

STRICT ∞ -GROUPOIDS ARE GROTHENDIECK ∞ -GROUPOIDS

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ABSTRACT. We show that there exists a canonical functor from the category of strict ∞ -groupoids to the category of Grothendieck ∞ -groupoids and that this functor is fully faithful. As a main ingredient, we prove that free strict ∞ -groupoids on a globular pasting scheme are weakly contractible.

INTRODUCTION

The purpose of this paper is to prove that strict ∞ -groupoids are Grothendieck ∞ -groupoids or, more precisely, that there exists a canonical fully faithful functor from the category of strict ∞ -groupoids to the category of Grothendieck ∞ -groupoids (the morphisms of Grothendieck ∞ -groupoids we are considering in this article are the strict ones).

The notion of Grothendieck ∞ -groupoid was introduced by Grothendieck in his famous letter to Quillen (this letter constitutes the first thirteen sections of [13]). Roughly speaking, a Grothendieck ∞ -groupoid is an ∞ -graph (or globular set) endowed with operations similar to the one of strict ∞ -groupoids, with coherences (making it a strict ∞ -groupoid up to these coherences), with coherences between these coherences, and so on. Grothendieck explained in the letter how to construct an ∞ -groupoid $\Pi_\infty(X)$ out of a topological space X and he conjectured that this ∞ -groupoid $\Pi_\infty(X)$ (up to some notion of weak equivalence) classifies the homotopy type of X . This conjecture is now often referred to as the *homotopy hypothesis*. A precise statement of the conjecture is given at the very end of [4]. (It is well-known that strict ∞ -groupoids are not sufficient for this purpose. See for instance [10, Example 6.7], [20, Chapter 4] or [3].)

To define precisely Grothendieck ∞ -groupoids, one has to give a description of these higher coherences. It seems hopeless to describe them explicitly. Even for 3-groupoids, the explicit description of the coherences (see [12]) is very involved. Grothendieck's main insight is that one can generate all the coherences by induction in a simple way. An intuitive explanation of this inductive description is given in the introductions of [1] and [4].

The definition of Grothendieck ∞ -groupoids we use in this paper is not exactly the original one (this original definition is explained in [16]). First, we use the simplification introduced by Maltsiniotis in [17] and [18]. Second, we use the slight modification we introduced in [4]. The purpose of this modification was precisely to make canonical the inclusion functor from strict ∞ -groupoids to Grothendieck ∞ -groupoids as announced in Remark 2.8.1 of loc. cit.

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Let us come back to the main statement of the paper. Consider for the moment the following weaker statement: every strict ∞ -groupoid can be endowed with the structure of a Grothendieck ∞ -groupoid. This statement might seem tautological at first: it says, in some sense, that strict ∞ -groupoids are weak ∞ -groupoids. This apparent paradox comes from the fact that the coherences which are part of the algebraic structure of Grothendieck ∞ -groupoid are not explicitly defined (in the sense that they are generated by induction) and it is thus not clear that strict ∞ -groupoids admit such coherences. Of course, if it was not the case, the notion of Grothendieck ∞ -groupoid would have to be corrected.

It turns out that this weaker statement is equivalent to the following statement: free strict ∞ -groupoids on a globular pasting scheme are weakly contractible.

To prove the weak contractibility of these strict ∞ -groupoids, we use the following strategy. First, we reduce to the case of the free strict ∞ -groupoid on the n -disk by using the Brown-Golasinski model category structure on strict ∞ -groupoids (see [8] and [5]). The case of the n -disk is then proved using the path object of cylinders introduced by Métayer in [19] and studied in details in [15]. The idea of using these cylinders was suggested to us by Yves Lafont.

This Grothendieck ∞ -groupoid structure on a strict ∞ -groupoid is not unique if we stick to the Grothendieck's original definition. We show that, if we use the modified definition we introduced in [4], this structure becomes unique. This shows that there exists a canonical functor from the category of strict ∞ -groupoids to the category of Grothendieck ∞ -groupoids, as announced. Finally, we show that this functor is fully faithful.

Our paper is organized as follows. In the first section, we recall basic definitions and facts about strict ∞ -groupoids and their weak equivalences. In the second section, we recall the definition of Grothendieck ∞ -groupoids and we state our main result. In particular, we introduce the fundamental notions of globular extension, contractible extension, globular presheaf and coherator. In the third section, we introduce the globular extension $\tilde{\Theta}$, making the link between the world of strict ∞ -groupoids and the world of Grothendieck ∞ -groupoids. We explain the relation between the properties of $\tilde{\Theta}$ and our main result. The fourth section is dedicated to the homotopy theory of strict ∞ -groupoids. In particular, we introduce the Brown-Golasinski model structure and Métayer's path object of cylinders. The fifth section is the technical heart of the article. We show that the free strict ∞ -groupoid on the n -disk (seen as an ∞ -graph) is weakly contractible. To do so, we exhibit an n -cylinder leading to a homotopy between the identity functor of this ∞ -groupoid and a constant functor. In the sixth section, we prove that the globular extension $\tilde{\Theta}$ is canonically contractible. This means in particular that free strict ∞ -groupoids on a globular pasting scheme are weakly contractible. In the seventh section, we study conditions on a globular extension to get a fully faithful functor from strict ∞ -groupoids to globular presheaves on this globular extension. Finally, in the last section, we deduce from the previous sections the existence of a canonical fully faithful functor from strict ∞ -groupoids to Grothendieck ∞ -groupoids.

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Notation. If C is a category, we will denote by \widehat{C} the category of presheaves on C . If

$$\begin{array}{ccccccc} X_1 & & X_2 & & \cdots & & X_n \\ & \searrow f_1 & & \swarrow g_1 & \searrow f_2 & & \swarrow g_{n-1} \\ & & Y_1 & & Y_2 & \cdots & Y_{n-1} \end{array}$$

is a diagram in C , we will denote by

$$(X_1, f_1) \times_{Y_1} (g_1, X_2, f_2) \times_{Y_2} \cdots \times_{Y_{n-1}} (g_{n-1}, X_n)$$

its limit. Dually, we will denote by

$$(X_1, f_1) \amalg_{Y_1} (g_1, X_2, f_2) \amalg_{Y_2} \cdots \amalg_{Y_{n-1}} (g_{n-1}, X_n)$$

the colimit of the corresponding diagram in the opposite category.

1. STRICT ∞ -GROUPOIDS AND THEIR WEAK EQUIVALENCES

1.1. The globe category. We will denote by \mathbb{G} the *globe category*, that is, the category generated by the graph

$$D_0 \begin{array}{c} \xrightarrow{\sigma_1} \\ \xrightarrow{\tau_1} \end{array} D_1 \begin{array}{c} \xrightarrow{\sigma_2} \\ \xrightarrow{\tau_2} \end{array} \cdots \begin{array}{c} \xrightarrow{\sigma_{i-1}} \\ \xrightarrow{\tau_{i-1}} \end{array} D_{i-1} \begin{array}{c} \xrightarrow{\sigma_i} \\ \xrightarrow{\tau_i} \end{array} D_i \begin{array}{c} \xrightarrow{\sigma_{i+1}} \\ \xrightarrow{\tau_{i+1}} \end{array} \cdots$$

under the coglobular relations

$$\sigma_{i+1}\sigma_i = \tau_{i+1}\sigma_i \quad \text{and} \quad \sigma_{i+1}\tau_i = \tau_{i+1}\tau_i, \quad i \geq 1.$$

For $i \geq j \geq 0$, we will denote by σ_j^i and τ_j^i the morphisms from D_j to D_i defined by

$$\sigma_j^i = \sigma_i \cdots \sigma_{j+2}\sigma_{j+1} \quad \text{and} \quad \tau_j^i = \tau_i \cdots \tau_{j+2}\tau_{j+1}.$$

1.2. Globular sets. The category of *globular sets* or *∞ -graphs* is the category $\widehat{\mathbb{G}}$ of presheaves on \mathbb{G} . A globular set X thus consists of a diagram of sets

$$\cdots \begin{array}{c} \xrightarrow{s_{i+1}} \\ \xrightarrow{t_{i+1}} \end{array} X_i \begin{array}{c} \xrightarrow{s_i} \\ \xrightarrow{t_i} \end{array} X_{i-1} \begin{array}{c} \xrightarrow{s_{i-1}} \\ \xrightarrow{t_{i-1}} \end{array} \cdots \begin{array}{c} \xrightarrow{s_2} \\ \xrightarrow{t_2} \end{array} X_1 \begin{array}{c} \xrightarrow{s_1} \\ \xrightarrow{t_1} \end{array} X_0$$

satisfying the globular relations

$$s_i s_{i+1} = s_i t_{i+1} \quad \text{and} \quad t_i s_{i+1} = t_i t_{i+1}, \quad i \geq 1.$$

For $i \geq j \geq 0$, we will denote by s_j^i and t_j^i the maps from X_i to X_j defined by

$$s_j^i = s_{j+1} \cdots s_{i-1} s_i \quad \text{and} \quad t_j^i = t_{j+1} \cdots t_{i-1} t_i.$$

If X is a globular set, we will call X_0 the set of *objects* of X and X_i , for $i \geq 0$, the set of *i -arrows*. If u is an i -arrow of X for $i \geq 1$, $s_i(u)$ (resp. $t_i(u)$) will be called the *source* (resp. the *target*) of u . We will often denote an arrow u of X whose source is x and whose target is y by $u : x \rightarrow y$. We will say that two n -arrows u and v are *parallel* if either $n = 0$, or $n \geq 1$ and u, v have same source and same target.

