# MAT 1341C Final Exam (Winter 2016)

April $20^{th}$, Professor: Marc Bagnol

*Family Name:* ______________________________

*First Name:* ______________________________

*Student Number:* ______________________________

| Enter Multiple Choice Answers Here | |
|---|---|
| **1** | **C** |
| **2** | **C** |
| **3** | **B** |
| **4** | **B** |
| **5** | **D** |
| **6** | **C** |
| **7** | **A** |
| **8** | **E** |
| **9** | **B** |
| **10** | **C** |

| Marker's Use Only | |
|---|---|
| **11** | |
| **12** | |
| **13** | |
| **14** | |
| **15** | |
| **16[Bonus]** | |
| **Total** | |

**Some advice**

Read the whole exam before starting to write, and start by doing the questions you know how to do. The long answer questions are worth much more — spend more time on those.

1. You have 3 hours to complete this exam.
2. Cellular phones, unauthorized electronic devices or course notes are not allowed during this exam. **Phones and devices must be turned off and put away in your bag. Do not keep them in your possession, such as in your pockets.** If caught with such a device or document, the following may occur: you will be asked to leave immediately the exam and academic fraud allegations will be filed which may result in you obtaining a 0 (zero) for the exam.

   **By signing the attendance sheet you acknowledge that you will comply with these conditions.**
3. Questions 1 through 10 are multiple choice. They are worth 1 point each, you do not have to show work, and no part marks will be given. Please record your answers in the table to the right.
4. Questions 11 through 15 require a complete solution, and are worth 6 points each.
5. Question 16 is a bonus question worth 3 points and should only be attempted after all other questions have been completed and checked, since bonus marks are much harder to earn. Spend your time accordingly.
6. **The correct answers for questions 11–16 require justification written legibly and logically: you must convince the reader that you know why your solution is correct. You must answer these questions in the space provided.** Use the backs of pages if necessary.
7. Where it is possible to check your work, do so.
8. Good luck! Bonne chance!

**1.** If the determinant of $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & k \end{bmatrix}$ is 2, find the determinant of

$$\begin{bmatrix} b+5c & e+5f & h+5k \\ \text{-}4c & \text{-}4f & \text{-}4k \\ 3a & 3d & 3g \end{bmatrix}$$

**A.** -120

**B.** 120

**C.** -24

**D.** 24

**E.** 30

**F.** 2

**2.** What is the dimension of the column-space of this matrix?

$$\begin{bmatrix} 1 & -1 & 2 & -2 & 1 \\ 1 & -2 & 1 & 3 & 1 \\ -2 & 6 & -3 & -9 & -2 \\ 2 & -3 & 3 & 1 & 2 \end{bmatrix}$$

**A.** 1

**B.** 2

**C.** 3

**D.** 4

**E.** 5

**F.** 0

**3.** Let $A$ be a an $n \times n$ matrix. Which of the following statements is **NOT equivalent** to

$$\text{“The rank of } A \text{ is equal to } n\text{”}$$

**A.** The columns of $A$ are a basis of $\mathbb{R}^n$.

**B.** 0 is an eigenvalue of $A$.

**C.** $A$ is invertible.

**D.** $\det(A) \neq 0$.

**E.** The system $Ax = 0$ has a unique solution.

**F.** The rows of $A$ are a basis for $\mathbb{R}^n$.

**4.** Let $U = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 \;\middle|\; 2x - 3y = z \right\}$. *Only one* of the following statements is true, which one?

**A.** $U$ is a subspace of $\mathbb{R}^3$ and $\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ is a basis of $U$.

**B.** $U$ is a subspace of $\mathbb{R}^3$ and $\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix}$ is a basis of $U$.

**C.** $U$ is a subspace of $\mathbb{R}^3$ and $\begin{bmatrix} 2 \\ 0 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ is a basis of $U$.

