Extinction and Survival of Branching Random Walk in Random Environment

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1 Introduction

Consider particles that move in $\mathbb{Z}^d$ according to a random walk, branching into independent copies at each time. It starts with one particle at the origin of $\mathbb{Z}^d$. If we have a particle at time t in position x, it chooses at random one of its $2d$ nearest neighbosr, moves there and dies, leaving k children with probability $q_{t,x}(k)$. Then each particle follows the same behaviour independently. The offspring distributions $q_{t,x}$ are assumed to be i.i.d. in time and space following the common law γ. We refer to $(q_{t,x})_{t,x}$ as the environment. We call this model the branching random walk in random environment and we abbreviate it by BRWRE. It was introduced by Birkner in his thesis [5]. It is a case of stochastic interacting system where the only interaction between particles is the sharing of the same environment. It is a population model that could be used in biology. It generalises other simpler branching processes such as the Galton-Watson process and the Smith-Wilkinson process.

We will focus on the question of extinction and survival of this process. Let B_t be the set of particles in BWRE at time t, and denote the survival event by

$$\{\text{survival}\} = \{B_t \neq \varnothing \text{ for all } t \in \mathbb{N}\}$$

This dissertation will begin with the study of two articles, one by Comets and Yoshida [8] giving a characterisation of survival cases, but leaving open the critical case, and the other by Garet and Marchand [10] taking care of the critical case. This will be followed by some development about an open case for extinction and survival of BRWRE which will be the focal point of this dissertation.

In [8], the authors use the link between the well-studied directed polymers in random environment (abbreviated by DPRE) and the BRWRE to show that survival of the BRWRE is linked to the free energy of the associated directed polymer. If the free energy is positive, survival happens with positive probability and extinction is sure if it is negative. This article also shows that the exponential growth rate of the population in case of survival is given by the free energy.

In [10], the authors prove the conjecture made by [8] that in the critical case there is extinction. This is done by adapting the techniques developed in [4] and expounded in [13] to prove the extinction of the critical contact process.

Interestingly, one of the case discussed in the proof of the second paper is excluded by the hypotheses that are made in both papers. It is the case of possible sterile environment where with positive probability the reproduction law is simply δ_0, in other words, particles that move there die with no children. This case is excluded in the first paper because it enables to use the link between directed polymers in random environment and BRWRE very productively. A few results will be proven here that are, at the same time, original and easy. A few unsucessful leads for further results will also be expounded here. The characterisation of survival and extinction in that case is an open problem.

Notations : We write $\mathbf{P}[X]$ for $\int X d\mathbf{P}$ and $\mathbf{P}[X : A]$ for $\int_A X d\mathbf{P}$ for a random variable X and an event A on a probability space. For $x \in \mathbb{R}^d$, $|x| = |x_1| + \cdots + |x_d|$

1.1 Classical branching processes

We start by recalling the results on the simpler Galton-Watson process and Smith-Wilkinson process of which the BRWRE is a generalisation. This is useful not only to appreciate the growing complexity of the branching processes but also for embedding and comparison arguments that will be important later on.

The Galton-Watson process is the number N_t of members of the t^{th} generation of a population where we start with $N_0 = 1$ and each member of a generation reproduces independently according to the same offspring distribution q. We have the following well-known theorem. Here, $m = \sum\limits_{k=0}^{\infty} kq(k)$ is the mean number of children an individual has. We assume $q(1) < 1$ and $m < \infty$. See for reference [1].

Theorem 1.1 *One has*

a) $\mathbf{P}(\text{survival}) > 0$ *if and only if* $m > 1$.

b) *If* $m > 1$ *and* $\sum_{k=1}^{\infty} k \ln(k) q(k) < \infty$, *then*

$$\{\text{survival}\} = \{\lim_{t\to\infty} N_t/m^t \in]0,\infty[\}, \mathbf{P}-a.s.$$

The Smith-Wilkinson process is the simplest branching process with random environment. The offspring distribution is now random : $(q_t)_{t\in\mathbb{N}}$ is a sequence of i.i.d. probability measures on $\mathbb{N}$. We denote the common law γ. The law under a given environment $\mathbf{q} = (q_t)$ is $\mathbf{P}^{\mathbf{q}}$. We denote the law of $\mathbf{q}$ by $\mathbf{Q} = \gamma^{\otimes\mathbb{N}}$. The overall probability is $\mathbf{P}^{\gamma}$. We have

$$\mathbf{P}^{\gamma}(\cdot) = \int \mathbf{Q}(d\mathbf{q})\mathbf{P}^{\mathbf{q}}(\cdot)$$

We assume $q_t(1) < 1$ and $m_t < \infty$ where m_t is the mean of q_t.

Theorem 1.2 *Suppose that* $\mathbf{Q}[|\ln(m_t)|] < \infty$. *Then*

a)*[3, Theorems1-3] Either*

$$\mathbf{P}^{\mathbf{q}}(\text{survival}) > 0, \mathbf{Q}\text{-}a.s.$$

or

$$\mathbf{P}^{\mathbf{q}}(\text{survival}) = 0, \mathbf{Q}\text{-}a.s.$$

The former case holds if and only if

$$\mathbf{Q}[\ln(m_t)] > 0 \quad \textit{and} \quad \mathbf{Q}[\ln \frac{1}{1-q_t(0)}] < \infty \tag{1.1}$$

b)[2, Theorem 1] In addition to the latter condition, suppose that

$$\mathbf{Q}[m_t^{-1}\sum_{k\geq 1} q_t(k)k\ln k] < \infty \tag{1.2}$$

then

$$\{\text{survival}\} = \{\lim_{t\to\infty}\frac{N_t}{m_0 ... m_{t-1}} \in]0,\infty[\}, \mathbf{P}^\gamma\text{-}a.s.$$

1.2 Branching random walk in random environment

Different equivalent definitions of BRWRE describing the same dynamics as explained in the introduction are possible, namely in terms of number of particles at time-space locations as in [10] or a more precise definition where the genealogy of each particle is encoded as in [8]. A description purely in terms of number of particles would not be sufficient for this dissertation as we use monotonicity properties given by the genealogy both to prove the results of [8] and for a coupling argument in section 4.

We intoduce a tree whose vertices will encode the genealogy of each particle

$$\mathcal{T} = \bigcup_{t\in\mathbb{N}}\{1\}\times\mathbb{N}^t$$

For $\nu \in \mathcal{T}$, $|\nu|$ is the generation of ν, that is to say its length minus one. The children of the particle labeled $\nu = (1,\nu_1,..,\nu_t)$ will be labeled $((1,\nu_1,..,\nu_t,\nu_{t+1})$ with ν_{t+1} between 1 and k where k is the number of children of our particle. We denote $\mathcal{T}_t$ the set of all vertices of generation t and for $\nu \in \mathcal{T}_{t+s}$, $\nu_{|t} = (1,\nu_1,..,\nu_t)$ the ancestoral history of ν up to time t.

We now formalise the definition of our BRWRE. We first take care of the spatial motion. Let $p(\cdot,\cdot)$ be the transition probability for simple random walk on $\mathbb{Z}^d$ that is to say $p(x,y) = 1/2d$ if $|x-y| = 1$ and 0 otherwise. We define $(\Omega_X,\mathcal{F}_X)$ as the set $(\mathbb{Z}^d)^{\mathbb{N}\times\mathbb{Z}^d\times\mathcal{T}}$ with the product σ-field and for $X \in \Omega_X$, $X \to X_{t,x,\nu}$ as the projection. We define $\mathbf{P}_X \in \mathcal{P}(\Omega_X,\mathcal{F}_X)$ as the product measure such that

$$\mathbf{P}_X(X_{t,x,\nu} = y) = p(x,y) \text{ for all } (t,x,\nu) \in \mathbb{N}\times\mathbb{Z}^d\times\mathcal{T} \text{ and } y \in \mathbb{Z}^d$$

$X_{t,x,\nu}$ will be the the position at time $t+1$ of the children of the particle labeled ν when it occupies the time-space location (t,x).

We define the offspring distribution. We set $\Omega_q = \mathcal{P}(\mathbb{N})^{\mathbb{N}\times\mathbb{Z}^d}$. Each $\mathbf{q} \in \Omega_q$ is an environment and $q_{t,x}$ is the offspring distribution of the particles in (t,x). The set $\mathcal{P}(\mathbb{N})$ is equipped by the natural Borel σ-field induced from that of $[0,1]^{\mathbb{N}}$. We define $\mathcal{F}_q$ as the product σ-field on Ω_q. The random environment is simply the choice of a law γ which will be the common law of the i.i.d. $q_{t,x}$, we denote $\mathbf{Q} = \gamma^{\otimes\mathbb{N}\times\mathbb{Z}^d}$, $\mathbf{Q} \in \mathcal{P}(\Omega_q,\mathcal{F}_q)$.

We then define, following [10], $\Omega_U =]0,1[^{\mathbb{N}\times\mathbb{Z}^d\times\mathcal{T}}$ with the Borel σ-field and for $U \in \Omega_U$, $U \to U_{t,x,\nu}$ as the projection. $\mathbf{P}_U$ is the product uniform measure on $]0,1[$. The $U_{t,x,\nu}$ are used to generate the progeny of the particle with genealogy ν at x at time t thanks to the law $q_{t,x}$. Another formulation, used in [8], would have been possible by considering integer valued variables of law $q_{t,x}$ but this means that the probability measure on this space would have depended on Ω_q. Finally we define $(\Omega,\mathcal{F})$,

$$\Omega = \Omega_X\times\Omega_U\times\Omega_q,\ \mathcal{F} = \mathcal{F}_X\otimes\mathcal{F}_U\otimes\mathcal{F}_q$$

and $\mathbf{P}^q, \mathbf{P}^\gamma \in \mathcal{P}(\Omega, \mathcal{F})$

$$\mathbf{P}^q = \mathbf{P}_X \otimes \mathbf{P}_U \otimes \delta_q, \mathbf{P}^\gamma = \mathbf{P}_X \otimes \mathbf{P}_U \otimes \mathbf{Q}$$

We define a Markov chain $(B_t)_{t\in\mathbb{N}}$ with values in finite subsets of $\mathbb{Z}^d \times \mathcal{T}$, inductively by $B_0 = (0,1)$ a single particle at the origin, the common ancestor of all other particles, and for $t \geq 1$,

$$B_t = \bigcup_{(x,\nu)\in B_{t-1}} \{(y,\mu) \in \mathbb{Z}^d \times \mathcal{T}_t; X_{t-1,x,\nu} = y, \mu_{|t-1} = \nu, \mu_t \text{ such as } \sum_{i=0}^{\mu_t} q_{t-1,x(i)} \leq U_{t-1,x,\nu}\} \tag{1.3}$$

We call this process the branching random walk in random environment (BRWRE). Writing $(x,\nu) \in B_t$ means that the particle with ancestoral history ν is in position x at time t. The number of children of a particle is $K_{t,x,\nu} = \max(k \in \mathbb{N}; \sum_{i=0}^k q_{t,x}(i) \leq U_{t,x,\nu})$ and its law is $q_{t,x}$. We define the set of particles at x at time t,

$$B_{t,x} = \{(y,\nu) \in B_t; y = x\}$$

We consider the filtration $\mathcal{F}_t = \sigma(X_{s,\cdot,\cdot}, U_{s,\cdot,\cdot}, q_{s,\cdot}; s \leq t-1)$ for $t \geq 1$, $\mathcal{F}_0 = \{\emptyset, \Omega\}$ to which B_t is adapted.