1.3. Strict ∞ -categories. An *∞ -precategory* is a globular set C endowed with maps

$$\begin{aligned} *_{j}^i : (C_i, s_j^i) \times_{C_j} (t_j^i, C_i) &\rightarrow C_i, \quad i > j \geq 0, \\ k_i : C_i &\rightarrow C_{i+1}, \quad i \geq 0, \end{aligned}$$

such that

- for every $i > j \geq 0$ and every (u, v) in $(C_i, s_j^i) \times_{C_j} (t_j^i, C_i)$, we have

$$s_i(u *_{j}^i v) = \begin{cases} s_i(v), & j = i - 1, \\ s_i(u) *_{j}^{i-1} s_i(v), & j < i - 1, \end{cases}$$

and

$$t_i(u *_{j}^i v) = \begin{cases} t_i(u), & j = i - 1, \\ t_i(u) *_{j}^{i-1} t_i(v), & j < i - 1; \end{cases}$$

- for every $i \geq 0$ and every u in C_i , we have

$$s_{i+1}k_i(u) = u = t_{i+1}k_i(u).$$

If C is an ∞ -precategory, for $i \geq j \geq 0$, we will denote by k_i^j the map from C_j to C_i defined by

$$k_i^j = k_{i-1} \cdots k_{j+1}k_j.$$

A *morphism of ∞ -precategories* is a morphism of globular sets between ∞ -precategories which is compatible with the $*_j^i$'s and the k_i 's in an obvious way. We will denote by $\infty\text{-PCat}$ the category of ∞ -precategories.

A *strict ∞ -category* is an ∞ -precategory C satisfying the following axioms:

- Associativity

for every $i > j \geq 0$ and every (u, v, w) in $(C_i, s_j^i) \times_{C_j} (t_j^i, C_i, s_j^i) \times_{C_j} (t_j^i, C_i)$, we have

$$(u *_{j}^i v) *_{j}^i w = u *_{j}^i (v *_{j}^i w);$$

- Exchange law

for every $i > j > k \geq 0$ and every (u, u', v, v') in

$$(C_i, s_j^i) \times_{C_j} (t_j^i, C_i, s_k^i) \times_{C_k} (t_k^i, C_i, s_j^i) \times_{C_j} (t_j^i, C_i),$$

we have

$$(u *_{j}^i u') *_{k}^i (v *_{j}^i v') = (u *_{k}^i v) *_{j}^i (u' *_{k}^i v');$$

- Identities

for every $i > j \geq 0$ and every u in C_i , we have

$$k_i^j t_j^i(u) *_{j}^i u = u = u *_{j}^i k_i^j s_j^i(u);$$

- Functoriality of identities

for every $i > j \geq 0$ and every (u, v) in $(C_i, s_j^i) \times_{C_j} (t_j^i, C_i)$, we have

$$k_i(u *_{j}^i v) = k_i(u) *_{j}^{i+1} k_i(v).$$

The *category of strict ∞ -categories* is the full subcategory of the category of ∞ -precategories whose objects are strict ∞ -categories.

If C is a strict ∞ -category and if u and v are two parallel n -arrows of C , the set of $(n+1)$ -arrows from u to v in C will be denoted by $\mathbf{Hom}_C(u, v)$.

1.4. Let C be a strict ∞ -category. To simplify the formulas involving the operations of C , we will adopt the two following conventions:

- If u is an i -arrow of C and v is a j -arrow of C such that $s_k^i(u) = t_k^j(v)$ for some k less than i and j , then we will denote by $u *_{k} v$ the m -arrow $k_m^i(u) *_{k}^m k_m^j(v)$, where m is the greatest integer between i and j .

- If u is a j -arrow of C , we will denote by 1_u the $(j+1)$ -arrow $k_j(u)$ or, more generally, the i -arrow $k_i^j(u)$ for any $i \geq j$ if the context makes the value of i clear.

1.5. Strict ∞ -groupoids. Let C be a strict ∞ -category and let $u : x \rightarrow y$ be an i -arrow of C for $i \geq 1$. A $*_{i-1}^i$ -inverse, or briefly an *inverse*, of u is an i -arrow $u^{-1} : y \rightarrow x$ of C such that

$$u^{-1} *_{i-1} u = 1_x \quad \text{and} \quad u *_{i-1} u^{-1} = 1_y.$$

If such an inverse exists, it is unique and the notation u^{-1} is thus unambiguous.

A *strict ∞ -groupoid* is a strict ∞ -category whose i -arrows, $i \geq 1$, are invertible. The existence of $*_{i-1}^i$ -inverses implies the existence of $*_j^i$ -inverses (in an obvious sense) for every $i > j \geq 0$ (see Proposition 2.3 of [5] for details).

The *category of strict ∞ -groupoids* is the full subcategory of the category of strict ∞ -categories whose objects are strict ∞ -groupoids. We will denote it by $\infty\text{-Gpd}_{\text{str}}$. Note that a morphism of strict ∞ -groupoids automatically preserves the inverses.

1.6. Homotopy groups of strict ∞ -groupoids. Let G be a strict ∞ -groupoid. An n -arrow u of G is *homotopic* to another n -arrow v of G if there exists an $(n+1)$ -arrow from u to v in G . This obviously implies that the arrows u and v are parallel. If u is homotopic to v , we will write $u \sim v$. The relation \sim is an equivalence relation on G_n . It is moreover compatible with the composition $*_{n-1}^n : G_n \times_{G_{n-1}} G_n \rightarrow G_n$.

The set of *connected components* of G is

$$\pi_0(G) = G_0 / \sim.$$

If $n \geq 1$ and u, v are two parallel n -arrows of G , we will denote

$$\pi_n(G, u, v) = \text{Hom}_G(u, v) / \sim \quad \text{and} \quad \pi_n(G, u) = \pi_n(G, u, u).$$

Note that the composition $*_{n-1}^n$ induces a group structure on $\pi_n(G, u)$. For $n \geq 1$ and x an object of G , the n -th *homotopy group of G at x* is

$$\pi_n(G, x) = \pi_n(G, k_{n-1}^0(x)).$$

It is immediate that π_0 induces a functor from the category of strict ∞ -groupoids to the category of sets, and that π_n , for $n \geq 1$, induces a functor from the category of pointed strict ∞ -groupoids to the category of groups. Moreover, by the Eckmann-Hilton argument, the groups $\pi_n(G, x)$ are abelian when $n \geq 2$.

1.7. Weak equivalences of strict ∞ -groupoids. A morphism $f : G \rightarrow H$ of strict ∞ -groupoids is a *weak equivalence* if

- the map $\pi_0(f) : \pi_0(G) \rightarrow \pi_0(H)$ is a bijection;
- for all $n \geq 1$ and all object x of G , the morphism

$$\pi_n(f, x) : \pi_n(G, x) \rightarrow \pi_n(H, f(x))$$

is a group isomorphism.

Proposition 1.8. *Let $f : G \rightarrow H$ be a morphism of strict ∞ -groupoids. The following conditions are equivalent:*

- (1) f is a weak equivalence of strict ∞ -groupoids;

(2) $\pi_0(f) : \pi_0(G) \rightarrow \pi_0(H)$ is a bijection and for all $n \geq 1$ and every $(n-1)$ -arrow u of G , f induces a bijection

$$\pi_n(G, u) \rightarrow \pi_n(H, f(u));$$

(3) $\pi_0(f) : \pi_0(G) \rightarrow \pi_0(H)$ is a bijection and for all $n \geq 1$ and every pair (u, v) of parallel $(n-1)$ -arrows of G , f induces a bijection

$$\pi_n(G, u, v) \rightarrow \pi_n(H, f(u), f(v));$$

(4) $\pi_0(f) : \pi_0(G) \rightarrow \pi_0(H)$ is surjective and for all $n \geq 1$ and every pair (u, v) of parallel $(n-1)$ -arrows of G , f induces a surjection

$$\pi_n(G, u, v) \rightarrow \pi_n(H, f(u), f(v)).$$

Proof. See Proposition 1.7 of [5]. □

1.9. Weakly contractible strict ∞ -groupoids. A strict ∞ -groupoid G is said to be *weakly contractible* if the unique morphism from G to the terminal strict ∞ -groupoid is a weak equivalence. In other words, G is weakly contractible if

- the set $\pi_0(G)$ is trivial;
- for all $n \geq 1$ and all object x of G , the group $\pi_n(G, x)$ is trivial.

