Fuglede–Kadison determinants over free groups and Lehmer's constants

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Overview of the talk

Fuglede–Kadison 1952: $\det_G(A) = e^{\mathrm{trace}(\ln(A))} \in \mathbb{R}_{\geqslant 0}$
for (∞-dimensional) positive G-**equivariant** operators A on $\ell^2(G)$.

$\rightarrow$ Building block of **L^2-Alexander torsions** of knots/3-manifolds, powerful **topological invariants** (Li-Zhang 2006, Dubois–Friedl–Lück 2014)

⚠ In general, $\det_G(A)$ is **hard to compute**.

Goal

New computations *of FK determinants via* **Cayley graphs**.

Theorem (B.A. 2022)

For the **free group** $\mathbb{F}_2 = \langle x, y | \rangle$, *we have* $\det_{\mathbb{F}_2}(1 + x + y) = \frac{2}{\sqrt{3}}$.

Corollary (B.A. 2022)

New bounds *on possible values of* $\det_G$ *for a* **large class** *of* G.

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First Part:

Fuglede–Kadison determinants and Cayley graphs

Motivation: The elusive L^2-Alexander invariant of knots

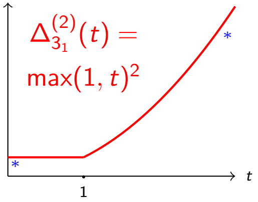
Li-Zhang 2006: The **L^2-Alexander invariant** of knots

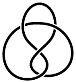
$\Delta_K^{(2)}(t) = \det_{\pi_1(S^3\setminus K)}(A(K,t))$ is a **continuous** function in $t > 0$.

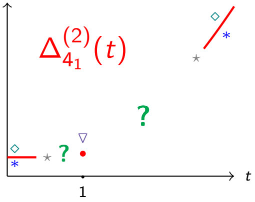
($\det_{\pi_1(S^3\setminus K)}$ = Fuglede–Kadison determinant and $A(K,t)$ = Fox matrix)

$\rightsquigarrow$ volume $\triangledown$, genus $*$, monodromy entropy $\star$, relative volume $\diamond$

(Lück–Schick 1999, Dubois–Friedl–Lück 2016, Liu 2017, Friedl-Lück 2019, B.A.-Friedl-Herrmann 2021)







Goal (Lück 2002, Dasbach–Lalin 2009, Kricker–Wong 2021, B.A. 2022)

New computations *of FK determinants via* **Cayley graphs**.

Fuglede–Kadison determinants

Let G be a finitely presented group (e.g. $G = \pi_1(S^3 \setminus K)$).

Regular representation $G \curvearrowright \ell^2(G) \rightsquigarrow$ Ring action $\mathbb{C}G \curvearrowright \ell^2(G)$.

Any $A = \sum a_g \cdot g \in \mathbb{C}G$, as a G-equivariant operator on $\ell^2(G)$, has a **trace** $\mathrm{tr}_G(A) := a_{1_G} \in \mathbb{C}$ and a **Fuglede–Kadison determinant**

$$\det{}_G(A) := \lim_{\epsilon \to 0^+} \exp\left(\frac{1}{2}\mathrm{tr}_G\left(\ln(A^* A + \epsilon\,\mathrm{Id})\right)\right) \in \mathbb{R}_{\geqslant 0}.$$

($\ln(A)$ = functional calculus, you can think "power series").

Ex: For G **finite**, $\ell^2(G) \cong \mathbb{C}^{|G|}$, and $\det_G(A) = \det(|A|)^{\frac{1}{|G|}}$.

Ex: $\det_G(a \cdot \mathrm{Id}) = |a|$.

Ex: $\det_G(\mathrm{Id} + a \cdot R_g) = \max(1, |a|)$ for g of infinite order.

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Computing $\det_G$ from Cayley graphs

Via $\ln(1-X) = -\sum_{n\geqslant 1} \frac{X^n}{n}$, for injective A and any $0 < \lambda < \|A\|^{-2}$,

$$\det{}_G(A) := \lim_{\epsilon \to 0^+} \exp\left(\frac{1}{2}\mathrm{tr}_G\left(\ln(A^*A + \epsilon\ \mathrm{Id})\right)\right)$$
$$= \ \ldots \ = \lambda^{-1/2} \exp\left(-\frac{1}{2}\sum_{n\geqslant 1}\frac{1}{n}\mathrm{tr}_G\left((\mathrm{Id} - \lambda A^*A)^n\right)\right),$$

which depends only on the sequence $c_n := \mathrm{tr}_G\left((A^*A)^n\right)$.