**D.** $U$ is a not a subspace of $\mathbb{R}^3$.

**E.** $U$ is a line through the origin in $\mathbb{R}^3$ with direction vector $\begin{bmatrix} 2 \\ -3 \\ -1 \end{bmatrix}$.

**F.** $U$ is a plane in $\mathbb{R}^3$ through the origin with normal vector $\begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$.

**5.** Let $A$ be a $2016 \times 2016$ matrix. Which of the following statements are true?

$\boxed{I}$ If $\text{rank}(A) = 2015$, then there are 2015 parameters in the general solution of $AX = \mathbf{0}$.

$\boxed{II}$ If $\text{rank}(A) = 2015$, then there is only one parameter in the general solution of $AX = \mathbf{0}$.

$\boxed{III}$ If $\text{rank}(A) \leq 2015$, then for any $B \in \mathbb{R}^{2016}$, the system $AX = B$ has a unique solution.

$\boxed{IV}$ If $\text{rank}(A) = 2016$, then the system $AX = \mathbf{0}$ has more than one solution.

$\boxed{V}$ If the system $AX = \mathbf{0}$ admits more than one solution, then $\text{rank}(A) \leq 2015$.

$\boxed{VI}$ If the system $AX = B$ is consistent for a particular $B \in \mathbb{R}^{2016}$, then $\text{rank}(A) = 2016$.

**A.** $\boxed{III}$ only
**B.** $\boxed{I}$ and $\boxed{IV}$
**C.** $\boxed{III}$ and $\boxed{V}$
**D.** $\boxed{II}$ and $\boxed{V}$
**E.** $\boxed{II}$ only
**F.** $\boxed{IV}$ only

**6.** Let $S = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \in \mathbb{M}_{22}$ (we write $\mathbb{M}_{22}$ for the vector space of $2 \times 2$ matrices).

Find the dimension of:

$$W = \left\{ A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbb{M}_{22} \;\middle|\; AS = SA \right\}$$

**A.** 0
**B.** 1
**C.** 2
**D.** 3
**E.** 4
**F.** $W$ is not a subspace, so we cannot speak of its dimension.

**7.** If $A = \begin{bmatrix} 2 & 1 & 3 \\ 2 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix}$, then the third row of $A^{-1}$ is

**A.** $\begin{bmatrix} 1 & -1 & 0 \end{bmatrix}$
**B.** $\begin{bmatrix} -1 & 2 & 0 \end{bmatrix}$
**C.** $\begin{bmatrix} 0 & -1 & 2 \end{bmatrix}$
**D.** $\begin{bmatrix} 3 & -1 & 1 \end{bmatrix}$
**E.** $\begin{bmatrix} -1 & 2 & 1 \end{bmatrix}$
**F.** $A$ is not invertible

**8.** Consider the vectors $u_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$, $u_2 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$, and $u_3 = \begin{bmatrix} 1 \\ 3 \\ \alpha \end{bmatrix}$ in $\mathbb{R}^3$. For which value(s) of $\alpha$ are $u_1$, $u_2$, $u_3$ linearly **dependent**?

**A.** $\alpha = -1$
**B.** $\alpha \neq -1$
**C.** $\alpha = 2$
**D.** $\alpha \neq 0$
**E.** $\alpha = 0$
**F.** None of the other responses are correct.

**9.** Define a linear transformation $f : \mathbb{R}^3 \to \mathbb{R}^2$ by

$$f\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 2x + y + z \\ 3y + z \end{bmatrix}$$

The standard matrix for $f$ is then:

**A.** $M_f = \begin{bmatrix} 2 & 1 \\ 1 & 3 \\ 1 & 3 \end{bmatrix}$

**B.** $M_f = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 3 & 1 \end{bmatrix}$

**C.** $M_f = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

**D.** $M_f = \begin{bmatrix} 2 & 1 & 1 \\ 3 & 1 & 0 \end{bmatrix}$

**E.** $M_f = \begin{bmatrix} 2 & 0 \\ 1 & 3 \\ 1 & 1 \end{bmatrix}$

**F.** None of the above are correct.

**10.** Which of the following statements are true? (we write $\mathbb{F}(\mathbb{R})$ the vector space of functions from $\mathbb{R}$ to $\mathbb{R}$; and $\mathbb{M}_{nn}$ for the vector space of $n \times n$ matrices)

$\boxed{I}$ $\{ f \in \mathbb{F}(\mathbb{R}) \mid f(\text{-}1)f(1) = 0 \}$ is a subspace of $\mathbb{F}(\mathbb{R})$.