We have the following recursion :

$$|B_{0,y}| = \delta_{0,y},\ |B_{t,y}| = \sum_{(x,\nu)\in B_{t-1}} \delta_y(X_{t-1,x,\nu})K_{t-1,x,\nu},\ t \geq 1, y \in \mathbb{Z}^d \tag{1.4}$$

1.3 The associated directed polymers in random environment

In this subsection, with $m_{t,x} = \sum_{k\in\mathbb{N}} k q_{t,x}(k)$, we assume

$$\mathbf{Q}[m_{0,0} + m_{0,0}^{-1}] < \infty \tag{1.5}$$

Note that the assumption $\mathbf{Q}[m_{0,0}^{-1}] < \infty$ forbids the possibility of

$$\mathbf{Q}(q_{0,0}(0) = 1) > 0 \tag{1.6}$$

which will be the condition that we will investigate in section 4.

Let $(S_t)_{t\in\mathbb{N}}$ be the symmetric simple random walk on $\mathbb{Z}^d$ starting from the origin defined on $(\Omega_S, \mathcal{F}_S, \mathbf{P}_S)$. We then introduce the partition function of directed polymer in random environment (DPRE),

$$Z_{t,x} = \mathbf{P}_S[\zeta_t : S_t = x] \text{ and } Z_t = \mathbf{P}_S[\zeta_t]$$

where

$$\zeta_t = \prod_{u=0}^{t-1} m_{u,S_u}$$

The polymer measure is the law with weights ζ_t on the path space, and a directed polymer in the environment $\mathbf{q}$ is a path sampled from this law. The link between the DPRE and the BRWRE is that

Proposition 1.1 $Z_{t,x} = \mathbf{P}^q[|B_{t,x}|]$ *and* $Z_t = \mathbf{P}^q[|B_t|]$

Proof : [14, Lemma 1.3.1] The second equality is obtained from the first by summing over all $x \in \mathbb{Z}^d$. For the first one, we compute the following conditional expectation thanks to (1.4).

$$\begin{aligned}
\mathbf{P}^q[|B_{t,x}| \mid \mathcal{F}_t] &= \sum_{(x_{t-1},\nu)\in B_{t-1}} \mathbf{P}^q[\delta_x(X_{t-1,x_{t-1},\nu})K_{t-1,x_{t-1},\nu}] \\
&= \sum_{(x_{t-1},\nu)\in B_{t-1}} p(x_{t-1},x)m_{t-1,x_{t-1}} = \sum_{x_{t-1}\in\mathbb{Z}^d} |B_{t-1,x_{t-1}}|p(x_{t-1},x)m_{t-1,x_{t-1}}
\end{aligned}$$

and then,

$$\mathbf{P}^q[|B_{t,x}|] = \sum_{x_1,..,x_{t-1}\in\mathbb{Z}^d} \prod_{i=0}^{t-1} p(x_i,x_{i+1})m_{i,x_i} = \mathbf{P}_S[\zeta_t : S_t = x]$$

□

This link is the most important tool there is to study the BRWRE. The DPRE is less random than the BRWRE as the offspring distributions are replaced by their means. A natural question is to determine what properties of the DPRE are transferred to the BRWRE. For example, it is known that the DPRE undergoes a phase transition from the weak disorder phase where it follows a law of large numbers and a central limit theorem (delocalization) to the strong disorder phase (localization) where it concentrates on a few trajectories. This phase transition also happens for the BRWRE, in the strong disorder phase, a positive proportion of the population will concentrate in certain space-time location. Those properties are not a consequence of the results for DPRE but rather the proofs are inspired by the corresponding proofs for DPRE. For reference, see [12, 11]. This link is fundamental but should not make us forget that the questions of interest are very different between the two models. In the BRWRE, the question of survival and extinction is a central question while it has no meaning for DPRE.

We now turn to the definition of the free energy for DPRE. We set $\Delta = \{\theta \in \mathbb{R}^d; |\theta| \leq 1\}$ and, for $\theta \in \Delta \cap \mathbb{Q}^d$, $\mathbb{N}^*(\theta) = \{t \in \mathbb{N}^*; t\theta \in \mathbb{Z}^d, t - t|\theta| \in 2\mathbb{N}\}$ and $\mathbb{N}(\theta) = \{0\} \cup \mathbb{N}^*(\theta$. It is the set of all times t when a simple walk on $\mathbb{Z}^d$ can verify $s_t = t\theta$. In particular, $\mathbf{P}_S(S_t = t\theta) \geq (2d)^{-t} > 0$ for all $t \in \mathbb{N}(\theta)$. We have $\mathbb{N}(\theta) = n(\theta)\mathbb{N}$.

Observe that (1.5) and $|\ln(u)| \leq \max(u, u^{-1})$ for $u > 0$ implies that $\mathbf{Q}|\ln Z_t| < \infty$ and $\mathbf{Q}|\ln Z_{t,x}| < \infty$ for all $(t,x) \in \mathbb{N}^* \times \mathbb{Z}^d$.

Proposition 1.2 *There exists a concave, upper semi-continuous function $\psi : \Delta \to \mathbb{R}$ such that for all $\theta \in \Delta \cap \mathbb{Q}^d$,*

$$\psi(\theta) = \lim_{\substack{t\to\infty \\ t\in\mathbb{N}^*(\theta)}} \frac{1}{t}\mathbf{Q}[\ln Z_{t,t\theta}]$$

ψ is symmetric in the sense that $\psi(\theta) = \psi(|\theta_{\sigma(1)}|,..,|\theta_{\sigma(d)}|)$ for any permutation σ of $\{1,..,d\}$. The maximum of this function is attained for$\theta = 0$. Moreover, the following limit

$$\Psi = \lim_{t\to\infty} \frac{1}{t}\mathbf{Q}[\ln Z_t]$$

also exists and equals $\psi(0)$.

This is a rewording by [8] of [7, Theorem 1.2]. The number Ψ is called the free energy of the polymer and $\psi(\theta)$ is the directional free energy.

1.4 Generating functions and comparisons between processes

The study of the Galton-Watson process can be done through the generating function of its fixed offsprong distribution,

$$\hat{q}(s) = \sum_{k=0}^{\infty} s^k q(k) = \mathbf{P}[s^{N_1}]$$

One can also define generating functions for the Smith-Wilkinson process and the BRWRE. Comparison between those generating functions leads to the following inequality between the survival probabilities of the BRWRE with offspring distribution γ, of the Galton-Watson process defined by $q(\cdot) = \mathbf{Q}(q_{0,0}(\cdot))$ and of the Smith-Wilkinson process with offspring distribution γ. We denote by σ^{GW} and σ^{SW} respectively the survival probability of that Galton-Watson process and of that Smith-Wilkinson process. One has

$$\sigma^{SW} \leq \mathbf{P}^{\gamma}(\text{survival}) \leq \sigma^{GW} \tag{1.7}$$

The associated Galton-Watson process is a BRWRE where the offspring distribution is deterministic while the Smith-Wilkinson process is obtained by replacing $\mathbb{Z}^d$ by a singleton. This inequality can be understood heuristically : compared to the Galton-Watson process, the Smith-Wilkinson process has a higher probability of extinction because catastrophic environment happens where there is a large probability of having no children. In such an environment, there is a large probability of extinction and this can happen even if the mean of the offspring distribution is large. The second condition in (1.1) can be interpreted in such a way, it makes sure that those catastrophic environment are not too frequent, allowing survival. For the BRWRE, catastrophic environment can cause extinction, for example, if $q_{0,0}$ is catastrophic then we can easily have extinction at the first step and therefore we have the second inequality. But after the second step, all the particles are not in the same location and thus the probability that they will all encounter a catastrophic environment is smaller which explains the first inequality. The fluctuation of the offspring distribution are modulated by the spatial movement.

We will use the second inequality of (1.7) in section 4 and therefore we include a proof following [8] in order to be as self-contained as possible concerning the results on BRWRE under confition condition(1.6), the focal point of this dissertation.

For $q \in \mathcal{P}(\mathbb{N})$, we denote as before

$$\hat{q}(s) = \sum_{k=0}^{\infty} s^k q(k) \text{ for } s \in [0,1].$$

We agree on the convention $0^0 = 1$. For a fixed $\mathbf{q} \in \Omega_q$ and $t \in \mathbb{N}$, we define $\Phi_t : [0,1]^{\mathbb{Z}^d} \to [0,1]^{\mathbb{Z}^d}$ by

$$\Phi_t(\xi) = (\Phi_{t,x}(\xi))_{x \in \mathbb{Z}^d},\ \Phi_{t,x}(\xi) = \sum_{y \in \mathbb{Z}^d} p(x,y)\hat{q}_{t,x}(\xi_y).$$

Therefore $(\Phi_t)_{t \in \mathbb{N}}$ is a sequence of i.i.d. random maps on the probability space $(\Omega_q, \mathcal{F}_q, \mathbf{Q})$.

Proposition 1.3 *For $\xi \in [0,1]^{\mathbb{Z}^d}$ and $t \in \mathbb{N}^*$,*

$$\mathbf{P}^q[\xi^{B_t} | \mathcal{F}_{t-1}] = \Phi_{t-1}(\xi)^{B_{t-1}} \tag{1.8}$$

with the notation

$$\xi^{B_t} = \prod_{x \in \mathbb{Z}^d} \xi_x^{|B_{t,x}|}$$

As a consequence,

$$\mathbf{P}^q[\xi^{B_t}] = \Phi_{0,0} \circ \Phi_1 \circ \ldots \circ \Phi_{t-1}(\xi)$$

This proposition allows us to call Φ_t the generating function of the branching random walk with environment $(q_{t,x})$.

