MAT 1341C Test #3 (Winter 2016)

March 14th — Professor: Marc Bagnol

Enter Multiple Choice Answers Here	
1	C
2	D
3	B

Family name: ______________________________

First name: ______________________________

Student number: ______________________________

DGD group:

☐

Marker's Use Only	
4	
5	
6	
7 [Bonus]	
Total	

PLEASE READ THESE INSTRUCTIONS VERY CAREFULLY.

1. You have 80 minutes to complete this exam.
2. This is a closed book exam, and no notes of any kind are permitted. The use of calculators, cell phones, or similar devices is not permitted. All cybernetic implants not necessary for life-support must be disabled at the beginning of the exam.
3. Read each question carefully, and **answer all questions in the space provided after each question.** For questions 4 to 7, you may use the backs of pages if necessary, but be sure to indicate to the marker that you have done this.
4. Questions 1 to 3 are multiple choice. These questions are worth 1 point each and no part marks will be given. **Please record your answers in the table above.**
5. Questions 4 to 6 and are worth 6 points each, and part marks can be earned. **The correct answers here require justification written legibly and logically: you must convince the marker that you know why your solution is correct.**
6. Question 7 is a challenging bonus question and is worth 3 points. It is *much* more difficult to obtain marks in the bonus question, so spend your time accordingly. You can earn 100% without attempting Q.7.
7. Where it is possible to check your work, do so.
8. Good luck! Bonne chance!

1. The linear system

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & -1 & 0 \\ 2 & 4 & -1 & 3 & 6 \\ -3 & -6 & 1 & 7 & 2 \end{array}\right]$$

has...

A. no solutions
B. a unique solution
C. an infinite number of solutions, with 1 free variable
D. an infinite number of solutions, with 2 free variables
E. an infinite number of solutions, with 3 free variables
F. none of the above are correct

2. If $A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}$, and B is a $3 \times n$ matrix then the third row of the matrix AB is

A. the same as the second row of A.
B. the same as the first row of B.
C. the same as the second row of B.
D. the sum of the first and the second rows of B.
E. the sum of the first and the third rows of B.
F. the sum of the second and third rows of B.

3. Find the value(s) of t for which $\begin{bmatrix} 3 \\ 1 \\ 4 \\ t \end{bmatrix}$ lies in the subspace spanned by $\begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \\ 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \\ 0 \\ 2 \end{bmatrix}$.

A. $t = 4$ or 8
B. $t = 8$ only
C. $t = 4$ only
D. $t =$ -8 or 8
E. $t = 0$ or 2
F. $t = 0$ only

4. Suppose we have parameters $\mathbf{c}, \mathbf{d} \in \mathbb{R}$ and consider the linear system with variables x, y and z:

$$\begin{cases} x + 2y - z = \text{-}3 \\ 2x - 2y + 4z = 0 \\ 2x + y + \mathbf{c}z = \mathbf{d} \end{cases}$$

(a) What are the values of $\mathbf{c}$ and $\mathbf{d}$ such that this system has

i. a unique solution?

ii. infinitely many solutions?

iii. no solutions?

ANSWERS:

Let us first put the system in REF:

$$\begin{bmatrix} 1 & 2 & \text{-}1 & | & \text{-}3 \\ 2 & \text{-}2 & 4 & | & 0 \\ 2 & 1 & \mathbf{c} & | & \mathbf{d} \end{bmatrix} \qquad L_2 \to L_2 + \text{-}2\,L_1 \qquad L_3 \to L_3 + \text{-}2\,L_1$$

$$\sim \begin{bmatrix} 1 & 2 & \text{-}1 & | & \text{-}3 \\ 0 & \text{-}6 & 6 & | & 6 \\ 0 & \text{-}3 & \mathbf{c}+2 & | & \mathbf{d}+6 \end{bmatrix} \qquad L_2 \to \text{-}1/6\,L_2$$

$$\sim \begin{bmatrix} 1 & 2 & \text{-}1 & | & \text{-}3 \\ 0 & 1 & \text{-}1 & | & \text{-}1 \\ 0 & \text{-}3 & \mathbf{c}+2 & | & \mathbf{d}+6 \end{bmatrix} \qquad L_3 \to L_3 + 3\,L_2$$

$$\sim \begin{bmatrix} 1 & 2 & \text{-}1 & | & \text{-}3 \\ 0 & 1 & \text{-}1 & | & \text{-}1 \\ 0 & 0 & \mathbf{c}-1 & | & \mathbf{d}+3 \end{bmatrix} \qquad L_3 \to L_3 + 3\,L_2$$

If $\underline{\mathbf{c} = 1}$, we have a line of 0s.

