

THE GROUPOIDAL ANALOGUE $\tilde{\Theta}$ TO JOYAL'S CATEGORY Θ IS A TEST CATEGORY

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ABSTRACT. We introduce the groupoidal analogue $\tilde{\Theta}$ to Joyal's cell category Θ and we prove that $\tilde{\Theta}$ is a strict test category in the sense of Grothendieck. This implies that presheaves on $\tilde{\Theta}$ model homotopy types in a canonical way. We also prove that the canonical functor from Θ to $\tilde{\Theta}$ is aspherical, again in the sense of Grothendieck. This allows us to compare weak equivalences of presheaves on $\tilde{\Theta}$ to weak equivalences of presheaves on Θ . Our proofs apply to other categories analogous to Θ .

1. INTRODUCTION

In *Pursuing Stacks*, Grothendieck defines a notion of weak ∞ -groupoid (see [11], [16], [2] and [17]) and conjectures that his weak ∞ -groupoids model homotopy types in a precise way. His definition of weak ∞ -groupoids is based on the notion of coherator. Roughly speaking, a coherator is a category encoding the algebraic theory of weak ∞ -groupoids. If C is a coherator, a weak ∞ -groupoid (of type C) is a presheaf on C satisfying some left exactness condition. A first step toward Grothendieck's conjecture would thus be to prove that presheaves on a coherator (without the exactness condition) model homotopy types.

This is where test categories enter the picture. Test categories were introduced by Grothendieck in [11] (see also [15] and [9]). The main property of these categories is that presheaves on a test category canonically model homotopy types. Therefore, to prove Grothendieck's conjecture, it is reasonable to start by trying to prove that every coherator is a test category. In [16], Maltsiniotis gave a series of conjectures implying Grothendieck's conjecture based on this idea.

Conjecturally, every coherator is (non canonically) endowed with a "forgetful" functor to $\tilde{\Theta}$. This is the reason why we are interested in understanding the homotopy theory of $\tilde{\Theta}$. In this article, we prove that $\tilde{\Theta}$ is a test category. Our hope is that we will be able to deduce that every coherator C is a test category from this result, using properties of the functor from C to $\tilde{\Theta}$.

As announced in the title, $\tilde{\Theta}$ is the groupoidal analogue to Joyal's cell category Θ . The category Θ was introduced by Joyal in [13] as the opposite category of the category of so-called finite disks. Batanin and Street conjectured in [6] that Θ could be seen as a

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full subcategory of the category of strict ∞ -categories. This was proved independently by Makkai and Zawadowski in [14] and by Berger in [7]. In the latter article, Berger also gave a nice combinatorial description of Θ . In our paper, the category Θ is introduced as the universal categorical globular extension. Roughly speaking, a categorical globular extension is a category endowed with operations dual to those of strict ∞ -categories and satisfying axioms dual to those of strict ∞ -categories. We show, starting from our definition, that Θ can be seen as the full subcategory of the category of strict ∞ -categories whose objects are free strict ∞ -categories on globular pasting schemes. This implies, using the result of Berger, Makkai and Zawadowski, that our category Θ is canonically isomorphic to Joyal's cell category.

The category $\tilde{\Theta}$ is defined in the same way by replacing strict ∞ -categories by strict ∞ -groupoids. In our terminology, $\tilde{\Theta}$ is the universal groupoidal globular extension. We prove that $\tilde{\Theta}$ can be seen as the full subcategory of the category of strict ∞ -groupoids whose objects are free strict ∞ -groupoids on globular pasting schemes. To our knowledge, there is no known combinatorial description of $\tilde{\Theta}$ analogous to Berger's description of Θ .

In [10], Cisinski and Maltsiniotis introduced the notion of décalage and used it to prove that Θ is a test category. They actually proved that Θ is a strict test category: a test category A is strict if the binary product of presheaves on A is compatible with the product of homotopy types in a strong sense. They proved that a small category satisfying some easy to check condition, plus the existence of a splittable décalage, is a strict test category. They then constructed a splittable décalage \mathcal{D}_Θ on Θ and applied this result to Θ . To construct this décalage, they used a beautiful description of Θ in terms of wreath products due to Berger (see [8]). Another proof of the fact that Θ is a test category is given in Section 7.2 of the PhD thesis [2] of the author.

Unfortunately, the category $\tilde{\Theta}$ cannot be obtained using wreath products and therefore the construction of the décalage \mathcal{D}_Θ does not apply to $\tilde{\Theta}$. In this article, we construct a splittable décalage $\mathcal{D}_{\tilde{\Theta}}$ on $\tilde{\Theta}$ "by hand". The formulas defining $\mathcal{D}_{\tilde{\Theta}}$ are inspired by the ones one can get by unfolding the definition of \mathcal{D}_Θ . In particular, the construction of our décalage $\mathcal{D}_{\tilde{\Theta}}$ will also apply to Θ and, in this case, we will get the décalage \mathcal{D}_Θ of [10].

We deduce from the existence of the splittable décalage $\mathcal{D}_{\tilde{\Theta}}$ that $\tilde{\Theta}$ is a strict test category. By a theorem of Cisinski, conjectured by Grothendieck, this implies that the category of presheaves on $\tilde{\Theta}$ is endowed with a model category structure whose homotopy category is the homotopy category of CW-complexes. There exists a canonical functor from Θ to $\tilde{\Theta}$. Using the fact that this functor is compatible with the décalages \mathcal{D}_Θ and $\mathcal{D}_{\tilde{\Theta}}$, we deduce that it is aspherical in the sense of Grothendieck. This implies that this functor induces a Quillen equivalence between the Grothendieck-Cisinski model category structures on presheaves on Θ and on $\tilde{\Theta}$. Note that the Grothendieck-Cisinski model structure on presheaves on Θ had already been obtained by Berger in [7] using topological techniques.

Moreover, our construction applies to other categories having similar universal properties. For instance, the category Θ_{lr} , which has a universal property related to "strict ∞ -categories not necessarily satisfying the axiom of functoriality of units", is also a strict test category and the canonical functor from Θ_{lr} to Θ is aspherical.

Most of the content of this article is extracted from the last chapter of the PhD thesis [2] of the author. The calculations have been entirely rewritten “using elements” (see paragraph 5.3 for details).

Our paper is organized as follows. In Section 2, we recall the definitions of strict ∞ -categories and strict ∞ -groupoids. We introduce the globular language and in particular globular sums and globular extensions. We also define the notion of categorical and groupoidal globular extensions, which are in a sense dual to those of strict ∞ -categories and strict ∞ -groupoids. In Section 3, we introduce the categories Θ_0 , Θ and $\tilde{\Theta}$. In Section 4, we give a brief introduction to the theory of test categories and we gather the definitions and results from [10] about décalages that we will need. We then enter the heart of the article. In Section 5, we explain how to construct a new globular extension, the twisted globular extension, from a globular extension endowed with some comultiplications. In Section 6, we apply this construction to a groupoidal globular extension and we show that the twisted globular extension is endowed with a structure of groupoidal globular extension. In Section 7, we use the results of Section 6 to build our décalage $\mathcal{D}_{\tilde{\Theta}}$. We show that $\mathcal{D}_{\tilde{\Theta}}$ is splittable. In the final Section, we draw the consequences of the previous Sections. We show that $\tilde{\Theta}$ is a strict test category and that the functor from Θ to $\tilde{\Theta}$ is aspherical. We explain how these results generalize to other analogous categories.

If C is a category, we will denote by C^o the opposite category. If

$$\begin{array}{ccccccc} X_1 & & X_2 & & \cdots & & X_n \\ & \searrow f_1 & & \swarrow g_1 \quad \searrow f_2 & & & \swarrow g_{n-1} \\ & & Y_1 & & \cdots & & Y_{n-1} \end{array}$$

is a diagram in C , we will denote by

$$(X_1, f_1) \times_{Y_1} (g_1, X_2, f_2) \times_{Y_2} \cdots \times_{Y_{n-1}} (g_{n-1}, X_n)$$

its projective limit. Dually, we will denote by

$$(X_1, f_1) \amalg_{Y_1} (g_1, X_2, f_2) \amalg_{Y_2} \cdots \amalg_{Y_{n-1}} (g_{n-1}, X_n)$$

the inductive limit of the corresponding diagram in C^o .

2. STRICT ∞ -CATEGORIES AND STRICT ∞ -GROUPOIDS

2.1. We will denote by \mathbb{G} the *globular category*, that is the category generated by the graph

$$D_0 \begin{array}{c} \xrightarrow{\sigma_1} \\ \xrightarrow{\tau_1} \end{array} D_1 \begin{array}{c} \xrightarrow{\sigma_2} \\ \xrightarrow{\tau_2} \end{array} \cdots \begin{array}{c} \xrightarrow{\sigma_{i-1}} \\ \xrightarrow{\tau_{i-1}} \end{array} D_{i-1} \begin{array}{c} \xrightarrow{\sigma_i} \\ \xrightarrow{\tau_i} \end{array} D_i \begin{array}{c} \xrightarrow{\sigma_{i+1}} \\ \xrightarrow{\tau_{i+1}} \end{array} \cdots$$

and the coglobular relations

$$\sigma_{i+1}\sigma_i = \tau_{i+1}\sigma_i \quad \text{and} \quad \sigma_{i+1}\tau_i = \tau_{i+1}\tau_i, \quad i \geq 1.$$

For $i \geq j \geq 0$, we will denote by σ_j^i and τ_j^i the morphisms from D_j to D_i defined by

$$\sigma_j^i = \sigma_i \cdots \sigma_{j+2} \sigma_{j+1} \quad \text{and} \quad \tau_j^i = \tau_i \cdots \tau_{j+2} \tau_{j+1}.$$

A *globular set* or ∞ -*graph* is a presheaf on \mathbb{G} . The datum of a globular set X amounts to the datum of a diagram of sets

$$\cdots \xrightarrow[t_{i+1}]{s_{i+1}} X_i \xrightarrow[t_i]{s_i} X_{i-1} \xrightarrow[t_{i-1}]{s_{i-1}} \cdots \xrightarrow[t_2]{s_2} X_1 \xrightarrow[t_1]{s_1} X_0$$

satisfying the globular relations

$$s_i s_{i+1} = s_i t_{i+1} \quad \text{and} \quad t_i s_{i+1} = t_i t_{i+1}, \quad i \geq 1.$$

For $i \geq j \geq 0$, we will denote by s_j^i and t_j^i the maps from X_i to X_j defined by

$$s_j^i = s_{j+1} \cdots s_{i-1} s_i \quad \text{and} \quad t_j^i = t_{j+1} \cdots t_{i-1} t_i.$$

A *morphism of globular sets* is a morphism of presheaves on \mathbb{G} .

2.2. An ∞ -*precategory* is a globular set X endowed with maps

$$\begin{aligned} *_{j}^i &: (X_i, s_j^i) \times_{X_j} (t_j^i, X_i) \rightarrow X_i, \quad i > j \geq 0, \\ k_i &: X_i \rightarrow X_{i+1}, \quad i \geq 0, \end{aligned}$$

such that

(1) for every (u, v) in $(X_i, s_j^i) \times_{X_j} (t_j^i, X_i)$ with $i > j \geq 0$, we have

$$s_i(u *_{j}^i v) = \begin{cases} s_i(v), & j = i - 1, \\ s_i(u) *_{j}^{i-1} s_i(v), & j < i - 1, \end{cases}$$

and

$$t_i(u *_{j}^i v) = \begin{cases} t_i(u), & j = i - 1, \\ t_i(u) *_{j}^{i-1} t_i(v), & j < i - 1; \end{cases}$$

(2) for every u in X_i with $i \geq 0$, we have

$$s_{i+1} k_i(u) = u = t_{i+1} k_i(u).$$

For $i \geq j \geq 0$, we will denote by k_i^j the map from $X_j \rightarrow X_i$ defined by

$$k_i^j = k_{i-1} \cdots k_{j+1} k_j.$$

A *morphism of ∞ -precategories* is a morphism of globular sets between ∞ -precategories which is compatible with the $*_{j}^i$'s and the k_i 's in an obvious way.

An ∞ -precategory X is a *strict ∞ -category* if it satisfies the following axioms:

- (Ass $_{i,j}$), $i > j \geq 0$,
for every (u, v, w) in $(X_i, s_j^i) \times_{X_j} (t_j^i, X_i, s_j^i) \times_{X_j} (t_j^i, X_i)$, we have

$$(u *_{j}^i v) *_{j}^i w = u *_{j}^i (v *_{j}^i w);$$

- (Exc $_{i,j,k}$), $i > j > k \geq 0$,
for every (u, u', v, v') in

$$(X_i, s_j^i) \times_{X_j} (t_j^i, X_i, s_k^i) \times_{X_k} (t_k^i, X_i, s_j^i) \times_{X_j} (t_j^i, X_i),$$

we have

$$(u *_{j}^i u') *_{k}^i (v *_{j}^i v') = (u *_{k}^i v) *_{j}^i (u' *_{k}^i v');$$

- (LUnit $_{i,j}$), $i > j \geq 0$,
for every u in X_i , we have

$$k_i^j t_j^i(u) *_{j}^i u = u;$$

- (RUnit $_{i,j}$), $i > j \geq 0$,
for every u in X_i , we have

$$u *_{j}^i k_i^j s_j^i(u) = u;$$

- (FUnit $_{i,j}$), $i > j \geq 0$,
for every (u, v) in $(X_i, s_j^i) \times_{X_j} (t_j^i, X_i)$, we have

$$k_i(u *_{j}^i v) = k_i(u) *_{j}^{i+1} k_i(v).$$

The *category of strict ∞ -categories* is the full subcategory of the category of ∞ -pre-categories whose objects are strict ∞ -categories. We will denote it by $\infty\text{-Cat}$.

2.3. An *∞ -pregroupoid* X is an ∞ -precategory endowed with maps

$$w_j^i : X_i \rightarrow X_i, \quad i > j \geq 0,$$

such that for every u in X_i for $i \geq 1$ and j such that $i > j \geq 0$, we have

$$s_i(w_j^i(u)) = \begin{cases} t_i(u), & j = i - 1, \\ w_j^{i-1}(s_i(u)), & j < i - 1, \end{cases}$$

and

$$t_i(w_j^i(u)) = \begin{cases} s_i(u), & j = i - 1, \\ w_j^{i-1}(t_i(u)), & j < i - 1. \end{cases}$$

A *morphism of ∞ -pregroupoids* is a morphism of ∞ -precategories between ∞ -groupoids which is compatible with the w_j^i 's in an obvious way.

An ∞ -pregroupoid X is a *strict ∞ -groupoid* if it is a strict ∞ -category and if it satisfies the following axioms:

- (LInv $_{i,j}$), $i > j \geq 0$,
for every u in X_i , we have

$$w_j^i(u) *_{j}^i u = k_i^j(s_j^i(u));$$

- (RInv $_{i,j}$), $i > j \geq 0$,
for every u in X_i , we have

$$u *_{j}^i w_j^i(u) = k_i^j(t_j^i(u)).$$

The *category of strict ∞ -groupoids* is the full subcategory of the category of ∞ -pregroupoids whose objects are strict ∞ -groupoids. We will denote it by $\infty\text{-Grpd}$. Note that a morphism of strict ∞ -categories between strict ∞ -groupoids is automatically a morphism of strict ∞ -groupoids.

Although it is not clear from our definition, being a strict ∞ -groupoid is a property of a strict ∞ -category. More precisely, if X is a strict ∞ -category such that for every $i > j \geq 0$, every i -arrow u admits a $*_{j}^i$ -inverse (i.e., an i -arrow $w_j^i(u)$ satisfying the axioms (LInv $_{i,j}$) and (RInv $_{i,j}$)), then X is endowed with a unique structure of ∞ -groupoid. Note also that our axioms for strict ∞ -groupoids are highly redundant. For instance, for a

strict ∞ -category to be a strict ∞ -groupoids, it suffices to ask for $*_0^i$ -inverses or for $*_{i-1}^i$ -inverses for every $i \geq 1$ (see Proposition 2.3 of [3]).

One can easily check that a strict ∞ -groupoid automatically satisfies the following additional axiom:

- (FInv $_{i,j,j'}$), $i > j, j' \geq 0$,
for every (u, v) in $(X_i, s_j^i) \times_{X_j} (t_j^i, X_i)$, we have

$$w_{j'}^i(u *_j^i v) = \begin{cases} w_{j'}^i(v) *_j^i w_{j'}^i(u), & j = j', \\ w_{j'}^i(u) *_j^i w_{j'}^i(v), & j \neq j'. \end{cases}$$

More precisely, if an ∞ -pregroupoid satisfies Axioms (Ass), (Exc), (LUnit), (RUnit) and (RInv), then it satisfies Axiom (FInv) (where by the name of an axiom without subscripts, we denote the conjunction on all meaningful subscripts of this axiom).

