MAT 2348 — assignment #2

Due: february 23rd. Numeric answers need to be justified. Don't Panic.

A. Let A and B be two finite sets with $\mathtt{card}(A) = n$ and $\mathtt{card}(B) = m \geq n$. How many injections from A to B are there?

B. We consider a set H and a binary relation $\perp$ on H, that is to say a subset $\perp \subseteq H \times H$. We write $a \perp b$ for $(a, b) \in \perp$. **We suppose moreover that the relation is symmetric: $a \perp b$ if and only if $b \perp a$.** Now, for any subset $A \subseteq H$ we can define

$$A^{\perp} = \{ b \in H \mid \text{for all } a \in A,\ a \perp b \}$$

then of course we can apply this $(\cdot)^{\perp}$ operation again to get for instance $(A^{\perp})^{\perp}, ((A^{\perp})^{\perp})^{\perp} \ldots$ in that case, we do not write the parenthesis to make it more readable: $A^{\perp\perp}, A^{\perp\perp\perp} \ldots$

1. Show that for any subset $A \subseteq H$, we have $A \subseteq A^{\perp\perp}$.
2. Show that if $A \subseteq B$, then $B^{\perp} \subseteq A^{\perp}$.
3. Show that if $A \subseteq B$, then $A^{\perp\perp} \subseteq B^{\perp\perp}$.
4. Show that for any subset $A \subseteq H$, we have $A^{\perp} = A^{\perp\perp\perp}$.

This situation with an "orthogonality relation" occurs in various domains: linear algebra, logic … and it always present this basic set of properties. The operation $(\cdot)^{\perp\perp}$ is usually seen as a "closure" operation, so the subsets such that $A = A^{\perp\perp}$ are called "closed".

C. We let the snake from the path problem move in 3D now:

1. Starting at the point $(0, 0, 0)$ in cartesian coordinates, how many paths are there that reach $(5, 5, 5)$ using only the moves $X = (+1, 0, 0)$, $Y = (0, +1, 0)$ and $Z = (0, 0, +1)$?
2. How many such path always stay below the plane $x = z$?

   (hint: remember that the Catalan numbers $b_n = \frac{1}{n+1}\binom{2n}{n}$ count the number of path that remain below the diagonal in the 2D version of the problem. Draw a picture!)

D. How many 6 letter words written with the letters "A,B,C" that use at least each letter once are there? Can you explain how this is related to Stirling numbers?

E. Consider a square with sides of length 2. How much points can we place inside the square without having two of them at a distance $\sqrt{2}$ or less of each other?

F. Let $f : \mathbb{N} \to \mathbb{N}$ a function such that $f(n + 1) = 2f(n) + 1$ and $f(0) = 0$. Let $g : \mathbb{N} \to \mathbb{N}$ defined as $g(n) = 2^n - 1$ for all n. Show by induction that $f = g$.