DISSERTATION / DOCTORAL THESIS

Titel der Dissertation / Title of the Doctoral Thesis

Contribution to multiplicative chaos theory

verfasst von / submitted by

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angestrebter akademischer Grad / in partial fulfilment of the requirements for the degree of

Doktor der Naturwissenschaften (Dr. rer. nat.)

Wien, 2021 / Vienna, 2021

Studienkennzahl lt. Studienblatt / degree programme code as it appears on the student record sheet:	UA 796 605 405
Studienrichtung lt. Studienblatt / degree programme as it appears on the student record sheet:	Mathematik
Betreut von / Supervisor:	Univ.-Prof. Dr. Nathanaël Berestycki

Acknowledgements

Firstly, to my supervisor, Nathanaël. Thank you for all the time you dedicated to me, for all the discussions and valuable advice. I have learnt a lot from you. Thank you! Thanks also to my mathematics teachers, Marc Lorenzi, Frédéric Normandin, Thomas Duquesne and Raphaël Cerf; to my ENS supervisor Julien Barral for sound advice; and to my collaborators Elie Aïdékon, Juhan Aru, Janne Junnila, Titus Lupu for sharing your time and ideas.

Thanks to my friends at CCA and King's in Cambridge and to the stochastics and financial mathematics group in Vienna. Special thanks go to *blend*: Alexandre, Marie and Victor, for welcoming me so many times in Paris; to Hugo for all our chats and to *Sbraaaa* for all the diverse and exciting experiences in Vienna. To Marie: thank you for bringing so much joy in my life! Finally, thanks to my family, to my mother and sister for having always believed in me, to my grandfather Victor *pour m'avoir communiqué son goût des maths dès le plus jeune âge.*

List of included articles

Chapters 2, 3 and 4 are respectively based on the following articles

- Juhan Aru, Antoine Jego and Janne Junnila, *Density of imaginary multiplicative chaos via Malliavin calculus.* ArXiv e-prints, June 2021. (submitted)
- Antoine Jego, *Critical Brownian multiplicative chaos.* Probab. Theory Relat. Fields 180 (2021), 495–552.
- Élie Aïdékon, Nathanaël Berestycki, Antoine Jego, and Titus Lupu, *Multiplicative chaos of the Brownian loop soup*, ArXiv e-prints, July 2021. (submitted)

Abstract

Gaussian multiplicative chaos theory studies properties of random measures formally defined by exponentiating a real parameter γ times a logarithmically correlated Gaussian field. Introduced by Kahane in the eighties, this theory has attracted a lot of attention in the past decade and has been shown to be related to many areas in mathematics. This thesis starts by studying the imaginary multiplicative chaos where the parameter γ is chosen to be a purely imaginary complex number. Compared to the real case, the resulting object is rougher and is not a (complex) measure any more. The goal of this first part is to prove a basic density result, showing that for any nonzero continuous test function f, the complex-valued random variable obtained by integrating the imaginary chaos against f has a smooth density w.r.t. Lebesgue measure on $\mathbb{C}$. Somewhat surprisingly, basic density results are not easy to prove for imaginary chaos and one of the main contributions of this part is introducing Malliavin calculus to the study of (complex) multiplicative chaos.

The second part of this thesis is concerned with Brownian multiplicative chaos measure. This measure has been introduced very recently and is an instance of multiplicative chaos associated to a non-Gaussian field: it is formally defined by exponentiating γ times the square root of the local times of planar Brownian motion. So far, only the subcritical measures where the parameter γ is less than 2 were studied. Chapter 3 considers the critical case where $\gamma = 2$, using three different approximation procedures which all lead to the same universal measure. On the one hand, we exponentiate the square root of the local times of small circles and show convergence in the Seneta–Heyde normalisation as well as in the derivative martingale normalisation. On the other hand, we construct the critical measure as a limit of subcritical measures. This is the first example of a non-Gaussian critical multiplicative chaos.

Finally, we construct a multiplicative chaos measure associated to a Brownian loop soup in a bounded domain D of the plane with given intensity $\theta > 0$, which is formally obtained by exponentiating the square root of its occupation field. The measure is constructed via a regularisation procedure, in which loops are killed at a fix rate, allowing us to make use of the Brownian multiplicative chaos measures. At the critical intensity $\theta = 1/2$, it is shown that this measure coincides with the hyperbolic cosine of the Gaussian free field, which is closely related to Liouville measure. This allows us to draw several conclusions which elucidate connections between Brownian multiplicative chaos, Gaussian free field and Liouville measure. For instance, it is shown that Liouville-typical points are of infinite loop multiplicity, with the relative contribution of each loop to the overall thickness of the point being described by the Poisson–Dirichlet distribution with parameter $\theta = 1/2$. Conversely, the Brownian chaos associated to each loop describes its microscopic contribution to Liouville measure. Along the way, our proof reveals a surprising exact integrability of the multiplicative chaos associated to a killed Brownian loop soup. We also obtain some estimates on the loop soup which may be of independent interest.

Zusammenfassung

Die Theorie des multiplikativen Gaußsches-Chaos untersucht die Eigenschaften von Zufallsmaßen, die formal durch die Potenzierung eines reellen Parameters γ mal einem logarithmisch korrelierten Gaußschen-Feld definiert sind. Diese von Kahane in den achtziger Jahren eingeführte Theorie hat in

den letzten zehn Jahren viel Aufmerksamkeit auf sich gezogen und es hat sich gezeigt, dass sie mit vielen Bereichen der Mathematik in Verbindung steht. In dieser Arbeit wird zunächst das imaginäre multiplikative Chaos untersucht, wobei der Parameter γ als rein imaginäre komplexe Zahl gewählt wird. Im Vergleich zum reellen Fall ist das resultierende Objekt gröber und ist kein (komplexes) Maß mehr. Ziel dieses ersten Teils ist es, ein grundlegendes Dichteergebnis zu beweisen, das zeigt, dass für jede stetige Testfunktion f ungleich Null die komplexwertige Zufallsvariable, die man durch Integration des imaginären Chaos gegen f erhält, eine glatte Dichte bezüglich des Lebesgue-Maßes auf $\mathbb{C}$ hat. Überraschenderweise sind die grundlegenden Dichteergebnisse für imaginäres Chaos nicht einfach zu beweisen, und einer der Hauptbeiträge dieses Teils ist die Einführung des Malliavin-Calculus in die Untersuchung des (komplexen) multiplikativen Chaos.