In that case, the system has solutions if and only if $\mathbf{d} = \text{-}3$ (the constant facing it is also 0) and then we have infinitely many solutions since column 3 misses a pivot.

If $\underline{\mathbf{c} \neq 1}$, we have no line of 0s and one pivot in each column. In that case the solution is unique.

So let's recap:

i. Unique solution: $\mathbf{c} \neq 1$, any value of $\mathbf{d}$

ii. Infinitely many solutions: $\mathbf{c} = 1$ and $\mathbf{d} = \text{-}3$

iii. No solution: $\mathbf{c} = 1$ and $\mathbf{d} \neq \text{-}3$

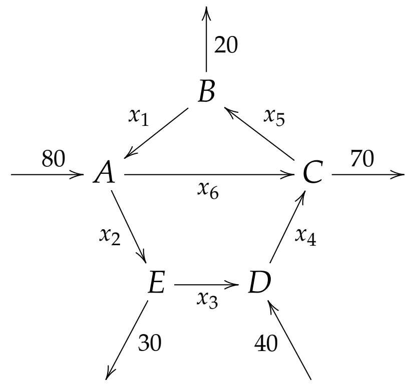
(b) In case (a)ii. of the previous question, find the general solution.
In that case the REF is

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & -3 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad L_1 \to L_1 + -2\, L_2$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & -1 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad \text{which is now in CREF.}$$

We have one free variable x_3 and the general solution is: $\begin{bmatrix} -1-x_3 \\ -1+x_3 \\ x_3 \end{bmatrix}$

5. Consider the network of streets with intersections A, B, C, D, E below. The arrows indicate the direction of traffic flow along the *one-way streets*, and the numbers refer to the exact number of cars observed to enter or leave A, B, C, D, E during one minute. Each x_i denotes the unknown number of cars which passed along the indicated streets during the same period.



(a) Write down a system of linear equations which describes the the traffic flow, together with all the constraints on the variables $x_1, x_2, x_3, x_4, x_5, x_6$.

(You do not have to perform any operations on your equations: this is done for you in (b). Do not simply copy out the equations from (b), you will not get any marks if you do this)

ANSWER:

Let's write down the equation of each intersection:

$A : x_1 + 80 = x_2 + x_6$
$B : x_5 = x_1 + 20$
$C : x_4 + x_6 = x_5 + 70$
$D : x_3 + 40 = x_4$
$E : x_2 = x_3 + 30$

This gives us the linear system:

$$\begin{cases} x_1 - x_2 - x_6 = \text{-}80 \\ \text{-}x_1 + x_5 = 20 \\ x_4 - x_5 + x_6 = 70 \\ x_3 - x_4 = \text{-}40 \\ x_2 - x_3 = 30 \end{cases}$$

Moreover, because the streets are one-way, we have the extra constraint that

$$x_1, x_2 \ldots, x_6 \geq 0$$

(the flows cannot be negative).

(b) The reduced row-echelon form of the augmented matrix of the system in part (a) is

$$\left[\begin{array}{cccccc|c} 1 & 0 & 0 & 0 & -1 & 0 & -20 \\ 0 & 1 & 0 & 0 & -1 & 1 & 60 \\ 0 & 0 & 1 & 0 & -1 & 1 & 30 \\ 0 & 0 & 0 & 1 & -1 & 1 & 70 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Give the general solution. *(you can ignore the other constraints from (a) at this point.)*

ANSWER:

From the CREF we can see there are two free variables x_5, x_6 and the general solution is

$$\begin{bmatrix} \text{-20} + x_5 \\ 60 + x_5 - x_6 \\ 30 + x_5 - x_6 \\ 70 + x_5 - x_6 \\ x_5 \\ x_6 \end{bmatrix}$$

(we ignore the positivity constraint from (a) at that point)

(c) If AC were closed due to roadwork, find the minimum flow along ED, using your results from (b) and constraints from (a).