$\boxed{II}$ $\{ A \in \mathbb{M}_{nn} \mid A^{\dagger} = A \}$ is a subspace of $\mathbb{M}_{nn}$.

$\boxed{III}$ $\left\{ \begin{bmatrix} a & b \\ a+b & 1 \end{bmatrix} \in \mathbb{M}_{22} \;\middle|\; a, b \in \mathbb{R} \right\}$ is a subspace of $\mathbb{M}_{22}$.

$\boxed{IV}$ $\left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 \;\middle|\; 2x - z = 1 \right\}$ is a subspace of $\mathbb{R}^3$.

$\boxed{V}$ $\{ f \in \mathbb{F}(\mathbb{R}) \mid f(0) \geq 0 \}$ is a subspace of $\mathbb{F}(\mathbb{R})$.

$\boxed{VI}$ $\{ f \in \mathbb{F}(\mathbb{R}) \mid f(\text{-}x) = \text{-}f(x) \}$ is a subspace of $\mathbb{F}(\mathbb{R})$.

**A.** $\boxed{II}$ and $\boxed{IV}$

**B.** $\boxed{I}$ and $\boxed{IV}$

**C.** $\boxed{II}$ and $\boxed{VI}$

**D.** $\boxed{III}$ and $\boxed{V}$

**E.** $\boxed{II}$ only

**F.** $\boxed{VI}$ only

**11.** Let $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$.

(a) Compute the characteristic polynomial of $A$ and factor it to show that the only eigenvalues of $A$ are 2 and -1.

(b) Find a basis of $E_2 = \{ v \in \mathbb{R}^3 \mid Av = 2v \}$.

(c) Find a basis of $E_{-1} = \{ v \in \mathbb{R}^3 \mid Av = -v \}$.

(d) Explain why $A$ is diagonalizable.

(e) Find an invertible matrix $P$ (justify *why* it is invertible) and a diagonal matrix $D$ such that $A = PDP^{-1}$.

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*ANSWERS:*

(a) We compute $\chi_A = \det(A - xI_3) = \det \begin{bmatrix} -x & 1 & 1 \\ 1 & -x & 1 \\ 1 & 1 & -x \end{bmatrix} = \cdots \text{(row/column expansion)} \cdots = x^3 - 3x - 2 = (x-2)(x^2+2x+1) = (x-2)(x+1)^2$

So the eigenvalues are 2 (multiplicity 1) and -1 (multiplicity 2).

(b) For this we solve $(A - 2I_3)X = \mathbf{0}$:

$$\begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \sim \cdots \text{(row-reduction)} \cdots \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

One free variable $x_3$, general solution $\begin{bmatrix} x_3 \\ x_3 \\ x_3 \end{bmatrix}$.

The space $E_2$ has dimension 1 and we have the vector $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ as a basis.

(c) For this we solve $(A + I_3)X = \mathbf{0}$:

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Two free variables $x_2, x_3$, general solution $\begin{bmatrix} -x_2 - x_3 \\ x_2 \\ x_3 \end{bmatrix}$.

The space $E_{-1}$ has dimension 2 and we have the vectors $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ as a basis.

(d) For each eigenvalue, the dimension of the corresponding space of eigenvectors matches the multiplicity, so by the diagonalization theorem $A$ is diagonalizable.