Der zweite Teil dieser Arbeit befasst sich mit dem Brownschen multiplikativen Chaosmaß. Dieses Maß wurde erst kürzlich eingeführt und ist ein Beispiel für multiplikatives Chaos, das mit einem nicht-Gaußschen Feld assoziiert ist: Es ist formal definiert durch Potenzierung von γ mal der Quadratwurzel der lokalen Zeiten der zweidimensionalen Brownschen Bewegung. Bislang wurden nur die subkritischen Maße untersucht, bei denen der Parameter γ kleiner als 2 ist. Kapitel 3 betrachtet den kritischen Fall, in dem $\gamma = 2$ ist, unter Verwendung von drei verschiedenen Approximationsverfahren, die alle zum gleichen universellen Maß führen. Einerseits exponentiieren wir die Quadratwurzel der lokalen Zeiten kleiner Kreise und zeigen Konvergenz in der Seneta-Heyde-Normalisierung sowie in der abgeleiteten Martingal-Normalisierung. Andererseits konstruieren wir das kritische Maß als einen Grenzwert von subkritischen Maßen. Dies ist das erste Beispiel für ein nicht-Gaußsches kritisches multiplikatives Chaos.

Schließlich konstruieren wir ein multiplikatives Chaosmaß, das mit einer Brownschen Schleifensuppe in einem begrenzten Bereich D der Ebene mit gegebener Intensität $\theta > 0$ assoziiert ist und formal durch Potenzierung der Quadratwurzel ihres Besetzungsfeldes erhalten wird. Das Maß wird über ein Regularisierungsverfahren konstruiert, bei dem Schleifen mit einer festen Rate abgetötet werden, was uns erlaubt, die Brownschen multiplikativen Chaosmaße zu nutzen. Für die kritischen Intensität $\theta = 1/2$ wird gezeigt, dass dieses Maß mit dem hyperbolischen Kosinus des Gaußschen freien Feldes übereinstimmt, das eng mit dem Liouville-Maß verwandt ist. Daraus lassen sich mehrere Schlussfolgerungen ziehen, die den Zusammenhang zwischen dem multiplikativen Brownschen Chaos, dem freien Gaußschen Feld und dem Liouville-Maß verdeutlichen. So wird beispielsweise gezeigt, dass Liouville-typische Punkte von unendlicher Schleifenvielfalt sind, wobei der relative Beitrag jeder Schleife zur Gesamtdicke des Punktes durch die Poisson–Dirichlet-Verteilung mit dem Parameter $\theta = 1/2$ beschrieben wird. Umgekehrt beschreibt das Brownsche Chaos, das jeder Schleife zugeordnet ist, ihren mikroskopischen Beitrag zum Liouville-Maß. Nebenbei enthüllt unser Beweis eine überraschende exakte Integrierbarkeit des multiplikativen Chaos, das einer getöteten Brownschen Schleifensuppe zugeordnet ist. Wir erhalten auch einige Abschätzungen über die Schleifensuppe, die von unabhängigem Interesse sein könnten.

Contents

Chapter 1

Introduction

1.1 Multiplicative cascades

Random multiplicative cascades measures were introduced by Mandelbrot [Man72, Man74a, Man74b] as a toy model for energy dissipation in a turbulent flow. They have been extensively studied and can be seen as being discrete counterparts of Gaussian multiplicative chaos. The introduction of [BKN$^+$14] gives a good account of the references on this topic. The exact tree structure underlying multiplicative cascades makes them easier to analyse and many properties are first discovered in the context of multiplicative cascades and then proved to hold as well in the more delicate setting of Gaussian multiplicative chaos. For this reason, we decided to present first the multiplicative cascades. Since they are obtained by exponentiating a branching random walk, we start by introducing this latter object.

Branching random walk For simplicity, we will only consider binary branching random walks. An introduction to this topic can be found in the book [Shi15]. Let ξ be a real-valued random variable such that

$$\mathbb{E}\left[e^{\xi}\right] = \frac{1}{2} \quad \text{and} \quad \mathbb{E}\left[\xi e^{\xi}\right] = 0. \tag{1.1}$$

Some further integrability conditions are also needed for the results that we mention below to hold. We do not want to enter into these technical details and we will simply assume that

$$\mathbb{E}\left[e^{(1+\varepsilon)|\xi|}\right] < \infty \quad \text{for some } \varepsilon > 0.$$

The binary branching random walk associated to ξ is the process which can be described as follows. A particle starts at the origin. At time 1, it dies and gives birth to two independent children that are located at a random position distributed according to the law of ξ. These two children then evolves independently of each other in a similar manner as the initial particle: at time 2, the particles die and each of them gives birth to two independent children whose displacements with respect to their respective parent are independent copies of ξ. The process keeps evolving in this way and is depicted in Figure 1.1.

More formally, let $\mathbb{T} = \bigcup_{n\geq 0}\{0,1\}^n$ be the binary tree, with the convention that the root $\{0,1\}^0$ is

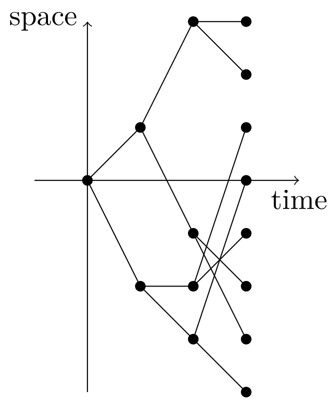


Figure 1.1: First three generations of a binary branching random walk.

