MALL^{-} proof equivalence is Logspace-complete, via binary decision diagrams

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Rule permutations and the equivalence problem

Permutations: in sequent calculus presentation of a logic we have a number of permutation equivalences, such as

$$\dfrac{\dfrac{\dfrac{\langle\pi\rangle}{A, B, C, D}}{A \parr B, C, D}\,\parr}{A \parr B, C \parr D}\,\parr \quad\sim\quad \dfrac{\dfrac{\dfrac{\langle\pi\rangle}{A, B, C, D}}{A, B, C \parr D}\,\parr}{A \parr B, C \parr D}\,\parr$$

or more specifically (**MALL⁻**)

$$\dfrac{\dfrac{\dfrac{\langle\pi\rangle}{A, C} \quad \dfrac{\langle\mu\rangle}{B, C}}{A \,\&\, B, C}\,\& \quad \dfrac{\langle\nu\rangle}{D}}{A \,\&\, B, C \otimes D}\,\otimes \quad\sim\quad \dfrac{\dfrac{\dfrac{\langle\pi\rangle}{A, C} \quad \dfrac{\langle\nu\rangle}{D}}{A, C \otimes D}\,\otimes \quad \dfrac{\dfrac{\langle\mu\rangle}{B, C} \quad \dfrac{\langle\nu\rangle}{D}}{B, C \otimes D}\,\otimes}{A \,\&\, B, C \otimes D}\,\&$$

These permutations mirror the commutative conversions of the cut-elimination procedure in sequent calculus.

Equivalence problem (MALL⁻equiv)

Given two **MALL⁻** proofs π and μ, are they related by permutations?

Proofnets

Proofnets (Girard): canonical combinatorial objects for equivalence classes.

Motivation: better understanding of the logic, finer study of its cut-elimination procedure.

A few logics (**MLL**$^-$) enjoy a completely satisfactory notion of proofnet.
In many cases: open problem.

Heijltjes, Houston: settled the case of **MLL** with unit, negatively.

Method: study and determine the complexity of the equivalence problem. In the case of **MLL** with units it is **Pspace**-complete.

Proofnets for MALL^-

Monomial nets (Girard): based on boolean weights attached to the edges of the net, **Ptime** but not canonical.

Slice nets (Hughes & Van Glabbeek): set of slices, i.e. different "versions" of the net. Canonical but exponential blow-up.

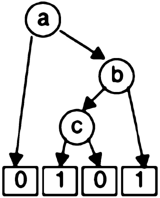
In an unpublished note, Hughes argues that a notion of proofnet for MALL^- *both* **Ptime** *and canonical is unlikely to exist.*

We study the equivalence problem to determine whether we get the same type of impossibility result as in **MLL** with units.

Binary Decision Diagrams

First (wrong) intuition: the missing step in canonicity for monomial nets amounts to equivalence of boolean formulas (**coNP**-complete).

A closer look reveals that these net involve only a specific type of formulas, that we call binary decision diagrams (BDD).



Definition

A BDD **is a binary tree with nodes labeled by boolean variables and leaves labelled by $\mathbf{1}$ and $\mathbf{0}$.**

Subclass: °BDD **read the variables in a specified order.**

(can be seen as boolean formulas built only with a $\texttt{If}\ x\ \texttt{Then}\ \cdot\ \texttt{Else}\ \cdot$ *construction)*

Two BDD are *equivalent* ($\phi \sim \psi$) if they give the same answers for all assignments of variables (e.g. $\texttt{If}\ x\ \texttt{Then}\ \mathbf{1}\ \texttt{Else}\ \mathbf{1} \sim \mathbf{1}$.)

BDD slicings

An intermediate notion between monomial nets and slice nets.

$$\begin{array}{c}\langle\pi\rangle\\ \Gamma\end{array} \quad\mapsto\quad \mathcal{B}_\pi$$

Basic idea: to each $\invamp$ connective in Γ associate a boolean variable. To each pair of dual atoms $\alpha, \alpha^\star$ in Γ associate a BDD.

Example:

$$\pi = \cfrac{\cfrac{\alpha, \alpha^\star}{\alpha \oplus \beta, \alpha^\star}\oplus_{\mathtt{l}} \qquad \cfrac{\beta, \beta^\star}{\alpha \oplus \beta, \beta^\star}\oplus_{\mathtt{r}}}{\alpha \oplus \beta, \alpha^\star \,\&_x\, \beta^\star}\&_x \qquad \begin{array}{l}\mathcal{B}_\pi[\alpha, \alpha^\star] = \texttt{If } x \texttt{ Then } \mathbf{1} \texttt{ Else } \mathbf{0}\\ \mathcal{B}_\pi[\beta, \beta^\star] = \texttt{If } x \texttt{ Then } \mathbf{0} \texttt{ Else } \mathbf{1}\end{array}$$

Equivalence: define $\mathcal{B} \sim \mathcal{B}'$ as $\mathcal{B}[\alpha, \alpha^\star] \sim \mathcal{B}'[\alpha, \alpha^\star]$ for all $\alpha, \alpha^\star$.

Theorem

We have $\pi \sim \mu$ if and only if $\mathcal{B}_\pi \sim \mathcal{B}_\mu$.

A chain of reductions (I)

The notion of BDD slicing gives a reduction:

$$\textsf{MALL}^{\text{-}}\textbf{equiv} \rightarrow \mathsf{BDD}\textbf{equiv}$$

Then we can obtain

$$^{\circ}\mathsf{BDD}\textbf{equiv} \rightarrow \textsf{MALL}^{\text{-}}\textbf{equiv}$$

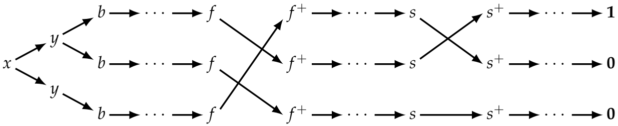
by encoding ordered BDD with a **MALL⁻** proof mimicking the tree structure.
Relies on the $\otimes / \&$ rule commutation.

A chain of reductions (II)

We complete the chain by considering the **Logspace**-complete problem **ORD**:

> Given a line graph G, two vertices f and s, do we have $f < s$ in the induced total order?

It reduces to °BDD**equiv**:



Lemma

The following chain of ($\mathbf{AC}_0$) reductions holds:

ORD $\rightarrow$ °BDD**equiv** $\rightarrow$ **MALL**$^{-}$**equiv** $\rightarrow$ BDD**equiv**

The main theorem

The restricted nature of BDD makes their equivalence sit far below **coNP**.

Lemma

The equivalence of BDD **problem (**BDD**equiv) is in Logspace.**

Summing up, the complexity of $\mathbf{MALL}^{-}$**equiv**:

Theorem

ORD	$\rightarrow$	$^{\circ}$BDD**equiv**	$\rightarrow$	$\mathbf{MALL}^{-}$**equiv**	$\rightarrow$	BDD**equiv**
(**Logspace**-hard)						($\in$ **Logspace**)

Therefore the equivalence problem for $\mathbf{MALL}^{-}$ is Logspace-complete.

Conclusion

Equivalence can be decided efficiently.

This does not settle the question of proofnets: building a canonical representative efficiently is stronger than solving equivalence efficiently.

Some ideas for an impossibility result: (?)

- canonical representative
- cut-elimination
- optimization problems for BDD

... Thank you for your attention !