ANSWER:

AC closed means that $x_6 = 0$, in that case the general solution becomes

$$\begin{bmatrix} \text{-20} + x_5 \\ 60 + x_5 \\ 30 + x_5 \\ 70 + x_5 \\ x_5 \\ 0 \end{bmatrix}$$

The positivity constraints gives us that x_5 must be at least 20 to get $x_1 = \text{-20} + x_5 \geq 0$. Since the flow on ED is $x_3 = 30 + x_5$ it is minimal when $x_5 = 20$. So the minimal flow on ED when AC is closed is 50.

6. State whether each of the following statements is (always) true [**T**], or is (possibly) false [**F**], in the box after the statement.

- If you say the statement may be false, you must give an explicit counter-example with numbers, matrices, or functions, as is appropriate!
- If you say the statement is always true, you must give a clear explanation.

A. If A is a 2×3 matrix, then $\operatorname{rank}(A) = 2$.

EXPLANATION:
Take the zero 2×3 matrix:

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

its rank is 0 (it is already in REF and has no pivot).

ANSWER: $\boxed{\text{F}}$

B. If $u = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$, $v = \begin{bmatrix} 3 \\ 5 \\ 8 \end{bmatrix}$, and $w = \begin{bmatrix} -2 \\ 1 \\ 7 \end{bmatrix}$, then the only linear combination $au + bv + cw$ equal to the zero vector is the one where $a = b = c = 0$.

EXPLANATION:
Finding the a, b, c giving the zero vector is the same as solving the homogeneous system

$\begin{bmatrix} 1 & 3 & -2 \\ 1 & 5 & 1 \\ 2 & 8 & 7 \end{bmatrix}$ $\quad L_2 \to L_2 + -1\, L_1 \qquad L_3 \to L_3 + -2\, L_1$

$\sim \begin{bmatrix} 1 & 3 & -2 \\ 0 & 2 & 3 \\ 0 & 2 & 11 \end{bmatrix}$ $\quad L_3 \to L_3 + -1\, L_2$

$\sim \begin{bmatrix} 1 & 3 & -2 \\ 0 & 2 & 3 \\ 0 & 0 & 8 \end{bmatrix}$ we have one pivot in each column so the system has a unique solution: $a = b = c = 0$.

ANSWER: $\boxed{\text{T}}$

6 (cont.)

C. If A is a 2×2 matrix and $A^2 = 0$, then $A = 0$.

EXPLANATION:
The matrix $\left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]$ squares to

$$\left[\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right]$$

.

ANSWER: $\boxed{\text{F}}$

D. If $[\,A \,|\, b\,]$ is a linear system with $b \in \mathbb{R}^3$ and A a 3×4 matrix with $\mathrm{rank}(A) = 2$, then $[\,A \,|\, b\,]$ has infinitely many solutions.

EXPLANATION:
The system

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

has A of rank 2 (two pivots) and yet has no solutions (we have a line of 0s facing a non-0 constant).

ANSWER: $\boxed{\text{F}}$

7. [Challenge/Bonus]

The *trace* of a 2×2 matrix is defined as

$$\mathbf{Tr}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = a + d.$$

Show that, for any 2×2 matrices A and B, we have

$$\mathrm{Tr}(AB) = \mathrm{Tr}(BA).$$

(Note: your proof must work for **any** 2×2 *matrices* A *and* B*, you cannot choose them!)*

ANSWER:

Let's take two 2×2 matrices $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $B = \begin{bmatrix} x & y \\ z & t \end{bmatrix}$.

We can compute $AB = \begin{bmatrix} ax + bz & ay + bt \\ cx + dz & cy + dt \end{bmatrix}$ and $BA = \begin{bmatrix} xa + yc & xb + yd \\ za + tc & zb + td \end{bmatrix}$.

From that we get $\mathrm{Tr}(AB) = ax + bz + cy + dt$
and $\mathrm{Tr}(BA) = xa + yc + zb + td = ax + bz + cy + dt = \mathrm{Tr}(AB)$.

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