reduced to the empty set $\varnothing$. Let $n \geq 1$ and $w = (w_1, \ldots, w_n) \in \{0,1\}^n$. n represents the generation of w that we denote by $|w| = n$ and for any $k \leq n$, we denote by $w|k$ the ancestor of w in generation k, i.e. $w|k = (w_1, \ldots, w_k)$. Let $(\xi_w, w \in \mathbb{T})$ be independent copies of ξ. For any $w \in \mathbb{T}$, define

$$V(w) := \sum_{k \leq |w|} \xi_{w|k}.$$

The collection of spatial positions $(V(w), w \in \mathbb{T})$ defines our branching random walk. Note that these spatial positions form along any branch of the tree a one-dimensional random walk with increments being independent copies of ξ. We encode the spatial positions of the n-th generation in a random field $\Gamma_n = (\Gamma_n(x), x \in [0,1])$ as follows. For any $x \in [0,1]$, let $x = \sum_{k=1}^{\infty} w_k 2^{-k}$ be its decomposition in base 2 and define

$$\Gamma_n(x) := V(w_1, \ldots, w_n).$$

Subcritical multiplicative cascades The multiplicative cascade built from this branching random walk is defined as follows. Let $\gamma > 0$ and define for any $n \geq 1$, the random Borel measure

$$\mu_\gamma^n(dx) = \frac{e^{\gamma \Gamma_n(x)}}{\mathbb{E}\left[e^{\gamma \xi}\right]^n} \mathbf{1}_{\{x \in [0,1]\}} \mathrm{d}x.$$

For any Borel set $I \subset [0,1]$, $(\mu_\gamma^n(I), n \geq 1)$, is a nonnegative martingale and therefore it almost surely converges as $n \to \infty$. Standard arguments then imply the almost sure convergence of the measure μ_γ^n for the topology of weak convergence. Determining whether the limiting measure is trivial (i.e. almost surely equal to zero) or not is not a simple task. Kahane and Peyrière [KP76] showed that the answer depends on γ: with our normalisation (1.1), their result states that the limiting measure is nondegenerate if, and only if, $\gamma \in (0,1)$. The multiplicative chaos measure μ_γ, $\gamma \in (0,1)$, is then defined to be the limiting measure.

Critical multiplicative cascades It is still possible to make sense of a multiplicative cascade associated to $\gamma = 1$. As explained above, normalising the measure by its first moment leads to a vanishing measure when $\gamma = 1$. Two different successful normalisations have been considered. The first one blows up the subcritical normalisation by a deterministic factor, whereas the second one uses a random factor: it is shown in [BK04] and [AS14] that

$$\sqrt{n}\mu_n^{\gamma=1} \quad \text{and} \quad -\Gamma_n(x)\mu_n^{\gamma=1}(dx)$$

both converge towards some nondegenerate Borel measures. Moreover, the two limiting measures coincide up to some deterministic multiple constant. The first normalisation is known as the Seneta-Heyde normalisation, whereas the second one is called the derivative martingale. This latter name stems from the fact that this second approximation is actually a martingale and that it is formally the derivative of μ_n^γ with respect to γ:

$$\frac{\mathrm{d}\mu_n^\gamma(dx)}{\mathrm{d}\gamma}\Big|_{\gamma=1} = \frac{e^{\gamma\Gamma_n(x)}}{\mathbb{E}\left[e^{\gamma\xi}\right]^n}\left(\Gamma_n(x) - n\frac{\mathbb{E}\left[\xi e^{\gamma\xi}\right]}{\mathbb{E}\left[e^{\gamma\xi}\right]}\right)dx\Big|_{\gamma=1} = \Gamma_n(x)\frac{e^{\Gamma_n(x)}}{\mathbb{E}\left[e^{\xi}\right]^n}dx.$$

This last equality comes from the normalisation (1.1). Note that, although the derivative martingale is a signed measure at the level of the approximation, the limiting measure is a positive measure.

[Mad16] also shows that the critical cascade measure can be obtained from the subcritical ones, i.e.

$$\frac{1}{1-\gamma}\mu_\gamma$$

converges as $\gamma \to 1^-$ to a multiple of the critical cascade measure.

Maximum of branching random walk The large values of the branching random walk Γ_n are well-described by the cascade measures. For instance, the celebrated result of Aïdékon [Aï13] states that

$$\sup_{x\in[0,1]} \Gamma_n(x) - \frac{3}{2}\log n \to G + \log \mu_1([0,1]) \quad \text{in distribution} \tag{1.2}$$

where G is a Gumbel random variable independent of the critical cascade measure μ_1.

Complex multiplicative cascades A natural extension to the theory of multiplicative cascades consists in allowing the random variable ξ to take complex values. Such an extension has been studied in [BJM10] (see also [DES93, Big92, HK15, HK18]). The limiting object exhibits very different properties compared to the real case. We will discuss more thoroughly the complex case in the Gaussian multiplicative chaos context in Section 1.2.2.

The next section will present the main character of this thesis: Gaussian multiplicative chaos. This object will be the analogue of multiplicative cascades where the branching random walk will be replaced by a Gaussian field with logarithmic correlations. Even though branching random walk is not necessarily Gaussian (the variable ξ was not assumed to be Gaussian), it is log-correlated. Indeed, a simple computation shows that if $x = \sum_{k=1}^\infty w_k 2^{-k}$ and $y = \sum_{k=1}^\infty w_k' 2^{-k}$ are two elements of $[0,1]$, then

$$\mathrm{Cov}(\Gamma_n(x), \Gamma_n(y)) = \mathrm{Var}(\xi)\min(k,n)$$

where k is the maximum integer such that for all $i = 1 \dots k$, $w_i = w'_i$. In other words,

$$\mathrm{Cov}(\Gamma_n(x), \Gamma_n(y)) = \mathrm{Var}(\xi) \min\left(n, \frac{-\log|x-y|}{\log 2}\right) + O(1).$$

1.2 Gaussian multiplicative chaos

Since Gaussian multiplicative chaos is defined as the exponential of a (complex) parameter γ times a logarithmically-correlated Gaussian field, we first need to recall what these fields are. This is the purpose of the following section.

1.2.1 Logarithmically-correlated Gaussian fields

A log-correlated Gaussian field Γ on a given domain $U \subset \mathbb{R}^d$ is formally a Gaussian vector indexed by points in U whose correlations blow up logarithmically on the diagonal

$$\mathbb{E}\left[\Gamma(x)\Gamma(y)\right] \sim -\log|x-y| \qquad \text{as} \qquad |x-y| \to 0.$$

Because of the blow-up on the diagonal, these fields cannot be well-defined pointwise and their definitions require some care.

Let U be a bounded domain in $\mathbb{R}^d$, $d \geq 1$, and let C be a positive definite kernel of the form

$$C(x,y) = -\log|x-y| + g(x,y).$$

C will capture the correlations of our Gaussian field and we make the following integrability and regularity assumptions on g: g is bounded from above and g belongs to the Sobolev space $H^{d+\varepsilon}_{\mathrm{loc}}(U \times U) \cap L^2(U \times U)$.

