MAT 2348 (winter 2015) final exam — teacher: Marc Bagnol

Closed book exam. No calculators allowed.

Numeric answers need to be justified. You can use the result of a previous question in an answer even you did not answer it. *More difficult questions are marked with a *, do what you know how to do first.* **Don't Panic.**

A. How many distinct words can be formed with the letters of *OTTAWA*? How many such words have the two *T* side by side? How many do not start by the letter *O*?

B. Give the coefficient of X^n of the following generating functions:

$$\frac{X}{1+2X} \qquad \frac{1}{(1+X)(2-3X)} \qquad (e^X-1)^2$$

C. Give the generating functions for the sequences $a_n = n^2(n+1)$ and $b_n = \frac{3^n}{n!}$.

D. Compute (using the exponential generating function method) how many k-letter words made of the letters A, B, C, D there are that use A and B at least once.

E. We consider for some positive integer k the equation

$$x_1 + x_2 + x_3 = k$$

with x_1, x_2, x_3 positive integers.

1. How many solutions are there?
2. How many solutions such that x_1, x_2, x_3 are odd are there?
3. How many solutions such that $x_1 \geq 1$ and $x_2 \leq 3$ are there?

F.

1. How many numbers from 1 to 100 are not multiples of 4, 5 or 6?
2. How many numbers from 1 to 100 are multiples of exactly one of these three numbers?
   (for instance, we count the number 8 *because it is a multiple of* 4 *but not of* 5 *nor* 6*)*

G.

1. Show that if we take 15 different numbers from 1 to 100, at least two of them must have a difference of strictly less than 7.

*2. Consider a standard 8×8 chessboard (with the usual layout of alternating black and white squares) from which the top-left and bottom-right corner squares have been removed. Is it possible to cover this board entirely by non-overlapping 1×2 dominos?

H. Solve the following recurrences:

1. $a_0 = 0$ and $2a_{n+1} - a_n = 5^n$
2. $b_0 = 0$, $b_1 = 2$ and $b_{n+2} - 4b_{n+1} + 4b_n = 0$

*3. The Fibonacci sequence: $c_0 = 1$, $c_1 = 1$, $c_{n+2} - c_{n+1} - c_n = 0$

Give the generating function for the following recurrences *(you do not have to solve them)*:

4. $d_0 = d_1 = d_2 = 3$ and $d_{n+3} - d_{n+2} + 5d_{n+1} + d_n = 0$
5. $d_0 = d_1 = d_2 = d_3 = 0$ and $d_{n+4} + 6d_n = n$

I. Considering the identity $(X+1)^{\alpha+2} = (X+1)^\alpha + 2X(X+1)^\alpha + X^2(X+1)^\alpha$ (for any real α) show (using generating functions) that $\binom{\alpha+2}{k} = \binom{\alpha}{k} + 2\binom{\alpha}{k-1} + \binom{\alpha}{k-2}$ for all integers $k \geq 2$.

J. Let G be the directed graph (V, E) defined as:

$V = \{\mathtt{a}, \mathtt{b}, \mathtt{c}, \mathtt{d}\}$
$E = \big\{(\mathtt{a}, \mathtt{a})\ (\mathtt{a}, \mathtt{b})\ (\mathtt{d}, \mathtt{a})\ (\mathtt{b}, \mathtt{c})\ (\mathtt{d}, \mathtt{c})\ (\mathtt{c}, \mathtt{a})\big\}$

(in this exercise, relatively informal justifications are accepted)

1. Draw the graph G.
2. Give a walk (that is not a trail), a trail (that is not a path) and a path from `d` to `c`.
3. List the cycles of G.
4. We define G' from G by replacing the edge $(\mathtt{a}, \mathtt{b})$ by the edge $(\mathtt{b}, \mathtt{a})$. Are G and G' isomorphic?
5. We define G'' from G' by replacing the edge $(\mathtt{c}, \mathtt{a})$ by the edge $(\mathtt{a}, \mathtt{c})$. Are G and G'' isomorphic?

K. We fix a finite set E. For any subset $C \subseteq E$ we define the function $\chi_C : E \to \{0, 1\}$ as follows:

$$\begin{cases} \chi_C(x) = 1 & \text{if } x \in C \\ \chi_C(x) = 0 & \text{if } x \notin C \end{cases}$$

(χ_C is called the characteristic function of C)

1. Express $\chi_{A \cap B}$ in terms of χ_A and χ_B.
2. Is there a subset $C \subseteq E$ such that $\chi_C = 1 - \chi_A$?
3. Express $\chi_{A \cup B}$ in terms of χ_A and χ_B.
4. To what quantity does the sum $\sum_{x \in E} \chi_A(x)$ correspond?