_{p\geqslant 0}(-1)^p p \left(\frac{d}{ds}|_{s=0} \left(\frac{\int_0^{\epsilon} t^{s-1}(\theta_p(M)(t) - \theta_p(M)(\infty))dt}{\Gamma(s)} \right) + \int_{\epsilon}^{\infty} t^{-1}(\theta_p(M)(t) - \theta_p(M)(\infty))dt \right).$$

Example : $M = S^1 = \mathbb{R}/\mathbb{Z}$, $\theta_p(M)(t) = \frac{1}{\sqrt{4\pi t}}$, $\rho^{(2)}(S^1) = 0$.

Connexion with hyperbolic volume

The steps

Theorem (Lück-Schick 99, others)

Let M^3 be a complete hyperbolic manifold (closed or cusped) with finite volume $\mathrm{vol}(M)$. Then:

- *its L^2-Betti numbers $b_*^{(2)}(M)$ are all 0,*
- *its topological and analytic L^2-torsions ($T^{(2)}$ and $\rho^{(2)}$) satisfy*

$$T^{(2)}(M) = \exp\left(\rho^{(2)}(M)\right) = \exp\left(\frac{\mathrm{vol}(M)}{6\pi}\right).$$

This theorem was proven with several steps:

- Dodziuk : Analytic L^2-Betti numbers are zero.
- Burghelea-Friedlander-Kappeler-McDonald : (M closed) $T^{(2)} = e^{\rho^{(2)}}$.
- Lott-Mathai : (M closed) $\rho^{(2)}(M) = \frac{\mathrm{vol}(M)}{6\pi}$.
- Lück-Schick : extension to M cusped.

L^2-analytic torsion and volume for M^3 closed hyperbolic

$$\rho^{(2)}(M) = \frac{1}{2}\sum_{p\geqslant 0}(-1)^p p\left(\frac{d}{ds}|_{s=0}\left(\frac{\int_0^\epsilon t^{s-1}(\theta_p(M)(t)-\theta_p(M)(\infty))dt}{\Gamma(s)}\right) + \int_\epsilon^\infty t^{-1}(\theta_p(M)(t)-\theta_p(M)(\infty))dt\right).$$

$\widetilde{M} = \mathbb{H}^3$ is **homogeneous**, thus $\tau_p(t) := \mathrm{tr}_{\mathbb{C}}\left(e^{-t\Delta_p}(x,x)\right)$ is **independent** of $x \in \mathbb{H}^3$, and its integral is simply

$$\theta_p(M)(t) = \int_{\mathcal{F}} \tau_p(t)dt = \tau_p(t)\cdot\int_{\mathcal{F}} 1 = \tau_p(t)\cdot \mathrm{vol}(M).$$

Hence $$\frac{\rho^{(2)}(M)}{\mathrm{vol}(M)} = \frac{1}{2}\sum_{p\geqslant 0}(-1)^p p\left(\frac{d}{ds}|_{s=0}\left(\frac{\int_0^\epsilon t^{s-1}(\tau_p(t)-\tau_p(\infty))dt}{\Gamma(s)}\right) + \int_\epsilon^\infty t^{-1}(\tau_p(t)-\tau_p(\infty))dt\right) = \frac{1}{6\pi}.$$ (Lott-Mathai)

Genralizations

In dimension 3, it generalizes to M irreducible with empty or toroidal boundary:

$$T^{(2)}(M) = \exp\left(\frac{\mathrm{vol}(M)}{6\pi}\right),$$

where $\mathrm{vol}(M)$ is the **Gromov norm**, i.e. the sum of the hyperbolic volumes in the JSJ decomposition of M.

It also generalizes to bigger odd dimensions.