To define a Gaussian field Γ with covariance C, one can proceed as follows. By spectral theorem, there exists a sequence of strictly positive eigenvalues $\lambda_1 \geq \lambda_2 \geq \dots > 0$ and corresponding orthogonal eigenfunctions $(f_k)_{k\geq 1}$ spanning the subspace $(\mathrm{Ker}\, C)^\perp$ in $L^2(\mathbb{R}^d)$. The log-correlated field Γ can then be defined via its Karhunen–Loève expansion

$$\Gamma = \sum_{k\geq 1} A_k C^{1/2} f_k = \sum_{k \geq 1} A_k \sqrt{\lambda_k} f_k,$$

where $(A_k)_{k\geq 1}$ is an i.i.d. sequence of standard normal random variables. It has been shown in [JSW20, Proposition 2.3] that the above series converges in $H^{-\varepsilon}(\mathbb{R}^d)$ for any fixed $\varepsilon > 0$. The log-correlated field Γ is therefore well-defined as a random generalised function which belongs to $H^{-\varepsilon}(\mathbb{R}^d)$ for all $\varepsilon > 0$. In fact, it barely fails to being a true function since it can be shown that Γ belongs to the Hölder space $C^{-\varepsilon}(\mathbb{R}^d)$ with negative index $-\varepsilon$, for any $\varepsilon > 0$; see [JSW20, Lemma 2.5]. This definition gives a natural notion of Gaussian field with covariance C since one can easily compute that, for all test functions $\varphi, \psi : U \to \mathbb{R}$,

$$\mathbb{E}\left[\left(\int_U \Gamma(x)\varphi(x)dx\right)\left(\int_U \Gamma(x)\psi(x)dx\right)\right] = \int_{U\times U} \varphi(x)C(x,y)\psi(y)dxdy.$$

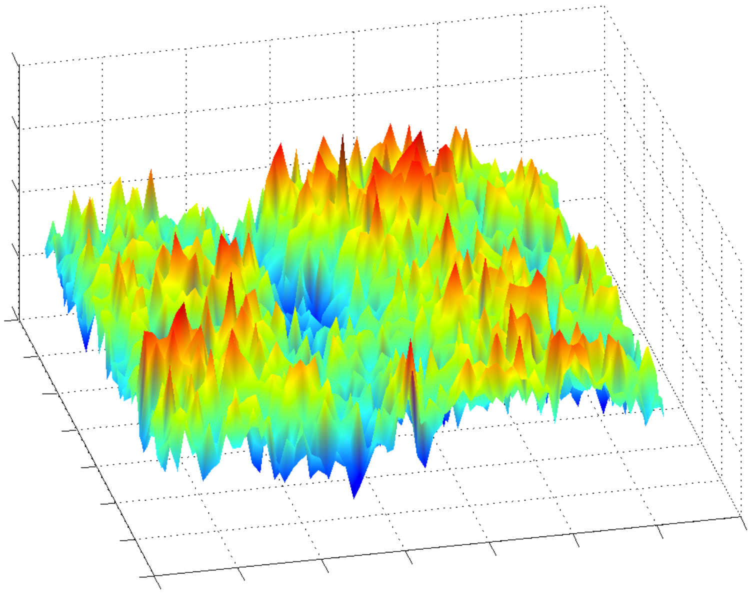


Figure 1.2: Simulation of 2D Gaussian free field in a square made by R. Rhodes and V. Vargas.

Gaussian free field An important example of log-correlated Gaussian field is the so-called two-dimensional Gaussian free field (GFF). It can be thought of as being the analogue of Brownian bridge where the time interval has been replaced by a two-dimensional domain. It pops up in many different contexts. For instance, it arises as a universal scaling limit of a wide range of models such as the height function of dimer models [Ken01], the characteristic polynomial of large random matrices [HKO01, RV07, FKS16] and the Ginzburg–Landau model [Mil11, NS97] (see the review [Pow20a] for more references). The GFF corresponds to the log-correlated field whose covariance is given by the Green function of the Laplacian that we define now.

Let $U \subset \mathbb{R}^2$ be a bounded simply connected domain and for any $t > 0$ and $x, y \in D$, let $p_t^U(x,y)$ be the transition probability of Brownian motion killed at the boundary of U. This transition probability can be expressed as $p_t^U(x,y) = p_t(x,y)\pi_t^U(x,y)$ where $p_t(x,y) = 1/(2\pi t)\exp\left(-|x-y|^2/(2t)\right)$ is the heat kernel and $\pi_t^U(x,y)$ is the probability for a Brownian bridge from x to y of duration t to stay in U. The Green function G_U with zero-boundary condition is then defined as

$$G_U(x,y) = \pi \int_0^\infty p_t^U(x,y)dt, \qquad x, y \in U.$$

It can be shown that the Green function is a positive definite kernel satisfying the assumptions of the above paragraph. The GFF is then simply the log-correlated field associated to this specific kernel. See Figure 1.2 for a simulation and see [Ber16, WP21, BP21] for more on the GFF.

1.2.2 Gaussian multiplicative chaos

Let $U \subset \mathbb{R}^d$ and let Γ be a log-correlated Gaussian field in U as in Section 1.2.1. The Gaussian multiplicative chaos μ_γ associated to Γ and to a complex-valued parameter γ is formally defined as

$$\mu_\gamma(x) = e^{\gamma\Gamma(x) - \frac{\gamma^2}{2}\mathbb{E}[\Gamma(x)^2]}.$$

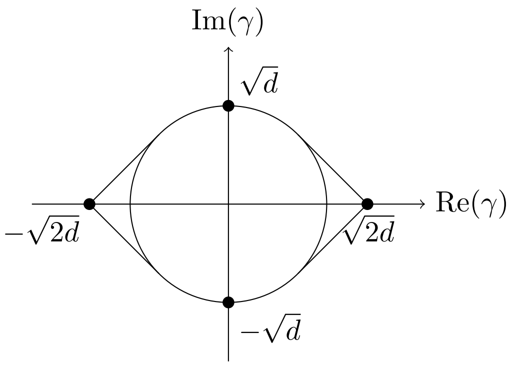


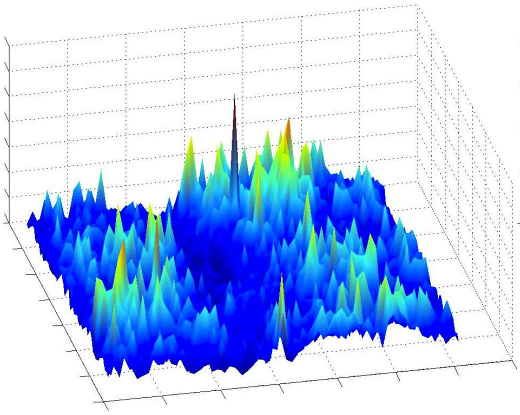
Figure 1.3: Range of parameter γ for which an associated Gaussian multiplicative chaos is defined.

