MAT 2348 — assignment #4 (solutions)

A.

1. The exponential GF is $(e^X - 1 - X - X^2/2!)e^X e^X e^X = e^{4X} - e^{3X} - Xe^{3X} - X^2e^{3X}/2!$ so the number of n-letter words, equal to the coefficient of $X^n/n!$, is $4^n - 3^n - 3^{n-1}n - 3^{n-2}\binom{n}{2}$.

2. The exponential GF is $(1 + X + X^2/2!)e^X e^X e^X = e^{3X} + Xe^{3X} + X^2e^{3X}/2! + X^3e^{3X}/3!$ so the number of n-letter words, equal to the coefficient of $X^n/n!$, is $3^n + 3^{n-1}n + 3^{n-2}\binom{n}{2}$.

3. The exponential GF is $\frac{e^X+e^{-X}}{2}\frac{e^X+e^{-X}}{2}\frac{e^X-e^{-X}}{2}\frac{e^X-e^{-X}}{2} = \frac{1}{16}(e^{2X} - e^{-2X})^2 = \frac{1}{16}(e^{4X} - 2 + e^{-4X})$ so the number of n-letter words, equal to the coefficient of $X^n/n!$, is 0 for $n = 0$ and $\frac{4^n(1+(-1)^n)}{16}$ for $n \geq 1$.

B.

1. Computing the GF $f(X)$ for a_n:
$$\begin{array}{rclllllll} f(X) - a_0 & = & a_1X & + & a_2X^2 & + & a_3X^3 & + & \cdots \\ Xf(X) & = & a_0X & + & a_1X^2 & + & a_2X^3 & + & \cdots \\ \frac{X^2}{(1-X)^2} & = & 0X & + & 1X^2 & + & 2X^3 & + & \cdots \end{array}$$
therefore $(f(X) - a_0) = 5Xf(X) - \frac{X^2}{(1-X)^2}$ and finally: $f(X) = \frac{1}{1-5X} + \frac{-X^2}{(1-X)^2(1-5X)}$.

   We can apply the partial fraction decomposition method to get $\frac{1}{(1-X)^2(1-5X)} = \frac{25/16}{1-5X} + \frac{-5/16}{1-X} + \frac{-1/4}{(1-X)^2}$.

   So in the end: $f(X) = \frac{1+(25/16)X^2}{1-5X} + \frac{-(5/16)X^2}{1-X} + \frac{-(1/4)X^2}{(1-X)^2}$ and $a_0 = 1$, $a_1 = 5$, then for $n \geq 2$, $a_n = 5^n + (25/16)5^{n-2} - (5/16) - (1/4)(n-1)$.

2. Computing the GF $f(X)$ for a_n:
$$\begin{array}{rclllll} f(X) - a_0 - a_1X & = & a_2X^2 & + & a_3X^3 & + & \cdots \\ X(f(X) - a_0) & = & a_1X^2 & + & a_2X^3 & + & \cdots \\ X^2f(X) & = & a_0X^2 & + & a_1X^3 & + & \cdots \end{array}$$
therefore $2(f(X) - a_0 - a_1X) = -(X(f(X) - a_0)) + X^2f(X)$ and finally: $f(X) = \frac{-2X}{X^2-X-2}$.

   Partial fraction decomposition: $\frac{1}{X^2-X-2} = \frac{1}{(X+1)(X-2)} = \frac{1/3}{X-2} + \frac{-1/3}{X+1} = \frac{-1/6}{1-(X/2)} + \frac{-1/3}{1+X}$.

   So $f(X) = \frac{(1/3)X}{1-(X/2)} + \frac{(2/3)X}{1+X}$ and we get: for $n \geq 2$ $a_n = (1/3)(1/2)^{n-1} + (2/3)(-1)^{n-1}$.

C.

2. $d(\mathtt{a}) = 2$, $d(\mathtt{b}) = 2$, $d(\mathtt{c}) = 3$, $d(\mathtt{d}) = 2$, $d(\mathtt{e}) = 3$, $d(\mathtt{f}) = 2$

3. Doing this, we see that the degree of d in G' is 1. As there are no vertex of degree 1 in G, these graphs cannot be isomorphic (because isomorphism preserves degrees).

D.

2. The walk $\mathtt{b} - \mathtt{d} - \mathtt{c} - \mathtt{c} - \mathtt{e} - \mathtt{b} - \mathtt{d}$. It is not a trail because $(\mathtt{b}, \mathtt{d})$ is used twice.

   The trail $\mathtt{b} - \mathtt{c} - \mathtt{e} - \mathtt{b} - \mathtt{d}$. It is not a path because we go through $\mathtt{b}$ twice.

   Finally $\mathtt{b} - \mathtt{d}$ is a path.

3. We have:

   $\mathtt{c} - \mathtt{c}$

   $\mathtt{b} - \mathtt{c} - \mathtt{e}$, $\mathtt{c} - \mathtt{e} - \mathtt{c}$, $\mathtt{e} - \mathtt{b} - \mathtt{c}$

   $\mathtt{b} - \mathtt{d} - \mathtt{c} - \mathtt{e}$, $\mathtt{d} - \mathtt{c} - \mathtt{e} - \mathtt{b}$, $\mathtt{c} - \mathtt{e} - \mathtt{b} - \mathtt{d}$, $\mathtt{e} - \mathtt{b} - \mathtt{d} - \mathtt{c}$

   (we regrouped on each lines the cycles that are morally the same: we are just shifting the starting point)

4. To make the graph acyclic, we must at least remove the $(\mathtt{c}, \mathtt{c})$ edge and if we do this the graph still has cycles. therefore this number is at least 2. Now if we remove the $(\mathtt{c}, \mathtt{e})$ edge we do not have any cycle anymore, so the minimal number of edges to remove to make the graph acyclic is 2.