Acceleration in special relativity : What is the meaning of "uniformly accelerated movement" ?

Julien Baglio* and Matthieu Bethermin†
Physics Department, ENS Cachan
61 avenue du Président Wilson, 94235 Cachan, France
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In Newtonian mechanics, no problems arise while thinking about the meaning of an uniformly accelerated movement. Thus, if the movement is viewed in a frame $\mathcal{R}$, considering a movement with a constant acceleration causes no particular problem. In particular, it is equivalent to view it as a movement due to a constant force, according to Newton's second law. How this can be seen in Einstein's special relativity ? This article shows that the first consideration is no longer valid, while that of the second still holds. We will also present another kinematic point of view which gives exactly the same results as the dynamical one, but is unnatural considering the common sense.

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I. INTRODUCTION

Until the beginning of the XXe century, the paradigm underlying mechanics had been that of Newton : time was absolute and switching between two inertial frames should follow Galilean transformation. In a modern point of view, laws of physics were to follow an invariance under the Galilean group.

But it was not the case of Maxwell's laws of electrodynamics. Einstein, following the work of Poincaré and Lorentz, developped the special relativity to cure the paradox : laws of physics are invariant under Lorentz-Poincaré group, which means in particular that Galilean transformation have to be replaced by that of Lorentz. The most important consequence is the vanishing of absolute time.

In this article, we come back to a familiar notion : the uniformly accelerated movement. In Newtonian mechanics, an uniformly accelerated movement in a frame noted $\mathcal{R}$ means that the acceleration mesured in that frame is constant :

$$\boldsymbol{a} = \mathbf{cst} \tag{1}$$

Integrating this equation it is easiest to obtain

$$\boldsymbol{x}(t) = \frac{1}{2}\boldsymbol{a}t^2 + \boldsymbol{v_0}t + \boldsymbol{x_0} \tag{2}$$

where $\boldsymbol{v_0}$ and $\boldsymbol{x_0}$ stand for the initial velocity and position of the system. By Newton's second law of mechanics, this means that if $\mathcal{R}$ is an inertial frame, a constant action is applied on the system. In what follows, $\mathcal{R}$ will always be assumed as an inertial frame.

If we boost to the frame, associated with the system, where its speed is null, the acceleration is not the same. Adding the inertia force $-m\boldsymbol{a}$ and using the second Newtonian law, we obtain $m\boldsymbol{a'} = \boldsymbol{F} - m\boldsymbol{a}$. Thus, as $\boldsymbol{a}$ is a constant, $\boldsymbol{a'}$ remains constant and the movement is still uniformly accelerated. Applying well a Galilean transformation, we even obtain that $\boldsymbol{a'} = \mathbf{0}$, which is not a surprise after all.

II. ACCELERATION IN SPECIAL RELATIVITY

How can the work done above can be generalised in special relativity ?

1. *A naive point of view which fails*

The first idea to give sense to the idea of an uniformy accelerated movement would be holding our definition (1).

Consider a mobile in an inertial frame $\mathcal{R}$. If we consider the meaning of "uniformly accelerated movement" as $\boldsymbol{a} = \mathbf{cst}$, we therefore obtain exactly the solution (2). But a problem is immediately raised, if the speed of the mobile is considered : nothing prevents it to be greater than the speed of light, which is inconsistant with special relativity itself !

Thus we have proved that the definition (1) has to be abandonned in special relativity : the kinematic notion of "uniformly accelerated movement" seems not to hold.

2. *The dynamical meaning*

Before coming back to the kinematic point of view, let's investigate the behaviour of that of the dynamical.

In special relativity, the impulsion is defined through a quadrivector :

*Also at SLAC Theoy Group, Stanford University.
†Also at SLAC Babar Group B, Stanford University.

$$P^\mu = \frac{dU^\mu}{d\tau} = \left(\frac{E}{c}, \gamma m \boldsymbol{v}\right) \tag{3}$$

where U^μ is the speed quadrivector, γ is the Lorentz factor and τ is the proper time of the mobile studied. We then define $\boldsymbol{p} = \gamma m \boldsymbol{v}$ as the relativistic impulsion, which is equivalent to the newtonian impulsion as $v \to 0$ (i.e. $v << c$).