Exponentiating a generalised function is not *a priori* a well-defined operation and making sense of such an object requires some non trivial work. In fact, this will be possible only in the restricted range of γ depicted in Figure 1.3. This eye-shaped domain is defined as the open convex hull of the union of the interval $(-\sqrt{2d}, \sqrt{2d})$ and the disc centred at the origin with radius $\sqrt{d}$. The construction and the study of GMC was first carried out in the real case $\gamma \in (-\sqrt{2d}, \sqrt{2d})$. It was initiated by Kahane [Kah85] and then extensively studied in the last decade [RV10, DS11, RV11, Sha16, Ber17]. The outcome of these works is that, when $\gamma \in (-\sqrt{2d}, \sqrt{2d})$, it is possible to make sense of μ_γ as a random Borel measure. The complex case $\gamma \notin (-\sqrt{2d}, \sqrt{2d})$ was then studied in [AJKS11, JSW19, JSW20, Lac20, AJ21, AJJ21]; see also [LRV15a] for a variant of this model. In that case, it is still possible to make sense of the exponential of γ times a log-correlated field, but the resulting object is not a random (complex) measure any more, but a rougher generalised function; see [JSV19].

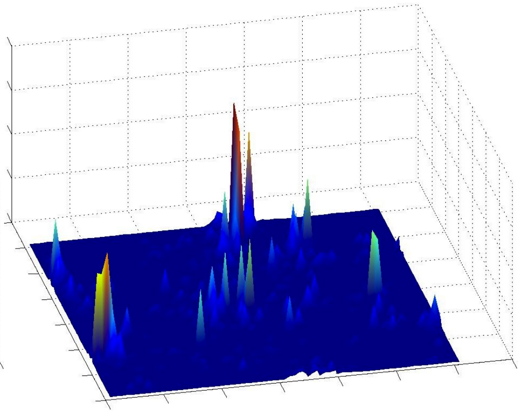
Interesting phenomena happen when γ is on, or tends to, the boundary of the eye-shaped domain represented in Figure 1.3. In this thesis we will only discuss the real cases $\gamma = \pm\sqrt{2d}$. In this case and as in multiplicative cascades, it is still possible to make sense of an associated exponential of γ times the log-correlated field Γ as a non degenerate random Borel measure. This delicate situation has been studied in [DRSV14b, DRSV14a, JS17, JSW19, Pow18, APS19, APS20]; see [Pow20b] for a review.

Liouville measure Simulations of the Gaussian multiplicative chaos associated to a 2D Gaussian free field can be found in Figure 1.4 and 1.5. This special instance of Gaussian multiplicative chaos is of prime importance and shows up in many different contexts; see the introduction of Section 1.3 for more about this.

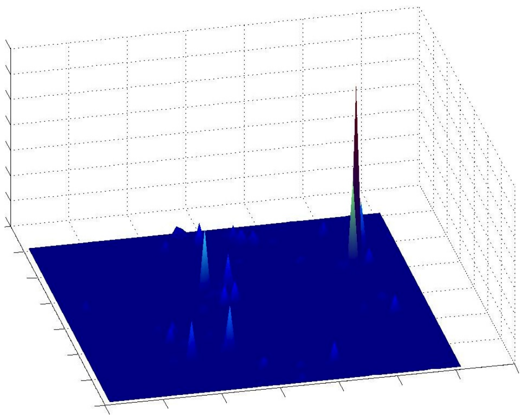
We now give a few details on the construction of μ_γ. The standard way to proceed goes via an approximation procedure. Let Γ_ε be a smooth approximation of the log-correlated field Γ, and define an approximation version $\mu_\gamma^\varepsilon(x) = e^{\gamma\Gamma_\varepsilon(x) - \frac{\gamma^2}{2}\mathbb{E}[\Gamma_\varepsilon(x)^2]}$ of μ_γ. To conclude, one needs to show that μ_γ^ε converges in a suitable space and that the limiting object does not depend on the specific choice of regularisation. As we are about to argue, this convergence is fairly direct in the so-called L^2-phase $\{|\gamma| < \sqrt{d}\}$. Let $\varphi : U \to \mathbb{R}$ be a test function. In order to show that $\int_U \varphi(x)\mu_\gamma^\varepsilon(x)dx$ converges as



(a) $\gamma = 0.2$



(b) $\gamma = 1$



(c) $\gamma = 1.8$

Figure 1.4: Simulation of Liouville measure made by R. Rhodes and V. Vargas.

$\varepsilon \to 0$, we compute for $\varepsilon, \delta > 0$,

$$\begin{aligned}
&\mathbb{E}\left[\left|\int_U \varphi(x)\mu_\gamma^\varepsilon(x)dx - \int_U \varphi(x)\mu_\gamma^\delta(x)dx\right|^2\right] \\
&= \int_{U\times U} \varphi(x)\varphi(y)\mathbb{E}\left[\left(\mu_\gamma^\varepsilon(x) - \mu_\gamma^\delta(x)\right)\left(\overline{\mu_\gamma^\varepsilon(y)} - \overline{\mu_\gamma^\delta(y)}\right)\right] dxdy. \qquad (1.3)
\end{aligned}$$