Proof : We only have to prove (1.8). We begin by writing :

$$\begin{aligned}
\xi^{B_t} = \prod_{y\in\mathbb{Z}^d} \xi_y^{|B_{t,y}|} = \prod_{y\in\mathbb{Z}^d} \prod_{(x,\nu)\in B_{t-1}} \xi_y^{K_{t-1,x,\nu}\delta_y(X_{t-1,x,\nu})} &= \prod_{(x,\nu)\in B_{t-1}} \prod_{y\in\mathbb{Z}^d} \xi_y^{K_{t-1,x,\nu}\delta_y(X_{t-1,x,\nu})} \\
&= \prod_{(x,\nu)\in B_{t-1}} \prod_{y\in\mathbb{Z}^d} \left(1 + (\xi_y^{K_{t-1,x,\nu}} - 1)\delta_y(X_{t-1,x,\nu})\right) \\
&= \prod_{(x,\nu)\in B_{t-1}} \left(1 + \sum_{y\in\mathbb{Z}^d} (\xi_y^{K_{t-1,x,\nu}} - 1)\delta_y(X_{t-1,x,\nu})\right) \\
&= \prod_{(x,\nu)\in B_{t-1}} \sum_{y\in\mathbb{Z}^d} \xi_y^{K_{t-1,x,\nu}} \delta_y(X_{t-1,x,\nu})
\end{aligned}$$

where on the third line, we have used that

$$\prod_{y\in Y}(1 + x_y) = 1 + \sum_{A\subset Y} \prod_{y\in A} x_y \text{ for any finite set } Y \text{ and } (x_y)_{y\in Y} \in \mathbb{R}^Y$$

The terms with $|A| \geq 2$ vanishes here. Since

$$\mathbf{P}^q[\sum_{y\in\mathbb{Z}^d} \xi_y^{K_{t-1,x,\nu}} \delta_y(X_{t-1,x,\nu})|\mathcal{F}_{t-1}] = \Phi_{x,t-1}(\xi),$$

we get

$$\mathbf{P}^q[\xi_t^B|\mathcal{F}_{t-1}] = \prod_{(x,\nu)\in B_{t-1}} \Phi_{x,t-1}(\xi) = \Phi_{t-1}(\xi)^{B_{t-1}}$$

□

Lemma 1.1 *Let $q(k) = \mathbf{Q}[q_{0,0}(k)]$, $k \in \mathbb{N}$. For $s \in [0,1]$,*

$$\mathbf{P}^\gamma[s^{|B_t|}] \geq (\hat{q} \circ \cdots \circ \hat{q})(s).$$

where $\hat{q}$ appears t times in the right hand-side.

Proof : It follows from (1.8) and Jensen's inequality that

$$\mathbf{P}\gamma[s^{|B_t|}|\mathcal{F}_{t-1}] = \prod_{x\in\mathbb{Z}^d} \mathbf{Q}[\hat{q}_{0,0}(s)^n]_{n=|B_{t-1,x}|} \geq \prod_{x\in\mathbb{Z}^d} \mathbf{Q}[\hat{q}_{0,0}(s)]^{|B_{t-1,x}|} = \hat{q}(s)^{|B_{t-1}|}$$

And the lemma follows by iteration.

□

We can now prove the second inequality in (1.7). It is well-known that, for N_t the Galton-Watson process we are interested in and $s \in [0,1]$,

$$\mathbf{P}[s^{N_t}] = (\hat{q} \circ \cdots \circ \hat{q})(s)$$

where $\hat{q}$ appears t times. We have $\mathbf{P}^\gamma[0^{|B_t|}] = \mathbf{P}^\gamma(B_t = \varnothing)$ and therefore $\mathbf{P}^\gamma(\text{survival}) = 1 - \lim_{t\to\infty} \mathbf{P}^\gamma[0^{|B_t|}]$ and similar formulae for σ^{GW}. Therefore the second inequality of (1.7) follows from lemma 1.1.

$\square$

Another result demonstrated thanks to the generating functions is the following zero-one law :

$$\text{Either, } \mathbf{P}^q(\text{survival}) > 0,\ \mathbf{Q}\text{-a.s. or } \mathbf{P}^q(\text{survival}) = 0,\ \mathbf{Q}\text{ - a.s..}$$

This of course does not remain true under (1.6), the best we can hope for being $\mathbf{P}^\gamma(\text{survival}) > 0$. This also tells us that under (1.6) we have a different behaviour than for the Smith-Wilkinson process.

2 Extinction and survival in the non-critical case

In this section we again assume condition (1.5)

$$\mathbf{Q}[m_{0,0} + m_{0,0}^{-1}] < \infty$$

in order to use proposition 1.2 on the free energy of the DPRE which provides the most important part of the proof. It will allow us to adapt the results of theorem 1.2 on the Smith-Wilkinson process to the BRWRE.

2.1 Criteria for global and local survival, and growth rates

Theorem 2.1 ***(global growth)*** *Let* $\varepsilon > 0$.

a) *We have,* $\mathbf{P}^\gamma$*-a.s.,*

$$|B_t| \le e^{(\Psi+\varepsilon)t} \textit{ for all large } t\textit{'s.}$$

In particular,

$$\mathbf{P}^\gamma(\text{survival}) = 0 \textit{ if } \Psi < 0$$

b) *Suppose* $\Psi > 0$ *and*

$$\mathbf{Q} \ln \frac{1}{1 - q_{0,0}(0)} < \infty \tag{2.1}$$

Then

$$\mathbf{P}^q(\text{survival}) > 0,\ \mathbf{Q}\text{-}a.s.$$

Suppose $\Psi > 0$, (2.1) *and*

$$\mathbf{Q}\left[\frac{\mathbf{P}^q[|B_{t,0}| \ln |B_{t,0}|]}{Z_{t,0}}\right] < \infty \textit{ for all } t \in 2\mathbb{N}^* \tag{2.2}$$

Then,

$$\{\text{survival}\} = \{\lim_{t\to\infty} \frac{1}{t} \ln |B_t| = \Psi\},\ \mathbf{P}^\gamma\text{-}a.s.$$

This theorem identifies the free energy as the growth rate of the BRWRE. It leaves open the case $\Psi = 0$. We see the local version of this theorem next.

Theorem 2.2 ***(local growth)*** *Let $\theta \in \Delta \cap \mathbb{Q}^d$ and $\varepsilon > 0$.*

a) *We have, $\mathbf{P}^\gamma$-a.s.,*

$$|B_{t,t\theta}| \leq e^{(\psi(\theta)+\varepsilon)t} \text{ for all large } t\text{'s}.$$

In particular,

$$\mathbf{P}^\gamma(B_{t,t\theta} \neq \varnothing \text{infinitely often}) = 0 \text{ if } \psi(\theta) < 0$$

b) *Suppose $\psi(\theta) > 0$ and (2.1) Then*

$$\mathbf{P}^q(B_{t,t\theta} \neq \varnothing \text{ for all } t \in \mathbb{N}(\theta)) > 0, \ \mathbf{Q}\text{-a.s.}$$

Suppose $\psi(\theta) > 0$, (2.1) and

$$\mathbf{Q}\left[\frac{\mathbf{P}^q[|B_{t,\theta}| \ln |B_{t,\theta}|]}{Z_{t,\theta}}\right] < \infty \text{ for all } t \in 2\mathbb{N}^* \tag{2.3}$$

Then,

$$\{B_{t,t\theta} \neq \varnothing \text{ for all } t \in \mathbb{N}(\theta)\} \subset \{\lim_{t\to\infty} \frac{1}{t} \ln |B_t| = \psi(\theta)\}, \ \mathbf{P}^\gamma\text{-a.s.}$$

Those two theorems are the main results from [8]. We will focus on the proof of the criteria for extinction and survival leaving out the proofs of the local results and of the growth rate.

2.2 Proof of 2.1 a) and 2.2 a)

We need a concentration result on the partition function of the DPRE in order to prove the first part of the result.

Proposition 2.1 *There are constants c_1, $c_2 \in]0, \infty[$, such that*

$$\mathbf{Q}\left(\frac{1}{t}|\ln Z_t - \mathbf{Q}[\ln Z_t]| > \varepsilon\right) \leq 2\exp(-c_1\varepsilon^2 t),$$

for all $\varepsilon \in]0, c_2]$ and $t \in \mathbb{N}$, and

$$\mathbf{Q}\left(\frac{1}{t}|\ln Z_{t,t\theta} - \mathbf{Q}[\ln Z_{t,t\theta}]| > \varepsilon\right) \leq 2\exp(-c_1\varepsilon^2 t),$$

for all $\varepsilon \in]0, c_2]$, $\theta \in \Delta \cap \mathbb{Q}^d$ and $t \in \mathbb{N}^*(\theta)$.

This is proven in [8] thanks to a series of general martingale inequalities. We can now turn to the proof of 2.1 a). It will slightly differ from the one given in [8] which contained a wrong inequality (this was easily corrected).

Thanks to proposition 1.2 and proposition 2.1, there exists T such that for all $t \geq T$,

$$\mathbf{Q}[\ln Z_t] \leq t(\Psi + \frac{\varepsilon}{4}) \text{ and } \mathbf{Q}\left(\ln Z_t \geq \mathbf{Q}[\ln Z_t] + \frac{\varepsilon}{4}t\right) \leq 2\exp(-c_1'\varepsilon^2 t).$$

and we have $\ln Z_t \geq (\Psi + \frac{\varepsilon}{2})t \Rightarrow \ln Z_t \geq \mathbf{Q}[\ln Z_t] + \frac{\varepsilon}{4}t$ and therefore,

$$\mathbf{Q}\left(Z_t \geq e^{(\Psi+\frac{\varepsilon}{2})t}\right) \leq 2\exp(-c_1'\varepsilon^2 t)$$

On the other hand, we have for any $\chi = \chi(\mathbf{q}) \in \{0,1\}$ that

$$\mathbf{P}^q\left(|B_t| \geq e^{(\Psi+\varepsilon)t}\right) \leq \mathbf{P}^q\left(|B_t| \geq e^{(\Psi+\varepsilon)t}\right)\chi + 1 - \chi \leq e^{-(\Psi+\varepsilon)t}Z_t\chi + 1 - \chi$$

by Chebyshev inequality. We take $\chi = \mathbf{1}\{Z_t < e^{(\Psi+\frac{\varepsilon}{2})t}\}$ and take $\mathbf{Q}$ expectation to obtain

$$\mathbf{P}^\gamma\left(|B_t| \geq e^{(\Psi+\varepsilon)t}\right) \leq e^{-\frac{\varepsilon}{2}t} + \mathbf{Q}\left(Z_t \geq e^{(\Psi+\frac{\varepsilon}{2})t}\right) \leq e^{-\frac{\varepsilon}{2}t} + 2\exp(-c_1'\varepsilon^2 t)$$

We therefore have $|B_t| \leq e^{(\Psi+\varepsilon)t}$ for all large t's by Borel-Cantelli lemma. This proves theorem 2.1 a). The proof of theorem 2.2 a) is similar.

2.3 Proof of survival

The main idea of the proof of survival is to embed a Smith-Wilkinson process in our BRWRE and then to use theorem 1.2. Schematically, the result on growth rate also follows from theorem 1.2 as (2.2) and (2.3) are the analogs of (1.2).

Lemma 2.1 ***(Survival along subsequence of times)***
Suppose $\theta \in \Delta \cap \mathbb{Q}^d$, (2.1) and that there exists $T \in \mathbb{N}^*(\theta)$ such that

$$\mathbf{Q}[\ln Z_{T,T\theta}] > 0 \tag{2.4}$$

Then

$$\mathbf{P}^q(B_{sT,sT\theta} \neq \varnothing \text{ for all } s \geq 1) > 0,\ \mathbf{Q}\text{-}a.s.$$

Let us first see how this lemma proves the first part of 2.1 b), the result on survival. If $\Psi = \psi(0) > 0$, then by proposition 1.2, for any large enough T, we have $\mathbf{Q}[\ln Z_{T,0}] > 0$ and we have survival of the BRWRE.

