MAT 1341C Test #4 (Winter 2016)

April 4th — Professor: Marc Bagnol

Enter Multiple Choice Answers Here	
1	C
2	E
3	D

Family name: ____________________

First name: ____________________

Student number: ____________________

DGD group: ☐

Marker's Use Only	
4	
5	
6	
7 [Bonus]	
Total	

PLEASE READ THESE INSTRUCTIONS VERY CAREFULLY.

1. You have 80 minutes to complete this exam.
2. This is a closed book exam, and no notes of any kind are permitted. The use of calculators, cell phones, or similar devices is not permitted. All cybernetic implants not necessary for life-support must be disabled at the beginning of the exam.
3. Read each question carefully, and **answer all questions in the space provided after each question.** For questions 4 to 7, you may use the backs of pages if necessary, but be sure to indicate to the marker that you have done this.
4. Questions 1 to 3 are multiple choice. These questions are worth 1 point each and no part marks will be given. **Please record your answers in the table above.**
5. Questions 4 to 6 and are worth 6 points each, and part marks can be earned. **The correct answers here require justification written legibly and logically: you must convince the marker that you know why your solution is correct.**
6. Question 7 is a challenging bonus question and is worth 3 points. It is *much* more difficult to obtain marks in the bonus question, so spend your time accordingly. You can earn 100% without attempting Q.7.
7. Where it is possible to check your work, do so.
8. Good luck! Bonne chance!

1. What is the dimension of the subspace of $\mathbb{R}^5$ spanned by $\begin{bmatrix} 1 \\ 1 \\ 2 \\ 3 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 2 \\ -3 \\ 5 \\ 1 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ -7 \\ 8 \\ -1 \\ 5 \end{bmatrix}$?

- **A.** 0
- **B.** 1
- **C.** 2
- **D.** 3
- **E.** 4
- **F.** 5

2. Let $B = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & -1 \\ 1 & 1 & -1 \end{bmatrix}$. The third row of B^{-1} is:

- **A.** $\begin{bmatrix} 0 & 1 & -1 \end{bmatrix}$
- **B.** $\begin{bmatrix} -1 & 1 & 0 \end{bmatrix}$
- **C.** $\begin{bmatrix} -2 & 0 & 1 \end{bmatrix}$
- **D.** $\begin{bmatrix} 1 & -1 & 0 \end{bmatrix}$
- **E.** $\begin{bmatrix} 1 & 0 & -1 \end{bmatrix}$
- **F.** B is not invertible.

3. The sequence of vectors $u_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, u_2 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, u_3 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$ is:

- **A.** Orthonormal
- **B.** Orthogonal, but not orthonormal because $u_1 \cdot u_2 \neq 0$
- **C.** Orthogonal, but not orthonormal because $u_3 \cdot u_3 \neq 1$
- **D.** Not orthogonal because $u_1 \cdot u_2 \neq 0$
- **E.** Not orthogonal because $u_1 \cdot u_3 \neq 0$
- **F.** None of the above is correct.

(Additional space for questions 1,2,3)

4. Let

$$A = \begin{bmatrix} 1 & -3 & 0 & 2 \\ 2 & -6 & 1 & 6 \\ 1 & -3 & 1 & 4 \end{bmatrix}$$

(a) Find a basis for the row-space of A.

(b) Find a basis for the column-space of A.

(c) Find a basis for $\text{Null}(A) = \{ X \in \mathbb{R}^4 \mid AX = \mathbf{0} \}$.

(d) Find a matrix B such that $\text{Col}(A) = \text{Null}(B)$.

ANSWERS:

(a) We apply the row-space algorithm, for this we apply Gauss reduction to A which gives us:

$$\begin{bmatrix} \boxed{1} & -3 & 0 & 2 \\ 0 & 0 & \boxed{1} & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

dropping the last row of 0s, we get that $\begin{bmatrix} 1 & -3 & 0 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & 2 \end{bmatrix}$ is a basis of $\text{Row}(A)$.

(b) On the CREF of A computed above we see that columns 1 and 3 are holding a pivot. The column-space algorithm tells us that the corresponding columns of A form a basis, that is $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ is a basis of $\text{Col}(A)$.

(c) For that we solve the linear system $AX = \mathbf{0}$, we already have reduced A so we can directly read the general solution off the CREF: we have two free variables x_2, x_4 and the solutions are $\begin{bmatrix} 3x_2 - 2x_4 \\ x_2 \\ -2x_4 \\ x_4 \end{bmatrix}$.

From this we get the basis setting $x_2 = 1, x_4 = 0$ and $x_2 = 0, x_4 = 1$ which gives us in the end that $\begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$ is a basis of $\text{Null}(A)$.

(d) We have our basis of $\text{Col}(A)$ from question (b): $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ so a vector $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ will be in $\text{Col}(A)$ iff the system $\begin{bmatrix} 1 & 0 & | & x \\ 2 & 1 & | & y \\ 1 & 1 & | & z \end{bmatrix}$ has a solution. We apply Gauss reduction to get: $\begin{bmatrix} 1 & 0 & | & x \\ 0 & 1 & | & y-2x \\ 0 & 0 & | & x-y+z \end{bmatrix}$. So this system is consistent iff $x - y + z = 0$ that is iff $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \text{Null}(\begin{bmatrix} 1 & \text{-1} & 1 \end{bmatrix})$.