To $A^*A = \sum_{\text{finite}} a_{g_i} \cdot g_i \in \mathbb{C}G$, associate the **weighted Cayley graph** Γ_{A^*A} with group $\langle g_i \rangle_G$, generating set the g_i, and weights a_{g_i}.

Now $c_n = \mathrm{tr}_G\left((A^*A)^n\right)$ is exactly the **number of loops** on Γ_{A^*A} based in 1 of **length** n (counted with **weights** $\prod a_{g_j}$).

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Computing $\det_{\mathbb{F}_d}$ from trees

In general, to compute **every** $c_n = \mathrm{tr}_G\left((A^*A)^n\right)$, we consider the **generating series** $u_{A^*A}(z) = \sum_{n\in\mathbb{N}} c_n \cdot z^n$.

Relators of G, topology of $\Gamma_{A^*A} \rightsquigarrow$ **functional equations** in u_{A^*A}.

Lemma (Bartholdi 1999, Dasbach–Lalin 2009)

For $G = \mathbb{F}_2 = \langle x, y\rangle$, *we have* $u_{(1+x^{-1}+y^{-1})(1+x+y)}(z) = \frac{4}{1+3\sqrt{1-8z}}$.

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For $G = \mathbb{F}_2 = \langle x, y\rangle$, *we have* $\det_{\mathbb{F}_2}(1+x+y) = \frac{2}{\sqrt{3}}$.

Helped by Γ being a **tree** and A^*A being very **symmetric**.

(Also works for $1 + x_1 + \ldots + x_d$ and $x_1 + x_1^{-1} + \ldots + x_d + x_d^{-1}$ in $\mathbb{Z}\mathbb{F}_d$)

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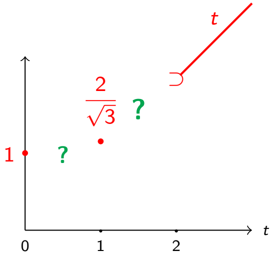
Possible hints for the L^2-Alexander invariant

What about $\det_{\mathbb{F}_2}(1 + x + t \cdot y)$? What we know so far:



Γ is the same, but the **functional equations** get more complicated.

Knowing $f(t) = \det_{\mathbb{F}_2}(1 + x + t \cdot y)$ could give **hints** of the **general form** of the L^2-**Alexander invariant** $\Delta_K^{(2)}(t)$.

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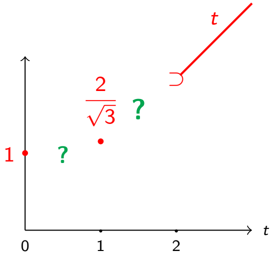

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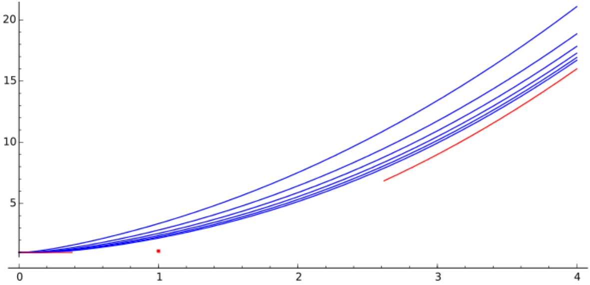
Upper approximations of $\det_G$

For any $N \geqslant 1$, since $\mathrm{Id} - \lambda A^* A > 0$, we get the **upper bounds**

$$\det{}_G(A) = \lambda^{-1/2} \exp\left(-\frac{1}{2} \sum_{n \geqslant 1} \frac{1}{n} \mathrm{tr}_G \left((\mathrm{Id} - \lambda A^* A)^n \right) \right)$$

$$\leqslant \lambda^{-1/2} \exp\left(-\frac{1}{2} \sum_{n=1}^{N} \frac{1}{n} \mathrm{tr}_G \left((\mathrm{Id} - \lambda A^* A)^n \right) \right).$$



Ex: $G = \pi_1(S^3 \setminus 4_1)$, $\Delta_{4_1}^{(2)}(t) = \det_G(1 - t \cdot g_1 - t \cdot g_2 - t \cdot g_3 + t^2 \cdot g_4) \leqslant f_N(t)$.