3. THE CATEGORIES Θ_0 , Θ AND $\tilde{\Theta}$

3.1. Let n be a positive integer. A *table of dimensions* of width n is the datum of integers $i_1, \dots, i_n, i'_1, \dots, i'_{n-1}$ such that

$$i_k > i'_k \quad \text{and} \quad i_{k+1} > i'_k, \quad 1 \leq k \leq n-1.$$

We will denote such a table of dimensions by

$$\begin{pmatrix} i_1 & i_2 & \cdots & i_n \\ i'_1 & i'_2 & \cdots & i'_{n-1} \end{pmatrix}.$$

Let (C, F) be a category under \mathbb{G} , that is a category C endowed with a functor $F : \mathbb{G} \rightarrow C$. We will denote in the same way the objects and morphisms of \mathbb{G} and their image by the functor F . Let

$$T = \begin{pmatrix} i_1 & i_2 & \cdots & i_n \\ i'_1 & i'_2 & \cdots & i'_{n-1} \end{pmatrix}$$

be a table of dimensions. The *globular sum* in C associated to T (if it exists) is the iterated amalgamated sum

$$(D_{i_1}, \sigma_{i'_1}^{i_1}) \amalg_{D_{i'_1}} (\tau_{i'_1}^{i_2}, D_{i_2}, \sigma_{i'_2}^{i_2}) \amalg_{D_{i'_2}} \cdots \amalg_{D_{i'_{n-1}}} (\tau_{i'_{n-1}}^{i_n}, D_{i_n})$$

in C . We will denote it briefly by

$$D_{i_1} \amalg_{D_{i'_1}} D_{i_2} \amalg_{D_{i'_2}} \cdots \amalg_{D_{i'_{n-1}}} D_{i_n}.$$

If the table of dimensions T is understood, for k such that $1 \leq k \leq n$, we will denote by ε_k the canonical morphism

$$\varepsilon_k : D_{i_k} \rightarrow D_{i_1} \amalg_{D_{i'_1}} D_{i_2} \amalg_{D_{i'_2}} \cdots \amalg_{D_{i'_{n-1}}} D_{i_n}.$$

The pair (C, F) is said to be a *globular extension* if for every table of dimensions T (of any width), the globular sum associated to T exists in C . We will often say, by abuse of language, that C is a globular extension.

Let C and D be two globular extensions. A *morphism of globular extensions* from C to D is a functor from C to D under \mathbb{G} (that is such that the triangle

$$\begin{array}{ccc} & \mathbb{G} & \\ & \swarrow & \searrow \\ C & \longrightarrow & D \end{array}$$

is commutative) which respects globular sums. We will also call such a functor a *globular functor*.

If C is a category under \mathbb{G} and D is a category, we will denote by $\underline{\mathbf{Hom}}_{\text{gl}}(C, D)$ the full subcategory of the category $\underline{\mathbf{Hom}}(C, D)$ of functors from C to D , whose objects are functors $C \rightarrow D$ such that $(D, \mathbb{G} \rightarrow C \rightarrow D)$ is a globular extension. In particular, when $C = \mathbb{G}$ (and $\mathbb{G} \rightarrow \mathbb{G}$ is the identity functor), the objects of $\underline{\mathbf{Hom}}_{\text{gl}}(\mathbb{G}, D)$ are the globular extension structures on D . We will also denote this category by $\text{Ext}_{\text{gl}}(D)$.

Let C be a globular extension. A *model* of C or *globular presheaf* on C is a presheaf $G : C^{\circ} \rightarrow \mathbf{Set}$ which respects globular products (i.e., limits dual to globular sums). We will denote by $\text{Mod}(C)$ the full subcategory of the category \hat{C} of presheaves on C whose objects are globular presheaves.

Proposition 3.2 (Universal property of Θ_0). *There exists a globular extension Θ_0 such that for every category C , the precomposition by the functor $\mathbb{G} \rightarrow \Theta_0$ induces an equivalence of categories*

$$\underline{\mathbf{Hom}}_{\text{gl}}(\Theta_0, C) \rightarrow \text{Ext}_{\text{gl}}(C).$$

Moreover, for every such Θ_0 , this equivalence of categories is surjective on objects.

Proof. Consider the Yoneda functor $\mathbb{G} \rightarrow \hat{\mathbb{G}}$. For each table of dimensions T , choose a globular sum S_T in $\hat{\mathbb{G}}$. Let Θ_0 be the full subcategory of $\hat{\mathbb{G}}$ whose objects are the S_T 's.

Before proving that Θ_0 has the desired universal property, let us introduce some notations. If A is a category, we will denote by \tilde{A} the category of copresheaves on A , that is the category $\underline{\mathbf{Hom}}(A, \mathbf{Set})^{\circ}$. If B is a second category, we will denote by $\underline{\mathbf{Hom}}_1(A, B)$ the full subcategory of $\underline{\mathbf{Hom}}(A, B)$ whose objects are functors preserving inductive limits.

Let C be a category. We will construct a quasi-inverse to the canonical functor

$$U : \underline{\mathbf{Hom}}_{\text{gl}}(\Theta_0, C) \rightarrow \text{Ext}_{\text{gl}}(C).$$

Let

$$U' : \underline{\mathbf{Hom}}_1(\hat{\mathbb{G}}, \tilde{C}) \rightarrow \underline{\mathbf{Hom}}(\mathbb{G}, \tilde{C})$$

be the functor induced by the Yoneda functor $\mathbb{G} \rightarrow \hat{\mathbb{G}}$. Since the category \tilde{C} is cocomplete, the universal property of $\hat{\mathbb{G}}$ gives us a quasi-inverse L' of U' . Consider now the functor G defined by the composition

$$\text{Ext}_{\text{gl}}(C) \rightarrow \underline{\mathbf{Hom}}(\mathbb{G}, \tilde{C}) \xrightarrow{L'} \underline{\mathbf{Hom}}_1(\hat{\mathbb{G}}, \tilde{C}) \rightarrow \underline{\mathbf{Hom}}(\Theta_0, \tilde{C}),$$

where the first and the last functors are respectively induced by the (contravariant) Yoneda functor $C \rightarrow \tilde{C}$ and the inclusion $\Theta_0 \rightarrow \hat{\mathbb{G}}$. Since the Yoneda functor $C \rightarrow \tilde{C}$ preserves inductive limits, the functor G factors through $\underline{\mathbf{Hom}}_{\text{gl}}(\Theta_0, C)$ and gives rise to a functor

$$L : \text{Ext}_{\text{gl}}(C) \rightarrow \underline{\mathbf{Hom}}_{\text{gl}}(\Theta_0, C).$$

One easily checks that L is a quasi-inverse of U .

Since the second assertion is invariant under equivalences of categories under \mathbb{G} , it suffices to prove it for the category Θ_0 we have just built. The assertion then follows from the fact that the functor U' is surjective on objects. \square

3.3. Two globular extensions satisfying the above universal property are uniquely equivalent up to a unique natural isomorphism. One can show that the objects of such a globular extension have no automorphisms. In particular, a skeletal version of such a globular extension (i.e., such that isomorphic objects are equal) is unique up to a unique isomorphism. We will denote by Θ_0 this globular extension.

Note that the above universal property states in particular that Θ_0 is the free globular completion of \mathbb{G} in the following sense: if $(C, F : \mathbb{G} \rightarrow C)$ is a globular extension, there exists a globular functor $\Theta_0 \rightarrow C$ unique up to a unique natural isomorphism. More precisely, the choice of such a functor $\Theta_0 \rightarrow C$ amounts to the choice of a globular sum for every table of dimensions.

The category Θ_0 defined above is canonically isomorphic to the category Θ_0 defined in terms of finite planar rooted trees by Berger in [7]. Berger's definition is explained in detail in Section 2.3 of [2]. See also Section 4 of [18] or [14] for a description of the bijection between tables of dimensions and finite planar rooted trees. Note that tables of dimensions are called zig-zag sequences in [18] and ud-vectors (standing for up and down vectors) in [14].

3.4. Let C be a globular extension. If X is a globular presheaf on C , then by restricting it to \mathbb{G} , we obtain a globular set. We thus have a canonical functor

$$\text{Mod}(C) \rightarrow \widehat{\mathbb{G}}.$$

Proposition 3.5. *The functor*

$$\text{Mod}(\Theta_0) \rightarrow \widehat{\mathbb{G}}$$

is an equivalence of categories.

Proof. This is exactly what the universal property of Θ_0 claims when applied to Set° . \square

3.6. If C is a globular extension, the Yoneda functor $C \rightarrow \widehat{C}$ factors through $\text{Mod}(C)$. We thus have a functor $C \rightarrow \text{Mod}(C)$.

By the previous proposition, the functor

$$\Theta_0 \rightarrow \text{Mod}(\Theta_0) \rightarrow \widehat{\mathbb{G}}$$

is fully faithful. A globular set which is in the image of this functor will be called a *globular pasting scheme*. We can thus view Θ_0 as the full subcategory of $\widehat{\mathbb{G}}$ whose objects are globular pasting schemes.

Note that in the bijection between tables of dimensions and finite planar rooted trees, the above functor from Θ_0 to $\widehat{\mathbb{G}}$ associates to a tree T the globular set T^* introduced by Batanin in [5]. The globular pasting schemes can also be characterized as the cardinals (in the sense of Definition 4.16 of [19]) of the free strict ∞ -category functor $\widehat{\mathbb{G}} \rightarrow \infty\text{-Cat}$ (see Section 9 of [18]).

3.7. A *globular extension under Θ_0* is a category C endowed with a functor $\Theta_0 \rightarrow C$ such that $(C, \mathbb{G} \rightarrow \Theta_0 \rightarrow C)$ is a globular extension. If C is a globular extension under Θ_0 , the globular sum associated to a table of dimensions is uniquely defined. A *morphism of*

globular extensions under Θ_0 is a functor under Θ_0 between globular extensions under Θ_0 . Note that such a functor automatically respects globular sums.

If C is a category under Θ_0 and D is a category, we will denote by $\underline{\mathbf{Hom}}_{\text{gl}_0}(C, D)$ the full subcategory of the category $\underline{\mathbf{Hom}}(C, D)$ whose objects are functors $C \rightarrow D$ such that $(D, \Theta_0 \rightarrow C \rightarrow D)$ is a globular extension under Θ_0 .

Proposition 3.8. *Let C be a category under Θ_0 . There exists a globular extension \overline{C} under Θ_0 , endowed with a functor $C \rightarrow \overline{C}$ under Θ_0 such that the functor $C \rightarrow \overline{C}$ induces an isomorphism of categories*

$$\underline{\mathbf{Hom}}_{\text{gl}_0}(\overline{C}, D) \rightarrow \underline{\mathbf{Hom}}_{\text{gl}_0}(C, D).$$

Proof. This is a special case of a standard categorical construction (see Proposition 3 of [4]). See also Section 2.6 of [2] and paragraph 3.10 of [17]. \square

3.9. If C is a category under Θ_0 , the globular extension \overline{C} of the previous proposition (which is unique up to a unique isomorphism) will be called the *globular completion* of C . Note that the functor $C \rightarrow \overline{C}$ is bijective on objects.

3.10. A *precategorical globular extension* is a globular extension C under Θ_0 endowed with morphisms

$$\begin{aligned} \nabla_j^i : D_i &\rightarrow D_i \amalg_{D_j} D_i, & i > j \geq 0, \\ \kappa_i : D_{i+1} &\rightarrow D_i, & i \geq 0, \end{aligned}$$

such that

(1) for every i, j such that $i > j \geq 0$, we have

$$\nabla_j^i \sigma_i = \begin{cases} \varepsilon_2 \sigma_i, & j = i - 1, \\ (\sigma_i \amalg_{D_j} \sigma_i) \nabla_j^{i-1} & j < i - 1, \end{cases}$$

and

$$\nabla_j^i \tau_i = \begin{cases} \varepsilon_1 \tau_i, & j = i - 1, \\ (\tau_i \amalg_{D_j} \tau_i) \nabla_j^{i-1} & j < i - 1, \end{cases}$$

where $\varepsilon_1, \varepsilon_2 : D_i \rightarrow D_i \amalg_{D_{i-1}} D_i$ denote the canonical morphisms;

(2) for every $i \geq 0$, we have

$$\kappa_i \sigma_{i+1} = 1_{D_i} \quad \text{and} \quad \kappa_i \tau_{i+1} = 1_{D_i}.$$

If C is a precategorical globular extension, for $i \geq j \geq 0$, we will denote by κ_i^j the morphism from D_i to D_j defined by

$$\kappa_i^j = \kappa_j \dots \kappa_{i-2} \kappa_{i-1},$$

and, for $i > 0$, we set

$$\nabla_i = \nabla_{i-1}^i.$$

A *morphism of precategorical globular extensions* is a morphism of globular extensions under Θ_0 between precategorical globular extensions preserving the ∇_j^i 's and the κ_i 's.

A precategorical globular extension is *categorical* if it satisfies the following axioms:

- (Ass_{*i,j*}), $i > j \geq 0$,
the following square commutes:

$$\begin{array}{ccc}
 D_i & \xrightarrow{\nabla_j^i} & D_i \amalg_{D_j} D_i \\
 \nabla_j^i \downarrow & & \downarrow 1_{D_i} \amalg_{D_j} \nabla_j^i \\
 D_i \amalg_{D_j} D_i & \xrightarrow{\nabla_j^i \amalg_{D_j} 1_{D_i}} & D_i \amalg_{D_j} D_i \amalg_{D_j} D_i \quad ;
 \end{array}$$

- (Exc_{*i,j,k*}), $i > j > k \geq 0$,
the following diagram commutes:

$$\begin{array}{ccc}
 & D_i & \\
 \nabla_k^i \swarrow & & \searrow \nabla_j^i \\
 D_i \amalg_{D_k} D_i & & D_i \amalg_{D_j} D_i \\
 \nabla_j^i \amalg_{D_k} \nabla_j^i \downarrow & & \downarrow \nabla_k^i \amalg_{\nabla_j^i} \nabla_k^i \\
 (D_i \amalg_{D_j} D_i) \amalg_{D_k} (D_i \amalg_{D_j} D_i) & \simeq & (D_i \amalg_{D_k} D_i) \amalg_{D_j \amalg_{D_k} D_j} (D_i \amalg_{D_k} D_i),
 \end{array}$$

where the left amalgamated sum is

$$(D_i \amalg_{D_k} D_i, \sigma_j^i \amalg_{D_k} \sigma_j^i) \amalg_{D_j \amalg_{D_k} D_j} (\tau_j^i \amalg_{D_k} \tau_j^i, D_i \amalg_{D_k} D_i) \quad ;$$

- (LUnit_{*i,j*}), $i > j \geq 0$,
the following triangle commutes:

$$\begin{array}{ccc}
 D_i & & \\
 \nabla_j^i \downarrow & \searrow \sim & \\
 D_i \amalg_{D_j} D_i & \xrightarrow{\kappa_i^j \amalg_{D_j} 1_{D_i}} & D_j \amalg_{D_j} D_i \quad ;
 \end{array}$$

- (RUnit_{*i,j*}), $i > j \geq 0$,
the following triangle commutes:

$$\begin{array}{ccc}
 D_i & & \\
 \nabla_j^i \downarrow & \searrow \sim & \\
 D_i \amalg_{D_j} D_i & \xrightarrow{1_{D_i} \amalg_{D_j} \kappa_i^j} & D_i \amalg_{D_j} D_j \quad ;
 \end{array}$$

- (FUnit $_{i,j}$), $i > j \geq 0$,
the following square commutes:

$$\begin{array}{ccc}
 D_{i+1} & \xrightarrow{\nabla_j^{i+1}} & D_{i+1} \amalg_{D_j} D_{i+1} \\
 \kappa_i \downarrow & & \downarrow \kappa_i \amalg_{D_j} \kappa_i \\
 D_i & \xrightarrow{\nabla_j^i} & D_i \amalg_{D_j} D_i \quad .
 \end{array}$$

The *category of categorical globular extensions* is the full subcategory of the category of precategoryal globular extensions whose objects are categorical globular extensions.

If C is a category, we will denote by $\text{Ext}_{\text{cat}}(C)$ the category whose objects are categorical globular extension structures on C , i.e., functors from Θ_0 to C endowed with ∇_j^i 's and κ_i 's making C a categorical globular extension, and whose morphisms are natural transformations.

Proposition 3.11 (Universal property of Θ). *There exists a categorical globular extension Θ such that for every category C , the precomposition by the functor $\Theta_0 \rightarrow \Theta$ induces an isomorphism of categories*

$$\underline{\text{Hom}}_{\text{gl}_0}(\Theta, C) \rightarrow \text{Ext}_{\text{cat}}(C).$$

Proof. Let Θ_{pcat} be the globular completion of the category obtained from Θ_0 by formally adjoining morphisms κ_i and ∇_j^i satisfying the relations of precategoryal globular extensions.

Let now Θ be the globular completion of the category obtained from Θ_{pcat} by formally imposing the commutativity of the diagrams appearing in the definition of categorical globular extensions.

It is clear that Θ has the desired universal property. □

3.12. We will denote by Θ the categorical globular extension of the previous proposition (which is unique up to a unique isomorphism). We will see that this category is canonically isomorphic to Joyal's cell category introduced in [13]. Note that the functor $\Theta_0 \rightarrow \Theta$ is bijective on objects.

3.13. Let C be a categorical globular extension. If X is a globular presheaf on C , the globular set obtained by restricting X to \mathbb{G} is canonically endowed with a structure of strict ∞ -category whose compositions are the

$$*_j^i = X(\nabla_j^i) : X_i \times_{X_j} X_i \rightarrow X_i, \quad i > j \geq 0,$$

and whose units are the

$$k_i = X(\kappa_i) : X_i \rightarrow X_{i+1}, \quad i > 0.$$

We thus have a canonical functor

$$\text{Mod}(C) \rightarrow \infty\text{-Cat}.$$

Proposition 3.14. *The functor*

$$\mathrm{Mod}(\Theta) \rightarrow \infty\text{-Cat}$$

is an equivalence of categories.

Proof. This is an immediate consequence of the universal property of Θ applied to $\mathcal{S}et^o$ and of Proposition 3.5. \square

Proposition 3.15. *The functor*

$$\Theta \rightarrow \mathrm{Mod}(\Theta) \rightarrow \infty\text{-Cat}$$

identifies Θ with the full subcategory of $\infty\text{-Cat}$ whose objects are free strict ∞ -categories on globular pasting schemes.

