MAT 2348 — midterm exam

B.

1. We can consider this as a "bars and bullets" problem with $3-1$ bars and 12 bullets, which gives $\binom{12+3-1}{12}$ possibilities.
2. A solution with even numbers is the same thing (setting $y_i = \frac{x_i}{2}$) as a solution to $y_1 + y_2 + y_3 = 6$, so there are $\binom{6+3-1}{6}$ possibilities.
3. A solution with numbers greater or equal to 1 is the same thing (setting $y_i = x_i - 1$) as a solution to $y_1 + y_2 + y_3 = 9$, so there are $\binom{9+3-1}{9}$ possibilities.
4. We will use the IE principle: setting F_i the set of solutions such that $x_i \geq 6$, we have by the IE principle that $|F_1 \cup F_2 \cup F_3| = S_1 - S_2 + S_3$ (notations as in the statement of the principle).

   Now by the same type of reasonning as in previous answers, $|F_i| = \binom{7+3-1}{7}$, $|F_i \cap F_j| = \binom{2+3-1}{2}$ for $i \neq j$ and $|F_1 \cap F_2 \cap F_3| = 0$.

   So we have $S_1 = 3\binom{7+3-1}{7}$, $S_2 = 3\binom{2+3-1}{2}$ and $S_3 = 0$. The number of solutions is then (by applying the sum principle)

   $$\binom{12+3-1}{12} - |F_1 \cup F_2 \cup F_3| = \binom{12+3-1}{12} - 3\binom{7+3-1}{7} + 3\binom{2+3-1}{2}$$

   Alternative solution (defeating my initial idea of testing if you know the how to apply IE principle):

   Set $y_i = 5 - x_i$, so that the problem becomes finding solutions to $y_1 + y_2 + y_3 = 3$ with $y_i \geq 0$ (and $y_i \leq 5$, but this does not matter…so this is a special case, the trick does not work with $= 7$ for instance) so that we have $\binom{3+3-1}{3} = 10$ possible solutions.

C.

1. By the binomial theorem $(x+y)^1 1 = \sum_{k=0}^{1} 1\binom{11}{k} x^k y^{n-k}$ and the coefficient of x^5y^6 is therefore $\binom{11}{5}$. Then, $(2x+3y)^1 1 = \sum_{k=0}^{1} 1\binom{11}{k}(2x)^k(3y)^{n-k}$ and the coefficient of x^5y^6 is $\binom{11}{5}2^5 3^6$.
2. By the binomial theorem, $\sum_{k=0}^{n} = \binom{n}{k}(-2)^k = (-2+1)^n = (-1)^n$.

D. If we associate to each line its number of black squares, we can apply the pigeonhole principle: there are $n+1$ possible number of black square per line (it can range from 0 to n) and $2n$ lines, as $2n > n+1$ (for $n \geq 1$) there must be two lines that hold the same number of black squares.

E.

1. $\binom{2n+2}{n+1} = \frac{(2n+2)!}{(n+1)!(n+1)!} = \frac{(2n+2)(2n+1)(2n)!}{n!n!(n+1)^2} = \frac{(2n+2)(2n+1)}{(n+1)^2}\binom{2n}{n}$.

2. We prove by induction
$$S[n] \ : \ f(n) = g(n)$$
Base case: by hypothesis $f(0) = 1 = 2.0 + 1 = g(0)$, so $S[0]$ holds.
Induction step: suppose $S[n]$ holds. Then we have
$$f(n+1) = f(n) + 2 =_{(\text{by } S[n])} g(n) + 2 = 2n + 1 + 2 = g(n+1)$$
so that $S[n+1]$ holds.
By induction $S[n]$ holds for all n.

3. We fix c and prove by induction
$$S[n] \ : \ \textstyle\sum_{k=c}^{n} \binom{k}{c} = \binom{n+1}{c+1}$$
Base case: $\sum_{k=c}^{c} \binom{k}{c} = \binom{c}{c} = 1 = \binom{c+1}{c+1}$, so $S[c]$ holds.
Induction step: suppose $S[n]$ holds. Then we have
$$\textstyle\sum_{k=c}^{n+1} \binom{k}{c} = \binom{n+1}{c} + \sum_{k=c}^{n} \binom{k}{c} =_{(\text{by } S[n])} \binom{n+1}{c} + \binom{n+1}{c+1} =_{(\text{Pascal's rule})} \binom{n+2}{c+2}$$
so that $S[n+1]$ holds.
By induction $S[n]$ holds for all n.

F.

1. If we have a triple, we can define a function as follows:
$$f(x) \begin{cases} 1 & \text{if } x \in A \\ 2 & \text{if } x \in B \\ 3 & \text{if } x \in C \end{cases}$$
Conversely, given a function we can define a triple as
$$A = [f]^{-1}(1) \qquad B = [f]^{-1}(2) \qquad C = [f]^{-1}(3)$$
This establishes a correspondance (which is clearly a bijection) between the triples considered and functions. By the isomorphism principle, there as many of each of them.

2. The number of functions from E to $\{1, 2, 3\}$ is 3^n, by the above question, this is also the number of triples.

3. Such a pair is the same thing as an above triple if we write it as $A \setminus B$, $B \setminus A$, $A \cap B$. So there are as many pairs as pairwise disjoint triples.
Alternative answer: apply the sum principle reasoning on the size of A. If A is of size k, then we have $\binom{n}{k}$ choices for A and then B contains all the elements that are not in A plus a subset of A, with 2^k possibilities. So in the end the total number of possibilities is $\sum_{k=0}^{n} \binom{n}{k} 2^k = 3^n$ by the binomial theorem.