Proposition 1.10. *A strict ∞ -groupoid G is weakly contractible if and only if G is non-empty and for every $n \geq 1$ and every pair (u, v) of parallel $(n-1)$ -arrows of G , there exists an n -arrow from u to v in G .*

Proof. This is exactly the content of the equivalence (1) \Leftrightarrow (4) of Proposition 1.8 applied to the unique morphism from G to the terminal strict ∞ -groupoid. □

2. GROTHENDIECK ∞ -GROUPOIDS

In this section, we recall briefly the definition of Grothendieck ∞ -groupoids and we state our main result. We encourage the reader to read Sections 1 and 2 of [4] for more explanation and examples.

2.1. Globular sums. A *table of dimensions* is a table

$$\begin{pmatrix} i_1 & i_2 & \cdots & i_k \\ i'_1 & i'_2 & \cdots & i'_{k-1} \end{pmatrix},$$

where $k \geq 1$, consisting of nonnegative integers satisfying

$$i_l > i'_l < i_{l+1}, \quad 1 \leq l \leq k-1.$$

The integer k is called the *width* of T . The *dimension* of such a table is the greatest integer appearing in the table.

Let (C, F) be a category under \mathbb{G} , i.e., a category C endowed with a functor $F : \mathbb{G} \rightarrow C$. We will often denote in the same way the objects and morphisms of \mathbb{G} and their image by the functor F . Let

$$T = \begin{pmatrix} i_1 & i_2 & \cdots & i_k \\ i'_1 & i'_2 & \cdots & i'_{k-1} \end{pmatrix}$$

be a table of dimensions. The *globular sum* in C associated to T (if it exists) is the iterated amalgamated sum

$$(D_{i_1}, \sigma_{i'_1}^{i_1}) \amalg_{D_{i'_1}} (\tau_{i'_1}^{i_2}, D_{i_2}, \sigma_{i'_2}^{i_2}) \amalg_{D_{i'_2}} \cdots \amalg_{D_{i'_{k-1}}} (\tau_{i'_{k-1}}^{i_k}, D_{i_k})$$

in C , i.e., the colimit of the diagram

$$\begin{array}{ccccccc} D_{i_1} & & D_{i_2} & & D_{i_3} & \cdots & D_{i_{k-1}} & & D_{i_k} \\ & \swarrow & \nearrow & \swarrow & \nearrow & & \swarrow & & \nearrow \\ & \sigma_{i'_1}^{i_1} & & \tau_{i'_1}^{i_2} & & & \sigma_{i'_{k-1}}^{i_{k-1}} & & \tau_{i'_{k-1}}^{i_k} \\ & & D_{i'_1} & & D_{i'_2} & & & & D_{i'_{k-1}} \end{array}$$

in C . We will denote it briefly by

$$D_{i_1} \amalg_{D_{i'_1}} D_{i_2} \amalg_{D_{i'_2}} \cdots \amalg_{D_{i'_{k-1}}} D_{i_k}.$$

2.2. Globular extensions. A category C under \mathbb{G} is said to be a *globular extension* if all the globular sums exist in C , i.e., if for every table of dimensions T , the globular sum associated to T exists in C .

If C and D are two globular extensions, a *morphism of globular extensions* from C to D is a functor from C to D under \mathbb{G} (that is, such that the triangle

$$\begin{array}{ccc} & \mathbb{G} & \\ & \swarrow & \searrow \\ C & \longrightarrow & D \end{array}$$

commutes) which preserves globular sums. Such a functor will also be called a *globular functor*.

2.3. The globular extension Θ_0 . We will consider the category $\widehat{\mathbb{G}}$ as a category under \mathbb{G} by using the Yoneda functor. If T is a table of dimensions, we will denote by G_T the globular sum associated to T in $\widehat{\mathbb{G}}$.

The category Θ_0 is the category defined in the following way:

- the objects of Θ_0 are the table of dimensions;
- if S and T are two objects of Θ_0 , then

$$\mathrm{Hom}_{\Theta_0}(S, T) = \mathrm{Hom}_{\widehat{\mathbb{G}}}(G_S, G_T).$$

By definition of Θ_0 , there is a canonical fully faithful functor $\Theta_0 \rightarrow \widehat{\mathbb{G}}$. This functor is moreover injective on objects and Θ_0 can be considered as a full subcategory of $\widehat{\mathbb{G}}$.

The functor $\mathbb{G} \rightarrow \widehat{\mathbb{G}}$ factors through Θ_0 and we get a functor $\mathbb{G} \rightarrow \Theta_0$. The category Θ_0 will always be considered as a category under \mathbb{G} using this functor. By definition, Θ_0 is a globular extension.

The globular extension Θ_0 is the initial globular extension in the following sense: for every globular extension C , there exists a globular functor $\Theta_0 \rightarrow C$, unique up to a unique natural transformation (see Proposition 3.2 and paragraph 3.3 of [2]). More precisely, if C is a globular extension, the choice of a globular functor $\Theta_0 \rightarrow C$ amounts to the choice, for every table of dimensions T , of a globular sum associated to T in C (this globular sum being only defined up to a canonical isomorphism).

Remark 2.4.

(1) The G_T 's are exactly the globular sets associated to finite planar rooted trees by Batanin in [6]. These globular sets are sometimes called *globular pasting schemes*.

(2) The category Θ_0 was first introduced by Berger in [7] in terms of planar rooted trees.

2.5. Globular theories. A globular extension C is called a *globular theory* if any globular functor $\Theta_0 \rightarrow C$ is bijective on objects. If C is a globular theory, then there exists a unique globular functor $\Theta_0 \rightarrow C$.

A *morphism of globular theories* is a morphism of globular extensions between globular theories. Note that if C and D are globular theories, a morphism from C to D is nothing but a functor from C to D under Θ_0 .

2.6. Globular presheaves. Let C be a globular theory. A *globular presheaf* on C , or *model* of C , is a presheaf X on C which sends globular sums to globular products (globular products being the notion dual to globular sums). In other words, a presheaf X is a globular presheaf if, for every table of dimensions

$$\begin{pmatrix} i_1 & i_2 & \cdots & i_k \\ i'_1 & i'_2 & \cdots & i'_{k-1} \end{pmatrix},$$

the canonical map

$$X(\mathbb{D}_{i_1} \amalg_{\mathbb{D}_{i'_1}} \cdots \amalg_{\mathbb{D}_{i'_{k-1}}} \mathbb{D}_{i_k}) \rightarrow X_{i_1} \times_{X_{i'_1}} \cdots \times_{X_{i'_{k-1}}} X_{i_k}$$

is a bijection. We will denote by $\text{Mod}(C)$ the full subcategory of the category of presheaves on C whose objects are the globular presheaves on C .

If $C \rightarrow D$ is a morphism of globular theories, then the inverse image functor from presheaves on D to presheaves on C restricts to a functor from globular presheaves on D to globular presheaves on C .

2.7. Globularly parallel arrows and liftings. Let C be a globular extension. Two morphisms $f, g : \mathbb{D}_n \rightarrow X$ of C are said to be globularly parallel if either $n = 0$, or $n \geq 1$ and

$$f\sigma_n = g\sigma_n \quad \text{and} \quad f\tau_n = g\tau_n.$$

Let now $(f, g) : \mathbb{D}_n \rightarrow X$ be a pair of morphisms of C . A *lifting* of the pair (f, g) is a morphism $h : \mathbb{D}_{n+1} \rightarrow X$ such that

$$h\sigma_n = f \quad \text{and} \quad h\tau_n = g.$$

The existence of such a lifting obviously implies that f and g are globularly parallel.

2.8. Admissible pairs. Let C be a globular extension. A pair of morphisms

$$(f, g) : \mathbb{D}_n \rightarrow S$$

is said to be $(\infty, 0)$ -*admissible*, or briefly *admissible*, if

- the morphisms f and g are globularly parallel;
- the object S is a globular sum;
- the dimension of S (as a globular sum) is less than or equal to $n + 1$.

2.9. Contractible globular extensions. A globular extension C is $(\infty, 0)$ -*contractible*, or briefly *contractible*, if every admissible pair of C admits a lifting. If moreover, these liftings are unique, we will say that C is *canonically $(\infty, 0)$ -contractible*, or briefly *canonically contractible*.