Proof. By the previous proposition, this functor is fully faithful. It thus suffices to describe its image.

If C is a globular extension, we will denote by

$$C \xrightarrow{i_C} \mathrm{Mod}(C) \xrightarrow{j_C} \widehat{C}$$

the canonical decomposition of the Yoneda functor. By Propositions 1.27 and 1.51 of [1], the functor i_C admits a left adjoint r_C .

Let now $u : C \rightarrow D$ be a morphism of globular extensions. Denote by $u^* : \widehat{D} \rightarrow \widehat{C}$ the restriction functor and by $u_! : \widehat{C} \rightarrow \widehat{D}$ its left adjoint. The functor u^* induces a functor $u^\bullet : \mathrm{Mod}(D) \rightarrow \mathrm{Mod}(C)$. Moreover, this functor admits $u_\bullet = r_D u_! j_C$ as a left adjoint and the square

$$\begin{array}{ccc} C & \xrightarrow{u} & D \\ i_C \downarrow & & \downarrow i_D \\ \mathrm{Mod}(C) & \xrightarrow{u_\bullet} & \mathrm{Mod}(D) \end{array}$$

is commutative up to isomorphism. In particular, if $k : \Theta_0 \rightarrow \Theta$ denotes the canonical morphism, the square

$$\begin{array}{ccc} \Theta_0 & \xrightarrow{k} & \Theta \\ i_{\Theta_0} \downarrow & & \downarrow i_\Theta \\ \mathrm{Mod}(\Theta_0) & \xrightarrow{k_\bullet} & \mathrm{Mod}(\Theta) \end{array}$$

is commutative up to isomorphism.

Let U be the forgetful functor $\infty\text{-Cat} \rightarrow \widehat{\mathbb{G}}$ and let $L : \widehat{\mathbb{G}} \rightarrow \infty\text{-Cat}$ be its left adjoint, i.e., the free strict ∞ -category functor. The square

$$\begin{array}{ccc} \mathrm{Mod}(\Theta_0) & \xleftarrow{k^\bullet} & \mathrm{Mod}(\Theta) \\ \downarrow & & \downarrow \\ \widehat{\mathbb{G}} & \xleftarrow{U} & \infty\text{-Cat} \end{array} ,$$

where the vertical functors are the equivalences of categories of Propositions 3.5 and 3.14, is obviously commutative. It follows that the square

$$\begin{array}{ccc} \text{Mod}(\Theta_0) & \xrightarrow{k_\bullet} & \text{Mod}(\Theta) \\ \downarrow & & \downarrow \\ \widehat{\mathbb{G}} & \xrightarrow{L} & \infty\text{-Cat} \end{array}$$

is commutative up to isomorphism.

We thus obtain that the diagram

$$\begin{array}{ccc} \Theta_0 & \xrightarrow{k} & \Theta \\ i_{\Theta_0} \downarrow & & \downarrow i_\Theta \\ \text{Mod}(\Theta_0) & \xrightarrow{k_\bullet} & \text{Mod}(\Theta) \\ \downarrow & & \downarrow \\ \widehat{\mathbb{G}} & \xrightarrow{L} & \infty\text{-Cat} \end{array}$$

is commutative up to isomorphism, hence the result. \square

Proposition 3.16. *The category Θ is canonically isomorphic to Joyal's cell category.*

Proof. By Theorem 5.10 of [14] (or Theorem 1.12 of [7]), Joyal's cell category is canonically isomorphic to the full subcategory of $\infty\text{-Cat}$ described in the previous proposition. Hence the result by this proposition. \square

3.17. A *pregroupoidal globular extension* is a precategorical globular extension endowed with morphisms

$$\Omega_j^i : D_i \rightarrow D_i, \quad i > j \geq 0,$$

such that for all i, j satisfying $i > j \geq 0$, we have

$$\Omega_j^i \sigma_i = \begin{cases} \tau_i & j = i - 1, \\ \sigma_i \Omega_j^{i-1} & j < i - 1, \end{cases}$$

and

$$\Omega_j^i \tau_i = \begin{cases} \sigma_i & j = i - 1, \\ \tau_i \Omega_j^{i-1} & j < i - 1. \end{cases}$$

A *morphism of pregroupoidal globular extensions* is a morphism of precategorical globular extensions between pregroupoidal globular extensions preserving the Ω_j^i 's.

A pregroupoidal globular extension is *groupoidal* if it is categorical and if it satisfies the following additional axioms:

- (LInv_{*i,j*}), $i > j \geq 0$,
the following square commutes:

$$\begin{array}{ccc} D_i & \xrightarrow{\kappa_i^j} & D_j \\ \nabla_j^i \downarrow & & \downarrow \sigma_j^i \\ D_i \amalg_{D_j} D_i & \xrightarrow{(\Omega_j^i, 1_{D_i})} & D_i \end{array} ;$$

- (RInv_{*i,j*}), $i > j \geq 0$,
the following square commutes:

$$\begin{array}{ccc} D_i & \xrightarrow{\kappa_i^j} & D_j \\ \nabla_j^i \downarrow & & \downarrow \tau_j^i \\ D_i \amalg_{D_j} D_i & \xrightarrow{(1_{D_i}, \Omega_j^i)} & D_i \end{array} .$$

The *category of groupoidal globular extensions* is the full subcategory of the category of pregroupoidal globular extensions whose objects are groupoidal globular extensions.

If C is a category, we will denote by $\text{Ext}_{\text{gr}}(C)$ the category whose objects are groupoidal globular extension structures on C , i.e., functors from Θ_0 to C endowed with ∇_j^i 's, κ_i 's and Ω_j^i 's making C a groupoidal globular extension, and whose morphisms are natural transformations.

Proposition 3.18 (Universal property of $\tilde{\Theta}$). *There exists a groupoidal globular extension $\tilde{\Theta}$ such that for every category C , the precomposition by the functor $\Theta_0 \rightarrow \tilde{\Theta}$ induces an isomorphism of categories*

$$\underline{\text{Hom}}_{\text{gl}_0}(\tilde{\Theta}, C) \rightarrow \text{Ext}_{\text{gr}}(C).$$

Proof. The proof is similar to the one of the categorical case (Proposition 3.11). \square

3.19. We will denote by $\tilde{\Theta}$ the groupoidal globular extension of the previous proposition (which is unique up to a unique isomorphism). The category $\tilde{\Theta}$ is the groupoidal analogue to Joyal's category Θ . Note that the functor $\Theta_0 \rightarrow \tilde{\Theta}$ is bijective on objects.

3.20. Let C be a groupoidal globular extension. As in the categorical case, if X is a globular presheaf on C , the globular set obtained by restricting X to \mathbb{G} is canonically endowed with a structure of strict ∞ -category, and this ∞ -category is a strict ∞ -groupoid whose inverses are given by the

$$w_j^i = X(\Omega_j^i) : X_i \rightarrow X_i, \quad i > j \geq 0.$$

We thus have a canonical functor

$$\text{Mod}(C) \rightarrow \infty\text{-Grpd}.$$

The two following propositions are proved exactly as in the categorical case.

Proposition 3.21. *The functor*

$$\mathrm{Mod}(\tilde{\Theta}) \rightarrow \infty\text{-Grpd}$$

is an equivalence of categories.

Proposition 3.22. *The functor*

$$\tilde{\Theta} \rightarrow \mathrm{Mod}(\tilde{\Theta}) \rightarrow \infty\text{-Grpd}$$

identifies $\tilde{\Theta}$ with the full subcategory of $\infty\text{-Grpd}$ whose objects are free strict ∞ -groupoids on globular pasting schemes.

4. TEST CATEGORIES AND DÉCALAGES

4.1. We recall that if A is a small category, we denote by \hat{A} the category of presheaves on A . Let $u : A \rightarrow B$ be a functor and b an object of B . We will denote by A/b the comma category whose objects are pairs $(a, f : u(a) \rightarrow b)$ where a is an object of A and f a morphism of B , and whose morphisms from an object (a, f) to an object (a', f') are morphisms $g : a \rightarrow a'$ of A such that $f'u(g) = f$. In particular, if A is a small category and F is a presheaf on A , the category A/F (where $u : A \rightarrow \hat{A}$ is the Yoneda functor) is the category of elements of F .

4.2. We will denote by \mathcal{Cat} the category of small categories. We recall that a *weak equivalence* of small categories is a functor which is sent by the nerve functor on a weak equivalence of simplicial sets. We will denote by \mathcal{W} the class of weak equivalences of small categories and by \mathbf{Hot} the Gabriel-Zisman localization $\mathcal{Cat}[\mathcal{W}^{-1}]$ of \mathcal{Cat} by \mathcal{W} . A famous theorem of Quillen states that \mathbf{Hot} is canonically equivalent to the homotopy category of simplicial sets (see Corollary 3.3.1 of [12]) and hence to the homotopy category of CW-complexes.

4.3. Let A be a small category. We have a pair of adjoint functors

$$\begin{array}{ccc} i_A : \hat{A} & \rightarrow & \mathcal{Cat} & & i_A^* : \mathcal{Cat} & \rightarrow & \hat{A} \\ & & F \mapsto A/F & & & & C \mapsto (a \mapsto \mathrm{Hom}_{\mathcal{Cat}}(A/a, C)). \end{array}$$

A morphism of presheaves on A is a *weak equivalence* if it is sent by i_A on a weak equivalence of small categories. We will denote by $\mathcal{W}_{\hat{A}}$ the class of weak equivalences of presheaves on A and by \mathbf{Hot}_A the Gabriel-Zisman localization of \hat{A} by $\mathcal{W}_{\hat{A}}$. The functor i_A induces a functor $\overline{i_A} : \mathbf{Hot}_A \rightarrow \mathbf{Hot}$. If $i_A^*(\mathcal{W}) \subset \mathcal{W}_{\hat{A}}$, i.e., if $i_A i_A^*(\mathcal{W}) \subset \mathcal{W}$, then the functor i_A^* induces a functor $\overline{i_A^*} : \mathbf{Hot} \rightarrow \mathbf{Hot}_A$. Moreover, if this condition is satisfied, the pair of adjoint functors (i_A, i_A^*) induces a pair of adjoint functors $(\overline{i_A}, \overline{i_A^*})$.

4.4. A small category A is a *weak test category* if the following conditions are satisfied:

- we have $i_A^*(\mathcal{W}) \subset \mathcal{W}_{\hat{A}}$;
- for every presheaf F on A , the unit morphism $\eta_F : F \rightarrow i_A^* i_A(F)$ belongs to $\mathcal{W}_{\hat{A}}$;
- for every small category C , the counit morphism $\varepsilon_C : i_A i_A^*(C) \rightarrow C$ belongs to \mathcal{W} .

The two last conditions are the obvious sufficient conditions for the adjunction $(\overline{i_A}, \overline{i_A^*})$ to be an equivalence adjunction. In particular, if A is a weak test category, the category \mathbf{Hot}_A is canonically equivalent to \mathbf{Hot} .

4.5. A small category A is a *local test category* if for every object a of A , the category A/a is a weak test category. A small category is a *test category* if it is a weak test category and a local test category.

A test category A is a *strict test category* if the functor i_A respects binary products up to weak equivalence, i.e., if for all presheaves F and G on A , the canonical functor

$$A/(F \times G) \rightarrow A/F \times A/G$$

is a weak equivalence.

Theorem 4.6 (Grothendieck-Cisinski). *Let A be a local test category. Then $(\widehat{A}, \mathcal{W}_{\widehat{A}})$ is endowed with a structure of model category whose cofibrations are the monomorphisms. This model category structure is cofibrantly generated and proper.*

Moreover, if A is a strict test category, weak equivalences are stable by binary products.

Proof. See Corollary 4.2.18 of [9] for the model category structure. The properness follows by Theorem 4.3.24 of [9] and by the case of simplicial sets.

The last assertion is obvious. \square

4.7. A small category A is *aspherical* if the unique functor from A to the terminal category is a weak equivalence. It is easy to check that categories admitting a terminal object are aspherical. One can prove (see Remark 1.5.4 of [15]) that a local test category is test if and only if it is aspherical. We will only need the following obvious case: a local test category with a terminal object is a test category.

Let $u : A \rightarrow B$ be a functor between small categories. The functor u is *aspherical* if for every object b of B , the category A/b is aspherical.

Let A be a small category. A presheaf F on A is *aspherical* if the category A/F is aspherical. Every representable presheaf is aspherical since for every object a of A , the category A/a admits a terminal object.

If $u : A \rightarrow B$ is a functor between small categories, we will denote by $u^* : \widehat{B} \rightarrow \widehat{A}$ the restriction functor and by $u_* : \widehat{A} \rightarrow \widehat{B}$ its right adjoint.

Proposition 4.8. *Let $u : A \rightarrow B$ be a functor between aspherical small categories. The following properties are equivalent:*

- (1) *the functor u is aspherical;*
- (2) *for every morphism $\varphi : F \rightarrow G$ of presheaves on B , the morphism φ is a weak equivalence of presheaves on B if and only if the morphism $u^*(\varphi)$ is a weak equivalence of presheaves on A .*

Proof. See [11] or Proposition 1.2.9 of [15]. \square

Proposition 4.9. *Let $u : A \rightarrow B$ be an aspherical functor between test categories. Then (u^*, u_*) is a Quillen equivalence (where \widehat{A} and \widehat{B} are endowed with the Grothendieck-Cisinski model structure of Theorem 4.6).*

Proof. See Proposition 4.2.24 of [9]. \square

4.10. Let A be a small category. Denote by $\emptyset_{\widehat{A}}$ the initial presheaf on A and by $e_{\widehat{A}}$ the terminal one. An *interval* $(I, \partial_0, \partial_1)$ on \widehat{A} consists of a presheaf I on A and two morphisms $\partial_0, \partial_1 : e_{\widehat{A}} \rightarrow I$. Such an interval is *separating* if the equalizer of ∂_0 and ∂_1 is $\emptyset_{\widehat{A}}$.

4.11. Let A be a small category. A *décalage* on A consists of an endofunctor $D : A \rightarrow A$, an object a_0 of A and two natural transformations

$$1_A \xrightarrow{\alpha} D \xleftarrow{\beta} a_0$$

(where a_0 denotes the constant endofunctor whose value is a_0). We will denote by $(A, a_0, D, \alpha, \beta)$ such a *décalage*. A *splitting* of $(A, a_0, D, \alpha, \beta)$ consists of a retraction $r_a : D(a) \rightarrow a$ of α_a for every object a of A . Note that the r_a 's are *not* asked to be functorial in a . A *décalage* is *splittable* if it admits a splitting.

Proposition 4.12. *Let A be a small category. If A admits a splittable *décalage* and \widehat{A} admits a separating interval $(I, \partial_0, \partial_1)$ such that I is aspherical, then A is a strict test category.*

Proof. See Proposition 3.6 and Corollary 3.7 of [10]. □

4.13. Let $\mathcal{D}_A = (A, a_0, D, \alpha, \beta)$ and $\mathcal{D}_B = (B, b_0, E, \gamma, \delta)$ be two *décalages*. A *morphism of décalages* from \mathcal{D}_A to \mathcal{D}_B is a functor $u : A \rightarrow B$ such that

$$uD = Eu, \quad u(a_0) = b_0, \quad u * \alpha = \gamma * u, \quad \text{and} \quad u * \beta = \delta * u.$$

Proposition 4.14. *Let $u : A \rightarrow B$ be a functor between small categories. If there exists a *décalage* \mathcal{D}_A on A and a *splittable décalage* \mathcal{D}_B on B such that u induces a morphism of *décalages* from \mathcal{D}_A to \mathcal{D}_B , then u is aspherical.*

Proof. See Proposition 3.9 of [10]. □

5. SHIFTED GLOBULAR EXTENSIONS

5.1. In this section, we fix a globular extension (C, F) endowed with morphisms

$$\nabla_i : D_i \rightarrow D_i \amalg_{D_{i-1}} D_i, \quad i \geq 1,$$

such that

$$\nabla_i \sigma_i = \varepsilon_2 \sigma_i \quad \text{and} \quad \nabla_i \tau_i = \varepsilon_1 \tau_i,$$

where $\varepsilon_1, \varepsilon_2 : D_i \rightarrow D_i \amalg_{D_{i-1}} D_i$ denote the canonical morphisms.

The purpose of the section is to define a new structure of globular extension on C , i.e., a functor $K : \mathbb{G} \rightarrow C$ such that (C, K) is a globular extension, using the ∇_i 's. We will call (C, K) the *twisted globular extension* of (C, F) (by the ∇_i 's).

5.2. We set

$$\tilde{D}_i = D_1 \amalg_{D_0} D_2 \amalg_{D_1} \dots \amalg_{D_{i-1}} D_{i+1}, \quad i \geq 1.$$

Recall that we denote the canonical morphisms by

$$\varepsilon_k : D_k \rightarrow \tilde{D}_i, \quad 1 \leq k \leq i+1.$$

We define morphisms

$$\begin{aligned} \tilde{\sigma}_i &: \tilde{D}_{i-1} \rightarrow \tilde{D}_i, & i \geq 1, \\ \tilde{\tau}_i &: \tilde{D}_{i-1} \rightarrow \tilde{D}_i, & i \geq 1, \end{aligned}$$

by the formulas

$$\begin{aligned}\tilde{\sigma}_i &= (\varepsilon_1, \dots, \varepsilon_{i-1}, (\varepsilon_i, \varepsilon_{i+1}\tau_{i+1})\nabla_i), \\ \tilde{\tau}_i &= (\varepsilon_1, \dots, \varepsilon_i).\end{aligned}$$

(It is obvious that $\tilde{\tau}_i$ is well-defined and we will prove that $\tilde{\sigma}_i$ is well-defined in paragraph 5.3.)

