MAT 1341C Diagnostic Test (Winter 2016)

February 22nd — Professor: Marc Bagnol

Family name: ______________________________

First name: ______________________________

Student number: ______________________________

DGD group:

□

Enter Multiple Choice Answers Here	
1	C
2	E
3	C

θ	$\cos\theta$	$\sin\theta$
0	1	0
$\pi/6$	$\sqrt{3}/2$	$1/2$
$\pi/4$	$\sqrt{2}/2$	$\sqrt{2}/2$
$\pi/3$	$1/2$	$\sqrt{3}/2$
$\pi/2$	0	1

Marker's Use Only	
4	
5	
6	
7 [Bonus]	
Total	

PLEASE READ THESE INSTRUCTIONS VERY CAREFULLY.

1. You have 80 minutes to complete this exam.
2. This is a closed book exam, and no notes of any kind are permitted. The use of calculators, cell phones, or similar devices is not permitted. All cybernetic implants not necessary for life-support must be disabled at the beginning of the exam.
3. Read each question carefully, and **answer all questions in the space provided after each question.** For questions 4 to 7, you may use the backs of pages if necessary, but be sure to indicate to the marker that you have done this.
4. Questions 1 to 3 are multiple choice. These questions are worth 1 point each and no part marks will be given. **Please record your answers in the table above.**
5. Questions 4 to 6 and are worth 6 points each, and part marks can be earned. **The correct answers here require justification written legibly and logically: you must convince the marker that you know why your solution is correct.**
6. Question 7 is a challenging bonus question and is worth 3 points. It is *much* more difficult to obtain marks in the bonus question, so spend your time accordingly. You can earn 100% without attempting Q.7.
7. Where it is possible to check your work, do so.
8. Good luck! Bonne chance!

1. Let $W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 \mid xy = 0 \right\}$. Which of the following statements is true?

A. W is a subspace of $\mathbb{R}^3$

B. $0 \in W$ but W is not stable by scaling.

C. W stable by scaling but is not stable by addition.

D. $0 \in W$ and W is stable by addition.

E. W is stable by scaling and addition.

F. $0 \notin W$

2. Suppose that a vector space V can be spanned by a sequence of 45 vectors, and that V has a linearly independent sequence with 23 vectors. Then it is always true that:

A. $\dim(V) < 45$

B. $\dim(V) > 45$

C. $23 < \dim(V) \leq 45$

D. $23 \leq \dim(V) < 45$

E. $23 \leq \dim(V) \leq 45$

F. None of the above is true.

3. Which of the following is a spanning sequence of

$$\left\{ \begin{bmatrix} a \\ b \\ 0 \\ d \end{bmatrix} \in \mathbb{R}^4 \;\middle|\; a - b = 0 \right\}$$

A. $\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$

B. $\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

C. $\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

D. $\begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$

E. $\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$

F. None of the above are correct.

4. Consider the vector space $\mathbb{F}(\mathbb{R})$ consisting of all functions from $\mathbb{R}$ to $\mathbb{R}$.
Consider the vectors f, g, h, k defined by:

$$f(x) = 1 \qquad g(x) = \big(\cos(x)\big)^2 \qquad h(x) = \sin(x) \qquad k(x) = 4\big(\sin(x)\big)^2$$

and let $W = \text{Span}(f, g)$.

(a) Show that the sequence f, g is linearly independent.
(b) Show that $h \notin W$
(c) Use a trigonometric identity to show that $k \in W$.
(d) What is the dimension of W?

Remember that you must justify your answers.

ANSWERS:

(a) Suppose we have $a, b \in \mathbb{R}$ such that $af + bg = \mathbf{0}$.
This means that for all x, $a + b\big(\cos(x)\big)^2 = 0$. But then by taking x to be $\pi/2$ and 0 we get the equations

$$\begin{cases} a = 0 \\ a + b = 0 \end{cases}$$

which have for only solution $a = b = 0$.
We can therefore conclude (by the LI test) that f, g is LI.

(b) Suppose there are $a, b \in \mathbb{R}$ such that $h = af + bg$.
This means that for all x, $sin(x) = a + b\big(\cos(x)\big)^2$. Taking this time x to be $\pi/2$ and -$\pi/2$ gives us

$$\begin{cases} 1 = a \\ \text{-}1 = a \end{cases}$$

which has no solution.
We can therefore conclude that $h \notin \text{Span}(f, g)$.

(c) From the trigonometric identity $(\cos(x))^2 + (\sin(x))^2 = 1$ we can derive

$$g + \tfrac{1}{4}k = f \qquad \text{and therefore} \qquad k = 4f - 4g$$

that is to say $k \in \text{Span}(f, g)$.

(d) The sequence f, g spans W and is LI, therefore it is a basis of W (with two elements) and we can conclude that W is of dimension 2.