Remark 2.10. The last condition in the definition of an admissible pair was not part of Grothendieck's original definition and was introduced by us in [4]. This condition is needed to make canonical the functor from strict ∞ -groupoids to Grothendieck ∞ -groupoids obtained in Theorem 8.1 or, equivalently, to make the globular extension $\tilde{\Theta}$ (defined in paragraph 3.2) a *canonically contractible* globular extension (see Theorem 6.4 and Remark 6.5).

2.11. Free globular extensions. A *cellular tower* of globular extensions is a tower of globular extensions

$$C_0 = \Theta_0 \rightarrow C_1 \rightarrow \cdots \rightarrow C_n \rightarrow \cdots ,$$

endowed, for each $n \geq 0$, with a set A_n of admissible pairs of C_n , such that C_{n+1} is the globular extension obtained from C_n by formally adding a lifting to each admissible pair in A_n .

We will say that a globular extension C is *free* if C is the colimit of some cellular tower of globular extensions. If C is a free globular extension, then C is a globular theory.

Proposition 2.12. *Let C be a free globular extension.*

- (1) *For any contractible globular extension D , there exists a globular functor from C to D .*
- (2) *For any canonically contractible globular theory D , there exists a unique globular functor C to D .*

Proof. See Proposition 2.14 of [4] for the first point. The second point is related to Remark 2.14.1 of loc. cit. Let us prove it (this will also give a proof of the first point).

Let (C_n, A_n) be any cellular tower such that C is the colimit of the C_n 's. For $n \geq 0$, let us denote by i_{n+1} the globular functor from C_n to C_{n+1} . By the universal property of the colimit, a globular functor $C \rightarrow D$ is given by a system of globular functors $F_n : C_n \rightarrow D$ such that $F_{n+1}i_n = F_n$. (Note that all the functors involved are functors under Θ_0 and are hence automatically globular.) By the universal property defining C_{n+1} from C_n , such a system is uniquely determined by $F_0 : C_0 = \Theta_0 \rightarrow D$ and by the choice, for every pair (f, g) of A_n , of a lifting of the admissible pair $(F_n(f), F_n(g))$ in D .

It follows immediately that if D is contractible, such a system exists and that if F_0 is fixed (which is the case if D is a globular theory) and that D is canonically contractible, such a system is unique. \square

2.13. Coherators. An $(\infty, 0)$ -*coherator*, or briefly a *coherator*, is a globular extension which is free and $(\infty, 0)$ -contractible.

2.14. ∞ -groupoids of type C . Let C be a coherator. An *∞ -groupoid of type C* is a globular presheaf on C . The category of ∞ -groupoids of type C is the category $\text{Mod}(C)$. This category will be denoted more suggestively by $\infty\text{-Gpd}_C$.

We can now state our main result:

Theorem. *Let C be a coherator. There exists a canonical functor*

$$\infty\text{-}\mathcal{Gpd}_{\text{str}} \rightarrow \infty\text{-}\mathcal{Gpd}_C.$$

Moreover, this functor is fully faithful.

This result will be proved in the very last section of the article.

3. THE GLOBULAR EXTENSION $\tilde{\Theta}$

3.1. Free strict ∞ -groupoids. Let $U : \infty\text{-}\mathcal{Gpd}_{\text{str}} \rightarrow \widehat{\mathbb{G}}$ be the forgetful functor sending a strict ∞ -groupoid to its underlying globular set. This functor U admits a left adjoint $L : \widehat{\mathbb{G}} \rightarrow \infty\text{-}\mathcal{Gpd}_{\text{str}}$ which by definition sends a globular set to the *free strict ∞ -groupoid* on this globular set.

3.2. The globular extension $\tilde{\Theta}$. Recall that if T is a table of dimensions, we defined in paragraph 2.3 an associated globular set G_T .

The category $\tilde{\Theta}$ is the category defined in the following way:

- the objects of $\tilde{\Theta}$ are the table of dimensions;
- if S and T are two objects of $\tilde{\Theta}$, then

$$\text{Hom}_{\tilde{\Theta}}(S, T) = \text{Hom}_{\infty\text{-}\mathcal{Gpd}_{\text{str}}}(L(G_S), L(G_T)).$$

By definition of $\tilde{\Theta}$, there are canonical functors

$$\Theta_0 \rightarrow \tilde{\Theta} \rightarrow \infty\text{-}\mathcal{Gpd}_{\text{str}}.$$

The functor $\tilde{\Theta} \rightarrow \infty\text{-}\mathcal{Gpd}_{\text{str}}$ is by definition fully faithful. It is easily seen to be injective on objects and $\tilde{\Theta}$ can thus be considered as a full subcategory of the category of strict ∞ -groupoids.

The category $\tilde{\Theta}$ will always be considered as a category under \mathbb{G} by using the functor $\mathbb{G} \rightarrow \Theta_0 \rightarrow \tilde{\Theta}$. It follows immediately from the fact that L is a left adjoint and hence preserves colimits that $\tilde{\Theta}$ is a globular extension and hence a globular theory.

Proposition 3.3. *The category of globular presheaves on $\tilde{\Theta}$ is canonically equivalent to the category of strict ∞ -groupoids.*

Proof. See Propositions 3.21 and 3.22 of [2]. □

3.4. Towards our canonical fully faithful functor. The next three sections are dedicated to proving that the globular extension $\tilde{\Theta}$ is canonically contractible (see the introduction for details on the different steps). Assuming this fact, we can easily get half of our main result. Indeed, by applying Proposition 2.12 to $\tilde{\Theta}$, we get that if C is a coherator, there exists a unique globular functor $\tilde{\Theta} \rightarrow C$. This globular functor induces a functor

$$\infty\text{-}\mathcal{Gpd}_{\text{str}} \cong \text{Mod}(\tilde{\Theta}) \rightarrow \text{Mod}(C) = \infty\text{-}\mathcal{Gpd}_C,$$

and we thus obtain a canonical functor from strict ∞ -groupoids to ∞ -groupoids of type C .

The second half of the result, namely the fact that this functor is fully faithful, will follow from the developments of Section 7.

4. HOMOTOPY THEORY OF STRICT ∞ -GROUPOIDS

4.1. Trivial fibrations of strict ∞ -groupoids. Recall that a morphism of presheaves is said to be a *trivial fibration* if it has the right lifting property with respect to every monomorphism. This defines in particular a notion of *trivial fibration* of globular sets.

A morphism of strict ∞ -groupoids is said to be a *trivial fibration* if its underlying morphism of globular sets is a trivial fibration.

4.2. Free strict ∞ -groupoids. Recall that we denote by $U : \infty\text{-Gpd}_{\text{str}} \rightarrow \widehat{\mathbb{G}}$ the forgetful functor from strict ∞ -groupoids to globular sets and by $L : \widehat{\mathbb{G}} \rightarrow \infty\text{-Gpd}_{\text{str}}$ its left adjoint.

4.3. Disks. We will consider the Yoneda functor $\mathbb{G} \rightarrow \widehat{\mathbb{G}}$ as an inclusion. In particular, for every $n \geq 0$, we have a globular set D_n . If X is a globular set, a morphism $D_n \rightarrow X$ corresponds, by the Yoneda lemma, to an n -arrow of X .

For $n \geq 0$, the strict ∞ -groupoid $L(D_n)$ will be denoted by \widetilde{D}_n . It follows from the fact that L is a left adjoint to U that if G is a strict ∞ -groupoid, a morphism $\widetilde{D}_n \rightarrow G$ corresponds to an n -arrow of G .

For $i \geq j \geq 0$, we have two morphisms $\widetilde{\sigma}_j^i, \widetilde{\tau}_j^i : \widetilde{D}_j \rightarrow \widetilde{D}_i$ defined by

$$\widetilde{\sigma}_j^i = L(\sigma_j^i) \quad \text{and} \quad \widetilde{\tau}_j^i = L(\tau_j^i).$$

When $j = i - 1$, we will denote $\widetilde{\sigma}_i = \widetilde{\sigma}_{i-1}^i$ and $\widetilde{\tau}_i = \widetilde{\tau}_{i-1}^i$.

4.4. Spheres. We define by induction on $n \geq 0$ a globular set S^{n-1} endowed with a morphism $i_n : S^{n-1} \rightarrow D_n$ in the following way. For $n = 0$, we set

$$S^{-1} = \emptyset,$$

the empty globular set, and we define

$$i_0 : S^{-1} \rightarrow D_0$$

as the unique morphism from the initial object to D_0 . For $n \geq 1$, we set

$$S^{n-1} = (D_{n-1}, i_{n-1}) \amalg_{S^{n-2}} (i_{n-1}, D_{n-1})$$

and

$$i_n = (\tau_n, \sigma_n) : S^{n-1} \rightarrow D_n.$$

The globular set S^{n-1} can be described concretely as the sub-globular set of D_n obtained from D_n by removing the unique n -arrow. In particular, if X is a globular set and $n \geq 1$, a morphism $S^{n-1} \rightarrow X$ corresponds to a pair of parallel $(n-1)$ -arrows of X .