(e) Again by the diagonalization theorem, we know that if we put together the basis found in (b) and (c) we have a basis of $\mathbb{R}^3$ made of eigenvectors of $A$. Therefore the matrix

$P = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ is invertible and we have

$$A = P \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} P^{-1}$$

**12.** Let $W = \left\{ \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} \in \mathbb{R}^4 \;\middle|\; -x + y + z + t = 0 \right\}$.

(a) Show that $W$ is a subspace of $\mathbb{R}^4$.

(b) Find a basis for $W$.

(c) Apply the Gram-Schmidt procedure to the basis found in (b) to find an *orthogonal* basis for $W$.

(d) Find the projection on $W$ of the vector $\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$.

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*ANSWERS:*

(a) $W$ is the null space of the matrix $\begin{bmatrix} -1 & 1 & 1 & 1 \end{bmatrix}$ so it is a subspace of $\mathbb{R}^4$.

(b) We can use $x = y + z + t$ to rewrite $W$ as:

$$W = \left\{ \begin{bmatrix} y+z+t \\ y \\ z \\ t \end{bmatrix} \;\middle|\; y, z, t \in \mathbb{R} \right\} = \text{Span}\left( \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right)$$

Clearly $u_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, u_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, u_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ are LI so they form a basis of $W$.

(c) We apply Gram-Schmidt to $u_1, u_2, u_3$:

$u_1' = u_1$

$$u_2' = u_2 - \frac{u_1' \cdot u_2}{u_1' \cdot u_1'} u_1' = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/2 \\ -1/2 \\ 1 \\ 0 \end{bmatrix} \text{ (we rescale it to } u_2' = \begin{bmatrix} 1 \\ -1 \\ 2 \\ 0 \end{bmatrix} \text{)}$$

$$u_3' = u_3 - \frac{u_1' \cdot u_3}{u_1' \cdot u_1'} u_1' - \frac{u_2' \cdot u_3}{u_2' \cdot u_2'} u_2' = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} - \frac{1}{6} \begin{bmatrix} 1 \\ -1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 2/6 \\ -2/6 \\ -2/6 \\ 1 \end{bmatrix} \text{ (we rescale it to } u_3' = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 3 \end{bmatrix} \text{)}$$

In the end we have the orthogonal basis $u_1' = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, u_2' = \begin{bmatrix} 1 \\ -1 \\ 2 \\ 0 \end{bmatrix}, u_3' = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 3 \end{bmatrix}$ of $W$.

(d) We compute the projection of $v = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ using our OG basis:

$$\text{proj}_W(v) = \frac{u_1' \cdot v}{u_1' \cdot u_1'} u_1' + \frac{u_2' \cdot v}{u_2' \cdot u_2'} u_2' + \frac{u_3' \cdot v}{u_3' \cdot u_3'} u_3' = 0 + \frac{2}{6} \begin{bmatrix} 1 \\ -1 \\ 2 \\ 0 \end{bmatrix} + \frac{-1}{12} \begin{bmatrix} 1 \\ -1 \\ -1 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 3/12 \\ -3/12 \\ 9/12 \\ -3/12 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 1 \\ -1 \\ 3 \\ -1 \end{bmatrix}$$

**13.** Consider the matrix

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & 0 \\ 1 & 1 & 2 \\ 0 & 4 & 0 \end{bmatrix}$$

(a) Find a basis for $\text{Row}(A)$.
(b) Find a basis for $\text{Col}(A)$.
(c) Find a basis of $\text{Null}(A)$.
(d) Extend the basis you found in (b) to a basis of $\mathbb{R}^4$.

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*ANSWERS:*

(a) We apply the row-space algorithm, we begin by reducing the matrix:

$$A \sim \begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 4 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} \boxed{1} & 0 & 2 \\ 0 & \boxed{1} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Then we can get rid of the last two rows so that $\begin{bmatrix} 1 & 0 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}$ is a basis of $\text{Row}(A)$.