5. Let $W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 \;\middle|\; x+y+z=0 \right\}$

(a) Show that W is a subspace. *(note: you can avoid using the subspace test here)*

(b) Find a basis for W.

(c) What is the dimension of W?

(d) Is $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$ a basis of W?

Remember that you must justify your answers.

ANSWERS:

(a) W is a plane in $\mathbb{R}^3$ going through the origin and we saw that in that case it is a subspace of $\mathbb{R}^3$.

(b) Let's consider the vectors $u = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ and $v = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$. They both satisfy the equation so they are in W.
We can show that u, v is LI: if we have $a, b \in \mathbb{R}^2$ such that $au + bv = \mathbf{0}$ (that is to say $a\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$) then we must have (looking line by line)

$$\begin{cases} a = 0 \\ -a + b = 0 \\ -b = 0 \end{cases}$$

and the only possibility is $a = b = 0$ so indeed u, v is LI.
Let us now show that u, v spans W. Let $h = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ be a vector of W: we need to show that h is a linear combination of u, v. We look at the combination $xu - zv = \begin{bmatrix} x \\ -x-z \\ z \end{bmatrix}$.
Because $h \in W$ we have $y = -x - z$, and hence $xu - zv = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = w$. So w is indeed a linear combination of u, v.
Conclusion: the sequence of vectors $u, v \in W$ is LI, and spans W, it is a basis of W.

(c) We built a basis of W with two vectors, the dimension of W is therefore 2.

(d) Because W has dimension 2 all its bases are sequences of 2 vectors so this cannot be a basis of W.

6. State whether each of the following statements is (always) true [**T**], or is (possibly) false [**F**], in the box after the statement.

- If you say the statement may be false, you <u>must give an explicit counter-example with numbers, matrices, or functions, as is appropriate!</u>
- If you say the statement is always true, you must give a clear explanation.

(a) $X = \{ f \in \mathbb{F}(\mathbb{R}) \mid f(0) = 1 \}$ is a subspace of the function space $\mathbb{F}(\mathbb{R})$ of functions from $\mathbb{R}$ to $\mathbb{R}$.

EXPLANATION:

The set X does not contain the zero function (which is the zero vector of $\mathbb{F}(\mathbb{R})$) so by the subspace test it cannot be a subspace of $\mathbb{F}(\mathbb{R})$.

ANSWER: **F**

(b) If u, v, w is linearly dependent, then $u \in \operatorname{Span}(v, w)$

EXPLANATION:

Take for instance in $\mathbb{R}^2$ the vectors $u = \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right], v = \left[\begin{smallmatrix}0\\1\end{smallmatrix}\right], w = \left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]$. Then, because of $w = \mathbf{0}$ the sequence is LD, but u is not in $\operatorname{Span}(v, w) = \operatorname{Span}(v, \mathbf{0}) = \operatorname{Span}(v)$.

ANSWER: **F**

6. (cont.)

(c) Any set of three vectors in $\mathbb{R}^2$ is linearly dependent.

EXPLANATION:
We know that $\mathbb{R}^2$ has dimension 2, therefore the maximum size of any LI sequence in $\mathbb{R}^2$ is 2, hence a sequence of 3 vectors in $\mathbb{R}^2$ is necessarily LD.

ANSWER: T

(d) If v_1, v_2, v_3 and v_4 are nonzero vectors in a vector space V, and $U = \text{Span}(v_1, v_2, v_3, v_4)$ then $\dim(U) = 4$.

EXPLANATION:
If the vectors are LD, the dimension of their span can be much smaller, *eg.* in $\mathbb{R}^2$ we can take $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \end{bmatrix}$ which spans $\mathbb{R}^2$, of dimension 2.

ANSWER: F

7. [Challenge/Bonus]

Suppose that u, v, w are nonzero vectors in $\mathbb{R}^4$ and that the following are all true:

$$u \cdot v = 0$$

$$u \cdot w = 0$$

$$v \cdot w = 0$$

Show that u, v, w is linearly independent.

ANSWER:

First note that the fact that vectors are nonzero implies $\|u\| \neq 0, \|v\| \neq 0, \|w\| \neq 0$.

Now suppose we have $a, b, c \in \mathbb{R}$ such that $au + bv + cw = \mathbf{0}$. Then by taking the dot product with u on both sides we get

$$\begin{aligned}
& u \cdot (au + bv + cw) = u \cdot \mathbf{0} \\
\Leftrightarrow \quad & a(u \cdot u) + b(u \cdot v) + c(u \cdot w) = 0 \\
\Leftrightarrow \quad & a\|u\|^2 + 0 + 0 = 0 \\
\Leftrightarrow \quad & a\|u\|^2 = 0 \qquad \text{(we have } \|u\| \neq 0 \text{ so we can simplify)} \\
\Leftrightarrow \quad & a = 0
\end{aligned}$$

doing the same with v and w we get $b = 0$ and $c = 0$.

Therefore u, v, w is LI.