# MAT 2348 — assignment #2 (solutions)

**A.** Choosing an injection from $A$ to $B$ is the same thing as choosing an arrangement without repettition of $n$ elements (one for each element in $A$) among $m$. That is to say there are $\mathbf{A}_n^m = \frac{m!}{n!}$ choices.

**B.**

1. A $x$ is in $A^{\perp\perp} = (A^\perp)^\perp$ by definition if $x \perp y$ for any $y \in A^\perp$. If $x \in A$, we have $x \perp y$ (and hence $y \perp x$) for all $y \in A^\perp$, therefore $x \in A^{\perp\perp}$. That is to say $A \subseteq A^{\perp\perp}$
2. Let $x \in B^\perp$. By definition we have $x \perp y$ for all $y$ in $B$. Since $A \subseteq B$, $x \perp y$ for all $y \in A$ and therefore $x \in A^\perp$.
3. The above property gives us already $B^\perp \subseteq A^\perp$. We can then apply it a second time, to $B^\perp$ and $A^\perp$, which gives $A^{\perp\perp} \subseteq B^{\perp\perp}$.
4. The first question gives us $A \subseteq A^{\perp\perp}$ and $A^\perp \subseteq (A^\perp)^{\perp\perp}$. Applying the second question to $A \subseteq A^{\perp\perp}$, we get $(A^{\perp\perp})^\perp \subseteq A^\perp$. Therefore $A^\perp = A^{\perp\perp\perp}$.

**C.** If we forget about the second coordinate (*i.e.* we project on $X/Z$ axis) we are looking at the 2D version of the problem with $b_5$ possibilities. By inserting five $Y$ moves in a ten $X/Z$ sequence, we get a 3D path as required. There are $\binom{11+5-1}{5}$ ways of inserting the $Y$ moves (11 containers, 5 elements).

By the product principle, we get a total of $b_5\binom{11+5-1}{5}$ possibilities.

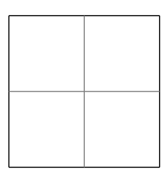
**D.** Indeed a 6 letter word is the same thing as a function from $\{1,2,3,4,5,6\}$ to $\{\mathrm{A},\mathrm{B},\mathrm{C}\}$: to each position, associate a letter. The word uses each letter at least once if and only if the function is surjective, hence the relation to Stirling numbers.

We can solve this directly: let us call $F_\mathrm{A}, F_\mathrm{B}, F_\mathrm{C}$ the set of words that do not use A, B, C respectively. By the inclusion-exclusion principle, the cardinal of the set of words that miss one of the letters is

$$\begin{aligned}|F_\mathrm{A} \cup F_\mathrm{B} \cup F_\mathrm{C}| &= |F_\mathrm{A}| + |F_\mathrm{B}| + |F_\mathrm{B}| - (|F_\mathrm{A} \cap F_\mathrm{B}| + |F_\mathrm{B} \cap F_\mathrm{C}| + |F_\mathrm{A} \cap F_\mathrm{C}|) + |F_\mathrm{A} \cap F_\mathrm{B} \cap F_\mathrm{C}| \\ &= 3(2^6) - 3(1^6) + 0\end{aligned}$$

By the sum principle, the number of words that miss none of the three letters is $3^6 - 3(2^6) + 3(1^6)$.

**E.** A square of side length 2 can be cut into four squares of side length 1 as follows:



with each of the small squares having side length $1$, so that two points inside such a square are at a distance of at most $\sqrt{2}$.

By the pigeonhole principle, if we place five points in the square, two of them will end up in the same small square, thus being at a distance at most $\sqrt{2}$.

On the other hand, if we place only four points, we can put them for instance at the angles of the square so that they are at a distance at least $2$ of each other. Hence $5$ is the lowest possible answer.

**F.** We prove by induction:

$$\text{for all } n\text{, the statement “}S[n]\ :\ f(n) = g(n)\text{” holds}$$

**Base case:** $f(0) = 0$ and $g(0) = 2^0 - 1 = 0$. So $S[0]$ holds.

**Induction step:** suppose $S[n]$ holds *i.e.* $f(n) = g(n)$. Then

$$f(n+1) = 2f(n) + 1 = 2g(n) + 1 = 2(2^n - 1) + 1 = 2^{n+1} - 1 = g(n+1)$$

that is to say $S[n+1]$ holds.

**Conclusion:** we have $S[0]$ and $S[n] \Rightarrow S[n+1]$. By induction, $S[n]$ holds for all $n$.