MAT 2348 (winter 2015) final exam — solutions

Closed book exam. No calculators allowed.

Numeric answers need to be justified. You can use the result of a previous question in an answer even you did not answer it. ***More difficult questions are marked with a *, do what you know how to do first.*** **Don't Panic.**

A. This is an arrangement with types problem, with two elements of type A, T and one of type O, W: $\frac{6!}{2!2!1!1!}$ possibilities.

If the two T need to be side by side, we can consider them as a single letter, and we get $\frac{5!}{2!1!1!1!}$ possibilities.

There are $\frac{5!}{2!2!1!}$ words starting by O (as much as letters made with the letters of $TTAWA$, hence by the sum principle there are $6! - \frac{5!}{2!2!1!}$ words that do not start by O.

B.

$\frac{X}{1+2X} = \sum_{n=0}^{\infty}(-2)^n X^{n+1}$.

$\frac{1}{(1+X)(2-3X)} = \frac{1/5}{1+X} + \frac{3/5}{2-3X} = \frac{1/5}{1+X} + \frac{3/10}{1-(3/2)X} = \sum_{n=0}^{\infty}(1/5)(-1)^n X^n + (3/10)(3/2)^n$.

$(e^X - 1)^2 = e^{2X} - 2e^X + 1 = 1 + \sum_{n=0}^{\infty}(2^n - 2)\frac{X^n}{n!}$

C. $f(X) = \frac{1}{(1-X)^2}$ generates $n+1$, derive it then multiply the result by X to get $Xf'(X) = \frac{2X}{(1-X)^3}$ generates $n(n+1)$, then perform the same operation again to get that $g(X) = \frac{2X+4X^2}{(1-X)^4}$ generates $n^2(n+1)$.

$h(X) = e^{3X}$ generates $\frac{3^n}{n!}$.

D. The exponential generating function for this problem is

$$(e^X - 1)(e^X - 1)e^X e^X = (e^{2X} - 2e^X + 1)e^{2X} = e^{4X} - 2e^{3X} + e^{2X} = \sum_{n=0}^{\infty}(4^n - 2.3^n + 2^n)\frac{X^n}{n!}$$

The coefficient of $\frac{X^n}{n!}$ (and therefore the number of n-letter words we look for) is $(4^n - 2.3^n + 2^n)$.

E.

1. The generating function of the problem is $\frac{1}{(1-X)^3} = \sum_{k=0}^{\infty}\binom{k+3-1}{k}X^k$ and therefore we have $\binom{k+3-1}{k}$ solutions.
2. In that case the generating function is $(X + X^3 + X^5 + \cdots)^3 = \frac{X^3}{(1-X^2)^3} = \sum_{k=0}^{\infty}\binom{k+3-1}{k}X^{2k+3}$. So the coefficient of X^k (which equals the number of solutions) is 0 when k is even or smaller than 3 and $\binom{k'+3-1}{k'}$ when $k = 2k' + 3$.

3. The generating function in that case is $\frac{X}{(1-X)}(1+X+X^2+X^3)\frac{1}{(1-X)} = \frac{X(1-X^4)}{(1-X)^3} = \sum_{k=0}^{\infty}\binom{k+3-1}{k}X^{k+1} - \sum_{k=0}^{\infty}\binom{k+3-1}{k}X^{k+5}$. Hence the number of solutions for $k \geq 5$ is $\binom{k-1+3-1}{k-1} - \binom{k-5+3-1}{k-5} = \binom{k+1}{k-1} - \binom{k-3}{k-5}$, 0 for $k=0$ and $\binom{k+1}{k-1}$ for $1 \leq k \leq 4$.

F.

1. We apply the inclusion-exclusion principle: let F_i be the set of multiples of i. We have $|F_4| = 25$, $|F_5| = 20$ and $|F_6| = 16$. Moreover $|F_4 \cap F_5| = |F_{20}| = 5$, $|F_4 \cap F_6| = |F_{12}| = 8$ and $|F_5 \cap F_6| = |F_{30}| = 3$. Finally, $|F_4 \cap F_5 \cap F_6| = |F_{60}| = 1$.

   By the inclusion-exclusion principle $|F_4 \cup F_5 \cup F_6| = 25+20+16-5-8-3+1 = 46$. Then by the sum principle, there are $100-46 = 54$ non-multiples.