Let X be a globular presheaf on C . We set

$$\tilde{X}_i = X(\tilde{D}_i) = X_1 \times_{X_0} X_2 \times_{X_1} \dots \times_{X_{i-1}} X_{i+1}, \quad i \geq 1.$$

For k such that $1 \leq k \leq i+1$, we will denote by p_k the canonical projection

$$p_k : \tilde{X}_i \rightarrow X_k.$$

We will often denote by \bar{x} an element of \tilde{X}_i and by x_1, \dots, x_{i+1} the components of \bar{x} .

We define maps

$$\begin{aligned}\tilde{s}_i : \tilde{X}_i &\rightarrow \tilde{X}_{i-1}, \quad i \geq 1, \\ \tilde{t}_i : \tilde{X}_i &\rightarrow \tilde{X}_{i-1}, \quad i \geq 1,\end{aligned}$$

by the formulas dual to the ones defining $\tilde{\sigma}_i$ and $\tilde{\tau}_i$:

$$\begin{aligned}\tilde{s}_i(x_1, \dots, x_{i+1}) &= (x_1, \dots, x_{i-1}, x_i *_{i-1}^i t_{i+1}(x_{i+1})), \\ \tilde{t}_i(x_1, \dots, x_{i+1}) &= (x_1, \dots, x_i).\end{aligned}$$

In particular, once we have proved that $\tilde{\sigma}_i$ is well-defined, we will have

$$\tilde{s}_i = X(\tilde{\sigma}_i) \quad \text{and} \quad \tilde{t}_i = X(\tilde{\tau}_i).$$

5.3. Let $i \geq 1$. Let us prove that $\tilde{\sigma}_i$ is well-defined. We need to show that

$$\varepsilon_{i-1}\sigma_{i-1} = (\varepsilon_i, \varepsilon_{i+1}\tau_{i+1})\nabla_i\tau_i\tau_{i-1}.$$

But

$$\begin{aligned}(\varepsilon_i, \varepsilon_{i+1}\tau_{i+1})\nabla_i\tau_i\tau_{i-1} &= (\varepsilon_i, \varepsilon_{i+1}\tau_{i+1})\varepsilon_1\tau_i\tau_{i-1} \\ &= \varepsilon_i\tau_i\tau_{i-1} \\ &= \varepsilon_{i-1}\sigma_{i-1}.\end{aligned}$$

This calculation was straightforward. However, in the sequel of this paper, we will need to prove more and more complicated identities. For this reason, we will prove our identities “using elements”. In [2], we gave proofs without using this technique. The result is barely readable.

Let us explain what we mean by “using elements”. Let $f, g : S \rightarrow T$ be two parallel morphisms of C . Suppose we want to prove that f is equal to g . By the (contravariant) Yoneda lemma, it suffices to check that for every object U of C , the two maps

$$\mathbf{Hom}_C(T, U) \rightarrow \mathbf{Hom}_C(S, U),$$

induced by f and g , are equal. Since every representable presheaf on C is globular, it suffices to prove that for every globular presheaf X on C , the two maps

$$\mathbf{Hom}_{\hat{C}}(T, X) \rightarrow \mathbf{Hom}_{\hat{C}}(S, X),$$

induced by f and g , are equal. But by the Yoneda lemma, these maps correspond to the maps

$$X(f), X(g) : X(T) \rightarrow X(S).$$

In conclusion, the morphisms f and g are equal if and only the maps $X(f)$ and $X(g)$ are equal for every globular presheaf X .

Let us apply this to

$$f = \varepsilon_{i-1}\sigma_{i-1} \quad \text{and} \quad g = (\varepsilon_i, \varepsilon_{i+1}\tau_{i+1})\nabla_i\tau_i\tau_{i-1}.$$

Let X be a globular presheaf on C . For \bar{x} in \tilde{X}_i , we have

$$X(f)(\bar{x}) = s_{i-1}(x_{i-1}) \quad \text{and} \quad X(g)(\bar{x}) = t_{i-1}t_i(x_i *_{i-1}^i t_{i+1}(x_{i+1})).$$

But

$$t_{i-1}t_i(x_i *_{i-1}^i t_{i+1}(x_{i+1})) = t_{i-1}t_i(x_i) = s_{i-1}(x_{i-1}).$$

We have thus given another proof of the well-definedness of $\tilde{\sigma}_i$.

From now on, we fix a globular presheaf X on C .

Proposition 5.4. *The maps*

$$D_i \mapsto \tilde{D}_i, \quad \sigma_{i+1} \mapsto \tilde{\sigma}_{i+1}, \quad \tau_{i+1} \mapsto \tilde{\tau}_{i+1}, \quad i \geq 0,$$

define a functor $\mathbb{G} \rightarrow C$.

In the sequel of this section, we will denote this functor by K .

Proof. We need to prove that the $\tilde{\sigma}_i$'s and $\tilde{\tau}_i$'s satisfy the coglobular relations. By paragraph 5.3, it suffices to show that the \tilde{s}_i 's and \tilde{t}_i 's satisfy the globular relations.

Let $i \geq 2$ and \bar{x} in \tilde{X}_i . We have

$$\begin{aligned} \tilde{s}_{i-1}\tilde{s}_i(\bar{x}) &= \tilde{s}_{i-1}(x_1, \dots, x_{i-1}, x_i *_{i-1}^i t_{i+1}(x_{i+1})) \\ &= (x_1, \dots, x_{i-2}, x_{i-1} *_{i-2}^{i-1} t_i(x_i *_{i-1}^i t_{i+1}(x_{i+1}))) \\ &= (x_1, \dots, x_{i-2}, x_{i-1} *_{i-2}^{i-1} t_i(x_i)) \\ &= \tilde{s}_{i-1}(x_1, \dots, x_i) \\ &= \tilde{s}_{i-1}\tilde{t}_i(\bar{x}) \end{aligned}$$

and

$$\begin{aligned} \tilde{t}_{i-1}\tilde{s}_i(\bar{x}) &= \tilde{t}_{i-1}(x_1, \dots, x_{i-1}, x_i *_{i-1}^i t_{i+1}(x_{i+1})) \\ &= (x_1, \dots, x_{i-1}) \\ &= \tilde{t}_{i-1}\tilde{t}_i(\bar{x}), \end{aligned}$$

hence the result. \square

We collect in the following lemma two identities related to the structure of \tilde{X}_i that we will use several times.

Lemma 5.5. *Let \bar{x} in \tilde{X}_i . We have*

$$\begin{aligned} s_l^{l+2}(x_{l+2}) &= s_l^{i+1}(x_{i+1}), \quad 0 \leq l \leq i-1, \\ s_{l+1}(x_{l+1}) &= t_l^{i+1}(x_{i+1}), \quad 0 \leq l \leq i-1. \end{aligned}$$

Proof. We have

$$\begin{aligned}
s_l^{l+2}(x_{l+2}) &= s_{l+1}s_{l+2}(x_{l+2}) \\
&= s_{l+1}t_{l+2}t_{l+3}(x_{l+3}) \\
&= s_l^{l+3}(x_{l+3}) \\
&= \dots \\
&= s_l^{i+1}(x_{i+1}),
\end{aligned}$$

and

$$\begin{aligned}
s_{l+1}(x_{l+1}) &= t_l^{l+2}(x_{l+2}) \\
&= t_{l+1}t_{l+2}(x_{l+2}) \\
&= t_{l+1}s_{l+2}(x_{l+2}) \\
&= t_{l+1}t_{l+1}^{l+3}(x_{l+3}) \\
&= t_l^{l+3}(x_{l+3}) \\
&= \dots \\
&= t_l^{i+1}(x_{i+1}).
\end{aligned}$$

□

5.6. Let us introduce some more notations. We set

$$\tilde{D}_{j,i} = D_{j+1} \amalg_{D_j} D_{j+2} \amalg_{D_{j+1}} \dots \amalg_{D_{i-1}} D_{i+1}, \quad i \geq j \geq 0.$$

In particular, we have

$$\begin{aligned}
\tilde{D}_{0,i} &= \tilde{D}_i, \quad i \geq 0, \\
\tilde{D}_i &= \tilde{D}_{0,k} \amalg_{D_k} \tilde{D}_{k+1,i}, \quad i > k \geq 0, \\
\tilde{D}_{j,i} &= \tilde{D}_{j,k} \amalg_{D_k} \tilde{D}_{k+1,i}, \quad i > k \geq j \geq 0.
\end{aligned}$$

Dually, we set

$$\tilde{X}_{j,i} = X(\tilde{D}_{j,i}) = X_{j+1} \times_{X_j} X_{j+2} \times_{X_{j+1}} \dots \times_{X_{i-1}} X_{i+1}, \quad i \geq j \geq 0,$$

and we have

$$\begin{aligned}
\tilde{X}_{0,i} &= \tilde{X}_i, \quad i \geq 0, \\
\tilde{X}_i &= \tilde{X}_{0,k} \times_{X_k} \tilde{X}_{k+1,i}, \quad i > k \geq 0, \\
\tilde{X}_{j,i} &= \tilde{X}_{j,k} \times_{X_k} \tilde{X}_{k+1,i}, \quad i > k \geq j \geq 0.
\end{aligned}$$

We will now prove that (C, K) is a globular extension.

Lemma 5.7. *Let \mathcal{C} be a category and let $f : X \rightarrow Y$, $g_X : A \rightarrow X$, $g_Y : A \rightarrow Y$ and $g_Z : A \rightarrow Z$ be morphisms of \mathcal{C} . Suppose that the amalgamated sums*

$$X \amalg_A Z = (X, g_X) \amalg_A (g_Z, Z) \quad \text{and} \quad Y \amalg_A Z = (Y, g_Y) \amalg_A (g_Z, Z)$$

exist in \mathcal{C} , and that we have $fg_X = g_Y$, so that the morphism

$$X \amalg_A Z \xrightarrow{f \amalg_A Z} Y \amalg_A Z$$

is well-defined. Then the square

$$\begin{array}{ccc} X & \longrightarrow & X \amalg_A Z \\ f \downarrow & & \downarrow f \amalg_A Z \\ Y & \longrightarrow & Y \amalg_A Z \end{array}$$

is cocartesian.

Proof. The square of the statement is the coproduct in $A \setminus \mathcal{C}$ of the squares

$$\begin{array}{ccc} X & \xrightarrow{1_X} & X \\ f \downarrow & & \downarrow f \\ Y & \xrightarrow{1_Y} & Y \end{array} \quad \text{and} \quad \begin{array}{ccc} A & \xrightarrow{g_Z} & Z \\ 1_A \downarrow & & \downarrow 1_Z \\ A & \xrightarrow{g_Z} & Z \end{array} ,$$

which are both cocartesian in $A \setminus \mathcal{C}$. □

Proposition 5.8. *Let $T = \begin{pmatrix} i & j \\ k & \end{pmatrix}$ be a table of dimensions of width 2. The globular sum $\tilde{D}_i \amalg_{\tilde{D}_k} \tilde{D}_j$ associated to T in (C, K) exists and is canonically isomorphic to*

$$\tilde{D}_i \amalg_{D_k} \tilde{D}_{k+1,j} = (\tilde{D}_i, \varepsilon_{i+1} \sigma_k^{i+1}) \amalg_{D_k} (\varepsilon_1 \tau_k^{k+2}, \tilde{D}_{k+1,j}).$$

Proof. We prove that the square

$$\begin{array}{ccc} \tilde{D}_k & \longrightarrow & \tilde{D}_j = \tilde{D}_k \amalg_{D_k} \tilde{D}_{k+1,j} \\ \tilde{\sigma}_k^i \downarrow & & \downarrow \tilde{\sigma}_k^i \amalg_{D_k} \tilde{D}_{k+1,j} \\ \tilde{D}_i & \longrightarrow & (\tilde{D}_i, \varepsilon_{i+1} \sigma_k^{i+1}) \amalg_{D_k} (\varepsilon_1 \tau_k^{k+2}, \tilde{D}_{k+1,j}) \end{array}$$

is cocartesian by applying the previous lemma to

$$X = \tilde{D}_k, \quad Y = \tilde{D}_i, \quad Z = \tilde{D}_{k+1,j}, \quad A = D_k,$$

and

$$f = \tilde{\sigma}_k^i, \quad g_X = \varepsilon_{k+1} \sigma_{k+1}, \quad g_Y = \varepsilon_{i+1} \sigma_k^{i+1}, \quad g_Z = \varepsilon_1 \tau_k^{k+2}.$$

The two amalgamated sums appearing in the square exist since they are globular sums in (C, F) . Hence, to apply the lemma, it suffices to check that

$$\tilde{\sigma}_k^i \varepsilon_{k+1} \sigma_{k+1} = \varepsilon_{i+1} \sigma_k^{i+1}.$$

Let us prove this identity using elements. Let \bar{x} in \tilde{X}_i . We need to prove that

$$s_{k+1} p_{k+1} \tilde{s}_k^i(\bar{x}) = s_k^{i+1}(x_{i+1}).$$

But

$$\begin{aligned}
s_{k+1}p_{k+1}\tilde{s}_k^i(\bar{x}) &= s_{k+1}p_{k+1}\tilde{s}_{k+1}\tilde{t}_{k+1}^i(\bar{x}) \\
&= s_{k+1}p_{k+1}\tilde{s}_{k+1}(x_1, \dots, x_{k+2}) \\
&= s_{k+1}(x_{k+1} *_k^{k+1} t_{k+2}(x_{k+2})) \\
&= s_{k+1}t_{k+2}(x_{k+2}) \\
&= s_k^{k+2}(x_{k+2}) \\
&= s_k^{i+1}(x_{i+1}),
\end{aligned}$$

where the last equality follows from Lemma 5.5. \square

Proposition 5.9. *Let*

$$T = \begin{pmatrix} i_1 & i_2 & \cdots & i_n \\ i'_1 & i'_2 & \cdots & i'_{n-1} \end{pmatrix}$$

be a table of dimensions. The globular sum $\tilde{D}_{i_1} \amalg_{\tilde{D}_{i'_1}} \cdots \amalg_{\tilde{D}_{i'_{n-1}}} \tilde{D}_{i_n}$ associated to T in (C, K) exists and is canonically isomorphic to

$$\tilde{D}_{i_1} \amalg_{D_{i'_1}} \tilde{D}_{i'_1+1, i_2} \amalg_{D_{i'_2}} \tilde{D}_{i'_2+1, i_3} \amalg_{D_{i'_3}} \cdots \amalg_{D_{i'_{n-1}}} \tilde{D}_{i'_{n-1}+1, i_n}.$$

In particular, (C, K) is a globular extension.

As announced at the beginning of this section, we will call (C, K) the *twisted globular extension* of (C, F) (by the ∇_i 's).

Proof. We prove the result by induction on the width n of the table of dimensions. Suppose

$$\begin{aligned}
&\tilde{D}_{i_2} \amalg_{\tilde{D}_{i'_2}} \cdots \amalg_{\tilde{D}_{i'_{n-1}}} \tilde{D}_{i_n} \\
&= \tilde{D}_{i_2} \amalg_{D_{i'_2}} \tilde{D}_{i'_2+1, i_3} \amalg_{D_{i'_3}} \cdots \amalg_{D_{i'_{n-1}}} \tilde{D}_{i'_{n-1}+1, i_n} \\
&= \tilde{D}_{i'_1} \amalg_{D_{i'_1}} \left(\tilde{D}_{i'_1+1, i_2} \amalg_{D_{i'_2}} \tilde{D}_{i'_2+1, i_3} \amalg_{D_{i'_3}} \cdots \amalg_{D_{i'_{n-1}}} \tilde{D}_{i'_{n-1}+1, i_n} \right).
\end{aligned}$$

As in the proof of the previous proposition, by using Lemma 5.7, we obtain that the square

$$\begin{array}{ccc}
\tilde{D}_{i'_1} & \longrightarrow & \tilde{D}_{i'_1} \amalg_{D_{i'_1}} \left(\tilde{D}_{i'_1+1, i_2} \amalg_{D_{i'_2}} \amalg_{D_{i'_2}} \tilde{D}_{i'_2+1, i_3} \cdots \amalg_{D_{i'_{n-1}}} \tilde{D}_{i'_{n-1}+1, i_n} \right) \\
\tilde{\sigma}_{i'_1}^{i_1} \downarrow & & \downarrow \tilde{\sigma}_{i'_1}^{i_1} \amalg_{D_{i'_1}} 1 \\
\tilde{D}_{i_1} & \longrightarrow & \tilde{D}_{i_1} \amalg_{D_{i'_1}} \left(\tilde{D}_{i'_1+1, i_2} \amalg_{D_{i'_2}} \amalg_{D_{i'_2}} \tilde{D}_{i'_2+1, i_3} \cdots \amalg_{D_{i'_{n-1}}} \tilde{D}_{i'_{n-1}+1, i_n} \right)
\end{array}$$

is cocartesian. Hence the result. \square

5.10. Dually, if

$$\begin{pmatrix} i_1 & i_2 & \cdots & i_n \\ i'_1 & i'_2 & \cdots & i'_{n-1} \end{pmatrix}$$

is a table of dimensions, the globular product $\tilde{X}_{i_1} \times_{\tilde{X}_{i'_1}} \dots \times_{\tilde{X}_{i'_{n-1}}} \tilde{X}_{i_n}$ exists and is canonically isomorphic to

$$\tilde{X}_{i_1} \times_{X_{i'_1}} \tilde{X}_{i'_1+1, i_2} \times_{X_{i'_2}} \tilde{X}_{i'_2+1, i_3} \times_{X_{i'_3}} \dots \times_{X_{i'_{n-1}}} \tilde{X}_{i'_{n-1}+1, i_n}.$$