Proof of the lemma : We will embed a Smith-Wilkinson process into $(B_{sT,sT\theta})$. The embedded process is simply the process of the particles that go from $(0,0)$ to $(T,T\theta)$ to $(2T,2T\theta)$ and so on. We introduce some notation to write this down precisely. For $(s,z,\lambda) \in \mathbb{N} \times \mathbb{Z}^d \times \mathcal{T}_s$, we define the (s,z,λ)-branch $(B_t^{s,z,\lambda})_{t\in\mathbb{N}}$ of $(B_t)_{t\in\mathbb{N}}$ inductively by $B_0^{s,z,\lambda} = (z,\lambda)$ and for $t \geq 1$,

$$B_t^{s,z,\lambda} = \bigcup_{(x,\nu)\in B_{t-1}^{s,z,\lambda}} \{(y,\mu) \in \mathbb{Z}^d \times \mathcal{T}_{s+t}; X_{s+t-1,x,\nu} = y, \mu_{|s+t-1} = \nu, \mu_s + t \leq K_{s+t-1,x,\nu}\}$$

This means restarting a BRWRE from a single particle in (s,z) whose ancestoral history is λ. If $(z,\lambda) \in B_s$, then $B_t^{s,z,\lambda} \subset B_{s+t}$ for all t's.

Note also that $B_\cdot^{s,z,\lambda}$ is a function of $\{(X_{s+\cdot,\cdot,\nu}, K_{s+\cdot,\cdot,\nu}); \nu_{|s} = \lambda\}$ and therefore $\{B_\cdot^{s,z,\lambda}\}_{\lambda\in\mathcal{T}_s}$ are i.i.d. under $\mathbf{P}^q$.

As for the BRWRE, the set of particles of $B_t^{s,z,\lambda}$ which occupy the site x by $B_{t,x}^{s,z,\lambda}$.

We fix a T such that (2.4) holds. We define a markov chain (B^*_s) with values in finite subsets of $\mathcal{T}$. We start by $B^*_0 = \{1\}$ and for $s \geq 1$,

$$B^*_s = \bigcup_{\lambda \in B^*_{s-1}} B^*_{s,\lambda}, \text{ with } B^*_{s,\lambda} = \{\nu \in \mathcal{T}_{sT}\,;\,(ST\theta, \nu) \in B_T^{(s-1)T,(s-1)T\theta,\lambda}\}$$

B^*_s is the set of all genealogies that were in $tT\theta$ at time tT, for all $t \leq s$. It is clear that $|B^*_s| \leq |B_{sT,sT\theta}|$ for all $s \in \mathbb{N}$. We now prove that $|B^*_\cdot|$ is a Smith-Wilkinson process.

For a fixed s, $\{B^*_{s,\lambda}\}_{\lambda \in \mathcal{T}_{(s-1)T}}$ are i.i.d. under $\mathbf{P}^q$ because as noted earlier $\{B_\cdot^{s,z,\lambda}\}_{\lambda \in \mathcal{T}_s}$ are i.i.d. under $\mathbf{P}^q$.

Furthermore, if we denote q^*_s the common distribution of $|B^*_{s,\lambda}|$, in other words the random offspring ditribution of $|B^*_s|$, then, by time-space shift invariance, we see that $(q^*_s)_{s \in \mathbb{N}}$ is an i.i.d. sequence. In other words q^*_s depends only on what happens between time $(s-1)T$ and sT in a finite space box of constant size.

We now have to see that $|B^*_\cdot|$ verifies the hypotheses of theorem 1.2 :

$$\mathbf{Q} \ln m^*_1 > 0 \text{ and } \mathbf{Q} \ln \frac{1}{1 - q^*_1(0)} < \infty$$

where m^*_1 is the mean of q^*_1.

We have that $|B^*_1| = |B_{T,T\theta}|$ and hence $m^*_1 = \mathbf{P}^q[|B_{T,T\theta}|] = Z_{T,T\theta}$. It follows from (2.4) that

$$\mathbf{Q} \ln m^*_1 = \mathbf{Q} \ln Z_{T,T\theta} > 0$$

For the second part, we take a path $x_0, x_1, \cdots, x_T$ on $\mathbb{Z}^d$ with $x_0 = 0$ and $x_T = T\theta$. Then,

$$1 - q^*_1(0) = \mathbf{P}^q(B_{T,T\theta} \neq \varnothing) \geq \prod_{t=0}^{T-1} \frac{1 - q_{t,x_t}(0)}{2d}$$

and,

$$\mathbf{Q} \ln \frac{1}{1 - q^*_1(0)} \leq T\mathbf{Q} \ln \frac{1}{1 - q_{0,0}(0)} + T \ln(2d) < \infty$$

Therefore, we have $\mathbf{P}^\gamma(\text{survival of } |B^*_\cdot|) > 0$, $\mathbf{P}^\gamma(\text{survival of } |B_{sT,ST\theta}|) > 0$ and finally $\mathbf{P}^\gamma(\text{survival}) > 0$.

$\square$

3 Extinction in the critical case

In [8] Comets and Yoshida conjectured that for the critical case $\Psi = 0$ of the BRWRE, there was extinction. This conjecture was based on the similarity between the BRWRE and another well-known model, the contact process for which extinction is known in the critical case.

The contact process consists of a time-continuous modelisation of the spread of an infectious disease on $\mathbb{Z}^d$. An infected particle heals after an exponential time of parameter 1 and the disease spreads to any neighbor after an exponential time of parameter λ if the particle is still infected. The critical case for this process is when $2d\lambda = 1$. Bezuidenhout and Grimett in their important article [4] proved that the critical contact process dies out. A conceptual step is to see the process in dimension $d+1$ where the last dimension is time.

Garet and Marchand adapted the techniques of [4] to the case of the BRWRE in [10] to prove the following theorem :

Theorem 3.1 *Assume* (1.5). *Then*

$$\Psi(\gamma) = 0 \Rightarrow \mathbf{P}^\gamma(\text{survival}) = 0$$

This was proven under the condition

$$\mathbf{P}^\gamma(q_{0,0}(0) > 0) > 0 \text{ and } \mathbf{P}^\gamma(q_{0,0}(0) + q_{0,0}(1) < 1) > 0$$

meant to avoid trivial cases of extinction and survival. But the result remains true without them except for the case $\mathbf{P}^\gamma(q_{0,0}(1) = 1) = 1$ where we have obvious survival.

As this is a quite technical proof, and that we cannot add much to it, we will not give a full proof here but rather try and explain the main parts of the proof and the way they fit together.

3.1 Strategy of proof

The main idea in [4] is to characterise survival by a local event. To say this in the setting of the BRWRE, for a fixed law γ allowing the BRWRE to survive, there exists a local event A, that only depends on a finite space-time box $[\![0, T]\!] \times [\![-L, L]\!]^d$ where $[\![a, b]\!]$ is the set of integers from a to b, and a level $p_0 < 1$ such that

$$\mathbf{P}^\gamma(A) > p_0 \tag{3.1}$$

and

$$\text{For every law } \gamma' \text{ on } \mathcal{P}(\mathbb{N}),\ \mathbf{P}^{\gamma'}(A) > p_0 \Rightarrow \mathbf{P}^{\gamma'}(\text{survival}) > 0 \tag{3.2}$$

Then for $\rho \in [0, 1]$, we define γ^ρ, a perturbation of γ, with less particles. For ρ close enough to 1, we will show in subsection 3.2 thanks to (3.1) and (3.2) that we have survival. The fact that the event is local is crucial as we will see. But if γ was critical then γ^ρ is subcritical and therefore goes extinct. This is a contradiction and we have that the critical BRWRE dies out.

We now turn to the definition of the block event. First, some notations. We define $\underline{B}_t$ the set of all sites occupied by the BRWRE at time t. For an initial configuration $C \subset \mathbb{Z}^d$, we denote B_t^C the BRWRE starting from $C = \underline{B}_0^C$ with one particle in each point of C. For $L \in \mathbb{N}$ we define ${}_LB_t$ the BRWRE restricted to the space box $[\![-L, L]\!]^d$ meaning we forget all particles that go out of this box. Finally, set for every integer $n \geq 1$,

$$D_n = \{x \in \mathbb{Z}^d\, ;\, |x| \leq n \text{ and } \sum_{i=1}^{n} x_i \in 2\mathbb{Z}\}$$

For n even, D_n is the maximal set reachable by the BRWRE can reach at time n.

For any positive integers n, L, T, with $n \leq L$, we define

$$A_{n,L,T} = \{\exists x \in [\![L + n, 2L + n]\!] \times [\![0, 2L]\!]^{d-1}, t \in [\![T, 2T]\!],\ x + D_n \subset {}_{2L+2n}\underline{B}_t^{D_n}\}$$

This is our local event. It means that if we start from D_n, we get another D_n at a later time and translated on the right in the first dimension and with positive other $d - 1$ dimensions. And this only depends on what happens on a larger space-time box, hence we can say the event is local. Of course, if we change the sign of one dimension, the results on the event remain true. We need to rewrite (3.1) and (3.2) for more clarity. What is shown in the proof is

Proposition 3.1 *One has*

a) *For a given γ that allows survival, for every ε, there exists integers n, L, T, with $n \leq L$, such that*

$$\mathbf{P}^{\gamma}(A_{n,L,T}) \geq 1-\varepsilon$$

b) *There exists an ε such that for any γ' and any integers n, L, T, with $n \leq L$,*

$$\mathbf{P}^{\gamma'}(A_{n,L,T}) \geq 1-\varepsilon \Rightarrow \mathbf{P}^{\gamma'}(\text{survival}) > 0$$

In [10], only 3.1a) was proved arguing that the proof of 3.1b) in the case of the BRWRE is the same as the one given in [4] and also expounded in [13]. Following the latter reference, we can only agree with [10]. We now expound a brief outline of the proof of 3.1b) and then 3.1a).

The idea for 3.1b) is that starting from D_n, we can get with great probability a new D_n translated on the right and on the left (on the first coordinate) at a later time. Each of those D_n, can in their turn create two new D_n. This gives rise to an oriented percolation. Actually, if we use the raw event $A_{n,L,T}$, we cannot make sure we have a macroscopic grid to have a percolation on. Therefore, we do several steps, several events $A_{n,L,T}$ translated in time and space starting from the D_n given by the earlier event/step, compensating the irregularity of the steps thanks to symetry (we can change the sign of any coordinate in a given step) to obtain that with great probability if we start with a D_n in a space-time box, we obtain a new D_n at another space-time box of the same size and at fixed distance on the right (or on the left) and this only depends on what happens in a larger space-time box (cf figure 3.1). This allows us to imbed an oriented percolation in our BRWRE in which a vertex will be open if we succeeded to go from an earlier space-time box to the space-time box corresponding to the vertex. Unfortunately, we don't have independence between adjacent variables as the larger space-time boxes that we are restricted to in order to go from one space-time box to another in our grid are overlapping. But thanks to localisation, variables for a given time are k-dependent, which means that they are independent if they are associated to points that are far enough. Thanks to [13, Theorem B26], we know that for a k-dependent family of Bernoulli variables, the variables stochastically dominates a family of independent Bernoulli variables of parameter p. Our oriented percolation can then be made supercritical for a good choice of ε and this provides survival.