If $n \geq 0$, the strict ∞ -groupoid $L(S^{n-1})$ will be denoted by \widetilde{S}^{n-1} . Since L is a left adjoint and hence preserves pushouts, the \widetilde{S}^{n-1} 's can be constructed from the \widetilde{D}_m 's using a similar induction. In particular, we have

$$\widetilde{S}^{n-1} = \widetilde{D}_{n-1} \amalg_{\widetilde{S}^{n-2}} \widetilde{D}_{n-1}.$$

It follows from the fact that L is a left adjoint to U that if G is a strict ∞ -groupoid and $n \geq 1$, a morphism $\widetilde{S}^{n-1} \rightarrow G$ corresponds to a pair of parallel $(n-1)$ -arrows of G .

Proposition 4.5.

- (1) A morphism $X \rightarrow Y$ of globular sets is a trivial fibration if and only if it has the right lifting property with respect to the morphisms $S^{n-1} \rightarrow D_n$, $n \geq 0$.
- (2) A morphism $G \rightarrow H$ of strict ∞ -groupoids is a trivial fibration if and only if it has the right lifting property with respect to the morphisms $\tilde{S}^{n-1} \rightarrow \tilde{D}_n$, $n \geq 0$.

Proof. The category \mathbb{G} is a direct category. Moreover, the concrete description of S^{n-1} shows that the inclusion $S^{n-1} \rightarrow D_n$ is nothing but the inclusion $\partial D_n \rightarrow D_n$, where ∂D_n is the boundary of D_n defined in terms of the direct structure of \mathbb{G} (see for instance paragraph 8.1.30 of [11] for a definition). The first assertion then follows from Proposition 8.1.37 of [11].

The second assertion follows formally from the first one by using the fact that L is a left adjoint to U . \square

Remark 4.6. A similar result holds for strict ∞ -categories. In particular, the trivial fibrations of [15] can be described without reference to spheres and disks.

Theorem 4.7 (Brown-Golasinski, Ara-Métayer). *The weak equivalences and the trivial fibrations of strict ∞ -groupoids define a combinatorial model category structure on the category of strict ∞ -groupoids. Moreover, every strict ∞ -groupoid is fibrant in this model category structure.*

Proof. Theorem 3.19 of [5] asserts the existence of a combinatorial model category structure on $\infty\text{-Gpd}_{\text{str}}$. The weak equivalences of this model category structure coincide with the weak equivalences of strict ∞ -groupoids defined in this article by Proposition 4.1 of loc. cit. The trivial fibrations of this model category structure are the morphisms having the right lifting property with respect to the $\tilde{S}^{n-1} \rightarrow \tilde{D}_n$. They coincide with the trivial fibrations of strict ∞ -groupoids defined in this article by Proposition 4.5. The fact that every strict ∞ -groupoid is fibrant follows from Theorem 5.1 of [15]. \square

Remark 4.8. This model category was first defined by Brown and Golasinski in [8] in terms of crossed complexes (crossed complexes are equivalent to strict ∞ -groupoids by the main result of [9]). We will hence call it the Brown-Golasinski model category structure. An alternative proof and a direct description are given in [5] in terms of the model category structure on strict ∞ -categories defined in [15].

Remark 4.9. By Proposition 4.5, for every $n \geq 0$, the morphism $\tilde{S}^{n-1} \rightarrow \tilde{D}_n$ is a cofibration (in the Brown-Golasinski model category structure). It follows that for every $n \geq 1$, the morphisms $\tilde{\sigma}_n, \tilde{\tau}_n : \tilde{D}_{n-1} \rightarrow \tilde{D}_n$ are also cofibrations. Indeed, these morphisms are obtained as compositions

$$\tilde{D}_{n-1} \rightarrow \tilde{S}^{n-1} = \tilde{D}_{n-1} \amalg_{\tilde{S}^{n-2}} \tilde{D}_{n-1} \rightarrow \tilde{D}_n,$$

where the first arrow is one of the two canonical morphisms. But these canonical morphisms are both pushouts of $\tilde{S}^{n-2} \rightarrow \tilde{D}_{n-1}$.

4.10. Path objects. Let us fix some terminology about path objects. Let \mathcal{M} be a model category and let B be an object of \mathcal{M} . A *path object* of B in \mathcal{M} is an object P

of \mathcal{M} endowed with a factorization

$$B \xrightarrow{r} P \xrightarrow{(p_1, p_0)} B \times B$$

of the diagonal of B as a weak equivalence followed by a fibration.

Let P be such a path object and let $f, g : A \rightarrow B$ be two morphisms of \mathcal{M} . A *right homotopy* from f to g using P is a morphism $h : A \rightarrow P$ of \mathcal{M} such that $p_0 h = f$ and $p_1 h = g$. The existence of such a homotopy implies that f and g become equal in the homotopy category of \mathcal{M} . Note that we do not need the morphism $P \rightarrow B \times B$ to be a fibration for this property to hold.

The rest of this section is devoted to the description of a functorial path object for the Brown-Golasinski model category structure.

4.11. Notation for iterated sources and targets. Let G be a strict ∞ -groupoid and let u be an n -arrow of G . We will use the following notation for the iterated sources, targets and identities of the n -arrow u :

$$u_i^{\flat} = \begin{cases} s_i^n(u), & \text{if } i \leq n, \\ k_i^n(u), & \text{if } i > n, \end{cases} \quad \text{and} \quad u_i^{\sharp} = \begin{cases} t_i^n(u), & \text{if } i \leq n, \\ k_i^n(u), & \text{if } i > n. \end{cases}$$

Note that by definition, we have $u = u_n^{\flat} = u_n^{\sharp}$. Here is an illustration of this notation in the case $n = 3$:

$$\begin{array}{c} u_1^{\flat} \\ \curvearrowright \\ u_0^{\flat} \quad u_2^{\flat} \begin{array}{c} \Downarrow u \\ \Downarrow u \end{array} u_2^{\sharp} \quad u_0^{\sharp} \\ \curvearrowleft \\ u_1^{\sharp} \end{array} .$$

4.12. Cylinders. Let G be a strict ∞ -groupoid. Let $n \geq 0$ and let u, v be two n -arrows of G . An *n -cylinder* z from u to v in G , denoted by $z : u \curvearrowright v$, consists of:

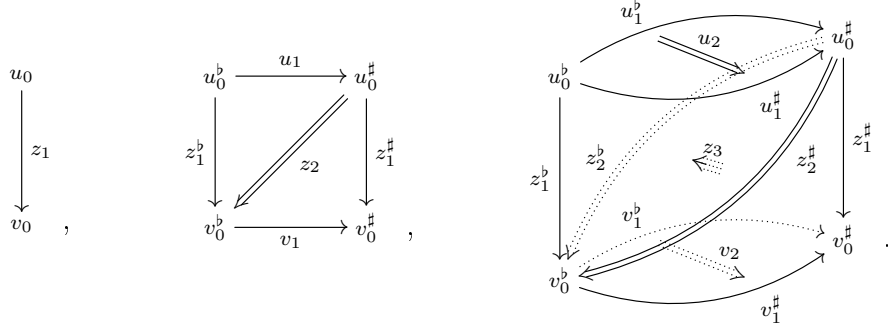
- for every $1 \leq i \leq n$, two i -arrows $z_i^{\flat}, z_i^{\sharp}$ of G ;
- an $(n+1)$ -arrow z_{n+1} of G (which will also be denoted by z_{n+1}^{\sharp} and z_{n+1}^{\flat} for the purpose of getting homogeneous formulas),

whose sources and targets are given, for $1 \leq i \leq n+1$ and $\varepsilon = \flat, \sharp$, by the following formulas:

$$\begin{aligned} s_i(z_i^{\varepsilon}) &= z_{i-1}^{\sharp} *_{i-2} (z_{i-2}^{\sharp} *_{i-3} (\cdots *_{i-1} (z_1^{\sharp} *_{i-1} u_{i-1}^{\varepsilon}))), \\ t_i(z_i^{\varepsilon}) &= (((v_{i-1}^{\varepsilon} *_{i-1} z_1^{\flat}) *_{i-2} z_2^{\flat}) *_{i-3} \cdots) *_{i-2} z_{i-1}^{\flat}. \end{aligned}$$

Note that if $i = n+1$, these formulas do not depend on the value of ε and the definition hence makes sense. We will denote by $\Gamma_n(G)$ the set of n -cylinders in G . Here are the

diagrams representing n -cylinders for $n = 0, 1, 2$:



Remark 4.13. This notion of n -cylinder was originally defined by Métayer (in the more general context of strict ∞ -categories) in [19]. In [14], Lafont and Métayer give a nice inductive reformulation of the definition (an n -cylinder is defined in terms of $(n-1)$ -cylinders in another strict ∞ -category). These n -cylinders are also studied in [15] where they play a crucial role.

