MAT 2348 — exercises #8

A Back to basics

Can you find an elementary solution (*i.e.* without using generating functions) to the composition problem? Remember from class: a composition of an integer is a way to write it as a sum of strictly positive integers. We want to know how many ways there are to do that. *(hint: there are many solutions, but the simpler is by remembering that 2^{n-1} is the number of subsets of a set of size $n-1$.)*

B Computations

1. Compute the coefficient of X^n in the following generating functions:

$$\frac{1}{1-3X^4} \qquad \frac{1}{2-X} \qquad \frac{X^2+1}{(X-1)} \qquad \frac{1}{(3-3X)(X-1)} \qquad \frac{1+2X+X^2}{(1+X)(1-X^3)}$$

2. We set $f(X) = \sum_{k=0}^{\infty} (-1)^k \frac{X^k}{k!}$. Show that $f'(X) = -f(X)$ using the derivation operation.

   Now if we fix $g(X) = \sum_{k=0}^{\infty} \frac{X^k}{k!}$, show that $f(X)g(X) = 1$.

   Can you find what simple function is $f(X)$ *(hint: use substitution)*? Explain how this gives alternative answers to the two previous questions.

C Generating functions

1. Give the generating function for the following problems:

$$x_1 + x_2 + x_3 + x_4 = n \qquad x_1 \text{ is a multiple of } 4,\ x_2 = 0 \text{ or } 2,\ x_3 = 1 \text{ or } 2,\ x_4 \geq 0$$

$$y_1 + y_2 = n \qquad y_1 \geq 1,\ y_2 \geq 0$$

   and show they are equal.

2. Try to find the "summation operator": given $f(X) = \sum_{k=0}^{\infty} a_k X^k$, how can we express the generating function $(\sum f)(X) = \sum_{k=0}^{\infty} (a_0 + a_1 + \cdots + a_k) X^k$.
   (hint: use the product, find a $g(X)$ such that $(\sum f)(X) = f(X)g(X)$)

Grimaldi's exercises 9.2: 3, 4, 17, 32.
Grimaldi's exercises 9.3: 4.