The idea of the proof of 3.1a) is that for a law γ such that survival is possible, if we start from D_n with n large, the probability of extinction will be small. But now, if there are not many particles going out of a space-time box, all of them dying happens with a reasonable probability and it implies extinction which has a small probability. This tells us that there are a lot of particles going out of a box with great probability. We prove this part here. Let us denote $N_{L,T}$ the number of particles on the top and sides of our space-time box, ε_0, α_0 such that $\mathbf{P}^{\gamma}(q_{0,0}(0) > \varepsilon_0) > \alpha_0$ and $\mathcal{F}_{L,T}$ the σ-algebra generated by all variables in the space-time box $[\![-L, L]\!]^d \times [\![0, T]\!]$.Let us take two increasing sequences L_j, T_j of integers.Observe that we have extinction if all the particles die when going out of the box. Then, if on every occupied site of the top and sides of the boxes we have an environment such that $q_{0,0}(0) > \varepsilon_0$, we get

$$\mathbb{1}_{\{N_{L_j,T_j}<k\}}(\varepsilon_0\alpha_0)^k \leq \mathbf{P}^{\gamma}(\text{extinction}|\mathcal{F}_{L_j,T_j})$$

since we have at most k occupied sites.

Thus, we have, by the martingale convergence theorem, that $\mathbf{P}^{\gamma}(\text{extinction}|\mathcal{F}_{L_j,T_j})$ almost surely converges to $\mathbb{1}_{\text{extinction}}$. And then $\limsup_{j\to\infty}\{N_{L_j,T_j} < k\} \subset \{\text{Extinction}\}$ and

$$\limsup_{j\to\infty} \mathbf{P}^{\gamma}(N_{L_j,T_j} < k) \leq \mathbf{P}^{\gamma}(\text{Extinction})$$

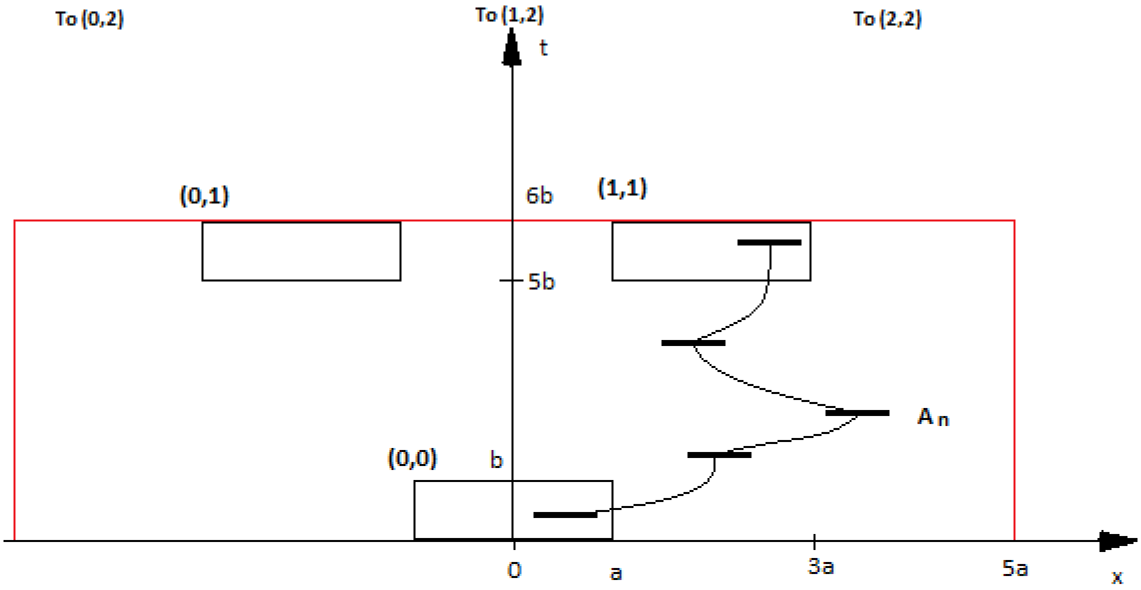

Figure 1 – The black boxes are the target boxes of the steps. The irregularity of the steps is compensated by the choice of the direction of the steps. The big red box is the box to which we are restricted. The red box centered on $(0,1)$ and $(1,1)$ would overlap and we therefore do not have independence. The labels (n,m) are the nodes of the grid we will have an oriented percolation on. The numbers a and b are two integers

This tells us that we can choose a space-time box such that with great probability we will have a lot of particles on the top and on the side of it. Thanks to the FKG inequality, we know that with great probability we will have a lot of particles on the top and side of any given orthant of our box. We add to this the fact that starting from one particle, we have with positive probability a D_n at a later time. Since we have a lot of particles on the top and the sides of our orthant, with great probability one of them will give birth to a D_n at a later time (see figure 2). But they have to be far in order for us to have independence between the events that one particle gives birth to a D_n. In order to have enough particles that are far enough, we only need to have so many occupied sites that we can find a subset of occupied sites that are far enough from each other.

This argument however can not be immediately applied to the BRWRE. This is where the proof differs the most from the one for the critical contact process. There can be a lot of particles in the BRWRE but concentrated on a few sites and therefore, we do not have independence between the differents attempts to give birth to an D_n. This is why in [10], the authors consider the case where $\mathbf{Q}(q_{0,0}(0) = 1) > 0$ as in this case, we can adapt the proof expounded just before to show that there were many particles on the top and sides of our space-time box to obtain that there are many occupied sites on the top and sides of the box. If we denote $\varepsilon_1 = \mathbf{P}^\gamma(q_{0,0}(0) = 1)$, and $\underline{N}_{L,T}$ the number of occupied sites on the top and sides of a space-time box, by setting sterile environment on the k occupied sites, we get

$$\mathbb{1}_{\{\underline{N}_{L_j,T_j} < k\}} \varepsilon_1^k \leq \mathbf{P}^\gamma(\text{extinction} | \mathcal{F}_{L_j,T_j})$$

and the end of the proof follows as before. In this case, it is possible to prove the result using only that there are a large number of occupied sites. In the complementary case, however we have to say that either a lot of sites will be occupied or that there will be a very large population on one site. It is not too hard to see that starting from a large population on one site if we don't have sterile environment, we get with great probability a D_n at a later time.

The condition (1.6) was excluded from [10] by making the same assumptions (1.5) as in [8] in order to prove the conjecture formulated in that article. We have seen that this is an important assumption to use the link between the DPRE and the BRWRE. However, to prove the result of [10] it is superfluous. Therefore, [10] proves more than theorem 3.1. They also prove that we will have extinction for critical cases under assumption (1.6).

3.2 Proof of extinction in the critical case

We will expound here how proposition 3.1 is used to prove extinction in the critical case as outlined in the previous subsection. We do this mainly because we will adapt the proof given here to the case (1.6) in section 4. Our presentation of this proof differs from that given in [10], where the authors defined the BRWRE without mentioning genealogies, instead simply counting the number of particles at each time-space location.

We consider γ that allows survival. By proposition 3.1b), we choose $\varepsilon > 0$ such that for any γ', $\mathbf{P}^{\gamma'}(A_{n,L,T}) \geq 1 - \varepsilon \Rightarrow \mathbf{P}^{\gamma'}(\text{survival}) > 0$. Then by proposition 3.1a), we choose positive integers n, L, T, with $n \leq L$ such that $\mathbf{P}^\gamma(A_{n,L,T}) > 1 - \varepsilon$. We denote by $C_n = [\![-(2L+2n), 2L+2n]\!]^d \times [\![0, T]\!]$ the space time box inside of which $A_{n,L,T}$ happens.

Then for $\rho \in [0,1]$, we define γ^ρ the image of γ by the map

$$q \mapsto \rho q + (1-\rho)\delta_0$$

where δ_0 is the Dirac mass in zero.

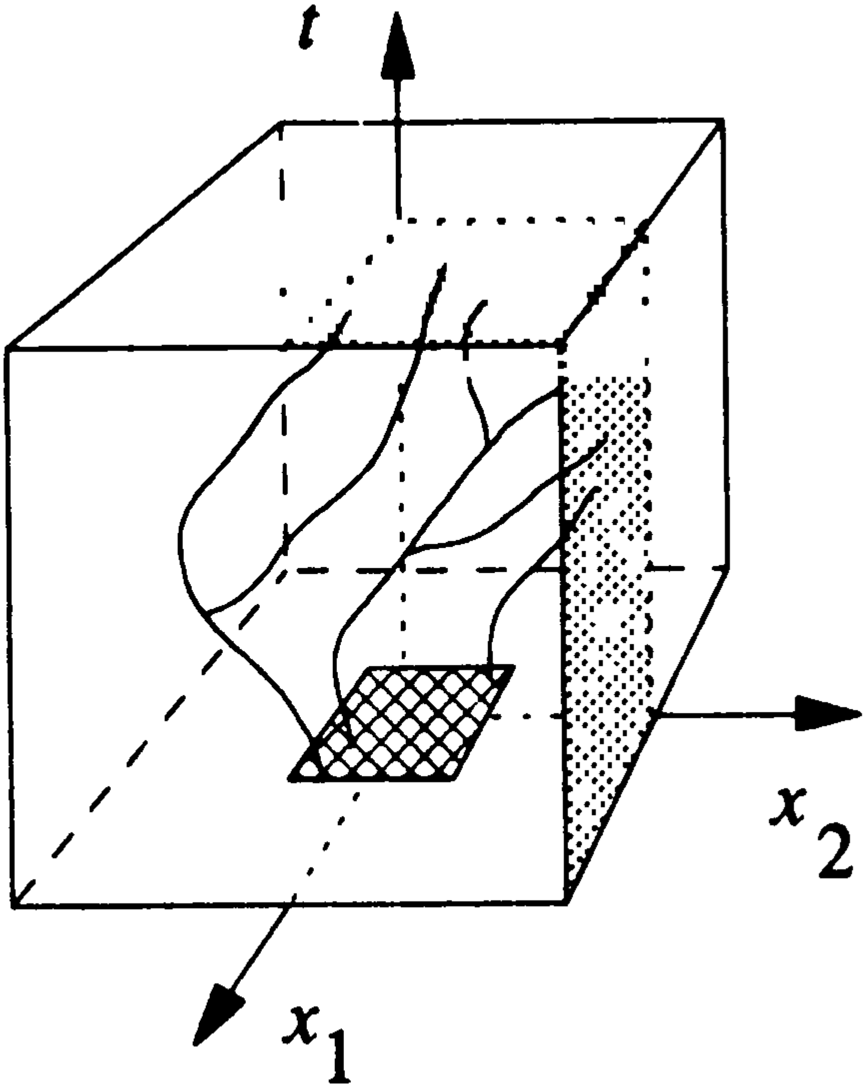

FIGURE 2 – If we start D_n with n large enough, with great probability we will have a lot of particles on the top and side of any given orthant. From there, we try to get a complete set D_n generated by one of those particles. Figure from [4, Figure 1]