4.14. The strict ∞ -groupoid of cylinders. Let G be a strict ∞ -groupoid. If $z : u \curvearrowright v$ is an n -cylinder with $n > 0$, we define two $(n-1)$ -cylinders

$$\sigma_n(z) : u_{n-1}^b \curvearrowright v_{n-1}^b \quad \text{and} \quad \tau_n(z) : u_{n-1}^\# \curvearrowright v_{n-1}^\#$$

by setting

$$\begin{aligned} \sigma_n(z)_i^\varepsilon &= \tau_n(z)_i^\varepsilon = z_i^\varepsilon, \quad 1 \leq i \leq n-1, \quad \varepsilon = b, \#, \\ \sigma_n(z) &= z_n^b \quad \text{and} \quad \tau_n(z) = z_n^\#. \end{aligned}$$

This defines the structure of a globular set on the $\Gamma_n(G)$'s. The resulting globular set will be denoted by $\Gamma(G)$. Métayer showed in an appendix to [19] that $\Gamma(G)$ is naturally endowed with the structure of a strict ∞ -category (see also Appendix A of [15]). Moreover, this strict ∞ -category is an ∞ -groupoid (see Lemma 3.11 of [5]). From now on, we will always consider $\Gamma(G)$ endowed with this structure of a strict ∞ -groupoid. The only result we will need that uses the precise definition of this structure is Theorem 4.16.

4.15. The path object of cylinders. Let G be a strict ∞ -groupoid. If u is an n -arrow of G , we define an n -cylinder $\tau u : u \curvearrowright u$ by setting:

$$(\tau u)_i^\varepsilon = 1_{u_{i-1}^\varepsilon}, \quad 1 \leq i \leq n+1, \quad \varepsilon = b, \#.$$

We obtain this way a factorization

$$G \rightarrow \Gamma(G) \rightarrow G \times G$$

of the diagonal of G in the category of globular sets. The first morphism sends an n -arrow u to the n -cylinder $\tau u : u \curvearrowright u$. The second morphism sends an n -cylinder $z : u \curvearrowright v$ to the pair of n -arrows (v, u) .

Theorem 4.16 (Lafont-Métayer-Worytkiewicz). *If G is a strict ∞ -groupoid, then the above factorization $G \rightarrow \Gamma(G) \rightarrow G \times G$ is actually a factorization in the category of strict ∞ -groupoids and $\Gamma(G)$ endowed with this factorization is a path object of G in the Brown-Golasiński model category structure.*

Proof. See Theorem 4.21 and Proposition 4.45 of [15]. \square

Remark 4.17. In this paper, we will not use the fact that $\Gamma(G) \rightarrow G \times G$ is a fibration. The version of the above theorem we will need can thus be stated only in terms of weak equivalences of strict ∞ -groupoids (and in particular, without reference to the Brown-Golasinski model category structure).

5. DISKS ARE CONTRACTIBLE

Throughout this section, we fix an integer $n \geq 0$. We will denote by u the n -arrow of \tilde{D}_n coming from the unique n -arrow of the globular set D_n . The goal of the section is to prove that the ∞ -groupoid \tilde{D}_n is weakly contractible. For this purpose, we will exhibit an n -cylinder from u to (an iterated identity of) an object of \tilde{D}_n .

5.1. A cylinder. For i such that $1 \leq i \leq n+1$, we define two i -arrows c_i^b and c_i^\sharp of \tilde{D}_n by

$$\begin{aligned} c_i^b &= 1_{u_0^b}, \\ c_i^\sharp &= (u_1^b)^{-1} *_0 ((u_2^b)^{-1} *_1 (\cdots *_i (\cdots *_i ((u_{i-1}^b)^{-1} *_i (u_i^b)^{-1}))). \end{aligned}$$

When $i = 1$, the definition of c_i^\sharp should be read as $c_1^\sharp = (u_1^b)^{-1}$. Note also that the $(n+1)$ -morphism u_{n+1}^b is by definition equal to 1_u . We leave to the reader the verification that the formula defining c_i^\sharp makes sense. By definition, for $1 \leq i \leq n$, we have

$$t_i(c_i^\sharp) = (u_1^b)^{-1} *_0 ((u_2^b)^{-1} *_1 (\cdots *_i (\cdots *_i ((u_{i-1}^b)^{-1} *_i u_{i-1}^b))).$$

We claim that this formula simplifies to

$$t_i(c_i^\sharp) = 1_{u_0^b}.$$

For $i = n+1$, we claim that we even have

$$c_{n+1}^\sharp = 1_{u_0^b}.$$

These two claims are precisely the content of the following easy lemma:

Lemma 5.2. *For i such that $1 \leq i \leq n+1$, we have*

$$(u_1^b)^{-1} *_0 ((u_2^b)^{-1} *_1 (\cdots *_i (\cdots *_i ((u_{i-1}^b)^{-1} *_i u_{i-1}^b))). = 1_{u_0^b}.$$

Proof. The result is obvious for $i = 1$. For $i \geq 2$, we have

$$\begin{aligned} &(u_1^b)^{-1} *_0 ((u_2^b)^{-1} *_1 (\cdots *_i (\cdots *_i ((u_{i-1}^b)^{-1} *_i (u_{i-1}^b)^{-1} *_i u_{i-1}^b)))) \\ &= (u_1^b)^{-1} *_0 ((u_2^b)^{-1} *_1 (\cdots *_i (\cdots *_i ((u_{i-1}^b)^{-1} *_i u_{i-1}^b)))) \end{aligned}$$

and the result follows by induction. \square

5.3. A cylinder (sequel). By the previous lemma, for i such that $1 \leq i \leq n+1$, the sources and targets of the c_i^\sharp 's are given by

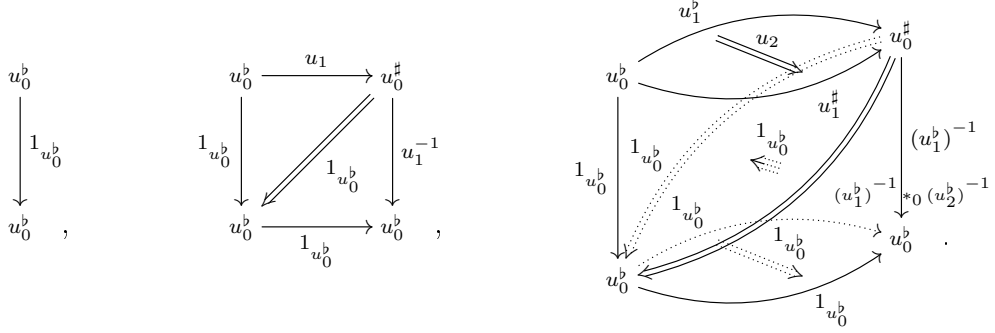
$$\begin{aligned} c_i^b &: 1_{u_0^b} \rightarrow 1_{u_0^b}, \\ c_i^\sharp &: (u_1^b)^{-1} *_0 ((u_2^b)^{-1} *_1 (\cdots *_i (\cdots *_i ((u_{i-1}^b)^{-1} *_i (u_{i-1}^\sharp)^{-1}))) \rightarrow 1_{u_0^b}. \end{aligned}$$

Moreover, we have

$$c_{n+1}^b = 1_{u_0^b} = c_{n+1}^\sharp.$$

This morphism will now simply be denoted by c_{n+1} .

