MAT 2348 — exercises #5

A Functions

Let $f : A \to B$ and $g : B \to C$ be two functions. Show the two lemmas we used in class:

- If $g \circ f$ is injective, then f is injective
- If $g \circ f$ is surjective, then g is surjective

B Induction

1. Show by induction that $\sum_{k=0}^{n} k(k!) = (n+1)! - 1$.
2. Let $f(n) = \sum_{k=0}^{n} 2^k$. Compute the first values of f $(f(1), f(2), f(3)\dots)$ and conjecture the general formula. Then prove it correct by induction.
3. Show that if A and B are finite sets and $\mathit{card}(A) \leq \mathit{card}(B)$ then there exist an injection $f : A \to B$. *(hint: as we did in class for bijections, induction on the cardinal of A)*
4. Show that $(\cos(\theta) + \sin(\theta))^n = \cos(n\theta) + \sin(n\theta)$ for all n.

   (hint: remember your trigonometry, $\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)\dots$*)*
5. Let $S(n)$ be the statement "*for all* n, $\sum_{k=0}^{n} k = \frac{(n+\frac{1}{2})^2}{2}$". Prove that for all n, $S(n) \Rightarrow S(n+1)$. However, does $S(0)$ hold? $S(1)$? $S(2)$?...

   (morality: be careful with the base case!)

C Induction and well ordering

(for the moment I leave it as a problem without intermediary questions, let's see how far you can get)

The well-ordering of $\mathbb{N}$ is the following property:

> "Given any subset A of $\mathbb{N}$, there exist an element m_A of A (we call it the *minimum* element of A) such that for any $x \in A$ we have $m_A \leq x$."

We want to show that the induction principle is a consequence of the well-ordering property. So suppose we don't know if the induction principle holds and that we have a sentence $S(n)$ that depends on an integer n, such that

$$S(0) \text{ holds} \quad \text{and} \quad S(n) \Rightarrow S(n+1) \text{ holds for all } n$$

Try to show that $S(n)$ holds for all n using the well-ordering property rather than the induction principle.

Grimaldi's exercises 4.1: 1, 2, 16, 23.