Moreover, the canonical isomorphism

$$c : \tilde{X}_{i_1} \times_{\tilde{X}_{i'_1}} \dots \times_{\tilde{X}_{i'_{n-1}}} \tilde{X}_{i_n} \rightarrow \tilde{X}_{i_1} \times_{X_{i'_1}} \tilde{X}_{i'_1+1, i_2} \times_{X_{i'_2}} \dots \times_{X_{i'_{n-1}}} \tilde{X}_{i'_{n-1}+1, i_n}$$

is given by the formula

$$\begin{aligned} c(x_1^1, \dots, x_{i_1+1}^1, x_1^2, \dots, x_{i_2+1}^2, \dots, x_1^n, \dots, x_{i_n+1}^n) \\ = (x_1^1, \dots, x_{i_1+1}^1, x_{i'_1+2}^2, \dots, x_{i_2+1}^2, \dots, x_{i'_{n-1}+2}^n, \dots, x_{i_n+1}^n). \end{aligned}$$

Let us describe the inverse of c starting by the case $n = 2$. Let $\binom{i}{k}^j$ be a table of dimensions of width 2 and let (\bar{x}, \bar{y}) be an element of $\tilde{X}_i \times_{\tilde{X}_k} \tilde{X}_j$. By definition, we have $\tilde{s}_k^i(\bar{x}) = \tilde{t}_k^j(\bar{y})$. Since $\tilde{s}_k^i = \tilde{s}_{k+1} \tilde{t}_{k+1}^i$, this means that

$$(x_1, \dots, x_k, x_{k+1} *_{k+1}^{k+1} t_{k+2}(x_{k+2})) = (y_1, \dots, y_{k+1}),$$

i.e., that

$$\begin{aligned} y_l &= x_l, \quad 1 \leq l \leq k, \\ y_{k+1} &= x_{k+1} *_{k+1}^{k+1} t_{k+2}(x_{k+2}). \end{aligned} \tag{*}$$

The inverse

$$c^{-1} : \tilde{X}_i \times_{X_k} \tilde{X}_{k+1, j} \rightarrow \tilde{X}_i \times_{\tilde{X}_k} \tilde{X}_j$$

is thus given by the formula

$$\begin{aligned} c^{-1}(x_1, \dots, x_{i+1}, y_{k+2}, \dots, y_{j+1}) \\ = (x_1, \dots, x_{i+1}, x_1, \dots, x_k, x_{k+1} *_{k+1}^{k+1} t_{k+2}(x_{k+2}), y_{k+2}, \dots, y_{j+1}). \end{aligned}$$

In the general case, the inverse

$$c^{-1} : \tilde{X}_{i_1} \times_{X_{i'_1}} \tilde{X}_{i'_1+1, i_2} \times_{X_{i'_2}} \dots \times_{X_{i'_{n-1}}} \tilde{X}_{i'_{n-1}+1, i_n} \rightarrow \tilde{X}_{i_1} \times_{\tilde{X}_{i'_1}} \dots \times_{\tilde{X}_{i'_{n-1}}} \tilde{X}_{i_n}$$

is given by the formula

$$\begin{aligned} c^{-1}(x_1^1, \dots, x_{i_1+1}^1, x_{i'_1+2}^2, \dots, x_{i_2+1}^2, \dots, x_{i'_{n-1}+2}^n, \dots, x_{i_n+1}^n) \\ = (x_1^1, \dots, x_{i_1+1}^1, x_1^2, \dots, x_{i_2+1}^2, \dots, x_1^n, \dots, x_{i_n+1}^n), \end{aligned}$$

where the

$$x_j^l, \quad 2 \leq l \leq n, \quad 1 \leq j \leq i'_l + 1,$$

are defined (by induction on l) by

$$\begin{aligned} x_j^{l+1} &= x_j^l, \quad 1 \leq j \leq i'_l, \\ x_{i'_l+1}^{l+1} &= x_{i'_l+1}^l *_{i'_l+1}^{i'_l+1} t_{i'_l+2}^l(x_{i'_l+2}^l). \end{aligned}$$

5.11. Since (C, K) is a globular extension, by the universal property of Θ_0 (Proposition 3.2), we can lift K to a globular functor $K_0 : \Theta_0 \rightarrow C$ defined up to a unique isomorphism. Suppose now that a globular lifting $F_0 : \Theta_0 \rightarrow C$ to F is given. Proposition 5.9 allows us to express globular sums of (C, K) in terms of those of (C, F) . The globular lifting $K_0 : \Theta_0 \rightarrow C$ is hence uniquely determined by F_0 . We will call (C, K_0) the *twisted globular extension under Θ_0* of (C, F_0) .

6. SHIFTED GROUPOIDAL GLOBULAR EXTENSIONS

6.1. In this section, we fix a pregroupoidal globular extension (C, F_0) . In particular, the globular extension C is endowed with morphisms

$$\nabla_i = \nabla_{i-1}^i, \quad i \geq 1,$$

and we can thus apply the previous section and in particular Proposition 5.9 and paragraph 5.11 to get a twisted globular extension (C, K_0) under Θ_0 .

The purpose of the section is to put (under some assumptions) a structure of pregroupoidal globular extension on (C, K_0) and to prove that if (C, F_0) is groupoidal, then so is (C, K_0) . In the latter case, we will call (C, K_0) (endowed with its additional structure) the *twisted groupoidal globular extension* of (C, K_0) .

6.2. We define morphisms

$$\begin{aligned} \tilde{\nabla}_j^i : \tilde{D}_i &\rightarrow \tilde{D}_i \amalg_{\tilde{D}_j} \tilde{D}_i = \tilde{D}_i \amalg_{D_j} \tilde{D}_{j+1,i}, \quad i > j \geq 0, \\ \tilde{\kappa}_i : \tilde{D}_{i+1} &\rightarrow \tilde{D}_i, \quad i \geq 0, \\ \tilde{\Omega}_j^i : \tilde{D}_i &\rightarrow \tilde{D}_i, \quad i > j \geq 0, \end{aligned}$$

by the formulas

$$\begin{aligned} \tilde{\nabla}_j^i &= \left(\varepsilon_1, \dots, \varepsilon_{j+1}, (\varepsilon_{j+2}, \varepsilon'_{j+2}) \nabla_j^{j+2}, \dots, (\varepsilon_{i+1}, \varepsilon'_{i+1}) \nabla_j^{i+1} \right), \\ &\text{where } \varepsilon'_k \text{ denotes } \varepsilon_{k+i-j}, \\ \tilde{\kappa}_i &= \left(\varepsilon_1, \dots, \varepsilon_{i+1}, \varepsilon_{i+1} \sigma_{i+1} \kappa_i \kappa_{i+1} \right), \\ \tilde{\Omega}_j^i &= \left(\varepsilon_1, \dots, \varepsilon_j, (\varepsilon_{j+1}, \varepsilon_{j+2} \tau_{j+2}) \nabla_{j+1}, \varepsilon_{j+2} \Omega_j^{j+2}, \dots, \varepsilon_{i+1} \Omega_j^{i+1} \right). \end{aligned}$$

Note that $\varepsilon'_k : D_k \rightarrow \tilde{D}_i \amalg_{D_j} \tilde{D}_{j+1,i}$ is the canonical morphism corresponding to the factor D_k of $\tilde{D}_{j+1,i}$. In the sequel of this section, (C, K_0) will denote the globular extension (C, K_0) under Θ_0 endowed with these $\tilde{\nabla}_j^i$'s, $\tilde{\kappa}_i$'s and $\tilde{\Omega}_j^i$'s.

Dually, we define maps

$$\begin{aligned} \tilde{*}_j^i : \tilde{X}_i \times_{\tilde{X}_j} \tilde{X}_i &= \tilde{X}_i \times_{X_j} \tilde{X}_{j+1,i} \rightarrow \tilde{X}_i, \quad i > j \geq 0, \\ \tilde{k}_i : \tilde{X}_i &\rightarrow \tilde{X}_{i+1}, \quad i \geq 0, \\ \tilde{w}_j^i : \tilde{X}_i &\rightarrow \tilde{X}_i, \quad i > j \geq 0, \end{aligned}$$

by the formulas

$$\bar{x} \tilde{*}_j^i \bar{y} = (x_1, \dots, x_{j+1}, x_{j+2} *_{j+2}^{j+2} y_{j+2}, \dots, x_{i+1} *_{j+1}^{i+1} y_{i+1}),$$

where (\bar{x}, \bar{y}) is in $\tilde{X}_i \times_{\tilde{X}_j} \tilde{X}_i$, and

$$\begin{aligned}\tilde{k}_i(\bar{x}) &= (x_1, \dots, x_{i+1}, k_{i+1}k_i s_{i+1}(x_{i+1})), \\ \tilde{w}_j^i(\bar{x}) &= (x_1, \dots, x_j, x_{j+1} *_{j+1}^{j+1} t_{j+2}(x_{j+2}), w_j^{j+2}(x_{j+2}), \dots, w_j^{i+1}(x_{i+1})),\end{aligned}$$

where \bar{x} is in \tilde{X}_i .

Proposition 6.3. *The $\tilde{\nabla}_j^i$'s are well-defined. Moreover, if (C, F_0) satisfies Axioms (Ass) and (Exc), then the $\tilde{\nabla}_j^i$'s have the desired globular source and target, i.e., they satisfy Condition (1) of the definition of a precatgegorical globular extension (see paragraph 3.10).*

Proof. Recall that we have fixed a globular presheaf X on C . Let $i > j \geq 0$. By paragraph 5.3, showing that ∇_j^i is well-defined is equivalent to showing that for every (\bar{x}, \bar{y}) in $\tilde{X}_i \times_{\tilde{X}_j} \tilde{X}_i$, the element $\bar{x} *_{j+1}^i \bar{y}$ belongs to \tilde{X}_i .

Let us show this. Let (\bar{x}, \bar{y}) be in $\tilde{X}_i \times_{\tilde{X}_j} \tilde{X}_i$. We need to check that

$$s_{j+1}(x_{j+1}) = t_{j+1}t_{j+2}(x_{j+2} *_{j+2}^{j+2} y_{j+2}),$$

and

$$s_l(x_l *_{j+1}^l y_l) = t_l t_{l+1}(x_{l+1} *_{j+1}^{l+1} y_{l+1}), \quad j+2 \leq l \leq i.$$

But

$$\begin{aligned}t_{j+1}t_{j+2}(x_{j+2} *_{j+2}^{j+2} y_{j+2}) &= t_{j+1}(t_{j+2}(x_{j+2}) *_{j+2}^{j+1} t_{j+2}(y_{j+2})) \\ &= t_{j+1}t_{j+2}(x_{j+2}) \\ &= s_{j+1}(x_{j+1}),\end{aligned}$$

and

$$\begin{aligned}s_l(x_l *_{j+1}^l y_l) &= s_l(x_l) *_{j+1}^{l-1} s_l(y_l) \\ &= t_l t_{l+1}(x_{l+1}) *_{j+1}^{l-1} t_l t_{l+1}(y_{l+1}) \\ &= t_l t_{l+1}(x_{l+1} *_{j+1}^{l+1} y_{l+1}).\end{aligned}$$

Again by paragraph 5.3, proving that ∇_j^i has the desired source and target is equivalent to proving the analogous result for $\bar{x} *_{j+1}^i \bar{y}$.

Let us prove this. If $j = i - 1$, we have

$$\begin{aligned}\tilde{s}_i(\bar{x} *_{i-1}^i \bar{y}) &= \tilde{s}_i(x_1, \dots, x_i, x_{i+1} *_{i-1}^{i+1} y_{i+1}) \\ &= (x_1, \dots, x_{i-1}, x_i *_{i-1}^i t_i(x_{i+1} *_{i-1}^{i+1} y_{i+1})) \\ &= (x_1, \dots, x_{i-1}, x_i *_{i-1}^i (t_i(x_{i+1}) *_{i-1}^i t_i(y_{i+1}))) \\ &= (x_1, \dots, x_{i-1}, (x_i *_{i-1}^i t_i(x_{i+1})) *_{i-1}^i t_i(y_{i+1})) \\ &\quad \text{(by Axiom (Ass}_{i,i-1})\text{)} \\ &= (y_1, \dots, y_{i-1}, y_i *_{i-1}^i t_i(y_{i+1})) \\ &\quad \text{(by equations (*) of paragraph 5.10)} \\ &= \tilde{s}_i(\bar{y}),\end{aligned}$$

and

$$\begin{aligned}\tilde{t}_i(\bar{x} \tilde{*}_{i-1}^i \bar{y}) &= \tilde{t}_i(x_1, \dots, x_i, x_{i+1} \tilde{*}_{i-1}^{i+1} y_{i+1}) \\ &= (x_1, \dots, x_i) \\ &= \tilde{t}_i(\bar{x}).\end{aligned}$$

If $j < i - 1$, we have

$$\begin{aligned}\tilde{s}_i(\bar{x} \tilde{*}_j^i \bar{y}) &= \tilde{s}_i(x_1, \dots, x_{j+1}, x_{j+2} \tilde{*}_j^{j+2} y_{j+2}, \dots, x_{i+1} \tilde{*}_j^{i+1} y_{i+1}) \\ &= (x_1, \dots, x_{j+1}, x_{j+2} \tilde{*}_j^{j+2} y_{j+2}, \dots, x_{i-1} \tilde{*}_j^{i-1} y_{i-1}, \\ &\quad (x_i \tilde{*}_j^i y_i) \tilde{*}_{i-1}^i t_{i+1}(x_{i+1} \tilde{*}_j^{i+1} y_{i+1})) \\ &= (x_1, \dots, x_{j+1}, x_{j+2} \tilde{*}_j^{j+2} y_{j+2}, \dots, x_{i-1} \tilde{*}_j^{i-1} y_{i-1}, \\ &\quad (x_i \tilde{*}_j^i y_i) \tilde{*}_{i-1}^i (t_{i+1}(x_{i+1}) \tilde{*}_j^i t_{i+1}(y_{i+1}))) \\ &= (x_1, \dots, x_{j+1}, x_{j+2} \tilde{*}_j^{j+2} y_{j+2}, \dots, x_{i-1} \tilde{*}_j^{i-1} y_{i-1}, \\ &\quad (x_i \tilde{*}_{i-1}^i t_{i+1}(x_{i+1})) \tilde{*}_j^i (y_i \tilde{*}_{i-1}^i t_{i+1}(y_{i+1}))) \\ &\quad (\text{by Axiom (Exc}_{i,i-1,j}\text{)}) \\ &= (x_1, \dots, x_{i-1}, x_i \tilde{*}_{i-1}^i t_{i+1}(x_{i+1})) \tilde{*}_j^i \\ &\quad (y_1, \dots, y_{i-1}, y_i \tilde{*}_{i-1}^i t_{i+1}(y_{i+1})) \\ &= \tilde{s}_i(\bar{x}) \tilde{*}_j^i \tilde{s}_i(\bar{y}),\end{aligned}$$

and

$$\begin{aligned}\tilde{t}_i(\bar{x} \tilde{*}_j^i \bar{y}) &= \tilde{t}_i(x_1, \dots, x_{j+1}, x_{j+2} \tilde{*}_j^{j+2} y_{j+2}, \dots, x_{i+1} \tilde{*}_j^{i+1} y_{i+1}) \\ &= (x_1, \dots, x_{j+1}, x_{j+2} \tilde{*}_j^{j+2} y_{j+2}, \dots, x_i \tilde{*}_j^i y_i) \\ &= (x_1, \dots, x_i) \tilde{*}_j^{i-1} (y_1, \dots, y_i) \\ &= \tilde{t}_i(\bar{x}) \tilde{*}_j^{i-1} \tilde{t}_i(\bar{y}),\end{aligned}$$

hence the result. □

Proposition 6.4. *If (C, F_0) satisfies Axiom (Ass), then so does (C, K_0) .*

Proof. Let $i > j \geq 0$ and let $(\bar{x}, \bar{y}, \bar{z})$ be in $\tilde{X}_i \times_{\tilde{X}_j} \tilde{X}_i \times_{\tilde{X}_j} \tilde{X}_i$. We have

$$\begin{aligned}(\bar{x} \tilde{*}_j^i \bar{y}) \tilde{*}_j^i \bar{z} &= (x_1, \dots, x_{j+1}, x_{j+2} \tilde{*}_j^{j+2} y_{j+2}, \dots, x_{i+1} \tilde{*}_j^{i+1} y_{i+1}) \tilde{*}_j^i \bar{z} \\ &= (x_1, \dots, x_{j+1}, \\ &\quad (x_{j+2} \tilde{*}_j^{j+2} y_{j+2}) \tilde{*}_j^{j+2} z_{j+2}, \dots, (x_{i+1} \tilde{*}_j^{i+1} y_{i+1}) \tilde{*}_j^{i+1} z_{i+1}) \\ &= (x_1, \dots, x_{j+1}, \\ &\quad x_{j+2} \tilde{*}_j^{j+2} (y_{j+2} \tilde{*}_j^{j+2} z_{j+2}), \dots, x_{i+1} \tilde{*}_j^{i+1} (y_{i+1} \tilde{*}_j^{i+1} z_{i+1})) \\ &\quad (\text{by Axiom (Ass}_{l,j}\text{) for } j+2 \leq l \leq i+1) \\ &= \bar{x} \tilde{*}_j^i (y_1, \dots, y_{j+1}, y_{j+2} \tilde{*}_j^{j+2} z_{j+2}, \dots, y_{i+1} \tilde{*}_j^{i+1} z_{i+1}) \\ &= \bar{x} \tilde{*}_j^i (\bar{y} \tilde{*}_j^i \bar{z}).\end{aligned}$$

