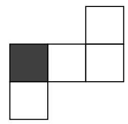
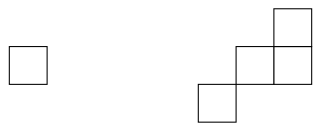
MAT 2348 — assignment #3 (solutions)

A. We decompose the board step by step using the ideas seen in class: we start by the square

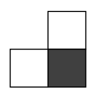


so the $P = XP_1 + P_2$ with P_1 and P_2 the polynomials of the respective boards



we see that $P_1 = 1 + X$. Now as we have two independent boards, $P_2 = P_3P_4$ with P_3 and P_4 the polynomials of the respective boards

so that $P_3 = 1 + X$. For P_4 we reason on the square



as in the beginning and carry on the computation to find that $P_4 = (1 + X)^2 + X(1)$.

Putting all this together, we get

$$\begin{aligned}
P &= X(1+X) + ((1+X)((1+X)^2 + X)) \\
&= X + X^2 + ((1+X)(1+3X+X^2)) \\
&= X + X^2 + (1+3X+X^2) + (X+3X^2+X^3) \\
&= 1 + 5X + 5X^2 + X^3
\end{aligned}$$

B. We start with $\frac{1+X}{(1-X)^3} = 1^2 + 2^2X + 3X^2 + \cdots$ obtained by derivations and multiplying by X (see early classes on GF).

Multiplying by X: $1^2X + 2^2X^2 + 3^2X^3 + \cdots = \frac{X+X^2}{(1-X)^3}$.

Deriving:

$$\begin{aligned}
1^3+2^3X+3^3X^2+\cdots &= \tfrac{(1+2X)(1-X)^3-(X+X^2)(-3)(1-X)^2}{(1-X)^6}\\
&= \tfrac{(1+2X)(1-X)^3-(X+X^2)(-3)(1-X)^2}{(1-X)^6}\\
&= \tfrac{(1+2X)(1-X)-(X+X^2)(-3)}{(1-X)^4}\\
&= \tfrac{(1+X-2X^2)+(3X+3X^2)}{(1-X)^4}\\
&= \tfrac{1+4X+X^2}{(1-X)^4}
\end{aligned}$$

Now we can apply the summation operator seen in class:

$$1^3+(1^3+2^3)X+(1^3+2^3+3^3)X^2+\cdots=\frac{1+4X+X^2}{(1-X)^4}\frac{1}{(1-X)}=\frac{1+4X+X^2}{(1-X)^5}$$

As $\frac{1}{(1-X)^5}=\sum_{n=0}^{\infty}\binom{n+5-1}{n}X^n$ we get that the coefficient of X^{n-1} in $\frac{1+4X+X^2}{(1-X)^5}$ is

$$\begin{aligned}
&\tbinom{(n-1)+5-1}{n-1}+4\tbinom{(n-2)+5-1}{n-2}+\tbinom{(n-3)+5-1}{n-3}\\
=\ &\tbinom{n+3}{n-1}+4\tbinom{n+2}{n-2}+\tbinom{n+1}{n-3}\\
=\ &\frac{1}{4!}(n(n+1)(n+2)(n+3)+4(n-1)n(n+1)(n+2)+(n-2)(n-1)n(n+1))\\
=\ &\frac{1}{24}n(n+1)((n+2)(n+3)+4(n-1)(n+2)+(n-2)(n-1))\\
=\ &\frac{1}{24}n(n+1)((n^2+5n+6)+4(n^2+n-2)+(n^2-3n+2))\\
=\ &\frac{1}{24}n(n+1)(6n^2+6n)\\
=\ &\frac{1}{4}n(n+1)n(n+1)\\
=\ &\left(\frac{n(n+1)}{2}\right)^2
\end{aligned}$$

and this coefficient is equal to the sum $1^3+2^3+3^3+\cdots+n^3$.

C.

$\frac{2X-3}{1+X}=(2X-3)(1-X+X^2+\cdots)=-3+\sum_{n=1}^{\infty}(2(-1^{n-1})-3(-1)^n)X^n=-3+\sum_{n=1}^{\infty}5(-1)^{n-1}X^n$

$\frac{1}{4-X}=\frac{1}{4}\frac{1}{1-X/4}=(1/4+X/4^2+X^2/4^3+X^3/4^4+\cdots$

In the next one we use partial fractions decomposition:

$\frac{1}{(1-X)(2+X)}=\frac{1/3}{1-X}+\frac{-1/3}{2+X}=(1/3)(\frac{1}{1-X}+\frac{-1/2}{1+X/2})=\sum_{n=0}^{\infty}(1/3)(1+(1/2)^{n+1}(-1)^n)X^n$

$\frac{1+X}{(1-X^2)}=\frac{1+X}{(1-X)(1+X)}=\frac{1}{1-X}=1+X+X^2+\cdots$

D.

1. $(X^2+X^3+X^4+\cdots)(X^4+X^5+X^6+\cdots)(1+X+X^2+\cdots) = \frac{X^6}{(1-X)^3}$
2. $(1+X+X^2+\cdots)(1+X+X^2+\cdots)(1+X+X^2+\cdots+X^9) = \frac{1-X^{10}}{(1-X)^3}$
3. $(X^3+X^5+X^7\cdots)(1+X+X^2+\cdots)(1+X+X^2+\cdots) = \frac{X^3}{(1-X^2)(1-X)^2}$

E.

1. The formula for product of GF gives

$$b_n = \sum_{i+j=n} a_i a_j = \sum_{k=0}^{n} a_k a_{n-k}$$

2. By the above formula this coefficient is equal to

$$\sum_{i+j=n} \binom{n}{i}\binom{n}{j} = \sum_{k=0}^{n} \binom{n}{k}\binom{n}{n-k} = \sum_{k=0}^{n} \binom{n}{k}^2$$

3. The binomial theorem tells us that this coefficient is $\binom{2n}{n}$
4. Therefore, as $(1+X)^n(1+X)^n = (1+X)^{2n}$, we have that $\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2$