Developing the product, we are left with four similar terms. For instance,

$$\begin{aligned}
&\int_{U\times U} \varphi(x)\varphi(y)\mathbb{E}\left[\mu_\gamma^\varepsilon(x)\overline{\mu_\gamma^\varepsilon(y)}\right] dxdy \\
&= \int_{U\times U} \varphi(x)\varphi(y)\mathbb{E}\left[e^{\gamma\Gamma_\varepsilon(x)+\bar\gamma\Gamma_\varepsilon(y)-\frac{\gamma^2}{2}\mathbb{E}\left[\Gamma_\varepsilon(x)^2\right]-\frac{\bar\gamma^2}{2}\mathbb{E}\left[\Gamma_\varepsilon(y)^2\right]}\right] dxdy \\
&= \int_{U\times U} \varphi(x)\varphi(y)e^{|\gamma|^2\mathbb{E}[\Gamma_\varepsilon(x)\Gamma_\varepsilon(y)]}dxdy.
\end{aligned}$$

If Γ_ε is a reasonable approximation of the field Γ, then $\mathbb{E}\left[\Gamma_\varepsilon(x)\Gamma_\varepsilon(y)\right] \to \mathbb{E}\left[\Gamma(x)\Gamma(y)\right]$ pointwise and also

$$\int_{U\times U} \varphi(x)\varphi(y)e^{|\gamma|^2\mathbb{E}[\Gamma_\varepsilon(x)\Gamma_\varepsilon(y)]}dxdy \to \int_{U\times U} \varphi(x)\varphi(y)e^{|\gamma|^2\mathbb{E}[\Gamma(x)\Gamma(y)]}dxdy.$$

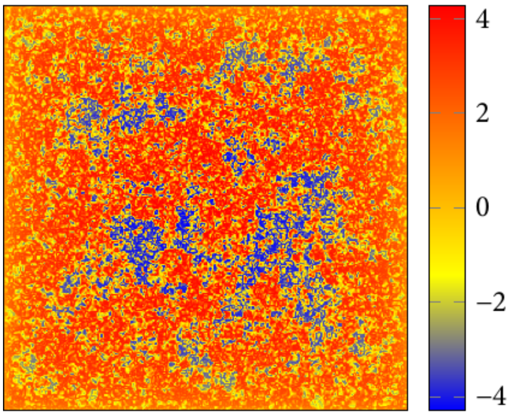


Figure 1.5: Simulation of the real part of the $\mu_\gamma = e^{\gamma \mathsf{GFF}}$ for $\gamma = i/\sqrt{2}$ made by J. Junnila, E. Saksman and C. Webb.

The same reasoning applies to the other three terms appearing in the development of (1.3). The condition $|\gamma| < \sqrt{d}$ ensures that we deal with finite integrals. Indeed, recalling that $\mathbb{E}\left[\Gamma(x)\Gamma(y)\right] \leq -\log|x-y| + O(1)$, we see that

$$\int_{U\times U} e^{|\gamma|^2 \mathbb{E}[\Gamma(x)\Gamma(y)]} dxdy \leq O(1) \int_{U\times U} |x-y|^{-|\gamma|^2} dxdy < \infty$$

as soon as $|\gamma| < \sqrt{d}$. Overall, this shows that in this regime of the parameter γ, $\left(\int_U \varphi(x)\mu_\gamma^\varepsilon(x)dx, \varepsilon > 0\right)$ is Cauchy in L^2. Lifting the convergence of $\left(\int_U \varphi(x)\mu_\gamma^\varepsilon(x)dx, \varepsilon > 0\right)$ for any test function φ to a convergence of $(\mu_\gamma^\varepsilon, \varepsilon > 0)$ in a suitable Sobolev space is then routine. Showing that two different approximations yield the same limiting measure can be done along similar lines. However, treating the case $|\gamma| \geq \sqrt{d}$ requires much more work.

1.2.3 Density of imaginary chaos: main result of Chapter 2

Although Gaussian multiplicative chaos has been thoroughly studied in the real case $\gamma \in [-\sqrt{2d}, \sqrt{2d}]$, the complex case remains much less understood. Chapter 2 will specifically be interested in the multiplicative chaos integrated against a nonnegative test function $f \neq 0$, formally written as

$$\mu_\gamma(f) := \int f(x) e^{\gamma\Gamma(x) - \frac{\gamma^2}{2}\mathbb{E}[\Gamma(x)^2]} \mathrm{d}x.$$

When $\gamma \in \mathbb{R}$, this random variable is very well-understood. For instance, it is known that [RV14]

$$\mathbb{E}\left[\mu_\gamma(f)^p\right] < \infty \qquad \iff \qquad p < \frac{2d}{\gamma^2},$$

the behaviour of its right tail $\mathbb{P}\left(\mu_\gamma(f) > t\right)$ as $t \to \infty$ is described by a power law with exponent $2d/\gamma^2$ [Won20, Won19], and the law of $\mu_\gamma(f)$ possesses a density w.r.t. Lebesgue measure on $\mathbb{R}$ [RV10]. This random variable even has an explicit distribution in some specific cases [Rem20] (specifying the log-correlated field Γ and the test function f). The complex case $\gamma \notin \mathbb{R}$ is not as much understood. Chapter 2 will be focused on the purely imaginary case $\gamma = i\beta \in i\mathbb{R}$. The imaginary axis is special since it is the only case for which $\mu_\gamma(f)$ has finite moments of all positive order [JSW20]. The article [JSW20] initiated the study of the variable $\mu_{i\beta}(f)$. In particular, they obtained estimates on the right tail of $\mu_{i\beta}(f)$ by controlling accurately the blow-up of $\mathbb{E}[\mu_\gamma(f)^p]$ as $p \to \infty$; see also the appendix of [LSZ17a] where very precise estimates were obtained in the case of the 2D GFF.

The main result of Chapter 2 is that $\mu_{i\beta}(f)$ has a smooth density w.r.t. Lebesgue measure on $\mathbb{C}$. The analogous result in the real case heavily relies on the positivity of the measure. A novel approach is needed in the complex setting and one of the main contributions of Chapter 2 can be seen as introducing Malliavin calculus to the study of multiplicative chaos.

We mention that in a companion paper we will show that this density is positive everywhere. As a corollary, we will obtain that

$$\mathbb{E}[|\mu_{i\beta}(f)|^p] < \infty \quad \iff \quad p > -2.$$

Chapter 2 therefore effectively controls the negative moments of $|\mu_{i\beta}(f)|$ which are much harder to control that the positive ones.

