MAT 2348 — assignment #1 (solutions)

A

1. A choice of postal code is a successive choice of 3 letters (26 possibilities each time) and 3 digits (10 possibilities each time). By the product principle we have therefore $26^3 10^3$ possibilities.
2. This is similar to the previous question, only the number of choices for letters is only 25, hence $25^3 10^3$ choices.
3. Such a choice can be decomposed in: first choose the letters, then choose the numbers. For the letters we already saw we have 26^3 possibilities. For the numbers, we have as much possibilities[1] as the number of solutions of the equation $x_1 + x_2 + x_3 = 7$, which is $\binom{7+3-1}{7}$ (remember it corresponds to combinations with repettition and the "bars and bullets" problem). By the product principle we have in the end $26^3\binom{3+7-1}{7}$ possibilities.

B

1. This is an arrangement with types situation with: 2 elements of type B; 4 elements of type A; 2 elements of type P; 1 element of type R. Therefore there are $\frac{(2+4+2+1)}{2!4!2!1!}$ possibilities.
2. Choosing such a word can be decomposed in two phases: first choose the order of the non-A letters (this is an arrangement with types with $\frac{(2+2+1)!}{2!2!1!}$ possibilities) which gives a 5 letter word; then choose a way to insert the A in this word so that they are not consecutive: we have 6 positions and we choose 4 of them, that is to say we have $\binom{6}{4}$ choices.

   In the end we get a total of $\frac{(2+2+1)!}{2!2!1!}\binom{6}{4}$ possibilities.
3. First note that there cannot be words with *both* sequences: there are only two B and in one case they are separated by a A, in the other they are not. Therefore we will be able to apply the sum principle.

   We count the number of words containing BABA: as we did in the previous question, we can consider that we first choose a word made of the remaining letters (2 P, 2 A, 1 R) with $\frac{(2+2+1)!}{2!2!1!}$ possibilities; then we insert the sequence BABA with 6 possible positions. Which gives $6\frac{(2+2+1)!}{2!2!1!}$ words containing BABA.

   We can do the same for ABBA, with the same result: $6\frac{(2+2+1)!}{2!2!1!}$ possibilities.

   Finally, by the sum principle we have a total of $6\frac{(2+2+1)!}{2!2!1!} + 6\frac{(2+2+1)!}{2!2!1!}$ possibilities.

C

1. Combinatorial argument: $\frac{(n+1)(n+2)\cdots(n+k)}{k!} = \binom{n+k}{n}$ and $\binom{n+k}{n}$ is certainly an integer since it is the number of combinations of n elements among $n+k$ elements.
2. n^k is the number of arrangements of n elements of length k, *with repettition*.

   $\mathbf{A}_k^n$ is the number of arrangements of n elements of length k, *without repettition*.

   As arrangement without reprettition is in particular an arrangement with repettition, there are more arrangements with repettition than arrangements without repettition,[2] so we must have

[1] Careful here: this is true because our digits can actually reach 7, think of the same problem (x_1, x_2, x_3 are digits, *i.e.* integers between 0 and 9) but with $x_1 + x_2 + x_3 = 15$.

[2] In the language of sets, there is an inclusion between the two sets

$n^k \geq \mathbf{A}_k^n$.

If we carry on this point of view, we see that as soon as $k \geq 2$ and $n \geq 2$ we have strictly more arrangements with repettition than arrangements without repettition; that if $k = 1$ then both numbers are equal; and that if $n = 1$ there are no arragement without repettition if $k > 1$.

D

1. $\binom{n-1}{k}n$ is the number of ways to choose one element among n, then to choose k elements among the remaining $n-1$.

   $\binom{n}{k}(n-k)$ is the number of ways to choose k elements among n, then to choose one elements among the remaining $n-k$.

   In both cases we are counting how many ways we have to choose k elements and one distinguised element from a set of n elements.

   (remark: here a computation is easier than a combinatorial argument)

2. The binomial theorems says that $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$, therefore if we take $y = 1$ and $x = 4 = 2^2$, we get $5^n = \sum_{k=0}^{n} \binom{n}{k} 2^{2k}$

3. $\binom{n+1}{k+1}$ is the number of selections of $k+1$ elements among $n+1$.

   Suppose now that we distinguish a specific element that we call s. A selection of $k+1$ elements can be of two distinct (*non-overlapping*) types: either they contain s or not.

   Choosing a combination of $k+1$ that contains s is the same thing as choosing k elements among the remaining n. So there are $\binom{n}{k}$ of these.

   Choosing a combination that does not contain s is the same thing as choosing $k+1$ elements among the n non-s elements. So there are $\binom{n}{k+1}$ of these.

   Now by the sum principle there are $\binom{n}{k} + \binom{n}{k+1}$ ways of choosing $k+1$ elements from the set with the distinguished s. But this is exactly the same as choosing without distinguised element so in the end $\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}$.