Here are the corresponding diagrams for $n = 0, 1, 2$:



Proposition 5.4. *The arrows $c_0^b, c_0^\sharp, \dots, c_n^b, c_n^\sharp, c_{n+1}$ define an n -cylinder $c : u \curvearrowright 1_{u_0^b}$.*

Proof. We have to check that for $1 \leq i \leq n+1$ and $\varepsilon = b, \sharp$, we have

$$\begin{aligned} s_i(c_i^\varepsilon) &= c_{i-1}^\sharp *_{i-2} (c_{i-2}^\sharp *_{i-3} (\cdots *_{i-1} (c_1^\sharp *_{i-1} u_{i-1}^\varepsilon))), \\ t_i(c_i^\varepsilon) &= (((1_{u_0^b} *_{i-1} c_1^b) *_{i-2} c_2^b) *_{i-3} \cdots) *_{i-2} c_{i-1}^b. \end{aligned}$$

The second equality expands to

$$1_{u_0^b} = 1_{u_0^b} *_{i-1}^{i-1} (1_{u_0^b} *_{i-1}^{i-1} (*_{i-2} \cdots (1_{u_0^b} *_{i-2}^{i-1} 1_{u_0^b}))),$$

which is obviously true. The first one is exactly the content (modulo Lemma 5.2 for the case $\varepsilon = b$) of the following lemma applied to $j = i - 1$: \square

Lemma 5.5. *For i and j such that $0 \leq j < i \leq n+1$ and $\varepsilon = b, \sharp$, we have*

$$\begin{aligned} &c_j^\sharp *_{j-1} (c_{j-1}^\sharp *_{j-2} (\cdots *_{i-1} (c_1^\sharp *_{i-1} u_{i-1}^\varepsilon))) \\ &= (u_1^b)^{-1} *_{i-1} ((u_2^b)^{-1} *_{i-1} (\cdots *_{j-2} ((u_j^b)^{-1} *_{j-1} u_{i-1}^\varepsilon))). \end{aligned}$$

Proof. Fix i such that $1 \leq i \leq n+1$. We prove the result by induction on j . For $j = 0$, the result is obvious. For $j \geq 1$, we have

$$\begin{aligned} &c_j^\sharp *_{j-1} (c_{j-1}^\sharp *_{j-2} (\cdots *_{i-1} (c_1^\sharp *_{i-1} u_{i-1}^\varepsilon))) \\ &= c_j^\sharp *_{j-1} ((u_1^b)^{-1} *_{i-1} ((u_2^b)^{-1} *_{i-1} (\cdots *_{j-3} ((u_{j-1}^b)^{-1} *_{j-2} u_{i-1}^\varepsilon)))) \\ &= ((u_1^b)^{-1} *_{i-1} ((u_2^b)^{-1} *_{i-1} (\cdots *_{j-3} ((u_{j-1}^b)^{-1} *_{j-2} (u_j^b)^{-1})))) *_{j-1} \\ &\quad ((u_1^b)^{-1} *_{i-1} ((u_2^b)^{-1} *_{i-1} (\cdots *_{j-3} ((u_{j-1}^b)^{-1} *_{j-2} u_{i-1}^\varepsilon)))). \end{aligned}$$

where the first equality is obtained by induction and the second one by expanding the definition of c_j^\sharp . The result is then obtained by applying $j - 1$ times the exchange law:

$$\begin{aligned}
 & ((u_1^b)^{-1} *_0 ((u_2^b)^{-1} *_1 (\cdots *_{j-3} ((u_{j-1}^b)^{-1} *_{j-2} (u_j^b)^{-1})))) *_{j-1} \\
 & \quad ((u_1^b)^{-1} *_0 ((u_2^b)^{-1} *_1 (\cdots *_{j-3} ((u_{j-1}^b)^{-1} *_{j-2} u_{i-1}^\varepsilon)))) \\
 &= (u_1^b)^{-1} *_0 \left[((u_2^b)^{-1} *_1 (\cdots *_{j-3} ((u_{j-1}^b)^{-1} *_{j-2} (u_j^b)^{-1}))) *_{j-1} \right. \\
 & \quad \left. ((u_2^b)^{-1} *_1 (\cdots *_{j-3} ((u_{j-1}^b)^{-1} *_{j-2} u_{i-1}^\varepsilon)))) \right] \\
 &= (u_1^b)^{-1} *_0 (u_2^b)^{-1} *_1 \left[((u_3^b)^{-1} *_2 (\cdots *_{j-3} ((u_{j-1}^b)^{-1} *_{j-2} (u_j^b)^{-1}))) *_{j-1} \right. \\
 & \quad \left. ((u_3^b)^{-1} *_2 (\cdots *_{j-3} ((u_{j-1}^b)^{-1} *_{j-2} u_{i-1}^\varepsilon)))) \right] \\
 &= \cdots \\
 &= (u_1^b)^{-1} *_0 ((u_2^b)^{-1} *_1 (\cdots *_{j-2} ((u_j^b)^{-1} *_{j-1} u_{i-1}^\varepsilon))).
 \end{aligned}$$

□

Theorem 5.6. *The strict ∞ -groupoid $\tilde{\mathcal{D}}_n$ is weakly contractible.*

Proof. It suffices to show that the identity morphism $\tilde{\mathcal{D}}_n \rightarrow \tilde{\mathcal{D}}_n$ is right homotopic to some constant morphism $\tilde{\mathcal{D}}_n \rightarrow \tilde{\mathcal{D}}_n$. We will denote by u_0^b the constant morphism $\tilde{\mathcal{D}}_n \rightarrow \tilde{\mathcal{D}}_n$ corresponding to the object u_0^b of $\tilde{\mathcal{D}}_n$. By Theorem 4.16, it suffices to define a morphism $k : \tilde{\mathcal{D}}_n \rightarrow \Gamma(\tilde{\mathcal{D}}_n)$ making the diagram

$$\begin{array}{ccc}
 & & \tilde{\mathcal{D}}_n \\
 & \nearrow^{1_{\tilde{\mathcal{D}}_n}} & \nearrow \\
 \tilde{\mathcal{D}}_n & \xrightarrow{k} & \Gamma(\tilde{\mathcal{D}}_n) \\
 & \searrow_{u_0^b} & \searrow_{p_1} \\
 & & \tilde{\mathcal{D}}_n
 \end{array}$$

commute. But the data of such a morphism k is clearly equivalent to the data of an n -cylinder from u to $1_{u_0^b}$. The result thus follows from Proposition 5.4. □

Remark 5.7. Let \mathcal{C} be a category and let \mathcal{W} be a class of morphisms of \mathcal{C} . Denote by $p : \mathcal{C} \rightarrow \mathcal{C}[\mathcal{W}^{-1}]$ the localization functor. Recall that \mathcal{W} is said to be *strongly saturated* if every morphism f of \mathcal{C} such that $p(f)$ is an isomorphism is in \mathcal{W} . A sufficient condition for \mathcal{W} to be strongly saturated is that the pair $(\mathcal{C}, \mathcal{W})$ is *Quillenisable*, i.e., that there exists a model category structure on \mathcal{C} whose weak equivalences are the elements of \mathcal{W} .

The proof that $\tilde{\mathcal{D}}_n$ is weakly contractible can be written so that the Brown-Golasiński model category structure is only used to prove that the class of weak equivalences of strict ∞ -groupoids is strongly saturated. In particular, its cofibrations and fibrations play no role in this proof.

6. THE GLOBULAR EXTENSION $\tilde{\Theta}$ IS CANONICALLY CONTRACTIBLE

Theorem 6.1. *Let T be an object of $\tilde{\Theta}$. Then the strict ∞ -groupoid $L(G_T)$ is weakly contractible.*

Proof. In the language of [4], the fact that the $\tilde{S}^{n-1} \rightarrow \tilde{D}_n$ are cofibrations and that the \tilde{D}_n 's are weakly contractible (Theorem 5.6) precisely means that the functor

$$\mathbb{G} \rightarrow \tilde{\Theta} \rightarrow \infty\text{-}\mathcal{Gpd}_{\text{str}}$$

is cofibrant and weakly contractible. The result then follows from Proposition 5.11 of loc. cit.

Let us briefly explain how to prove the result directly. Denote by k the width of T . If $k = 1$, then $L(G_T) = \tilde{D}_i$ for some i and the result follows from Theorem 5.6. If $k > 1$, then we have $T = S \amalg_{D_{i'}} D_i$ for some i, i' and some table of dimensions S of width $k - 1$. It follows that we have a cocartesian square of strict ∞ -groupoids

$$\begin{array}{ccc} \tilde{D}_{i'} & \longrightarrow & L(G_S) \\ \tilde{\tau}_{i'}^i \downarrow & & \downarrow \\ \tilde{D}_i & \longrightarrow & L(G_T) \end{array} .$$

By Remark 4.9, the left vertical morphism is a cofibration. Since $\tilde{D}_{i'}$ and \tilde{D}_i are both weakly contractible, this morphism is actually a trivial cofibration. It follows that the morphism $L(G_S) \rightarrow L(G_T)$ is a trivial cofibration and the result follows by induction on k . \square

Remark 6.2. The above proof is our first real use of the Brown-Golasinski model category structure (see Remark 5.7).

Remark 6.3. Theorem 6.1 can be reformulated by saying that free strict ∞ -groupoids on a globular pasting scheme are weakly contractible.

Theorem 6.4. *The globular extension $\tilde{\Theta}$ is canonically contractible.*

Proof. As we saw in the previous proof, in the language of [4], the functor $\mathbb{G} \rightarrow \infty\text{-}\mathcal{Gpd}_{\text{str}}$ is cofibrant and weakly contractible. Since every strict ∞ -groupoid is fibrant in the Brown-Golasinski model category structure, Proposition 5.12 of loc. cit. shows that the globular extension $\tilde{\Theta}$ is contractible.

