Université d'Ottawa · University of Ottawa
Faculté des sciences
Mathématiques et de statistique
Faculty of Science
Mathematics and Statistics

Discrete Mathematics - MAT 2348, Winter 2013
Instructor: Laura Dumitrescu
Final exam. Duration: 3 hours. Closed book examination

Instructions:

- This is a *closed book* exam, and no notes of any kind are allowed. The use of calculators, cell phones, pagers or any text storage or communication device **is not permitted.**
- The exam consists of 14 questions on 12 pages. Page 12 provides additional work space. Do not detach it.
- Questions are long-answer. The maximum number of points each of these questions has is given in the table below. To obtain the maximum number of points for any of these questions, a *proper justification* is required. **Your solution should be written legibly, logically and in an organized way. You must clearly show all relevant steps in your solution to receive full marks**. *Clearly indicate the final answer.*
- Read each question carefully before answering it. If time permits, verify your results.
- For rough work, you may use the back pages. *Do not use scrap paper of your own.*
- Good luck!

Last name:____________________ First name:____________________

Student number:____________________ Signature:____________________

Question	1	2	3	4	5	6	7	8	9	10	11	12	13	14	Total
Max	3	3	5	5	5	4	5	4	4	4	6	3	3	6	60
Marks															

585, av. King-Edward
Ottawa (Ontario) K1N 6N5 Canada
585 King Edward Avenue
Ottawa, Ontario K1N 6N5 Canada
(613) 562-5864 • Téléc./Fax (613) 562-5776
Courriel/Email: uomaths@science.uottawa.ca

1. How many distinct "words" can be formed with all the letters of MISSISSIPPI?

2. Find the number of integer solutions of the equation $y_1 + y_2 + y_3 = 20$, with restrictions $y_1 \geq 0,\ y_2 > 0,\ y_3 > 2$.

3. Use the method of mathematical induction to prove $\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$, where $n \geq 1$.

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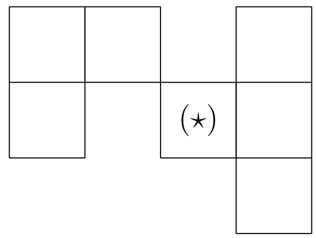
4. Suppose that nine distinct points with integer coordinates are randomly selected in the three-dimensional space $\mathbb{R}^3$. Prove that among these, there exist two points whose midpoint (of the line segment connecting them) also has integer coordinates.

5. (a) How many natural numbers, $1 \leq n \leq 600$ are not divisible by any of 2, 3, 6?

   (b) How many natural numbers, $1 \leq n \leq 600$ are divisible by exactly one of 2, 3, 6?

6. Use the recursive formula $r(C;x) = r(C_e;x) + xr(C_s;x)$ *one step* for the board given bellow. You must start with the indicated square. You do not need to find the final form of the associated rook polynomial.



7. Consider 4 integer numbers, y_1, y_2, y_3 and y_4 such that $y_1 \geq 0$, $2 \leq y_2 \leq 10$, $y_3 > 0$, y_3 is a multiple of 2 and $y_4 > 5$, y_4 is a multiple of 5.

   (a) Let a_r be the number of non-negative integer solutions of the equation

   $$y_1 + y_2 + y_3 + y_4 = r, \;\; r \geq 0,$$

   with the given restrictions. Write the generating function $F(x) = \sum_{r \geq 0} a_r y^r$ in closed form.

   (b) Let b_r be the number of non-negative integer solutions of the inequality

   $$y_1 + y_2 + y_3 + y_4 \leq r, \;\; r \geq 0,$$

   with the given restrictions. Write the generating function $G(x) = \sum_{r \geq 0} b_r y^r$ in closed form.

8. Let p_n be the number of partitions of the non-negative integer n such that each part is of the form 2^k, with $k \geq 0$. Find a closed form of the generating function $P(x) = \sum_{n\geq 0} p_n x^n$.

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9. Find a closed form of the generating function $A(x) = \sum_{n\geq 0} a_n x^n$, associated to the sequence $a_n = n^2 + n$, $n \geq 0$.

10. Use the method of generating function to prove

$$\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{r-k} \cdot \binom{n}{k}, \; r \geq 0, \; r \leq m, \; r \leq n.$$

Start from the identity $(1+x)^{m+n} = (1+x)^m \cdot (1+x)^n$, $m \geq 0$ and $n \geq 0$.

11. Find the solution for each of the following recurrence relations, with $n \geq 2$,

    (a) $a_n - 5a_{n-1} + 6a_{n-2} = 0$, $a_0 = 3$ and $a_1 = 7$

    (b) $a_n - 4a_{n-1} + 4a_{n-2} = 0$, $a_0 = 3$ and $a_1 = 8$

    (c) $a_n - 3a_{n-1} + 2a_{n-2} = 8 \cdot 3^{n-2}$, $a_0 = 5$ and $a_1 = 8$.

12. Let $a_n = 2^n + 3^n + 4^n$, $n \geq 0$. Find a homogenous recurrence relation satisfied by a_n and state the initial conditions that guarantee the uniqueness existence of the given solution.

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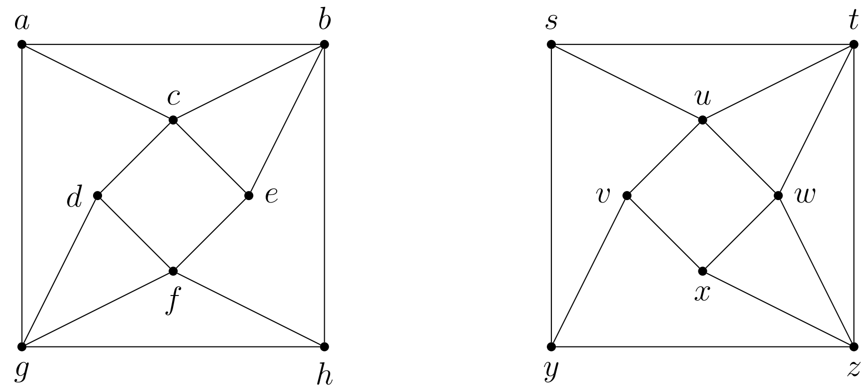
13. Let d_n be a determinant of order n, with $n \geq 1$ given by

$$d_n = \begin{vmatrix} 3 & 1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 1 & 3 & 1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 1 & 3 & 1 & 0 & \dots & 0 & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \dots & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & 0 & \dots & 1 & 3 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 1 & 3 \end{vmatrix}.$$

Find a recurrence formula satisfied by d_n and state the corresponding initial conditions.

14. Let G_1 and G_2 be the graphs appearing in the next figure.



(a) Give the degree of each vertex of G_1 and G_2.

(b) Find all subgraphs of G_1 that are isomorphic to K_3, the complete graph with 3 vertices (triangle). Do the same in the case of G_2.

(c) Are the two graphs isomorphic? Justify your answer.

Additional work space. Do not detach this page.