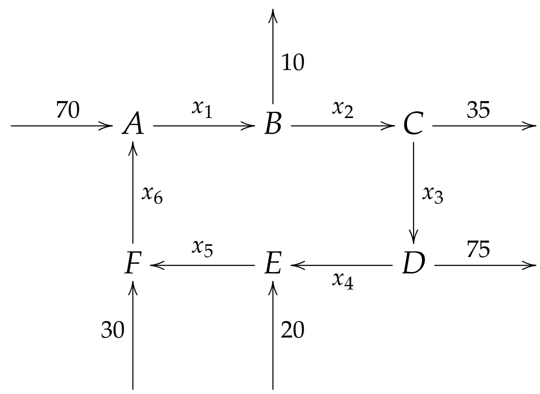
(b) From the CREF we see that the first two columns have a pivot, which means that the first two columns of $A$, $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 0 \\ 4 \end{bmatrix}$, will be a basis of $\text{Col}(A)$.

(c) From the CREF we have the solutions to $AX = \mathbf{0}$: one free variable $x_3$, general solution $\begin{bmatrix} -2x_3 \\ 0 \\ x_3 \end{bmatrix}$ so a basis of $\text{Null}(A)$ is $\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$.

(d) We put our two vectors as the rows of the matrix $B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 4 \end{bmatrix}$ and reduce it:

$B \sim \begin{bmatrix} \boxed{1} & 0 & 0 & 0 \\ 0 & \boxed{1} & 0 & 2 \end{bmatrix}$. To make it a rank 4 matrix, we need to add pivots in the third and fourth columns: $\left[\begin{array}{cccc} \boxed{1} & 0 & 0 & 0 \\ 0 & \boxed{1} & 0 & 2 \\ \hline 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & \boxed{1} \end{array}\right]$ so that $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ is a basis of $\mathbb{R}^4$.

**14.** Consider the network of *one-way* streets with intersections A, B, C, D, E and F pictured below. The arrows indicate the direction of traffic flow, and the numbers refer to the *exact* number of cars observed to enter or leave A, B, C, D, E and F during one minute. Each $x_i$ denotes the unknown number of cars which passed along the indicated streets during the same period.




(a) Write down a system of linear equations which describes the traffic flow, **together with all the additional constraints** on the variables $x_i$.
(do not perform any operations on your equations: this is done for you in (b); and *do not simply copy out the equations implicit in (b). You will not get any marks if you do this*)

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*ANSWER:*
We transcript the fact that flow-in = flow-out at each intersection:

$$\begin{cases} 70 + x_6 = x_1 \ (A) \\ x_1 = 10 + x_2 \ (B) \\ x_2 = 35 + x_3 \ (C) \\ x_3 = 75 + x_4 \ (D) \\ x_4 + 20 = x_5 \ (E) \\ x_5 + 30 = x_6 \ (F) \end{cases} \quad \text{that gives the system} \quad \begin{cases} x_1 - x_6 = 70 \\ x_1 - x_2 = 10 \\ x_2 - x_3 = 35 \\ x_3 - x_4 = 75 \\ -x_4 + x_5 = 20 \\ -x_5 + x_6 = 30 \end{cases}$$

Moreover, since the streets are one-way, we must have $x_i \geq 0$ and since the number of cars must be exact we have that the $x_i$ are integers.

(b) The reduced row-echelon form of the augmented matrix from part (a) is

$$\left[\begin{array}{cccccc|c} 1 & 0 & 0 & 0 & 0 & -1 & 70 \\ 0 & 1 & 0 & 0 & 0 & -1 & 60 \\ 0 & 0 & 1 & 0 & 0 & -1 & 25 \\ 0 & 0 & 0 & 1 & 0 & -1 & -50 \\ 0 & 0 & 0 & 0 & 1 & -1 & -30 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Give the general solution. (ignore the additional constraints from (a) at this point)

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*ANSWER:*
We have one free variable $x_6$ and the general solution is

$$\begin{bmatrix} 70 + x_6 \\ 60 + x_6 \\ 25 + x_6 \\ -50 + x_6 \\ -30 + x_6 \\ x_6 \end{bmatrix}$$