1.3 Applications

Gaussian multiplicative chaos shows up in a broad range of mathematical areas. For instance, the real chaos is instrumental in the mathematical construction of Liouville Conformal Field Theory (see the lecture notes [Var17]) and also describes the volume form of a surface chosen "uniformly at random" (see e.g. the lecture notes [BP21]). The imaginary chaos is related to the sine-Gordon model [LRV19] and encodes the scaling limit of the spin-field of the critical planar XOR-Ising model [JSW20]. A connection to the Brownian loop soup has also been established in [CGPR21]. In this section, we want to give some details concerning two other connections. We will in particular give some heuristics on why such links might exist. We will start with random matrices and we will then move on to the Riemann zeta function.

1.3.1 Random matrices

For a large class of random matrix models, powers of the characteristic polynomial are expected, and shown in some cases, to converge as the size of the matrix goes to infinity to some specific Gaussian multiplicative chaos. Results in this direction as well as references on this topic can be found in [FK14, Web15, NSW18, LOS18, BWW18, CN19, Kiv21]. We will present the specific case of the Complex Unitary Ensemble (CUE), but many other natural models have been studied, such as the complex Ginibre ensemble or the GUE.

Let M_n be a $n \times n$ random unitary matrix distributed according to the Haar measure on the unitary group $U(n)$. Since M_n is unitary, its eigenvalues live on the unit circle. The lack of boundary (compared to an interval for instance) and the rotational symmetry of the model makes it particularly nice to study. [Web15] and [NSW18] show that, for all $\gamma \in (0, \sqrt{2})$, the random measure

$$\frac{|\det(M_n - e^{i\theta})|^{\sqrt{2}\gamma}}{\mathbb{E}\left[|\det(M_n - e^{i\theta})|^{\sqrt{2}\gamma}\right]} \frac{\mathrm{d}\theta}{2\pi}$$

converges in distribution with respect to the topology of weak convergence to a Gaussian multiplicative chaos measure $e^{\gamma\Gamma(\theta) - \frac{\gamma^2}{2}\mathbb{E}[\Gamma(\theta)^2]} \frac{\mathrm{d}\theta}{2\pi}$ where Γ is the log-correlated Gaussian field on $[0, 2\pi]$ with covariance kernel

$$\mathbb{E}\left[\Gamma(\theta)\Gamma(\theta')\right] = -\log|e^{i\theta} - e^{i\theta'}|.$$

Note that here the underlying dimension is 1, so $\gamma \in (0, \sqrt{2})$ covers the whole subcritical regime.

What we would like to explain now is why one might expect the above specific log-correlated Gaussian field to show up. Define

$$\Gamma_n : \theta \in [0, 2\pi] \mapsto \sqrt{2}\log|\det(M_n - e^{i\theta})| \in [-\infty, \infty).$$

We are going to sketch the proof of the convergence in distribution $\Gamma_n \to \Gamma$ (in suitable Sobolev spaces). We start by writing Γ_n as a sum of traces of powers of M_n. Denoting $\lambda_j, j = 1 \dots n$, the eigenvalues of Γ_n, we can write

$$\Gamma_n(\theta) = \sqrt{2}\sum_{j=1}^n \log|\lambda_j - e^{i\theta}| = \frac{\sqrt{2}}{2}\sum_{j=1}^n \log((1 - \lambda_j e^{-i\theta})(1 - \overline{\lambda_j} e^{i\theta})).$$

Expanding the logarithm in a power series and then exchanging the two sums, we obtain that

$$\Gamma_n(\theta) = -\frac{\sqrt{2}}{2}\sum_{j=1}^n\sum_{k=1}^\infty \frac{1}{k}\left(\lambda_j^k e^{-ik\theta} + \overline{\lambda_j}^k e^{ik\theta}\right) = -\frac{\sqrt{2}}{2}\sum_{k=1}^\infty \frac{1}{k}\left(e^{-ik\theta}\,\mathrm{Tr}(M_n^k) + e^{ik\theta}\overline{\mathrm{Tr}(M_n^k)}\right).$$

The convergence of Γ_n to Γ then essentially boils down to the facts *(i)* and *(ii)* below. Let $Z_k, k \geq 1$, be i.i.d. standard complex normal variables, i.e. $\mathrm{Re}(Z_k)$ and $\mathrm{Im}(Z_k)$ are independent centred normal distributions with variance $1/2$.

(i) For any $K \geq 1$,

$$\left(\frac{1}{\sqrt{k}}\,\mathrm{Tr}(M_n^k), k = 1 \dots K\right) \to (Z_k, k = 1 \dots K)$$

in distribution.

(ii) As $K \to \infty$,

$$\theta \in [0, 2\pi] \mapsto \frac{\sqrt{2}}{2}\sum_{k=1}^K \frac{1}{\sqrt{k}}\left(Z_k e^{-ik\theta} + \overline{Z_k} e^{ik\theta}\right)$$

converges in distribution in the Sobolev space $H^{-\varepsilon}$ to Γ, for any $\varepsilon > 0$.

The proof of *(i)* comes from the striking observation that the mixed moments of $\left(\frac{1}{\sqrt{k}}\,\mathrm{Tr}(M_n^k), k = 1 \dots K\right)$ exactly coincide with those of k independent standard complex Gaussians as soon as the moment we are looking at is not too big, depending on n. More precisely, for all $K \geq 1$, $a_1, b_1 \dots, a_K, b_K \geq 0$ such that $\sum_{k=1}^K k a_k \leq n$ and $\sum_{k=1}^K k b_k \leq n$, we have

$$\mathbb{E}\left[\prod_{k=1}^K (\mathrm{Tr}(M_n^k))^{a_k}\overline{(\mathrm{Tr}(M_n^k))^{b_k}}\right] = \mathbb{E}\left[\prod_{k=1}^K (\sqrt{k}Z_k)^{a_k}\overline{(\sqrt{k}Z_k)^{b_k}}\right].$$

This result is due to [DS94] (see also [DE01]) and relies on computations specific to the unitary group. The observation *(ii)* which shows that *(i)* implies the convergence of Γ_n to the log-correlated Gaussian field Γ is due to [HKO01]. Underlying *(ii)* is the identity

$$\sum_{k=1}^\infty \frac{1}{k}\cos(k(\theta-\theta')) = -\log|e^{i\theta}-e^{i\theta'}|.$$

This line of argument hints at the log-correlated structure present in the CUE. Obtaining the convergence towards GMC measures is then far from simple, in particular because at the discrete level the field Γ_n is not Gaussian. This is the content of [Web15] and [NSW18].