Second Part:

Bounds on Lehmer's constants of groups

Mahler measures (Lehmer 1933, Mahler 1962)

The **Mahler measure** of a polynomial $P \in \mathbb{C}\left[X^{\pm 1}\right] \cong \mathbb{C}[\mathbb{Z}]$ is

$$\mathcal{M}(P) := \exp\left(\frac{1}{2\pi}\int_0^{2\pi} \ln\left|P\left(e^{i\theta}\right)\right| d\theta\right) \in \mathbb{R}_{\geqslant 0}.$$

Examples:

- $\mathcal{M}\left(C \cdot \prod_{i=1}^{r}(X-\alpha_i)\right) = |C| \cdot \prod_{i=1}^{r} \max(1, |\alpha_i|)$.
- $\mathcal{M}(L) = 1.176280818...$ for **Lehmer's polynomial** $L(X) = X^{10}+X^9-X^7-X^6-X^5-X^4-X^3+X+1$.

Lehmer's Problem (1933)

Do we have $\inf\{\mathcal{M}(P) > 1 \mid P \in \mathbb{Z}[X]\} > 1$? *or even* $= 1.176...$?

$\leftrightarrow$ algebraic numbers, hyperbolic systoles, ... (Smyth 2008, Pham-Thilmany 2021)

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Lehmer's constants of groups

Proposition (Schmidt 1995, Lück 2002)

For $G = \mathbb{Z}$, and $P \in \mathbb{C}\left[X^{\pm 1}\right] \cong \mathbb{C}[\mathbb{Z}]$, we have $\det_{\mathbb{Z}}(P) = \mathcal{M}(P)$.

Lück 2019: **Lehmer's constant** of the group G:
$\Lambda_1^w(G) := \inf\{\det_G(A) > 1 \mid A \in \mathbb{Z}G \curvearrowright \ell^2(G) \text{ injective}\} \in [1, 2]$.

Examples: $\Lambda_1^w(\{1\}) = 2$, $\Lambda_1^w(\mathbb{Z}/2) = \sqrt{3}$, $\Lambda_1^w(\mathbb{Z}/n) \underset{n\to\infty}{\to} 1$.

Lehmer-Lück Problem (Lück 2019)

For which G do we have $\Lambda_1^w(G) > 1$? or $\Lambda_1^w(G) = 1.176...$?

Lehmer-Lück for $G = \mathbb{Z} \Leftrightarrow$ Lehmer's Problem (since $\det_{\mathbb{Z}} = \mathcal{M}$).

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For $G \neq \{1\}$ **torsionfree**, $\Lambda_1^w(G) \leqslant \Lambda_1^w(\mathbb{Z}) \leqslant \mathcal{M}(L) = 1.176...$

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For $G = \mathbb{F}_2 = \langle x, y \rangle$, we have $\det_{\mathbb{F}_2}(1 + x + y) = \frac{2}{\sqrt{3}} = 1.15....$
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L^2-torsions and hyperbolic volumes

Let M be a 3-manifold. Let $C^{(2)}(M) := \ell^2(\pi_1 M) \underset{\mathbb{Z}\pi_1(M)}{\otimes} C_*(\widetilde{M}, \mathbb{Z})$.

<u>L^2-Torsion of M:</u> $T^{(2)}(M) := \prod_{k \in \mathbb{N}} \det_{\pi_1(M)} \left(\partial_k^{C^{(2)}(M)} \right)^{(-1)^k} \in \mathbb{R}_{>0}$.

Theorem (Burghelea-Friedlander-Kappeler-McDonald 1996, Lott-Mathai 1992, Lück-Schick 1999)

Let M be a (closed or cusped) hyperbolic 3-manifold. Then

$$T^{(2)}(M) = \exp\left(\frac{\mathrm{Vol}(M)}{6\pi} \right).$$

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For an infinite number of **hyperbolic 3-manifolds** *M and all $G > \pi_1(M)$, we have $\Lambda_1^w(G) \leqslant \exp\left(\frac{\mathrm{Vol}(M)}{6\pi} \right) < 1.15...$*

<u>Key:</u> $\mathrm{rk}(\pi_1(M)) = 2$ (e.g. Dehn fillings on the Whitehead link)

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Theorem (Burghelea-Friedlander-Kappeler-McDonald 1996, Lott-Mathai 1992, Lück-Schick 1999)

Let M be a (closed or cusped) hyperbolic 3-manifold. Then

$$T^{(2)}(M) = \exp\left(\frac{\mathrm{Vol}(M)}{6\pi}\right).$$

Theorem (B.A. 2022)

For an infinite number of **hyperbolic 3-manifolds** *M and all $G > \pi_1(M)$, we have* $\Lambda_1^w(G) \leqslant \exp\left(\frac{\mathrm{Vol}(M)}{6\pi}\right) < 1.15...$

Key: $\mathrm{rk}(\pi_1(M)) = 2$ (e.g. Dehn fillings on the Whitehead link)

Thank you for your attention!