Let us briefly explain how to prove the contractibility directly. Let $(f, g) : D_n \rightarrow T$ be an admissible pair of $\tilde{\Theta}$. Explicitly, f and g are morphisms of strict ∞ -groupoids from \tilde{D}_n to $L(G_T)$. Since f and g are globularly parallel, they can be glued along \tilde{S}^{n-1} and hence define a morphism

$$(f, g) : \tilde{S}^n = \tilde{D}_n \amalg_{\tilde{S}^{n-1}} \tilde{D}_n \rightarrow L(G_T).$$

One easily checks that a lifting of the pair (f, g) is exactly a morphism $h : \tilde{D}_{n+1} \rightarrow L(G_T)$ making the triangle

$$\begin{array}{ccc} \tilde{D}_{n+1} & & \\ \uparrow & \searrow h & \\ \tilde{S}^n & \xrightarrow{(f,g)} & L(G_T) \end{array}$$

commute or, in other words, a diagonal filler of the square

$$\begin{array}{ccc} \tilde{S}^n & \longrightarrow & L(G_T) \\ \downarrow & & \downarrow \\ \tilde{D}_{n+1} & \longrightarrow & * \end{array} ,$$

where $*$ denotes the terminal strict ∞ -groupoid. But such a diagonal filler exists since the left vertical morphism is a cofibration and the right vertical morphism is a trivial fibration by the previous theorem and the fact that every strict ∞ -groupoid is fibrant in the Brown-Golasinski model category structure.

Let us now prove that the globular extension $\tilde{\Theta}$ is *canonically* contractible. We have to prove that the lifting h constructed above is unique. Recall that if G is a strict ∞ -groupoid and $k \geq 0$, there is a canonical bijection between the morphisms $\tilde{D}_k \rightarrow G$ and the k -arrows of G . In this proof, we will identify these morphisms with their corresponding k -arrow. Using this identification, a lifting h of the pair (f, g) is nothing but an $(n+1)$ -arrow $h : f \rightarrow g$ of $L(G_T)$. Let $h' : f \rightarrow g$ be a second $(n+1)$ -arrow of $L(G_T)$. By the previous theorem, the ∞ -groupoid $L(G_T)$ is weakly contractible. It follows from Proposition 1.10 that there exists an $(n+2)$ -arrow $h \rightarrow h'$ in $L(G_T)$. But by definition of an admissible pair, the dimension of T is at most $n+1$. It follows that $L(G_T)$ has no non-trivial $(n+2)$ -arrows. This shows that $h = h'$, thereby proving the result. \square

Remark 6.5. In the above proof, we used our additional condition in the definition of admissible pairs (see Remark 2.10) in an essential way. If we drop this condition, then the globular extension $\tilde{\Theta}$ is no longer *canonically* contractible. Indeed, without this condition, the pair

$$(\tilde{\sigma}_0^2, \tilde{\tau}_0^2) : \tilde{D}_0 \rightarrow \tilde{D}_2$$

would be admissible. Nevertheless, it has two different liftings, namely $\tilde{\sigma}_2$ and $\tilde{\tau}_2$.

7. FULLY FAITHFUL FUNCTORS BETWEEN GLOBULAR PRESHEAF CATEGORIES

7.1. Precategorical globular extensions. Let C be a globular extension. A *precategorical structure* on C consists of the structure of a co- ∞ -precategory on the co- ∞ -graph defined by the functor $\mathbb{G} \rightarrow C$. More explicitly, such a structure is given by morphisms

$$\begin{aligned} \nabla_j^i : D_i &\rightarrow D_i \amalg_{D_j} D_i, & i > j \geq 0, \\ \kappa_i : D_{i+1} &\rightarrow D_i, & i \geq 0, \end{aligned}$$

such that

- for every $i > j \geq 0$, we have

$$\nabla_j^i \sigma_i = \begin{cases} \varepsilon_2 \sigma_i, & j = i - 1, \\ (\sigma_i \amalg_{D_j} \sigma_i) \nabla_j^{i-1} & j < i - 1, \end{cases}$$

and

$$\nabla_j^i \tau_i = \begin{cases} \varepsilon_1 \tau_i, & j = i - 1, \\ (\tau_i \amalg_{D_j} \tau_i) \nabla_j^{i-1} & j < i - 1, \end{cases}$$

where $\varepsilon_1, \varepsilon_2 : D_i \rightarrow D_i \amalg_{D_{i-1}} D_i$ denote the canonical morphisms;

- for every $i \geq 0$, we have

$$\kappa_i \sigma_{i+1} = 1_{D_i} \quad \text{and} \quad \kappa_i \tau_{i+1} = 1_{D_i}.$$

A *precategorical globular extension* is a globular extension endowed with a precategorical structure.

Proposition 7.2. *Any contractible globular extension can be endowed with a precategorical structure. Moreover, if the globular extension is canonically contractible, then this structure is canonical.*

Proof. Let C be a globular extension. The choice of a precategorical structure on C is equivalent to the choice of some liftings of admissible pairs of C (see Section 3 of [4] for details). In particular, if C is contractible, such a choice can always be made (see paragraph 3.11 of loc. cit.) and if C is canonically contractible, this choice is unique. \square

7.3. Let C be a precategorical globular theory. Then for any globular presheaf X on C , the underlying globular set of X (which is obtained by pre-composing by $\mathbb{G} \rightarrow C$) is canonically endowed with the structure of an ∞ -precategory. This defines a functor

$$\text{Mod}(C) \rightarrow \infty\text{-}\mathcal{P}\text{Cat}.$$

This functor is easily seen to be faithful.

Remark 7.4. This functor can also be described in the following way. Let Θ_{pcat} be the universal precategorical globular theory (its existence can be shown using the globular completion, see paragraph 3.9 of [2]). By definition of Θ_{pcat} , a precategorical structure on a globular theory C defines a globular functor $\Theta_{\text{pcat}} \rightarrow C$. This functor is bijective on objects and thus induces a faithful functor

$$\text{Mod}(C) \rightarrow \text{Mod}(\Theta_{\text{pcat}}) \cong \infty\text{-}\mathcal{P}\text{Cat}.$$

Proposition 7.5. *Let C be a precategorical globular theory. For any globular functor from C to $\tilde{\Theta}$, the induced functor*

$$\infty\text{-}\mathcal{G}\text{pd}_{\text{str}} \cong \text{Mod}(\tilde{\Theta}) \rightarrow \text{Mod}(C)$$

is fully faithful.

Proof. By Proposition 7.2 and Theorem 6.4, the globular extension $\tilde{\Theta}$ is endowed with a canonical precategorical structure. It follows that the functor $C \rightarrow \tilde{\Theta}$ respects the

precategorical structures. This implies that the triangle

$$\begin{array}{ccc} \text{Mod}(\tilde{\Theta}) & \longrightarrow & \text{Mod}(C) \\ & \searrow & \swarrow \\ & \infty\text{-}\mathcal{P}\text{Cat} & \end{array},$$

where the two oblique arrows are given by the respective precategorical structures of $\tilde{\Theta}$ and C , is commutative. Moreover, the functor $\infty\text{-}\mathcal{G}\text{pd}_{\text{str}} \cong \text{Mod}(\tilde{\Theta}) \rightarrow \infty\text{-}\mathcal{P}\text{Cat}$ is nothing but the forgetful functor and is hence fully faithful. The result then follows from the fact that the functor $\text{Mod}(C) \rightarrow \infty\text{-}\mathcal{P}\text{Cat}$ is faithful. \square

Corollary 7.6. *Let C be a contractible globular theory. For any globular functor from C to $\tilde{\Theta}$, the induced functor*

$$\infty\text{-}\mathcal{G}\text{pd}_{\text{str}} \cong \text{Mod}(\tilde{\Theta}) \rightarrow \text{Mod}(C)$$

is fully faithful.

Proof. By Proposition 7.2, C can be endowed with a precategorical structure. The result then follows from the previous proposition. \square

8. STRICT ∞ -GROUPOIDS ARE GROTHENDIECK ∞ -GROUPOIDS

Theorem 8.1. *Let C be a coherator. There exists a canonical functor*

$$\infty\text{-}\mathcal{G}\text{pd}_{\text{str}} \rightarrow \infty\text{-}\mathcal{G}\text{pd}_C.$$

Moreover, this functor is fully faithful.

Proof. By Theorem 6.4, the globular theory $\tilde{\Theta}$ is canonically contractible. It follows from Proposition 2.12 that there exists a unique globular functor $C \rightarrow \tilde{\Theta}$. This globular functor induces a functor

$$\infty\text{-}\mathcal{G}\text{pd}_{\text{str}} \cong \text{Mod}(\tilde{\Theta}) \rightarrow \text{Mod}(C) = \infty\text{-}\mathcal{G}\text{pd}_C,$$

thereby proving the first assertion. The second assertion follows immediately from Corollary 7.6. \square

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