□

Proposition 6.5. *If (C, F_0) satisfies Axiom (Exc), then so does (C, K_0) .*

Proof. Let $i > j > k \geq 0$ and let $(\bar{x}, \bar{y}, \bar{z}, \bar{t})$ be in

$$\tilde{X}_i \times_{\tilde{X}_j} \tilde{X}_i \times_{\tilde{X}_j} \tilde{X}_i \times_{\tilde{X}_j} \tilde{X}_i.$$

We have

$$\begin{aligned} & (\bar{x} \tilde{*}_j^i \bar{y}) \tilde{*}_k^i (\bar{z} \tilde{*}_j^i \bar{t}) \\ &= (x_1, \dots, x_{j+1}, x_{j+2} \tilde{*}_j^{j+2} y_{j+2}, \dots, x_{i+1} \tilde{*}_j^{i+1} y_{i+1}) \tilde{*}_k^i \\ & \quad (z_1, \dots, z_{j+1}, z_{j+2} \tilde{*}_j^{j+2} t_{j+2}, \dots, z_{i+1} \tilde{*}_j^{i+1} t_{i+1}) \\ &= (x_1, \dots, x_{k+1}, x_{k+2} \tilde{*}_k^{k+2} z_{k+2}, \dots, x_{j+1} \tilde{*}_k^{j+1} z_{j+1}, \\ & \quad (x_{j+2} \tilde{*}_j^{j+2} y_{j+2}) \tilde{*}_k^{j+2} (z_{j+2} \tilde{*}_j^{j+2} t_{j+2}), \dots, \\ & \quad (x_{i+1} \tilde{*}_j^{i+1} y_{i+1}) \tilde{*}_k^{i+1} (z_{i+1} \tilde{*}_j^{i+1} t_{i+1})) \\ &= (x_1, \dots, x_{k+1}, x_{k+2} \tilde{*}_k^{k+2} z_{k+2}, \dots, x_{j+1} \tilde{*}_k^{j+1} z_{j+1}, \\ & \quad (x_{j+2} \tilde{*}_k^{j+2} z_{j+2}) \tilde{*}_j^{j+2} (y_{j+2} \tilde{*}_k^{j+2} t_{j+2}), \dots, \\ & \quad (x_{i+1} \tilde{*}_k^{i+1} z_{i+1}) \tilde{*}_j^{i+1} (y_{i+1} \tilde{*}_k^{i+1} t_{i+1})) \\ & \quad \text{(by Axiom (Exc}_{l,j,k}) \text{ for } l \text{ such that } j+2 \leq l \leq i+1) \\ &= (x_1, \dots, x_{k+1}, x_{k+2} \tilde{*}_k^{k+2} z_{k+2}, \dots, x_{i+1} \tilde{*}_k^{i+1} z_{i+1}) \tilde{*}_j^i \\ & \quad (y_1, \dots, y_{k+1}, y_{k+2} \tilde{*}_k^{k+2} t_{k+2}, \dots, y_{i+1} \tilde{*}_k^{i+1} t_{i+1}) \\ &= (\bar{x} \tilde{*}_k^i \bar{z}) \tilde{*}_j^i (\bar{y} \tilde{*}_k^i \bar{t}). \quad \square \end{aligned}$$

Proposition 6.6. *The $\tilde{\kappa}_i$'s are well-defined. Moreover, if (C, F_0) satisfies Axiom (RUnit), then the $\tilde{\kappa}_i$'s have the desired globular source and target, i.e., they satisfy Condition (2) of the definition of a precategorical globular extension (see paragraph 3.10).*

Proof. Let $i \geq 0$ and let \bar{x} be in \tilde{X}_i . Let us first prove that $\tilde{k}_i(\bar{x})$ belongs to \tilde{X}_{i+1} . We need to show that

$$s_{i+1}(x_{i+1}) = t_{i+1} t_{i+2} k_{i+1} k_i s_{i+1}(x_{i+1}),$$

but this identity holds since $t_{l+1} k_l = 1_{X_l}$ for every $l \geq 0$.

Let us now prove that $\tilde{k}_i(\bar{x})$ has the desired globular source and target. We have

$$\begin{aligned} \tilde{s}_{i+1} \tilde{k}_i(\bar{x}) &= \tilde{s}_{i+1}(x_1, \dots, x_{i+1}, k_{i+1} k_i s_{i+1}(x_{i+1})) \\ &= (x_1, \dots, x_i, x_{i+1} \tilde{*}_i^{i+1} t_{i+2} k_{i+1} k_i s_{i+1}(x_{i+1})) \\ &= (x_1, \dots, x_i, x_{i+1} \tilde{*}_i^{i+1} k_i s_{i+1}(x_{i+1})) \\ &= (x_1, \dots, x_{i+1}) \\ & \quad \text{(by Axiom (RUnit}_{i+1,i})} \\ &= \bar{x}, \end{aligned}$$

and

$$\begin{aligned}\tilde{t}_{i+1}\tilde{k}_i(\bar{x}) &= \tilde{t}_{i+1}(x_1, \dots, x_{i+1}, k_{i+1}k_i s_{i+1}(x_{i+1})) \\ &= (x_1, \dots, x_{i+1}) \\ &= \bar{x},\end{aligned}$$

hence the result. \square

Proposition 6.7. *If (C, F_0) satisfies Axioms (LUnit) and (RUnit), then so does (C, K_0) .*

Proof. Let $j \geq 0$ and let \bar{y} be in \tilde{X}_j . Let us first prove by induction on $i > j$ that

$$\tilde{k}_i^j(\bar{y}) = (y_1, \dots, y_{j+1}, k_{j+2}^j s_{j+1}(y_{j+1}), \dots, k_{i+1}^j s_{j+1}(y_{j+1})).$$

For $i = j+1$, this identity holds by definition of \tilde{k}_j . Assume the result holds for an $i > j$. Then we have

$$\begin{aligned}\tilde{k}_{i+1}^j(\bar{y}) &= \tilde{k}_i \tilde{k}_i^j(\bar{y}) \\ &= \tilde{k}_i(y_1, \dots, y_{j+1}, k_{j+2}^j s_{j+1}(y_{j+1}), \dots, k_{i+1}^j s_{j+1}(y_{j+1})) \\ &= (y_1, \dots, y_{j+1}, k_{j+2}^j s_{j+1}(y_{j+1}), \dots, k_{i+1}^j s_{j+1}(y_{j+1}), \\ &\quad k_{i+1}k_i s_{i+1} k_{i+1}^j s_{j+1}(y_{j+1})),\end{aligned}$$

but

$$\begin{aligned}k_{i+1}k_i s_{i+1} k_{i+1}^j s_{j+1}(y_{j+1}) &= k_{i+1}k_i s_{i+1} k_i k_i^j s_{j+1}(y_{j+1}) \\ &= k_{i+1}k_i k_i^j s_{j+1}(y_{j+1}), \\ &= k_{i+2}^j s_{j+1}(y_{j+1}),\end{aligned}$$

hence the formula.

Let now $i > j$ and let \bar{x} be in \tilde{X}_i . We have

$$\begin{aligned}\tilde{k}_i^j \tilde{s}_j^i(\bar{x}) &= \tilde{k}_i^j \tilde{s}_{j+1} \tilde{t}_{j+1}^i(\bar{x}) \\ &= \tilde{k}_i^j \tilde{s}_{j+1}(x_1, \dots, x_{j+2}) \\ &= \tilde{k}_i^j(x_1, \dots, x_j, x_{j+1} *_{j+1}^{j+1} t_{j+2}(x_{j+2})) \\ &= (x_1, \dots, x_j, x_{j+1} *_{j+1}^{j+1} t_{j+2}(x_{j+2}), \\ &\quad k_{j+2}^j s_{j+1}(x_{j+1} *_{j+1}^{j+1} t_{j+2}(x_{j+2})), \dots, \\ &\quad k_{i+1}^j s_{j+1}(x_{j+1} *_{j+1}^{j+1} t_{j+2}(x_{j+2}))).\end{aligned}$$

But for l such that $j+2 \leq l \leq i+1$, we have

$$\begin{aligned}k_l^j s_{j+1}(x_{j+1} *_{j+1}^{j+1} t_{j+2}(x_{j+2})) &= k_l^j s_{j+1} t_{j+2}(x_{j+2}) \\ &= k_l^j s_j^{j+2}(x_{j+2}) \\ &= k_l^j s_j^l(x_l),\end{aligned}$$

where the last equality comes from Lemma 5.5. Hence the identity

$$\begin{aligned} \tilde{k}_i^j \tilde{s}_j^i(\bar{x}) &= (x_1, \dots, x_j, x_{j+1} *_{j+1}^{j+1} t_{j+2}(x_{j+2}), \\ &\quad k_{j+2}^j s_j^{j+2}(x_{j+2}), \dots, k_{i+1}^j s_j^{i+1}(x_{i+1})). \end{aligned} \quad (*_{ks})$$

Let us now compute $\tilde{k}_i^j \tilde{t}_j^i(\bar{x})$. We have

$$\begin{aligned} \tilde{k}_i^j \tilde{t}_j^i(\bar{x}) &= \tilde{k}_i^j(x_1, \dots, x_{j+1}) \\ &= (x_1, \dots, x_{j+1}, k_{j+2}^j s_{j+1}(x_{j+1}), \dots, k_{i+1}^j s_{j+1}(x_{j+1})). \end{aligned}$$

But by Lemma 5.5, we have

$$s_{j+1}(x_{j+1}) = t_j^l(x_l), \quad j+2 \leq l \leq i+1,$$

and so we obtain the formula

$$\tilde{k}_i^j \tilde{t}_j^i(\bar{x}) = (x_1, \dots, x_{j+1}, k_{j+2}^j t_j^{j+2}(x_{j+2}), \dots, k_{i+1}^j t_j^{i+1}(x_{i+1})). \quad (*_{kt})$$

We can now prove the proposition. We have

$$\begin{aligned} \bar{x} *_{j+1}^i \tilde{k}_i^j \tilde{s}_j^i(\bar{x}) &= \bar{x} *_{j+1}^i (x_1, \dots, x_j, x_{j+1} *_{j+1}^{j+1} t_{j+2}(x_{j+2}), \\ &\quad k_{j+2}^j s_j^{j+2}(x_{j+2}), \dots, k_{i+1}^j s_j^{i+1}(x_{i+1})) \\ &= (x_1, \dots, x_{j+1}, \\ &\quad x_{j+2} *_{j+2}^{j+2} k_{j+2}^j s_j^{j+2}(x_{j+2}), \dots, x_{i+1} *_{j+1}^{i+1} k_{i+1}^j s_j^{i+1}(x_{i+1})) \\ &= (x_1, \dots, x_{i+1}) \\ &\quad (\text{by Axioms (RUnit}_{l,j}) \text{ for } l \text{ such that } j+2 \leq l \leq i+1) \\ &= \bar{x}, \end{aligned}$$

and

$$\begin{aligned} \tilde{k}_i^j \tilde{t}_j^i(\bar{x}) *_{j+1}^i \bar{x} &= (x_1, \dots, x_{j+1}, k_{j+2}^j t_j^{j+2}(x_{j+2}), \dots, k_{i+1}^j t_j^{i+1}(x_{i+1})) *_{j+1}^i \bar{x} \\ &= (x_1, \dots, x_{j+1}, \\ &\quad k_{j+2}^j t_j^{j+2}(x_{j+2}) *_{j+2}^{j+2} x_{j+2}, \dots, k_{i+1}^j t_j^{i+1}(x_{i+1}) *_{j+1}^{i+1} x_{i+1}) \\ &= (x_1, \dots, x_{i+1}) \\ &\quad (\text{by Axioms (LUnit}_{l,j}) \text{ for } l \text{ such that } j+2 \leq l \leq i+1) \\ &= \bar{x}. \end{aligned} \quad \square$$

Proposition 6.8. *If (C, F_0) satisfies Axiom (FUnit), then so does (C, K_0) .*

Proof. Let $i > j \geq 0$ and let (\bar{x}, \bar{y}) be in $\tilde{X}_i \times_{\tilde{X}_j} \tilde{X}_i$. We have

$$\begin{aligned}
\tilde{k}_i(\bar{x} \tilde{*}_j^i \bar{y}) &= \tilde{k}_i(x_1, \dots, x_{j+1}, x_{j+2} *_{j+2}^{j+2} y_{j+2}, \dots, x_{i+1} *_{j+1}^{i+1} y_{i+1}) \\
&= (x_1, \dots, x_{j+1}, x_{j+2} *_{j+2}^{j+2} y_{j+2}, \dots, x_{i+1} *_{j+1}^{i+1} y_{i+1}, \\
&\quad k_{i+2}^i s_{i+1}(x_{i+1} *_{j+1}^{i+1} y_{i+1})) \\
&= (x_1, \dots, x_{j+1}, x_{j+2} *_{j+2}^{j+2} y_{j+2}, \dots, x_{i+1} *_{j+1}^{i+1} y_{i+1}, \\
&\quad k_{i+2}^i (s_{i+1}(x_{i+1}) *_{j+1}^i s_{i+1}(y_{i+1}))) \\
&= (x_1, \dots, x_{j+1}, x_{j+2} *_{j+2}^{j+2} y_{j+2}, \dots, x_{i+1} *_{j+1}^{i+1} y_{i+1}, \\
&\quad (k_{i+2}^i s_{i+1}(x_{i+1}) *_{j+1}^{i+2} k_{i+2}^i s_{i+1}(y_{i+1}))) \\
&\quad (\text{by Axioms (FUnit}_{i,j}) \text{ and (FUnit}_{i+1,j})) \\
&= (x_1, \dots, x_{i+1}, k_{i+2}^i s_{i+1}(x_{i+1})) \tilde{*}_j^i (y_1, \dots, y_{i+1}, k_{i+2}^i s_{i+1}(y_{i+1})) \\
&= \tilde{k}_i(\bar{x}) \tilde{*}_j^{i+1} \tilde{k}_i(\bar{y}). \quad \square
\end{aligned}$$

Proposition 6.9. *The $\tilde{\Omega}_j^i$'s are well-defined. Moreover, if (C, F_0) satisfies Axioms (Ass), (Exc), (LUnit), (RUnit) and (RInv), then the $\tilde{\Omega}_j^i$'s have the desired globular source and target, i.e., they satisfy the condition of the definition of a pregroupoidal globular extension (see paragraph 3.17).*

Proof. Note that by the remark at the end of paragraph 2.3, (C, F_0) also satisfies Axiom (FInv).

Let $i > j \geq 0$ and let \bar{x} be in \tilde{X}_i . Let us first prove that $\tilde{w}_j^i(\bar{x})$ belongs to \tilde{X}_i . We need to show that

$$\begin{aligned}
s_j(x_j) &= t_j t_{j+1} (x_{j+1} *_{j+1}^{j+1} t_{j+2}(x_{j+2})), \\
s_{j+1}(x_{j+1} *_{j+1}^{j+1} t_{j+2}(x_{j+2})) &= t_{j+1} t_{j+2} w_j^{j+2}(x_{j+2}),
\end{aligned}$$

and

$$s_l w_j^l(x_l) = t_l t_{l+1} w_j^{l+1}(x_{l+1}), \quad j+2 \leq l \leq i.$$

The first identity has already been proved in paragraph 5.3. The two others follow from the following calculations:

$$\begin{aligned}
s_{j+1}(x_{j+1} *_{j+1}^{j+1} t_{j+2}(x_{j+2})) &= s_{j+1} t_{j+2}(x_{j+2}) \\
&= t_{j+1} w_j^{j+1} t_{j+2}(x_{j+2}) \\
&= t_{j+1} t_{j+2} w_j^{j+2}(x_{j+2}),
\end{aligned}$$

and

$$\begin{aligned}
s_l w_j^l(x_l) &= w_j^{l-1} s_l(x_l) \\
&= w_j^{l-1} t_l t_{l+1}(x_{l+1}) \\
&= t_l t_{l+1} w_j^{l+1}(x_{l+1}).
\end{aligned}$$

Let us now prove that $\tilde{w}_j^i(\bar{x})$ has the desired globular source and target. For $j = i - 1$, we have

$$\begin{aligned}
 \tilde{s}_i \tilde{w}_{i-1}^i(\bar{x}) &= \tilde{s}_i(x_1, \dots, x_{i-1}, x_i *_{i-1}^i t_{i+1}(x_{i+1}), w_{i-1}^{i+1}(x_{i+1})) \\
 &= (x_1, \dots, x_{i-1}, (x_i *_{i-1}^i t_{i+1}(x_{i+1})) *_{i-1}^i t_{i+1} w_{i-1}^{i+1}(x_{i+1})) \\
 &= (x_1, \dots, x_{i-1}, (x_i *_{i-1}^i t_{i+1}(x_{i+1})) *_{i-1}^i w_{i-1}^i t_{i+1}(x_{i+1})) \\
 &= (x_1, \dots, x_{i-1}, x_i *_{i-1}^i (t_{i+1}(x_{i+1}) *_{i-1}^i w_{i-1}^i t_{i+1}(x_{i+1}))) \\
 &\quad \text{(by Axiom (Ass}_{i,i-1})} \\
 &= (x_1, \dots, x_{i-1}, x_i *_{i-1}^i k_{i-1} t_i t_{i+1}(x_{i+1})) \\
 &\quad \text{(by Axiom (RInv}_{i,i-1})} \\
 &= (x_1, \dots, x_{i-1}, x_i *_{i-1}^i k_{i-1} s_i(x_i)) \\
 &= (x_1, \dots, x_i) \\
 &\quad \text{(by Axiom (RUnit}_{i,i-1})} \\
 &= \tilde{t}_i(\bar{x}),
 \end{aligned}$$

and

$$\begin{aligned}
 \tilde{t}_i \tilde{w}_{i-1}^i(\bar{x}) &= \tilde{t}_i(x_1, \dots, x_{i-1}, x_i *_{i-1}^i t_{i+1}(x_{i+1}), w_{i-1}^{i+1}(x_{i+1})) \\
 &= (x_1, \dots, x_{i-1}, x_i *_{i-1}^i t_{i+1}(x_{i+1})) \\
 &= \tilde{s}_i(\bar{x}).
 \end{aligned}$$