We define a BRWRE $(B_t)_{t\in\mathbb{N}}$ with γ as common law of the offspring distribution as in section 1.3 that we start in D_n. We then enlarge our probability space and introduce the $(Y_{t,x,\nu})_{t\in\mathbb{N},x\in\mathbb{Z}^d,\nu\in\mathcal{T}_t}$ that are Bernoulli variables of parameter $\rho\in[0,1]$ and are independent of every variable introduced previously. We denote by $\mathbf{P}$ the law on this new probability space. We define a new BRWRE starting from D_n

$$B_t^\rho=\bigcup_{(x,\nu)\in B_{t-1}^\rho}\{(y,\mu)\in\mathbb{Z}^d\times\mathcal{T}_t; Y_{t-1,x,\nu}=1, X_{t-1,x,\nu}=y, \mu_{|t-1}=\nu, \mu_t\leq K_{t-1,x,\nu}\}$$

It is the process obtained by tossing a coin before deciding if a particle reproduces. Note that the genealogies are not complete any more : there can be a third child but no first or second one. This is of no concern for the rest of the proof. The common law of the offspring distribution of this process is γ^ρ under $\mathbf{P}$ and it is coupled with B_t. For any $t\in\mathbb{N}, x\in\mathbb{Z}^d$, the mean $m_{t,x}^\rho$ of the offspring distribution of (B_t^ρ) verifies $m_{t,x}^\rho=\rho m_{t,x}$ and then the partition function of the DPRE associated with (B_t^ρ) verifies $Z_t^\rho=\rho^t Z_t$ and

$$\Psi(\gamma^\rho)=\lim_{t\to\infty}\frac{1}{t}Z_t^\rho=\ln\rho+\Psi(\gamma)$$

This tells us that if we find a $\rho<1$ such that we have survival, we will have that if $\Psi(\gamma)=0$ then we have almost sure extinction as explained in the previous subsection.

Let $M\in\mathbb{N}$ and define $\mathcal{T}_t^M=\{1\}\times[\![0,M]\!]^t\subset\mathcal{T}_t$ and then the event

$$B_M=\{\forall(x,t)\in C_n,\forall\nu\in\mathcal{T}_t^M, K_{t,x,\nu}\leq M\}$$

on which all particles in C_n have less than M children.

On the event

$$B_M\cap\bigcap_{(x,t)\in C_n,\nu\in\mathcal{T}_t^M}\{Y_{t,x,\nu}=1\},$$

we have by induction that $B_t=B_t^\rho$ for every $(x,t)\in C_n$. Then

$$\begin{aligned}\mathbf{P}^{\gamma^\rho}(A_{n,L,T})&\geq\mathbf{P}\left(A_{n,L,T}\cap B_M\bigcap_{(x,t)\in C_n,\nu\in\mathcal{T}_t^M}\{Y_{t,x,\nu}=1\}\right)\\&=\rho^{|C_n||\mathcal{T}_{2T}^M|}\mathbf{P}(A_{n,L,T}\cap B_M)\\&=\rho^{|C_n||\mathcal{T}_{2T}^M|}\mathbf{P}^\gamma(A_{n,L,T}\cap B_M)\end{aligned}$$

Taking M, then ρ large enough, we get

$$\mathbf{P}^{\gamma^\rho}(A_{n,L,T})\geq 1-\varepsilon$$

and therefore

$$\mathbf{P}^{\gamma^\rho}(\text{survival})>0$$

□

4 Extinction and survival with possible sterile environment

In this section, we consider the case

$$\mathbf{Q}(q_{0,0}(0) = 1) > 0 \tag{4.1}$$

that is disjoint of the case (1.5) that allowed us to use the link between the DPRE and the BRWRE. It is not clear how one could use this link nevertheless as we will see in subsection 4.4. The result of theorem 2.1 can be seen as a consequence of theorem 1.2 on the Smith-Wilkinson process. Here, we will not be able to use Smith-Wilkinson process in the same fashion.

All of these remarks make that case interesting as it suggests that the BRWRE behaves differently here or at least that the techniques needed to study it are different.

This case is more tractable than the more general $\mathbf{Q}[m_{0,0}^{-1}] = \infty$ because we do not have to consider odd integrability problems.

4.1 Oriented percolation of fertile environment

Set $p = \mathbf{Q}(q_{0,0}(\mathbb{N}^*) > 0)$ so that (4.1) is equivalent to $p < 1$. Then we can write

$$\gamma = p\gamma' + (1-p)\delta_{\delta_0}$$

where γ' is the law of $q_{t,x}$ conditioned on $q_{t,x}(\mathbb{N}^*) > 0$. We say that (t, x) is a fertile site when $q_{t,x}(\mathbb{N}^*) > 0$. Each (t,x) has independently probability p of being fertile. If we consider the oriented percolation of fertile sites in $\mathbb{N} \times \mathbb{Z}^d$, if it does not percolate i.e. if there is no fertile infinite path starting from the origin then our BRWRE will go extinct. It is known [9] that there is a critical value for p which we call p_c (that depends on the dimension d) such that $p \leq p_c$ is equivalent to no percolation a.s. and $p > p_c$ is equivalent to percolation with a positive probability. Therefore we already have a sufficient result for extinction :

Proposition 4.1 *If $p \leq p_c$ then $\mathbf{P}^{\gamma}(survival) = 0$*

To have survival, we will need to be on the event where we have percolation. This is not a very pleasant situation as this means conditioning on an event that depends on infinitely many variables. We will to a large extent fail to overcome this difficulty.

There are a few results known on the oriented percolation model. For instance, in the case $d + 1 = 2$ that we will restrict ourselves to in 4.3, Durett showed in [9] that when percolation happens, it occurs in a cone i.e. if we denote the rightmost point at time t by r_t then on the event of percolation $\lim_{t\to\infty} \frac{r_t}{t} = \alpha(p)$ for an $\alpha \in]0, 1]$ and we have $\lim_{p\to p_c} \alpha(p) = 0$.

In the following, we will indifferently say open or fertile and closed or sterile.

4.2 Simple results on extinction and survival

We expose here four results that will set the landscape of survival and extinction for the BRWRE with possible sterile environment. The first comes from the comparison with the Galton-Watson process. Set $m = \mathbf{P}^{\gamma'}[m_{0,0}]$ then $\mathbf{P}^{\gamma}[m_{0,0}] = mp$.

Proposition 4.2 *If $mp \leq 1$ then $\mathbf{P}^{\gamma}(survival) = 0$*

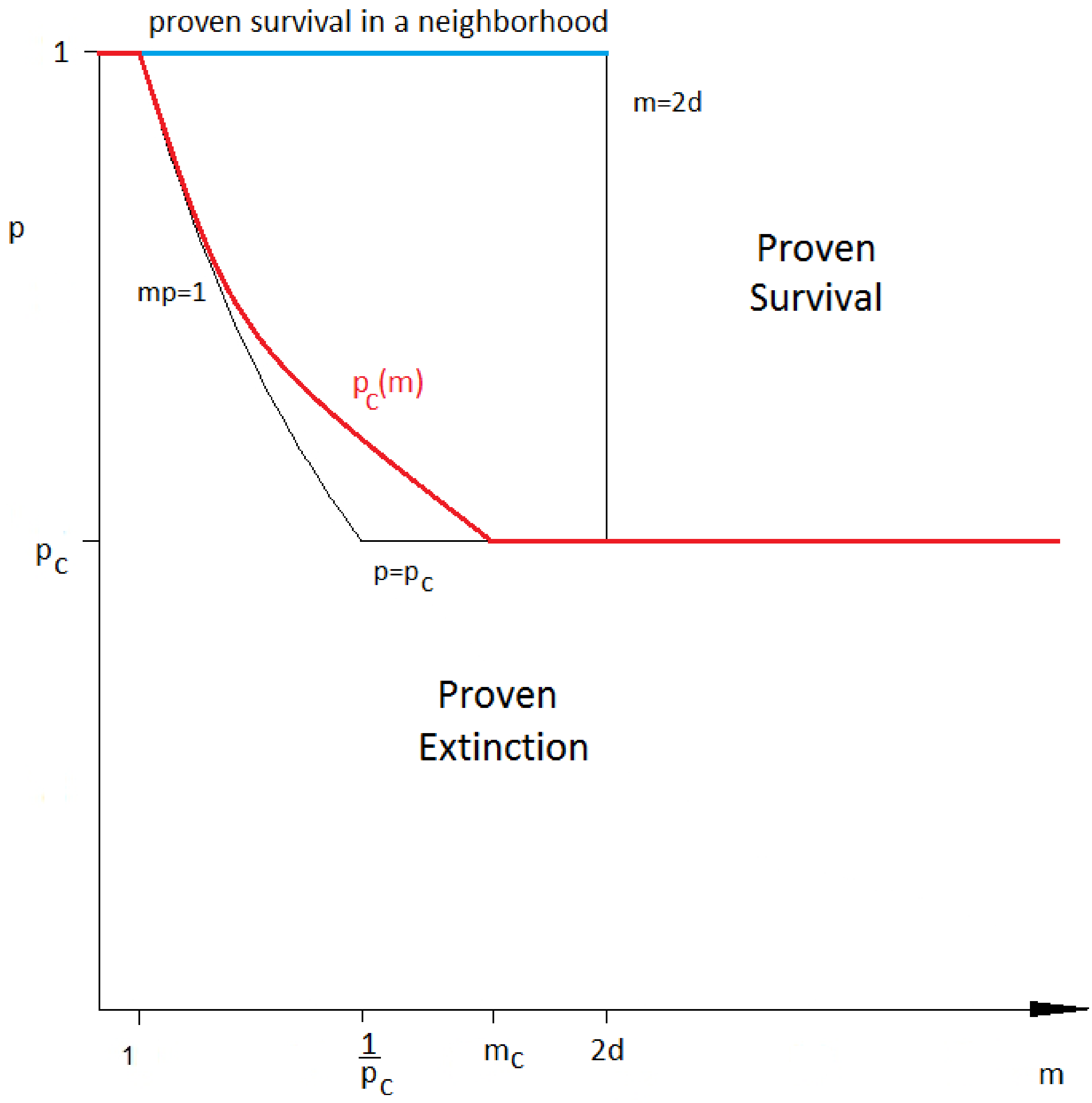

FIGURE 3 – This phase diagram contains all the results of subsection 4.1 and subsection 4.2. In red, we have the critical curve that separates survival on the upper side from extinction. The black lines indicate proven results. We also proved that we have survival in a neighborhood of the blue line.

This is an immediate consequence of (1.7). The mean of the offspring distribution of the Galton-Watson process is mp and thus the condition implies extinction for the Galton-Watson process and therefore also for the BRWRE. We can remark that the comparison with the Smith-Wilkinson process is of no use here as its probability of survival is 0 in the case of possible sterile environment.

We now consider a simpler case in which when we have fertile environment, we always have the same offspring distribution. When this is a superfluous hypothesis, we will say it but this will allow us to have simpler notations. We can write for $q \in \mathcal{P}(\mathbb{N})$ and $p \in [0,1]$

$$\gamma = \gamma(p,q) = p\delta_q + (1-p)\delta_{\delta_0} \tag{4.2}$$

We denote m the mean of q. The next result is the first result that gives a sufficient condition for survival.This result and its proof was communicated by Nobuo Yoshida.