2. We aply the formula for exactly one in the inclusion-exclusion situation: $E_1 = |F_4|+|F_5|+|F_6| - 2(|F_4 \cap F_6| + |F_4 \cap F_5| + |F_5 \cap F_6|) + 3|F_4 \cap F_5 \cap F_6| = 25+20+16-2(5+8+3)+3(1) = 32$.

G.

1. We apply the pidgeonhole principle: the "holes" being intervals $[1,7]$ $[8,14]$ $\cdots [92,98][99,100]$. There are 15 such intervals, and whenever two numbers belong to the same interval they are at a distance strictly less than 7. Now, if we pick 16 numbers between 1 and 100, two of them must be in the same interval by the pidgeonhole principle, and therefore be at a distance strictly less than 7.

*2. The corners removed are of the same solor, say black. Suppose we can cover the chessboard, which has $64-2 = 62$ squares: 32 white and 30 black. To do this we need 31 dominos which cover 2 squares each. Now each domino must cover exactly one black and one white square. In particular, there must be a bijection between dominos and black squares. But this is impossible since we have 31 dominos and 30 black squares.

H. We apply the GF method:

1. GF for a_n: $f(X) = \frac{X}{(2-X)(1-X)} = \frac{1/9}{1-5X} + \frac{-2/9}{2-X} = \frac{1/9}{1-5X} + \frac{-1/9}{1-X/2}$, hence $a_n = (1/9)(5^n - (1/2)^n)$.

2. GF for b_n: $g(X) = \frac{2X}{1-4X+4X^2} = \frac{2X}{(1-2X)^2}$. Hence $a_n = 2\binom{n}{n-1}2^{n-1} = n2^n$

*3. GF for c_n: $h(X) = \frac{1}{1-X-X^2} = \frac{1}{(\phi+X)(\psi+X)}$ with $\phi = \frac{1+\sqrt{5}}{2}$ and $\psi = \frac{1-\sqrt{5}}{2}$ (remember how to solve 2nd degree polynomials!). Then the partial fraction decomposition gives:

   $f(X) = (1/\sqrt{5})(\frac{\psi}{X+\psi} - \frac{\phi}{X+\phi}) = (1/\sqrt{5})(\frac{1}{1+X/\psi} - \frac{1}{1+X/\phi})$

   so that $c_n = (1/\sqrt{5})((-1/\psi)^n - (-1/\phi)^n)$. By the way, one can notice that $-1/\psi = \phi$, so that the expression simplifies as $c_n = (1/\sqrt{5})(\phi^{n+1} - \psi^{n+1})$.

4. $f(X) = \frac{3+15X^2}{1-X+5X^2+X^3}$

5. $f(X) = \frac{X^5}{(1+6X^4)(1-X)^2}$

I. We know that $(X+1)^{\alpha+2} = \sum_{n=0}^{\infty} \binom{\alpha+2}{n} X^n$, and moreover $(X+1)^{\alpha} + 2X(X+1)^{\alpha} + X^2(X+1)^{\alpha} = \sum_{n=0}^{\infty} \binom{\alpha}{n} X^n + 2\sum_{n=0}^{\infty} \binom{\alpha}{n} X^{n+1} + \sum_{n=0}^{\infty} \binom{\alpha}{n} X^{n+2}$. Looking at the coefficient of X^n for $n \geq 2$, we get the required equality.

J.

1.
2. `d` – `a` – `a` – `a` – `b` – `c` / `d` – `a` – `a` – `b` – `c` / `d` – `c`
3. `a` – `b` – `c` – `a` (and its 3 rotations) and `a` – `a` are the two cycles of G.
4. G' has only one cycle. The number of cycles in a graph is clearly preserved by isomorphism, so G and G' cannot be isomorphic.
5. Not isomorphic either (degree argument for instance), actually I made a mistake as I wanted them to be isomorphic. Well, you get points anyway.

K.

1. $\chi_{A\cap B}(x) = \chi_A(x).\chi_B(B)$
2. Yes, the complement of A in E: $1 - \chi_A = \chi_{E\setminus A}$
3. $\chi_{A\cup B} = \chi_A(x) + \chi_B(B) - \chi_A(x).\chi_B(B)$ (we use inclusion-exclusion here)
4. This is the cardinal of A: we count 1 for each x such that $\chi_A(x) = 1$, *i.e.* $x \in A$.