(c) Find the minimum flow along BC, **using your results from (b) and the constraints from (a).**

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*ANSWER:*
Since all flows must be positive integers, we have

$$\begin{cases} 70 + x_6 \geq 0 \\ 60 + x_6 \geq 0 \\ 25 + x_6 \geq 0 \\ -50 + x_6 \geq 0 \\ -30 + x_6 \geq 0 \\ x_6 \geq 0 \end{cases}$$

The most restrictive of these equations is $-50 + x_6 \geq 0$ which tells us that $x_6 \geq 50$. The flow on BC is $x_2$ so the minimal flow on BC is $x_2 = 60 + x_6 \geq 60 + 50 = 110$.

*(You must justify all your answers)*

**15.** State whether each of the following statements is (always) true [**T**], or is (possibly) false [**F**], in the box after the statement.

- If you say the statement may be false, you must give an explicit counter-example with numbers, matrices, or functions, as is appropriate!
- If you say the statement is always true, you must give a clear explanation.

(a) If $V$ is a vector space and $\dim(V) = n$, then every set of $n$ vectors in $V$ is linearly independent.

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*EXPLANATION:*

In $\mathbb{R}^2$ the two vectors $\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right], \left[\begin{smallmatrix}2\\2\end{smallmatrix}\right]$ are not LI.

ANSWER: F

(b) If $A$ is a $4 \times 6$ matrix such that the general solution of $AX = \mathbf{0}$ has 3 parameters, then the rank of $A$ is 3.

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*EXPLANATION:*

The rank theorem tells us that the number of columns is equal to the number of parameters plus the rank, so

$\text{rank}(A) + 3 = 6$ and therefore $\text{rank}(A) = 3$.

ANSWER: T

(c) If the vectors $u, v$ are linearly independent, then $u+v$, $2u-3v$ are also linearly independent.

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*EXPLANATION:*

If we have $a, b \in \mathbb{R}$ such that $a(u+v)+b(2u-3v) = \mathbf{0}$ then by developing and regrouping:

$(a+2b)u+(a-3b)v = \mathbf{0}$ because $u, v$ are LI, this means that

$\begin{cases} a+2b=0 \\ a-3b=0 \end{cases}$ and the only solution to this is $a=b=0$. So the two vectors are LI.

ANSWER: T

(d) If $A$ is a $2\times 2$ invertible matrix, then $A$ is diagonalizable.

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*EXPLANATION:*

The matrix $A = \left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right]$ is invertible (determinant $\neq 0$) but has characteristic polynomial $\chi_A = (1-x)^2$ with the eigenvalue 1 of multiplicity 2. Yet, solving $(A-I_2)X = \mathbf{0}$ gives a space of eigenvectors of dimension 1 so the matrix is not diagonalizable.

ANSWER: F

**16.** **[bonus]** We consider the matrix

$$A = \begin{bmatrix} 5 & \text{-}7 & 2 \\ 4 & \text{-}6 & 2 \\ 2 & \text{-}3 & 1 \end{bmatrix}$$

What is $A^{101}$?

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*ANSWER:*

Compute the characteristic polynomial and factor it: $\chi_A = x(x-1)(x+1)$

Then as we know that the spaces of eigenvectors have dimension at least one, we know the matrix is going to be diagonalizable. It means that there is an invertible $P$ such that:

$$A = P \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \text{-}1 \end{bmatrix} P^{\text{-}1}$$

Then we have

$$A^{101} = P \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \text{-}1 \end{bmatrix}^{101} P^{\text{-}1} = P \begin{bmatrix} 0^{101} & 0 & 0 \\ 0 & 1^{101} & 0 \\ 0 & 0 & (\text{-}1)^{101} \end{bmatrix} P^{\text{-}1} = P \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \text{-}1 \end{bmatrix} P^{\text{-}1} = A$$