Proposition 4.3 *If $m > 2d$ and $p > p_c$ then we have $\mathbf{P}^{\gamma(p,q)}(\text{survival}) > 0$*

Proof : We have $p > p_c$ therefore with positive probability, we have percolation. In particular, on that event, we can take a path from 0 to ∞ such that we only have fertile environment on the path. We consider N_t the number of particles that always stay on that path. The event of percolation is defined on $(\Omega_q, \mathcal{F}_q, \mathbf{Q})$ and therefore for any open path that we take, assuming we do the choice independently of Ω_U, we have that N_t is a Galton-Watson process and its offspring distribution is $q^*(k) = \frac{q(k)}{2d}$ for all $k \geq 1$. Therefore, its mean is $\frac{m}{2d}$ and thanks to theorem 1.1, we can conclude that if $m > 2d$ and $p > p_c$ then we have $\mathbf{P}^{\gamma(p,q)}(\text{survival}) > 0$.

□

The result is still true without hypothesis (4.2), if we replace the hypothesis $m > 2d$ by $\mathbf{Q}^{\gamma'}[\ln \frac{1}{1-q_{0,0}(0)}] < \infty$ and $\mathbf{Q}^{\gamma'}[\ln m_{0,0}] > \ln 2d$, the process we considered becomes a Smith-Wilkinson process that verifies the hypotheses of theorem 1.2. For the second condition, we have $1 - q_0^*(0) = \frac{1-q_0(0)}{2d}$.

The important thing here is that we have survival *for all $p > p_c$*. A natural question that we can ask is what is the best bound on m that we will call m_c such that, we still have survival for all $p > p_c$.

$$m_c = \inf_{m \in \mathbb{R}} \{\forall p > p_c, \forall q \in \mathcal{P}_m(\mathbb{N}); \mathbf{P}^{\gamma(p,q)}(\text{survival}) > 0\}$$

where $\mathcal{P}_m(\mathbb{N})$ denotes the set of offspring distributions with mean m.

A question of interest is the following

Question 1 *Are there choices of $q \in \mathcal{P}(\mathbb{N})$ such that $\mathbf{P}^{\gamma(p,q)}(\text{survival}) > 0$, $\forall p > p_c$ with the mean of q strictly smaller than m_c ?*

It seems unlikely because all the results we have seen on survival in this dissertation depended only on the means of the offspring distributions, except for the second condition for the Smith-Wilkinson process that propagated to theorem 2.1 but it seems unrelated. This is not intuitively clear and seems hard to answer to.

Of course, as we can see from the proof, we only used the existence of one path and therefore, there must be some room for improvement. This is however harder than it seems as we will see in 4.4.

Our two last results come from a coupling on different values of p. We construct $(\Omega_q, \mathcal{F}_q, \mathbf{Q})$ differently as with (4.2), it became much simpler. We define $\Omega_q = \Omega_V =]0,1[^{\mathbb{N} \times \mathbb{Z}^d}$ with the Borel σ-field and for

$V \in \Omega_V$, $V \to V_{t,x}$ as the projection. The probability measure $\mathbf{P}_V$ is the product uniform measure on $]0,1[$. For $p \in [0,1]$ and $q \in \mathcal{P}(\mathbb{N})$, $q \neq \delta_0$, we have $\gamma = \gamma(p,q)$ as defined in (4.2) for $\{q_{t,x} = q\} = \{V_{t,x} \leq p\}$ and $\{q_{t,x}(0) = 1\} = \{V_{t,x} > p\}$. We then define, for a fixed q, the coupled BRWRE that we denote by B_t^p with $B_0^p = \{(0,1)\}$ i.e. starting from one particle at the origin

$$B_t^p = \bigcup_{(x,\nu)\in B_{t-1}^p} \{(y,\mu) \in \mathbb{Z}^d \times \mathcal{T}_t; V_{t-1,x} \leq p, X_{t-1,x,\nu} = y, \mu_{|t-1} = \nu, \mu_t \leq K_{t-1,x,\nu}\}$$

The $K_{t,x}$ are simpler than before, they are i.i.d. random variables of law q. It is obvious that $p_1 < p_2$ implies $B_t^{p_1} \subset B_t^{p_2}$. We therefore have

Proposition 4.4 *For all $q \in \mathcal{P}(\mathbb{N})$, there is a $p_c(q) \in [0,1]$ such that for all $p > p_c(q)$,*

$$\mathbf{P}^{\gamma(p,q)}(\text{survival}) > 0$$

and for all $p \leq p_c(q)$,

$$\mathbf{P}^{\gamma(p,q)}(\text{survival}) = 0$$

This is also true without the hypothesis (4.2) if we replace q by γ'. $p_c(q)$ also depends on d.

A question that is similar to the previous question is the following

Question 2 *Do we have $p_c(q) = p_c(m)$? Does survival depend on something else than m ?*

The fact that we have extinction for the critical case comes from the last proposition of this subsection. It brings the results of [10] exposed in section 3 to the case (4.1).

Proposition 4.5 *For $\gamma = \gamma(p,q)$ such that $\mathbf{P}^{\gamma}(\text{survival}) > 0$, there exists $p' < p$ and $\rho \in [0,1[$ such that for $\gamma' = \gamma^{\rho}(p',q) = \gamma(p', \rho q + (1-\rho)\delta_0)$, we have $\mathbf{P}^{\gamma'}(\text{survival}) > 0$*

The proof of this proposition is very close to the one explained in 3.2. Similarly, we introduce the Y_t Bernoulli random variables of parameter ρ and we define the $\gamma(p',q)$-BRWRE

$$B_t^{p',\rho} = \bigcup_{(x,\nu)\in B_{t-1}^{p',\rho}} \{(y,\mu) \in \mathbb{Z}^d \times \mathcal{T}_t; Y_{t-1,x,\nu} = 1, V_{t-1,x} \leq p', X_{t-1,x,\nu} = y, \mu_{|t-1} = \nu, \mu_t \leq K_{t-1,x,\nu}\}$$

We deal with the $Y_{t,x,\nu}$ as earlier and at the same time, we have, $\mathbf{P}_V(\forall (t,x) \in C_n, V_{t,x} \notin]p',p]) = (p-p')^{|C_n|}$ and this is independent of everything else. We do not have to consider an analog of B_M here. We conclude the same way as earlier taking M and p large enough and then ρ.

One of the immediate consequence of this is that for any $q \in \mathcal{P}(\mathbb{N})$ with mean m such that $m > 1$, there exists $p < 1$ such that $\mathbf{P}^{\gamma(p,q)}(\text{survival}) > 0$

In order to draw all the information in a phase diagram in figure 4.2, taking into account the difficulty of putting $\mathcal{P}(\mathbb{N})$ on an axis, we admit that survival and extinction of a $\gamma(p,q)$-BRWRE depends of q only through m.

The curve $p_c(m)$ that is drawn is convex. Nothing ensures that this is the case but this seems reasonable and this question does not seem to be out of reach.

Question 3 *Let $\gamma_1 = \gamma(p_1,q_1)$ and $\gamma_2 = \gamma(p_2,q_2)$ that allow survival and let $\lambda \in]0,1[$. Denote $\gamma_\lambda = \lambda\gamma_1 + (1-\lambda)\gamma_2 = \gamma(\lambda p_1 + (1-\lambda)p_2, \lambda q_1 + (1-\lambda)q_2)$. Does γ_λ allow survival ?*

4.3 Towards a better bound for m_c

We assume for simplicity that $d = 1$. We want to find a better upper bound on m_c that is to say an m such that for any q of mean greater than m and for any $p > p_c$ we have

$$\mathbf{P}^{\gamma(p,q)}(\text{survival}) > 0$$

The important thing here is that we want it to be true for all $p > p_c$. We know from [9] that, if survival happens, it has to do so in a cone that may be arbitrarily narrow. The idea we will try to develop in order to get a better bound adresses that issue. We only need one open path as in 4.3 but now we will try to get more fertile environment by considering the fertile sites close to our path. This could lead to a less stringent condition on m. The main problem we have to deal with in order to achieve this is that we condition on the existence of an infinite cluster and this makes us lose the independence we need in the environment to use theorem 1.2. The idea is illustrated on figure 4.3.

We will now expound a simpler model allowing us to do explicit computations. We consider a directed path $(t, X_t)_{t\in\mathbb{N}}$ where X_t is a simple random walk on $\mathbb{Z}$. This will play the role of the path on the open cluster.

We know define the environment. All the sites in our path are open. We will call an inside site the site inside every turning point, the site visited by the blue line in figure 4.3. All the inside points are open with probability p, independently of X (see proposition 4.7 for motivation of this.). All other sites are closed.

We consider the branching random walk B_t with offspring distribution q of mean m on every open site. We define it in the same way as earlier including genealogies.

Proposition 4.6 *For such a branching random walk, we have survival with positive probability if* $m > 2^{\frac{2}{2+p}}$.

Our branching random walk will have two kinds of step. The first kind is when it does not do a turn or the inside site is not open. The second case is when it turns and the inside site is open. The idea is that in that case we get more particles as they can come back to the open path. The steps are not of same length.

We now define a markov chain (B^*_s) with values in finite subsets of $\mathcal{T}$ that will not have the same time as the branching random walk (B_t). We start by $B^*_0 = \{1\}$ and $t_0 = 0$. Assume we did s steps, we are at time s for B^* with corresponding time t_s for B, we have $\{X_{t_s}\} \times B^*_s \subset B_{t_s,X_{t_s}}$. If $X_{t_s+2} \neq X_{t_s}$ or the inside point is sterile, we set $t_{s+1} = t_s + 1$ and if $X_{t_s+2} = X_{t_s}$ and the inside point is open, we set $t_{s+1} = t_s + 2$. In both cases, we set

$$B^*_{s+1} = \{\nu \in B_{t_{s+1},X_{t_{s+1}}} \text{ such that } \nu_{|t_s} \in B^*_s\}$$

In the first case, every particle in B^*_s independently reproduce with law q_1 with $q_1(k) = \frac{q(k)}{2}$ of mean $\frac{m}{2}$ as in the proof of 4.3 (remember that we set $d = 1$). In the second case, every particle in B^*_s reproduces with law q_2 with

$$\hat{q}_2(s) = \sum_{k=1}^{\infty} q_2(k)s^k = \hat{q}_1 \circ \hat{q}(s)$$

This means that the particles at (t_s, X_{t_s}) reproduce with law q and go to either of the open sites at $t_s + 1$ and then with probability $\frac{1}{2}$ go to $(t_s + 2, X_{t_s+2})$. q_2 has mean $\frac{m^2}{2}$. The assumptions we made on