For $j < i - 1$, we have

$$\begin{aligned}
 \tilde{s}_i \tilde{w}_j^i(\bar{x}) &= \tilde{s}_i(x_1, \dots, x_j, x_{j+1} *_{j+1}^{j+1} t_{j+2}(x_{j+2}), w_j^{j+2}(x_{j+2}), \dots, w_j^{i+1}(x_{i+1})) \\
 &= (x_1, \dots, x_j, x_{j+1} *_{j+1}^{j+1} t_{j+2}(x_{j+2}), \\
 &\quad w_j^{j+2}(x_{j+2}), \dots, w_j^{i-1}(x_{i-1}), w_j^i(x_i) *_{i-1}^i t_{i+1} w_j^{i+1}(x_{i+1})) \\
 &= (x_1, \dots, x_j, x_{j+1} *_{j+1}^{j+1} t_{j+2}(x_{j+2}), \\
 &\quad w_j^{j+2}(x_{j+2}), \dots, w_j^{i-1}(x_{i-1}), w_j^i(x_i) *_{i-1}^i w_j^i t_{i+1}(x_{i+1})) \\
 &= (x_1, \dots, x_j, x_{j+1} *_{j+1}^{j+1} t_{j+2}(x_{j+2}), \\
 &\quad w_j^{j+2}(x_{j+2}), \dots, w_j^{i-1}(x_{i-1}), w_j^i(x_i *_{i-1}^i t_{i+1}(x_{i+1}))) \\
 &\quad \text{(by Axiom (FInv}_{i,i-1,j})} \\
 &= \tilde{w}_j^{i-1}(x_1, \dots, x_{i-1}, x_i *_{i-1}^i t_{i+1}(x_{i+1})) \\
 &= \tilde{w}_j^{i-1} \tilde{s}_i(\bar{x}),
 \end{aligned}$$

and

$$\begin{aligned}
 \tilde{t}_i \tilde{w}_j^i(\bar{x}) &= \tilde{t}_i(x_1, \dots, x_j, x_{j+1} *_{j+1}^{j+1} t_{j+2}(x_{j+2}), w_j^{j+2}(x_{j+2}), \dots, w_j^{i+1}(x_{i+1})) \\
 &= (x_1, \dots, x_j, x_{j+1} *_{j+1}^{j+1} t_{j+2}(x_{j+2}), w_j^{j+2}(x_{j+2}), \dots, w_j^i(x_i)) \\
 &= \tilde{w}_j^{i-1}(x_1, \dots, x_i) \\
 &= \tilde{w}_j^{i-1} \tilde{t}_i(\bar{x}),
 \end{aligned}$$

hence the result. \square

Proposition 6.10. *If (C, F_0) satisfies Axioms (LInv) and (RInv), then so does (C, K_0) .*

Proof. Let $i > j \geq 0$ and let \bar{x} be in \tilde{X}_i . We have

$$\begin{aligned}
\tilde{w}_j^i(\bar{x}) *_{\tilde{j}}^i \bar{x} &= (x_1, \dots, x_j, x_{j+1} *_{\tilde{j}}^{j+1} t_{j+2}(x_{j+2}), \\
&\quad w_j^{j+2}(x_{j+2}), \dots, w_j^{i+1}(x_{i+1})) *_{\tilde{j}}^i \bar{x} \\
&= (x_1, \dots, x_j, x_{j+1} *_{\tilde{j}}^{j+1} t_{j+2}(x_{j+2}), \\
&\quad w_j^{j+2}(x_{j+2}) *_{\tilde{j}}^{j+2} x_{j+2}, \dots, w_j^{i+1}(x_{i+1}) *_{\tilde{j}}^{i+1} x_{i+1}) \\
&= (x_1, \dots, x_j, x_{j+1} *_{\tilde{j}}^{j+1} t_{j+2}(x_{j+2}), \\
&\quad k_{j+2}^j s_j^{j+2}(x_{j+2}), \dots, k_{i+1}^j s_j^{i+1}(x_{i+1})) \\
&\quad (\text{by Axioms (LInv}_{l,j}) \text{ for } l \text{ such that } j+2 \leq l \leq i+1) \\
&= \tilde{k}_i^j \tilde{s}_j^i(\bar{x}),
\end{aligned}$$

where the last equality is equation $(*_{ks})$ (see the proof of Proposition 6.7), and

$$\begin{aligned}
\bar{x} *_{\tilde{j}}^i \tilde{w}_j^i(\bar{x}) &= \bar{x} *_{\tilde{j}}^i (x_1, \dots, x_j, x_{j+1} *_{\tilde{j}}^{j+1} t_{j+2}(x_{j+2}), \\
&\quad w_j^{j+2}(x_{j+2}), \dots, w_j^{i+1}(x_{i+1})) \\
&= (x_1, \dots, x_{j+1}, \\
&\quad x_{j+2} *_{\tilde{j}}^{j+2} w_j^{j+2}(x_{j+2}), \dots, x_{i+1} *_{\tilde{j}}^{i+1} w_j^{i+1}(x_{i+1})) \\
&= (x_1, \dots, x_{j+1}, k_{j+2}^j t_j^{j+2}(x_{j+2}), \dots, k_{i+1}^j t_j^{i+1}(x_{i+1})) \\
&\quad (\text{by Axioms (RInv}_{l,j}) \text{ for } l \text{ such that } j+2 \leq l \leq i+1) \\
&= \tilde{k}_i^j \tilde{t}_j^i(\bar{x}),
\end{aligned}$$

where the last equality is equation $(*_{kt})$ (see the proof of Proposition 6.7). \square

Corollary 6.11. *If (C, F_0) is groupoidal, then (C, K_0) (endowed with the $\tilde{\nabla}_j^i$'s, $\tilde{\kappa}_i$'s and $\tilde{\Omega}_j^i$'s) is a groupoidal globular extension.*

As announced at the beginning of this section, if (C, F_0) is a groupoidal globular extension, we will call (C, K_0) the *twisted groupoidal globular extension* of (C, F_0) .

7. THE DÉCALAGE ON $\tilde{\Theta}$

7.1. We now introduce the morphisms that will give rise to our décalage on $\tilde{\Theta}$.

Let (C, F) be a globular extension endowed with ∇_i 's as in Section 5 and let (C, K) be the twisted globular extension of (C, F) . We define morphisms

$$\begin{aligned}
\alpha_i &: D_i \rightarrow \tilde{D}_i, \quad i \geq 0, \\
\beta_i &: D_0 \rightarrow \tilde{D}_i, \quad i \geq 0,
\end{aligned}$$

by the formulas

$$\begin{aligned}
\alpha_i &= \varepsilon_{i+1} \sigma_{i+1}, \\
\beta_i &= \varepsilon_1 \tau_1.
\end{aligned}$$

Dually, we define maps

$$\begin{aligned} a_i &: \tilde{X}_i \rightarrow X_i, \quad i \geq 0, \\ b_i &: \tilde{X}_i \rightarrow X_0, \quad i \geq 0, \end{aligned}$$

by the formulas

$$\begin{aligned} a_i(x_1, \dots, x_{i+1}) &= s_{i+1}(x_{i+1}), \\ b_i(x_1, \dots, x_{i+1}) &= t_1(x_1). \end{aligned}$$

Proposition 7.2. *The maps*

$$\begin{aligned} D_i &\mapsto \alpha_i, \quad i \geq 0, \\ D_i &\mapsto \beta_i, \quad i \geq 0, \end{aligned}$$

define natural transformations

$$F \xrightarrow{\alpha} K \xleftarrow{\beta} D_0$$

(where D_0 denotes the constant functor $\mathbb{G} \rightarrow C$ of value D_0).

Proof. Let us first prove that α is a natural transformation. We must show that

$$\tilde{\sigma}_i \alpha_{i-1} = \alpha_i \sigma_i \quad \text{and} \quad \tilde{\tau}_i \alpha_{i-1} = \alpha_i \tau_i, \quad i \geq 1.$$

Let $i \geq 1$ and let \bar{x} be in \tilde{X}_i . We have

$$\begin{aligned} a_{i-1} \tilde{s}_i(\bar{x}) &= a_{i-1}(x_1, \dots, x_{i-1}, x_i *_{i-1}^i t_{i+1}(x_{i+1})) \\ &= s_i(x_i *_{i-1}^i t_{i+1}(x_{i+1})) \\ &= s_i t_{i+1}(x_{i+1}) \\ &= s_i s_{i+1}(x_{i+1}) \\ &= s_i a_i(\bar{x}), \end{aligned}$$

and

$$\begin{aligned} a_{i-1} \tilde{t}_i(\bar{x}) &= a_i(x_1, \dots, x_i) \\ &= s_i(x_i) \\ &= t_i t_{i+1}(x_{i+1}) \\ &= t_i s_{i+1}(x_{i+1}) \\ &= t_i a_i(\bar{x}), \end{aligned}$$

hence the naturality of α .

To prove the naturality of β , we must check that

$$\tilde{\sigma}_i \beta_{i-1} = \beta_i \quad \text{and} \quad \tilde{\tau}_i \beta_{i-1} = \beta_i, \quad i \geq 1.$$

This follows from the following calculations:

$$\begin{aligned} b_{i-1} \tilde{s}_i(\bar{x}) &= b_{i-1}(x_1, \dots, x_{i-1}, x_i *_{i-1}^i t_{i+1}(x_{i+1})) \\ &= t_1(x_1) \\ &= b_i(\bar{x}), \end{aligned}$$

and

$$\begin{aligned} b_{i-1}\tilde{t}_i(\bar{x}) &= b_{i-1}(x_1, \dots, x_i) \\ &= t_1(x_1) \\ &= b_i(\bar{x}). \end{aligned} \quad \square$$

7.3. Let now C be equal to $\tilde{\Theta}$. By the previous proposition, we have a diagram

$$F \xrightarrow{\alpha} K \xleftarrow{\beta} D_0$$

of functors from \mathbb{G} to $\tilde{\Theta}$. The functor F is globular by definition, the functor K is globular by Proposition 5.9 and the functor D_0 is trivially globular. This diagram thus lives in $\text{Ext}_{\text{gl}}(\tilde{\Theta})$. Let $F_0 : \Theta_0 \rightarrow \tilde{\Theta}$ be the canonical functor. By the universal property of Θ_0 (Proposition 3.2), we obtain a diagram

$$F_0 \xrightarrow{\alpha_0} K_0 \xleftarrow{\beta_0} D_0$$

in $\underline{\text{Hom}}_{\text{gl}}(\Theta_0, \tilde{\Theta})$. Note that for the same reason as in paragraph 5.11, this lifting is unique. But this diagram lives in $\text{Ext}_{\text{gr}}(\tilde{\Theta})$. Indeed, $(\tilde{\Theta}, F_0)$ is a groupoidal globular extension by definition, $(\tilde{\Theta}, K_0)$ is a groupoidal globular extension by Proposition 6.11 and $(\tilde{\Theta}, D_0)$ is trivially a groupoidal globular extension. Hence by the universal property of $\tilde{\Theta}$ (Proposition 3.18), this diagram lifts to a unique diagram

$$1_{\tilde{\Theta}} \xrightarrow{\tilde{\alpha}} \tilde{K} \xleftarrow{\tilde{\beta}} D_0$$

in $\underline{\text{Hom}}_{\text{gl}_0}(\tilde{\Theta}, \tilde{\Theta})$. This is our desired décalage on $\tilde{\Theta}$. We will denote it by $\mathcal{D}_{\tilde{\Theta}}$.

7.4. We will now construct a splitting to the décalage $\mathcal{D}_{\tilde{\Theta}}$. Let

$$\rho_i : \tilde{D}_i \rightarrow D_i, \quad i \geq 0,$$

be the morphism defined by the formula

$$\rho_i = (\tau_0^i \kappa_0, \dots, \tau_{i-1}^i \kappa_{i-1}, \kappa_i).$$

This morphism is not natural in i . For instance, the square

$$\begin{array}{ccc} \tilde{D}_0 & \xrightarrow{\rho_0} & D_0 \\ \tilde{\sigma}_1 \downarrow & & \downarrow \sigma_1 \\ \tilde{D}_1 & \xrightarrow{\rho_1} & D_1 \end{array}$$

is not commutative. Therefore, we cannot extend formally ρ to a general globular sum. Denote by

$$\rho_{j,i} : \tilde{D}_{j,i} \rightarrow D_i, \quad i \geq j \geq 0,$$

the composition of the canonical morphism $\tilde{D}_{j,i} \rightarrow \tilde{D}_i$ followed by ρ_i . If S is a globular sum whose table of dimensions is

$$\begin{pmatrix} i_1 & & i_2 & & \cdots & & i_n \\ & i'_1 & & i'_2 & & \cdots & & i'_{n-1} \end{pmatrix},$$

we define

$$\rho_S : \tilde{S} = \tilde{D}_{i_1} \amalg_{D_{i'_1}} \tilde{D}_{i'_1+1, i_2} \amalg_{D_{i'_2}} \cdots \amalg_{D_{i'_{n-1}}} \tilde{D}_{i'_{n-1}+1, i_n} \rightarrow S$$

by the formula

$$\rho_S = \rho_{i_1} \amalg_{D_{i'_1}} \rho_{i'_1+1, i_2} \amalg_{D_{i'_2}} \cdots \amalg_{D_{i'_{n-1}}} \rho_{i'_{n-1}+1, i_n}.$$

Dually, we define maps

$$\begin{aligned} r_i &: X_i \rightarrow \tilde{X}_i, \quad i \geq 0, \\ r_{j,i} &: X_i \rightarrow \tilde{X}_{j,i}, \quad i \geq j \geq 0, \\ r_S &: X(S) \rightarrow \tilde{X}(S), \quad S \text{ globular sum,} \end{aligned}$$

by the formulas

$$\begin{aligned} r_i(x_i) &= (k_0 t_0^i(x_i), \dots, k_{i-1} t_{i-1}^i(x_i), k_i(x_i)), \\ r_{j,i}(x_i) &= (k_j t_j^i(x_i), \dots, k_{i-1} t_{i-1}^i(x_i), k_i(x_i)), \\ r_S(x_{i_1}, \dots, x_{i_n}) &= (r_{i_1}(x_{i_1}), r_{i'_1+1, i_2}(x_{i_2}), \dots, r_{i'_{n-1}+1, i_n}(x_{i_n})). \end{aligned}$$

Proposition 7.5. *The ρ_S 's are well-defined. Moreover, for every object S of $\tilde{\Theta}$, we have*

$$\rho_S \tilde{\alpha}_S = 1_S.$$

In other words, ρ is a splitting of $\mathcal{D}_{\tilde{\Theta}}$.

Proof. Let $i \geq 0$ and let x_i be in X_i . To prove that $r_i(x_i)$ belongs to \tilde{X}_i , we need to check that

$$s_l k_{l-1} t_{l-1}^i(x_i) = t_l t_{l+1} k_l t_l^i(x_i), \quad 1 \leq l \leq i.$$

But using the identities $s_l k_{l-1} = 1_{X_{l-1}}$ and $t_{l+1} k_l = 1_{X_l}$, we get that both sides are equal to $t_{l-1}^i(x_i)$.

Let now S be a globular sum whose table of dimensions is

$$\begin{pmatrix} i_1 & i_2 & \cdots & i_n \\ i'_1 & i'_2 & \cdots & i'_{n-1} \end{pmatrix},$$

and let $(x_{i_1}, \dots, x_{i_n})$ be in $X(S)$. To prove that $r_S(x_{i_1}, \dots, x_{i_n})$ belongs to $\tilde{X}(S)$, we need to check that

$$s_{i'_l}^{i_l+1} k_{i_l}(x_{i_l}) = t_{i'_l}^{i'_l+2} k_{i'_l+1} s_{i'_l+1}^{i_l+1}(x_{i_{l+1}}), \quad 1 \leq l \leq n-1.$$

But this equality is equivalent to the equality

$$s_{i'_l}^{i_l}(x_{i_l}) = t_{i'_l}^{i'_l+1}(x_{i_{l+1}})$$

which holds by definition of $X(S)$.

Let us now prove that r_S is a section of a_S . We easily check that r_i is a section of a_i :

$$\begin{aligned} a_i r_i(x_i) &= (k_0 t_0^i(x_i), \dots, k_{i-1} t_{i-1}^i(x_i), k_i(x_i)) \\ &= s_{i+1} k_i(x_i) \\ &= x_i. \end{aligned}$$

More generally, if

$$a_{j,i} : \tilde{X}_{j,i} \rightarrow X_i, \quad i \geq j \geq 0,$$

is defined by the formula

$$a_{j,i}(x_{j+1}, \dots, x_{i+1}) = s_{i+1}(x_{i+1}),$$

the same calculation shows that $r_{j,i}$ is a section of $a_{j,i}$.