5. Let $e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, e_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, e_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$ and $U = \text{Span}(e_1, e_2, e_3)$.

(a) Use the Gram Schmidt algorithm to find an orthogonal basis of U.

(b) Compute the projection of $\begin{bmatrix} 1 \\ 1 \\ 2 \\ 2 \end{bmatrix}$ on U.

(c) Extend the basis found in (a) into a basis of $\mathbb{R}^4$.

ANSWERS:

(a) Start by setting $e'_1 := e_1$

Then $e'_2 = e_2 - \frac{e'_1 \cdot e_2}{e'_1 \cdot e'_1} e'_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} - (1/2) \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 0 \\ 0 \\ 1/2 \end{bmatrix}$

Then $e'_3 = e_3 - \frac{e'_1 \cdot e_3}{e'_1 \cdot e'_1} e'_1 - \frac{e'_2 \cdot e_3}{e'_2 \cdot e'_2} e'_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} - (0) \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} - (0) \begin{bmatrix} 1/2 \\ 0 \\ 0 \\ 1/2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$

So our basis is $e'_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, e'_2 = \begin{bmatrix} 1/2 \\ 0 \\ 0 \\ 1/2 \end{bmatrix}, e'_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$. To make our lives easier for next questions, let's rescale the second vector to $e'_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$.

(b) Since we have an OG basis of U the projection of $v = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 2 \end{bmatrix}$ on U is given by

$$\text{proj}_U(v) = \frac{e'_1 \cdot v}{e'_1 \cdot e'_1} e'_1 + \frac{e'_2 \cdot v}{e'_2 \cdot e'_2} e'_2 + \frac{e'_3 \cdot v}{e'_3 \cdot e'_3} e'_3 = (\text{-}1/2) \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} + (3/2) \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} + (3/2) \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 3/2 \\ 3/2 \\ 2 \end{bmatrix}$$

(c) We apply now the algorithm to complete a basis to e_1', e_2', e_3': we write these vectors as the rows of a 3 matrix:

$$\begin{bmatrix} 1 & 0 & 0 & -1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

and reduce it to get

$$\begin{bmatrix} \boxed{1} & 0 & 0 & -1 \\ 0 & \boxed{1} & 1 & 0 \\ 0 & 0 & 0 & \boxed{1} \end{bmatrix}$$

We are missing a pivot on the third column, so adding the vector $\begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}$ will give us a basis of $\mathbb{R}^4$.

6. State whether each of the following statements is (always) true [T], or is (possibly) false [F], in the box after the statement.

- If you say the statement may be false, you must give an explicit counter-example with numbers, matrices, or functions, as is appropriate!
- If you say the statement is always true, you must give a clear explanation.

(a) If A is an invertible matrix and $AB = 0$ then $B = 0$.

EXPLANATION:

If A^{-1} is defined then from $AB = 0$ we get by multiplying by A^{-1} on both sides:

$A^{-1}AB = A^{-1}0$

$(A^{-1}A)B = 0$

$IB = 0$

$B = 0$

ANSWER: T

(b) If A is a 5×6 matrix and the dimension of $\text{Null}(A)$ is 4, then the dimension of $\text{Col}(A)$ is 2.

EXPLANATION:

We have the rank theorem: (number of bound variables + number of free variables = number of columns)

$\text{rank}(A) + \dim(\text{Null}(A)) = 6$

$\text{rank}(A) + 4 = 6$

$\text{rank}(A) = 2$

Moreover we know that $\dim(\text{Col}(A)) = \text{rank}(A)$, so in the end $\dim(\text{Col}(A)) = 2$.

ANSWER: T

(c) A linearly independent sequence is always orthogonal.

EXPLANATION:

The basis of $\mathbb{R}^2$: $\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}$ is not orthogonal.

ANSWER: **F**

(d) If a 4×4 matrix has rank 4, then its rows are linearly independent.

EXPLANATION:

In that situation, the 4 rows of the matrix span a vector space of dimension 4 (remember that we know $\dim(\text{Row}(A)) = \text{rank}(A)$). This is only possible if they are linearly independent.

ANSWER: **T**

7. Let A be a 5×3 matrix. Show that there cannot be a 3×5 matrix B such that $AB = I_5$. (remember the notation I_5 is for the 5×5 identity matrix)

ANSWER:

Whatever B we choose columns of AB are all linear combinations of the columns of A.

If we could have $AB = \begin{bmatrix} 1&0&0&0&0 \\ 0&1&0&0&0 \\ 0&0&1&0&0 \\ 0&0&0&1&0 \\ 0&0&0&0&1 \end{bmatrix}$ then it would mean that $\begin{bmatrix} 1\\0\\0\\0\\0 \end{bmatrix}, \begin{bmatrix} 0\\1\\0\\0\\0 \end{bmatrix}, \begin{bmatrix} 0\\0\\1\\0\\0 \end{bmatrix}, \begin{bmatrix} 0\\0\\0\\1\\0 \end{bmatrix}, \begin{bmatrix} 0\\0\\0\\0\\1 \end{bmatrix}$ are all in $\text{Col}(A)$ and therefore that $\text{Col}(A) = \mathbb{R}^5$. But this is impossible since $\text{Col}(A)$ has dimension at most 3, since the rank of A is at most 3.