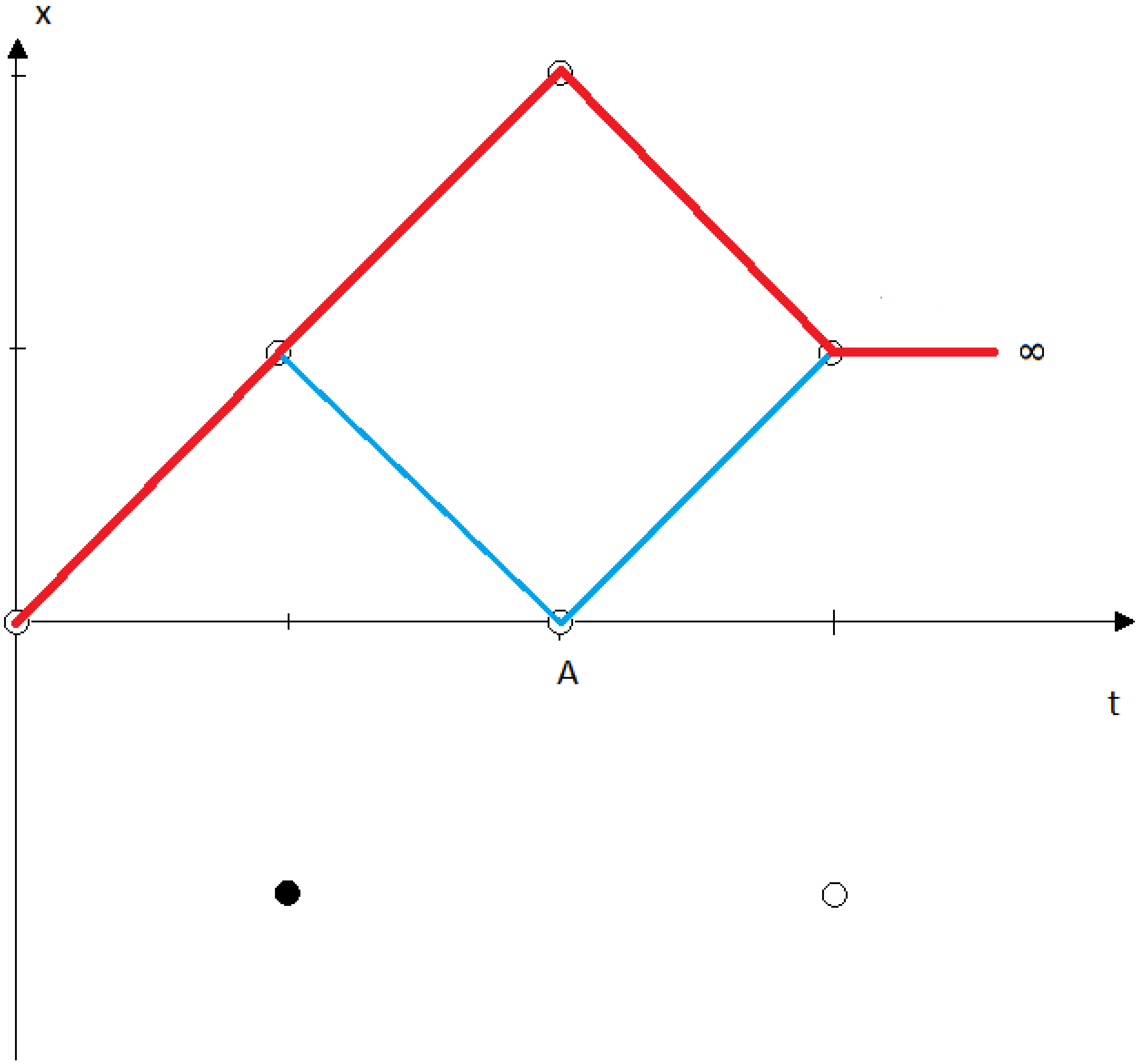

FIGURE 4 – This exposes the way we want to prove we have a better upper bound on m_c. The open circles denote fertile sites while the black circles denote sterile sites. We want as for the proof of 4.3 to consider a path of fertile sites in red that links 0 to ∞. But if we have an open site at the corner of a turning point of our path, we see that particles that leave the path one step before the turning point can with probability 1/2 come back to the path just after the turning point by following the blue line. We say that A is an inside point.

X and the inside points give us that the second case happens with probability $\frac{p}{2}$ and therefore the first one happens with probability $1-\frac{p}{2}$. B_s^* is only a subset of $B_{t_s,X_{t_s}}$ because if we turn twice in a row and that both inside points are open, we will take advantage only of the first, we lose the particles that go to the second inside point.

We have that $(|B_s^*|)_{s\in\mathbb{N}}$ is a Smith-Wilkinson process. We now look at the hypotheses of theorem 1.2. The second hypothesis is always verified as $q^*(0)$ takes only two non 1 values and $1-q^*(0)$ is therefore greater than a given constant. For the first one, we compute

$$\mathbf{Q}\ln(m_1) = \frac{p}{2}\ln\frac{m^2}{2} + (1-\frac{p}{2})\ln\frac{m}{2} = \frac{2+p}{2}\ln m - \ln 2$$

and

$$\mathbf{Q}\ln(m_1) > 0 \Leftrightarrow m > 2^{\frac{2}{2+p}}$$

□

This would give us the bound

$$m_c \leq 2^{\frac{2}{2+p_c}}$$

.

Now let's examine the two assumptions that we made. What we needed in our simple model was that the environment was i.i.d. and that we could easily compute the different probabilities of the environment. The latter can be left out if we accept a less explicit bound.

Our first hypothesis was replacing the open path on the infinite cluster by a simple random walk. There are different possibilities for us to choose the path we take. We could choose the leftmost or rightmost path connected to infinity (probably not the best possible choices). Or the path that stays closest to zero. A natural approach is to consider a random walk on the infinite cluster. We start at 0 under the condition that it is connected to ∞. This means that at least one of the two neighbors is open and connected to infinity at time 1. If there is only one, we choose X_1 to be that one, if there are two, X_1 is either with probability $\frac{1}{2}$. Unfortunately, such a path will be very far from having i.i.d. steps as every earlier step conditions what happens in the rest of time. In [6], such a random walk is defined and a law of large numbers and a central limit theorem are proven for it. There are interesting tools introduced in this article to prove those results such as a regeneration structure, a sequence $(\tau_i)_{i\in\mathbb{N}}$ such that $(\tau_i, X_{\tau_i} - X_{\tau_{i-1}})$ are i.i.d. Further work is needed to use this in our setting.

Our second assumption was based on the idea that once we know we have a point connected to ∞ the conditioning on being in an infinite cluster drops for earlier sites. We prove the following proposition on oriented percolation to illustrate this. We denote $\eta_{t,x} = 1$ if the site (t,x) is open, $\eta_{t,x} = 0$ if the site (t,x) is closed, $(t,x)\to(t',x')$ with $t<t'$ the event where there is an open path from x at time t to x' at time t', $(t,x)\to\infty$ if there is an open path starting from x at time t and of infinite length. We denote $\mathcal{F}_t = \sigma(\eta_{s,x}; s\leq t)$ and

$$A_\infty = \{(0,0)\to\infty\}. \tag{4.3}$$

Proposition 4.7 *Let* $(t,x)\in\mathbb{N}\times\mathbb{Z}^d$ *and* $A\in\mathcal{F}_t$ *with* $A\subset\{(0,0)\to(t,x)\}$, *then*

$$\mathbf{P}(A\cap\{(t,x)\to\infty\}|A_\infty) = \mathbf{P}(A).$$

*Proof :*One has

$$\mathbf{P}(A\cap\{(t,x)\to\infty\}|A_\infty) = \frac{\mathbf{P}(A\cap\{(t,x)\to\infty\}\cap A_\infty)}{\mathbf{P}(A_\infty)}$$

since $A \cap \{(t,x) \to \infty\} \subset A_\infty$, this is equal to

$$= \frac{\mathbf{P}(A \cap \{(t,x) \to \infty\})}{\mathbf{P}(A_\infty)} = \mathbf{P}(A)\frac{\mathbf{P}((t,x) \to \infty)}{\mathbf{P}(A_\infty)} = \mathbf{P}(A)$$

where we used the independence between what happens before and after t under $\mathbf{P}$ and shift-invariance.

□

This tells us that we can lose the infinite conditioning in certain conditions. This, however is far from what we would like to have. We would like to replace $(t,x) \to \infty$ by $X_t = x$ but this is not possible as X_t depends on $\mathcal{F}_t$. If we try to work without a random walk, using directly this proposition, the problem we encounter is that the negation of $(t,x) \to \infty$ is not easily dealt with.

4.4 Analogous quantities to free energy and conjecture

Under the assumption (4.1), the quantity $\Psi = \lim_{t\to\infty} \frac{1}{t}\mathbf{Q}[\ln Z_t]$ as defined in proposition 1.2 is always equal to $-\infty$ as on A_∞^c (where A_∞ is defined in4.3), $Z_t = 0$ for t large enough and this event has positive probability. We can however consider the quantity $\hat{\Psi}(\omega) = \lim_{t\to\infty} \frac{1}{t}\ln Z_t(\omega)$ for each $\omega \in \Omega$ if the limit exists. Under condition (4.2),

$$Z_t = \frac{m}{2d}^t N_t$$

with $N_t = |\{\text{open oriented paths from } 0,0) \text{ to } (y,t) \text{ for } y \in \mathbb{Z}^d\}|$. If we had a result of the form $\frac{1}{t}\ln N_t \to C$ when $t \to \infty$ with positive probability, then on this event, we would have $\hat{\Psi} = \ln m - \ln 2d + C$ and we could hope that if this quantity is positive, we would have survival. Unfortunately, the existence of the limit of $\frac{1}{t}\ln N_t$ is still an open problem for a large range of parameters. Nevertheless, in [15], Yoshida proved that

Theorem 4.1 *If $d \geq 3$, there exists p_0 such that if $p > p_0$ then, almost surely on the event of percolation,*

$$\lim_{t\to\infty} \frac{1}{t}\ln N_t = \ln 2d + \ln p$$

The result in [15] is actually stronger. It states that under the same hypotheses $\lim_{t\to\infty} \frac{N_t}{(2dp)^t}$ exists and is positive with positive probability.

This result tells us that if $d \geq 3$, and $p \geq p_0$, with positive probability, we have $\hat{\Psi} = \ln mp$. On this event, $\hat{\Psi} > 0 \Leftrightarrow mp > 1$. This reminds us of the condition of proposition 4.2. We have the following conjecture

Conjecture 1 *If $d \geq 3$ and $p \geq p_0$*

$$mp > 1 \Leftrightarrow \mathbf{P}^{\gamma(p,q)}(\text{survival}) > 0$$

The fact that survival implies $mp > 1$ is proposition 4.2. It corresponds to the "easy" case in theorem 2.1 (and many other problems in probability where the upper bound follows from straightforward inequalities), we have the "difficult" part of the result left to prove. This conjecture is based on the fact that 1.1 remains true even under (4.1). Therefore, if $\hat{\Psi} > 0$ with positive probability, it means that for a fixed environment on this event (the event depends only on Ω_q), the mean of the population grows exponentially, which suggests survival. To prove this, we cannot work in the same fashion as in section 2. Indeed, we do not have an analogous quantity of the directional free energy as, if we look at $(s, s\theta)$ for $\theta \in \Delta \cup \mathbb{Q}^d$, it might be closed.

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