Multiplicative-Additive Proof Equivalence is Logspace-complete

Marc Bagnol — *JSPS postdoc at MMM, University of Tokyo*

Proof equivalence and proofnets

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In certain cases, the notion can be captured syntactically by a list of similar *rule permuations*. Equivalence becomes a syntactic notion.

Equivalence problem

The equivalence problem of a logic $\mathbf{L}$ is the decision problem:

"*Given two* $\mathbf{L}$ *proofs* π *and* ν, *are they equivalent?*"

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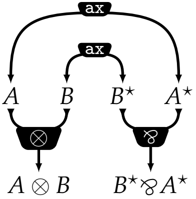

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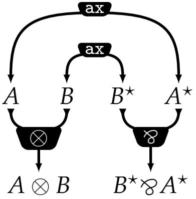

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$\hookrightarrow$ no notion of proofnet for **MLL** both canonical and low-complexity.

Multiplicative-additive logic

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A basic fragment of (intutionnistic) linear logic with

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(equates two objects of wildly different sizes)

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- Subsumed by `cut`-elimination equivalence, showed **coNP**-complete. (Mairson-Terui)

Binary Decision Slicing

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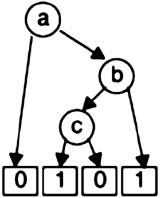

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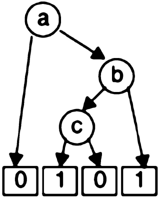

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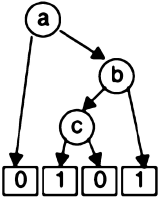

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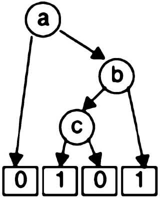

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Intermediate notion between monomial and slice nets:

To a proof π of $\Gamma \vdash A$ we associate a function $\mathcal{B}_\pi$ that maps each pair of atoms $[u,v]$ of $\Gamma \vdash A$ to a BDT $\mathcal{B}_\pi[u,v]$.

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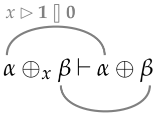
Main idea: label each $\oplus$ connectives in Γ with a boolean variable. The BDT associated to the pair $[u,v]$ tells the presence of an $[u,v]$ axiom link depending on which left/right ($\mathbf{0}$ / $\mathbf{1}$) branch of the $\oplus^\star$ rule we are sitting.

Example:

$$\pi = \dfrac{\dfrac{\alpha \vdash \alpha}{\alpha \vdash \alpha \oplus \beta} \qquad \dfrac{\beta \vdash \beta}{\beta \vdash \alpha \oplus \beta}}{\alpha \oplus_x \beta \vdash \alpha \oplus \beta}\,\oplus^\star_x \qquad \mapsto$$



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Definition (equivalence)

Define $\mathcal{B} \sim \mathcal{B}'$ as pointwise equivalence.
(for each pair of atoms $[\alpha,\beta]$, $\mathcal{B}[\alpha,\beta] \sim \mathcal{B}'[\alpha,\beta]$)

Binary Decision slicing and proof equivalence

The usefulness of this notion comes from:

Theorem

Equivalence of slicing captures proof equivalence: $\pi \sim \mu$ ***iff*** $\mathcal{B}_\pi \sim \mathcal{B}_\mu$ **.**

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An idea of the proof: for any valuation v of the variables define a slice

$$v(\mathcal{B}) = \{\ [\alpha,\beta] \mid v(\mathcal{B}[\alpha,\beta]) = \mathbf{1}\ \}$$

then show that the set of $v(\mathcal{B})$ slices is the same as the set of slices in the Hughes-van Glabbeek proofnets.

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$\hookrightarrow$ multiplicative-additive proof equivalence reduces to BDT equivalence.

Complexity

Equivalence of BDT

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The equivalence problem of BDT is in Logspace.
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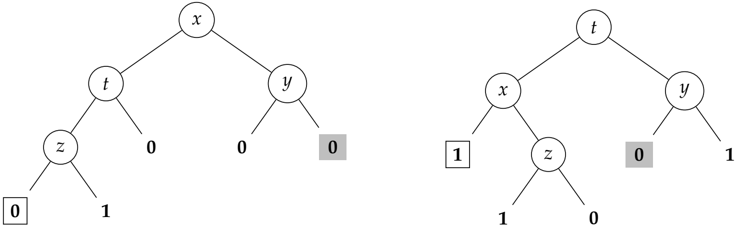

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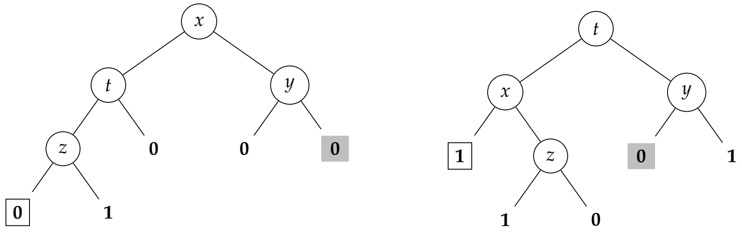

Two BDT are **not** equivalent *iff* they have compatible leafs holding opposite **0**/**1** values. Easily checked in **Logspace**.

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The equivalence problem is low-complexity (in contrast with **MLL***) and does not yield an impossibility result for proofnets. (⚠ does not solve the problem positively either, proofnets for this fragment could be impossible for other reasons)*

Hardness

Equivalence for multiplicative-additive linear logic is in **Logspace**. But is this result the best possible?

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MLL$^-$ proof equivalence is **Logspace**-hard.

Because **MLL$^-$** is a subsystem of multiplicative-additive linear logic, we get

Theorem

Proof equivalence of multiplicative-additive linear logic is Logspace-complete.

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... Thank you for your attention !