Let $\tilde{a}_S = X(\tilde{\alpha}_S)$ and let \tilde{a}'_S be the morphism \tilde{a}_S viewed as a morphism

$$X_{i_1} \times_{X_{i'_1}} \dots \times_{X_{i'_{n-1}}} X_{i_n} \rightarrow \tilde{X}_{i_1} \times_{\tilde{X}_{i'_1}} \dots \times_{\tilde{X}_{i'_{n-1}}} \tilde{X}_{i_n}.$$

By definition, we have

$$\tilde{a}'_S = r_{i_1} \times_{r_{i'_1}} \dots \times_{r_{i'_{n-1}}} r_{i_n}.$$

Let d be the canonical isomorphism

$$\tilde{X}_{i_1} \times_{X_{i'_1}} \tilde{X}_{i'_1+1, i_2} \times_{X_{i'_2}} \dots \times_{X_{i'_{n-1}}} \tilde{X}_{i'_{n-1}+1, i_n} \rightarrow \tilde{X}_{i_1} \times_{\tilde{X}_{i'_1}} \dots \times_{\tilde{X}_{i'_{n-1}}} \tilde{X}_{i_n}.$$

We recall that

$$\begin{aligned} d(x_1^1, \dots, x_{i_1+1}^1, x_{i'_1+2}^2, \dots, x_{i_2+1}^2, \dots, x_{i'_{n-1}+2}^n, \dots, x_{i_n+1}^n) \\ = (x_1^1, \dots, x_{i_1+1}^1, x_1^2, \dots, x_{i_2+1}^2, \dots, x_1^n, \dots, x_{i_n+1}^n), \end{aligned}$$

where the

$$x_j^l, \quad 2 \leq l \leq n, \quad 1 \leq j \leq i'_l + 1,$$

are defined by formulas given in paragraph 5.10.

We thus have

$$\begin{aligned} \tilde{a}_S(x_1^1, \dots, x_{i_1+1}^1, x_{i'_1+2}^2, \dots, x_{i_2+1}^2, \dots, x_{i'_{n-1}+2}^n, \dots, x_{i_n+1}^n) \\ = \tilde{a}'_S d(x_1^1, \dots, x_{i_1+1}^1, x_{i'_1+2}^2, \dots, x_{i_2+1}^2, \dots, x_{i'_{n-1}+2}^n, \dots, x_{i_n+1}^n) \\ = \tilde{a}'_S(x_1^1, \dots, x_{i_1+1}^1, x_1^2, \dots, x_{i_2+1}^2, \dots, x_1^n, \dots, x_{i_n+1}^n) \\ = (a_{i_1}(x_1^1, \dots, x_{i_1+1}^1), \dots, a_{i_n}(x_1^n, \dots, x_{i_n+1}^n)) \\ = (s_{i_1+1}(x_{i_1+1}^1), \dots, s_{i_n+1}(x_{i_n+1}^n)), \end{aligned}$$

hence the equality

$$\tilde{a}_S = a_{i_1} \times_{X_{i'_1}} a_{i'_1+1, i_2} \times_{X_{i'_2}} \dots \times_{X_{i'_{n-1}}} a_{i'_{n-1}+1, i_n}.$$

We can now compute $\tilde{a}_S r_S$:

$$\begin{aligned} \tilde{a}_S r_S(x_{i_1}, \dots, x_{i_n}) \\ = \tilde{a}_S(r_{i_1}(x_{i_1}), r_{i'_1+1, i_2}(x_{i_2}), \dots, r_{i'_{n-1}+1, i_n}(x_{i_n})) \\ = (a_{i_1} r_{i_1}(x_{i_1}), a_{i'_1+1, i_2} r_{i'_1+1, i_2}(x_{i_2}), \dots, a_{i'_{n-1}+1, i_n} r_{i'_{n-1}+1, i_n}(x_{i_n})) \\ = (x_{i_1}, \dots, x_{i_n}), \end{aligned}$$

where the last equality follows from the fact that $r_{j,i}$ is a section of $a_{j,i}$. We thus have shown that r_S is a section of \tilde{a}_S . \square

Remark 7.6. The category $\tilde{\Theta}$ has been defined by a universal property related to the notion of strict ∞ -groupoid. For each $n \geq 1$, we can define a category $\tilde{\Theta}_n$ enjoying a similar universal property with respect to the notion of strict n -groupoid. The category $\tilde{\Theta}_n$ can be seen as the full subcategory of $\tilde{\Theta}$ whose objects are globular sums of dimension at most n , i.e., globular sums

$$D_{i_1} \amalg_{D_{i'_1}} \cdots \amalg_{D_{i'_{n-1}}} D_{i_n},$$

with $i_k \leq n$ for all k such that $1 \leq k \leq n$. Let us denote by i_n the inclusion functor $\tilde{\Theta}_n \rightarrow \tilde{\Theta}$. This functor admits a left adjoint $p_n : \tilde{\Theta} \rightarrow \tilde{\Theta}_n$ which truncates globular sums in dimension n , i.e., which sends the globular sum

$$D_{i_1} \amalg_{D_{i'_1}} \cdots \amalg_{D_{i'_{n-1}}} D_{i_n}$$

to the (possibly degenerated) globular sum

$$D_{j_1} \amalg_{D_{j'_1}} \cdots \amalg_{D_{j'_{n-1}}} D_{j_n},$$

where

$$\begin{aligned} j_k &= \min(i_k, n), & 1 \leq k \leq n, \\ j'_k &= \min(i'_k, n), & 1 \leq k \leq n-1. \end{aligned}$$

Note that we have $p_n i_n = 1_{\tilde{\Theta}}$.

The décalage

$$\mathcal{D}_{\tilde{\Theta}} = 1_{\tilde{\Theta}} \xrightarrow{\tilde{\alpha}} \tilde{K} \xleftarrow{\tilde{\beta}} D_0$$

induces a décalage

$$\mathcal{D}_{\tilde{\Theta}_n} = 1_{\tilde{\Theta}_n} \xrightarrow{\tilde{\alpha}_n} \tilde{K}_n \xleftarrow{\tilde{\beta}_n} D_0$$

on $\tilde{\Theta}_n$, defined by

$$\tilde{K}_n = p_n \tilde{K} i_n, \quad \tilde{\alpha}_n = p_n * \tilde{\alpha} * i_n \quad \text{and} \quad \tilde{\beta}_n = p_n * \tilde{\beta} * i_n.$$

Moreover, every splitting of $\mathcal{D}_{\tilde{\Theta}}$ induces a splitting of $\mathcal{D}_{\tilde{\Theta}_n}$. Note that the inclusion functor $i_n : \tilde{\Theta}_n \rightarrow \tilde{\Theta}$ is *not* a morphism of décalages.

For each $n \geq 1$, the category $\tilde{\Theta}_n$ is canonically isomorphic to the full subcategory of the category of strict n -groupoids whose objects are free strict n -groupoids on globular pasting schemes of dimension at most n . In particular, $\tilde{\Theta}_1$ is canonically isomorphic to the category $\tilde{\Delta}$ defined as follows: the objects of $\tilde{\Delta}$ are the sets

$$\Delta_n = \{0, \dots, n\}, \quad n \geq 0,$$

and its morphisms are *all* the applications between these sets.

Let us now try to understand the induced décalage on $\tilde{\Delta} = \tilde{\Theta}_1$. The functor \tilde{K}_1 sends Δ_n to Δ_{n+1} and we thus set $\tilde{\Delta}_n = \Delta_{n+1}$. The functor $p_1 : \tilde{\Theta} \rightarrow \tilde{\Delta}$ sends the morphisms

$$\begin{aligned} \nabla_0^1 &: D_1 \rightarrow D_1 \amalg_{D_0} D_1, \\ \kappa_0 &: D_1 \rightarrow D_0, \\ \Omega_0^1 &: D_1 \rightarrow D_1, \end{aligned}$$

to the morphisms

$$\begin{aligned}\nabla &: \Delta_1 \rightarrow \Delta_1 \amalg_{\Delta_0} \Delta_1 = \Delta_2, \\ \kappa &: \Delta_1 \rightarrow \Delta_0, \\ \Omega &: \Delta_1 \rightarrow \Delta_1,\end{aligned}$$

defined by

$$\begin{aligned}\nabla &: 0 \mapsto 0, & 1 \mapsto 2, \\ \kappa &: 0 \mapsto 0, & 1 \mapsto 0, \\ \Omega &: 0 \mapsto 1, & 1 \mapsto 0.\end{aligned}$$

In the same way, the morphisms

$$\begin{aligned}\tilde{\nabla}_0^1 &: \tilde{D}_1 = D_1 \amalg_{D_0} D_2 \rightarrow \tilde{D}_1 \amalg_{\tilde{D}_0} \tilde{D}_1 = D_1 \amalg_{D_0} D_2 \amalg_{D_1} \amalg_{D_2}, \\ \tilde{\kappa}_0 &: \tilde{D}_1 = D_1 \amalg_{D_0} D_2 \rightarrow \tilde{D}_0 = D_1, \\ \tilde{\Omega}_0^1 &: \tilde{D}_1 = D_1 \amalg_{D_0} D_2 \rightarrow \tilde{D}_1 = D_1 \amalg_{D_0} D_2,\end{aligned}$$

are sent to the morphisms

$$\begin{aligned}\tilde{\nabla} &: \tilde{\Delta}_1 = \Delta_2 \rightarrow \tilde{\Delta}_1 \amalg_{\tilde{\Delta}_0} \tilde{\Delta}_1 = \Delta_3, \\ \tilde{\kappa} &: \tilde{\Delta}_1 = \Delta_2 \rightarrow \tilde{\Delta}_0 = \Delta_1, \\ \tilde{\Omega} &: \tilde{\Delta}_1 = \Delta_2 \rightarrow \tilde{\Delta}_1 = \Delta_2,\end{aligned}$$

defined by

$$\begin{aligned}\tilde{\nabla} &: 0 \mapsto 0, & 1 \mapsto 2, & 2 \mapsto 3, \\ \tilde{\kappa} &: 0 \mapsto 0, & 1 \mapsto 0, & 2 \mapsto 1, \\ \tilde{\Omega} &: 0 \mapsto 1, & 1 \mapsto 0, & 2 \mapsto 2.\end{aligned}$$

Let D be the endofunctor of $\tilde{\Delta}$ defined by

$$D(\Delta_n) = \tilde{\Delta}_n = \Delta_{n+1}$$

for every $n \geq 0$, and by

$$D(\varphi)(k) = \begin{cases} \varphi(k), & 0 \leq k \leq m, \\ n+1, & k = m+1, \end{cases}$$

for every morphism $\varphi : \Delta_m \rightarrow \Delta_n$ of $\tilde{\Delta}$. We have

$$\tilde{\nabla} = D(\nabla), \quad \tilde{\kappa} = D(\kappa) \quad \text{and} \quad \tilde{\Omega} = D(\Omega).$$

Thus the functors \tilde{K}_1 and D agree on objects and on the morphisms ∇ , κ and Ω . The universal property of $\tilde{\Delta} = \tilde{\Theta}_1$ then implies that $\tilde{K}_1 = D$. One can show in a similar way that the natural transformations $\tilde{\alpha}_1 : 1_{\tilde{\Delta}} \rightarrow \tilde{K}_1$ and $\tilde{\beta}_1 : \Delta_0 \rightarrow \tilde{K}_1$ are induced by the applications

$$\begin{array}{ccc} \Delta_n \rightarrow \Delta_{n+1} & & \Delta_0 \rightarrow \Delta_{n+1} \\ k \mapsto k & \text{and} & 0 \mapsto n+1. \end{array}$$

Note that this décalage restricts to the subcategory Δ of $\tilde{\Delta}$ whose objects are the Δ_n 's and whose morphisms are order-preserving maps. The induced décalage on Δ is precisely the one defined in Example 3.14 of [10].

8. $\tilde{\Theta}$ IS A TEST CATEGORY

Proposition 8.1. *The object D_0 is terminal in $\tilde{\Theta}$.*

Proof. This is an immediate consequence of Proposition 3.22. \square

Proposition 8.2. *(D_1, σ_1, τ_1) is a separating interval on $\hat{\tilde{\Theta}}$.*

Proof. We need to show that the equalizer of $\sigma_1, \tau_1 : D_0 \rightarrow D_1$ in $\hat{\tilde{\Theta}}$ is the initial presheaf, i.e., that there does not exist an object S in $\tilde{\Theta}$ such that the diagram

$$S \longrightarrow D_0 \begin{array}{c} \xrightarrow{\sigma_1} \\ \xrightarrow{\tau_1} \end{array} D_1$$

is commutative. Suppose that such an S exists. By precomposing with a morphism from D_0 , we can assume that S is D_0 . Since by the previous proposition, D_0 is a terminal object, that would imply that σ_1 and τ_1 are equal. By the universal property of $\tilde{\Theta}$ (Proposition 3.21), that would mean that s_1 and t_1 are equal for every strict ∞ -groupoid. This is obviously false. \square

Theorem 8.3. *The category $\tilde{\Theta}$ is a strict test category.*

Proof. By the previous proposition, (D_1, σ_1, τ_1) is a separating interval on $\hat{\tilde{\Theta}}$. Moreover, since D_1 is a representable presheaf, it is aspherical. Furthermore, by paragraph 7.3 and Proposition 7.5, $\tilde{\Theta}$ admits a splittable décalage. Hence the result by Proposition 4.12. \square

Corollary 8.4. *The pair $(\hat{\tilde{\Theta}}, \mathcal{W}_{\tilde{\Theta}})$ is endowed with a structure of model category whose cofibrations are the monomorphisms. This model category structure is cofibrantly generated, proper and the weak equivalences are stable by binary products.*

Moreover, the homotopy category $\text{Hot}_{\tilde{\Theta}}$ of $\hat{\tilde{\Theta}}$ is canonically equivalent to the homotopy category Hot .

Proof. This follows from the previous theorem by Theorem 4.6. \square

8.5. A similar proof (using the very same calculations) shows analogous results for the category Θ . Indeed, let (C, F_0) be a categorical globular extension. The definitions of the ∇_j^i 's and κ_i 's of paragraph 6.2 still make sense. Moreover, by Propositions 6.3, 6.4, 6.5, 6.6, 6.7 and 6.8, the twisted globular extension (C, K_0) under Θ_0 , endowed with these morphisms, is a categorical globular extension. By this result and the universal property of Θ , we can construct a décalage \mathcal{D}_Θ on Θ as we did in paragraph 7.3 for $\tilde{\Theta}$. The definition of the ρ_S 's of paragraph 7.4 still makes sense and the proof of Proposition 7.5 applies and shows that ρ is a splitting of \mathcal{D}_Θ . Moreover, the proof of Proposition 8.2 shows that (D_1, σ_1, τ_1) is a separating interval on $\hat{\tilde{\Theta}}$. We hence obtain by Theorem 4.12 that Θ is a strict test category. In particular, $\hat{\tilde{\Theta}}$ is endowed with a model category

structure as in Corollary 8.4. One can show that the décalage \mathcal{D}_Θ and its splitting are the same as those constructed in [10].

Moreover, since the décalages \mathcal{D}_Θ and $\mathcal{D}_{\tilde{\Theta}}$ are defined in a uniform way, the canonical functor $i : \Theta \rightarrow \tilde{\Theta}$ (obtained by the universal property of Θ) induces a morphism of décalages. Proposition 4.14 thus implies the following theorem.

Theorem 8.6. *The canonical functor $i : \Theta \rightarrow \tilde{\Theta}$ is aspherical.*

Corollary 8.7. *Let $i^* : \hat{\Theta} \rightarrow \hat{\tilde{\Theta}}$ be the restriction functor and let i_* be its right adjoint. Then (i^*, i_*) is a Quillen equivalence.*

Proof. This follows from the previous theorem by Proposition 4.9. \square

8.8. If I is a subset of $\{l, r, f\}$, let us denote by Θ_I the universal precategorical globular extension satisfying Axioms (Ass) and (Exc), plus Axiom (LUnit) (respectively (RUnit), respectively (FUnit)) if l (respectively r , respectively f) belongs to I . In particular, we have $\Theta = \Theta_{l,r,f}$.

In the same way, if J is a subset of $\{l, r, f, \tilde{l}, \tilde{r}\}$, let us denote by $\tilde{\Theta}_J$ the universal pregroupoidal globular extension satisfying the same axioms as $\Theta_{J \cap \{l,r,f\}}$ plus Axiom (LInv) (respectively (RInv)) if \tilde{l} (respectively \tilde{r}) belongs to J . In particular, we have $\tilde{\Theta} = \tilde{\Theta}_{l,r,f,\tilde{l},\tilde{r}}$.

A closer look at the calculations of the previous sections reveals that

$$\begin{array}{ccccc}
 & \Theta_{lr} & \longrightarrow & \tilde{\Theta}_{lr\tilde{r}} & \longrightarrow & \tilde{\Theta}_{lr\tilde{l}\tilde{r}} \\
 & \nearrow & & \searrow & & \searrow \\
 \Theta_r & & & & & \\
 & \searrow & & \nearrow & & \\
 & \Theta_{rf} & & \Theta = \Theta_{lrf} & \longrightarrow & \tilde{\Theta}_{lrf\tilde{r}} & \longrightarrow & \tilde{\Theta} = \tilde{\Theta}_{lrf\tilde{l}\tilde{r}}
 \end{array}$$

is a diagram of strict test categories and aspherical functors.

By duality, the diagram obtained by exchanging l and r , and \tilde{l} and \tilde{r} , is also a diagram of strict test categories and aspherical functors.

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