Then using

$$\frac{d\boldsymbol{p}}{dt} = \boldsymbol{F}$$

and using the dynamical definition of "uniformly accelerated movement", i.e. $\boldsymbol{F} = \boldsymbol{cst}$, we then obtain

$$m\frac{d\gamma \boldsymbol{v}}{dt} = m\boldsymbol{A} \tag{4}$$

where we have used $\boldsymbol{A} = \dfrac{\boldsymbol{F}}{m}$. As $\boldsymbol{A}$ is a constant, the integration of (4) is easiest :

$$\gamma \boldsymbol{v} = \boldsymbol{A}t + \gamma(0)\boldsymbol{v}(0) \tag{5}$$

In order to compare this calculus with that will follow, we assume that the initial speed is null, and we rotate our axis, in order to have $\boldsymbol{v} = v\hat{\boldsymbol{z}}$ (without loosing generality with the last restriction). We then reorganize the equation, to obtain

$$v(t) = \frac{At}{\sqrt{1 + \dfrac{A^2t^2}{c^2}}} \tag{6}$$

Thus, the dynamical meaning of "uniformly accelerated movement" still holds in special relativity.

III. HOW TO BUILD A KINETICAL MEANING WITHOUT BEING IN CONFLICT WITH SPECIAL RELATIVITY

We have seen in the previous section that the only definition which seems to hold in special relativity is the dynamical one. Here we present another kinetical definition which will be consistent with special relativity, and is equivalent to that of Newtonian in the $v << c$ limit.

Consider $\mathcal{R}$ as our inertial frame, and $\mathcal{R}'$ as the proper frame of the mobile studied. Recall that $\mathcal{R}'$ is at each time the tangential frame of the mobile, and thus is constantly changing. We will restrict ourselves to a privileged axis $\hat{\boldsymbol{z}}$. Then

Definition 1 *A mobile has a "uniformly accelerated movement" when a' = cst*

It means that at every moment, while boosting to the proper frame of the mobile, the value of a' is the same. Using Lorentz transformation (A.2) with $v = V$ it gives

$$a' = A = \frac{a}{\gamma}\frac{1}{\left(1 - \dfrac{v^2}{c^2}\right)^2}$$

Using the expression of the γ-factor, it reduces to

$$a' = A = \gamma^3 a = \gamma^3 \frac{dv}{dt} \tag{7}$$

Integrate equation (7) is done with the separation of variables method, and recalling that

$$\frac{dv}{\left(1 - \dfrac{v^2}{c^2}\right)^{3/2}} = d\left(\frac{v}{\sqrt{1 - \dfrac{v^2}{c^2}}}\right)$$

We then obtain

$$At = \frac{v}{\sqrt{1 - \dfrac{v^2}{c^2}}} + \text{cst} \tag{8}$$

With the initial condition $v(0) = 0$ and reorganizing the equation :

$$v(t) = \frac{At}{\sqrt{1 + \dfrac{A^2t^2}{c^2}}} \tag{9}$$

which is the same as (6).

Thus we have demonstrated that it is possible to hold a kinematical point of view in the meaning of "uniformly accelerated movement". But it is necessary to boost at every moment to the proper frame in order to have a correct definition.

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APPENDIX: LORENTZ TRANSFORMATION OF THE ACCELERATION

We present there the Lorentz transformation of the acceleration. Let's take an inertial frame $\mathcal{R}$ and another inertial frame $\mathcal{R}'$ in uniform rectilign translation at constant speed V along the $z-axis$ in $\mathcal{R}$-frame. Then with $\gamma = \frac{1}{\sqrt{1-\frac{V^2}{c^2}}}$, we have

$$
\begin{aligned}
a'_{x'} &= \frac{dv'_{x'}}{dt'} = a_x \\
a'_{y'} &= \frac{dv'_{y'}}{dt'} = a_y \\
a'_{z'} &= \frac{dv'_{z'}}{dt'} = \frac{a_z}{\gamma}\left(\frac{1}{\left(1-\frac{Vv_z}{c^2}\right)^2} + \frac{V(v_z-V)}{c^2\left(1-\frac{Vv_z}{c^2}\right)^3}\right)
\end{aligned}
\tag{A.1}
$$

[1] L. D. Landau and E. Lifshitz, *Classical Field Theory* (1964).

[2] FuturaSciences (2007), forums.futura-sciences.com/thread147381.html.