1.3.2 Riemann zeta function

In a different direction, it turns out that the statistics of the Riemann zeta function ζ on the critical line are closely related to Gaussian multiplicative chaos. More precisely, let us recall that the Riemann zeta function is defined for all $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$ by

$$\zeta(s) = \sum_{n=1}^\infty \frac{1}{n^s} = \prod_{p \text{ prime}} \left(1-\frac{1}{p^s}\right)^{-1} \tag{1.4}$$

and can be continued to a meromorphic function to the whole complex plane. This is of course an object of prime importance in number theory that is still actively studied. In particular, the following problem has attracted a lot of attention:

Let τ be a uniform random variable on $[1,2]$. What does

$$\log \zeta(1/2 + iT\tau + ix) \tag{1.5}$$

look like as $x \in \mathbb{R}$ ranges over some interval and as $T \to \infty$?

It is strongly believed that this function asymptotically behaves like a log-correlated Gaussian field. We present here a rigorous result of Saksman and Webb [SW20] that supports this picture by establishing a concrete link between the Riemann zeta function and Gaussian multiplicative chaos.

If (1.5) were indeed close to being a log-correlated Gaussian field, then its exponential would be closely related to some Gaussian multiplicative chaos. Indeed, Saksman and Webb [SW20] proved that

$$\zeta(1/2 + iT\tau + ix), \quad x \in \mathbb{R},$$

converges as $T \to \infty$ in distribution in a suitable Sobolev space towards some random generalised function. The limiting generalised function is explicitly expressed in terms of some complex Gaussian multiplicative chaos (although, the specific complex chaos therein differs slightly from the one introduced in Section 1.2.2). More generally, one can take powers of the ζ function. The article [SW20] focuses on the real part of the logarithm and formulates a very precise conjecture: for any $\gamma \in (0, \sqrt{2})$,

$$\frac{|\zeta(1/2 + iT\tau + ix)|^{\sqrt{2}\gamma}}{\mathbb{E}\left[|\zeta(1/2 + iT\tau + ix)|^{\sqrt{2}\gamma}\right]}, \quad x \in [0, 1],$$

is expected to converge to some (real) Gaussian multiplicative chaos measure. Although this latter result is only conjectured, they prove a result in this direction but with a slightly different flavour. Instead of directly randomly shifting the zeta function on the critical line, they first truncate the Euler product (1.4) and then let $T \to \infty$, i.e. they show that

$$\prod_{\substack{p \text{ prime} \\ p \leq N}} \left(1 - \frac{1}{p^{1/2 + iT\tau + ix}}\right)^{-1}, \quad x \in \mathbb{R},$$

converges as $T \to \infty$ to some randomised truncated Euler product $\zeta_{N,\mathrm{rand}}(1/2 + ix)$. Then they show that for any $\gamma \in (0, \sqrt{2})$, the random measure

$$\frac{|\zeta_{N,\mathrm{rand}}(1/2 + ix)|^{\sqrt{2}\gamma}}{\mathbb{E}\left[|\zeta_{N,\mathrm{rand}}(1/2 + ix)|^{\sqrt{2}\gamma}\right]} \mathbf{1}_{\{x \in [0,1]\}} \mathrm{d}x$$

converges to some Gaussian multiplicative chaos measure.

Many other results in the literature supports the idea that (1.5) behaves like a log-correlated field. For instance, [FHK12] and [FK14] predicted that

$$\max_{x \in [-1,1]} \log |\zeta(1/2 + iT\tau + ix)| - \left(\log \log T - \frac{3}{4} \log \log \log T\right) \tag{1.6}$$

converges as $T \to \infty$ towards some non degenerate random variable. The factor $3/4$ in front of the triple logarithm is specific to the log-correlated setting enhancing once more this connection (recall (1.2)). The upper bound of this conjecture was recently verified in [ABR20], meaning that the positive part of (1.6) is a tight sequence. More results and references on this topic can be found in the introduction of [AOR19] and in the review [BK21].

We would like to give in the rest of this section some heuristics hinting at the structure of log-correlated field present in the Riemann zeta function. Recall that from the Euler product (1.4), we have for all $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$,

$$\log |\zeta(s)| = \mathrm{Re} \log |\zeta(s)| = -\sum_p \mathrm{Re} \log(1 - p^{-s}).$$

Replacing $\log(1 - p^{-s})$ by $-p^{-s}$, we see that $\log |\zeta(s)|$ can be approximated by $\sum_p \mathrm{Re}\, p^{-s}$. When $s = 1/2 + it$ belongs to the critical line, the sum needs to be truncated and it can be shown that

$\log|\zeta(1/2+it)|$ is fairly well-approximated by

$$\log|\zeta(1/2+it)| \approx \sum_{p\leq X} \frac{\cos(t\log p)}{p^{1/2}}.$$

Controlling the error in the above approximation can be technically challenging. It depends on the choice of the cutoff X, but for the purpose of our discussion we will assume that we can take X very close to T. Assuming these heuristics, a small computation (see below) shows that the variance of $\log|\zeta(1/2+iT\tau)|$ asymptotically behaves like $1/2\log\log T$ which matches the rigorously proved central limit theorem of Selberg [Sel92]

$$\frac{\log|\zeta(1/2+iT\tau)|}{\sqrt{1/2\log\log T}} \to \mathcal{N}(0,1) \quad \text{in distribution.}$$

We now group the prime numbers as follows: for all $1\leq \ell \leq \log\log T$, let

$$Y_\ell(x) = \sum_{e^{\ell-1}<\log p\leq e^\ell} \frac{\cos((T\tau+x)\log p)}{p^{1/2}}.$$

This grouping is motivated by the fact that for all ℓ,

$$\mathbb{E}\left[Y_\ell(x)^2\right] = \frac{1}{2}\sum_{e^{\ell-1}<\log p\leq e^\ell} \frac{1}{p} + o(1) = \frac{1}{2} + o(1)$$

where the last equality follows from the prime number theorem. Overall we have decomposed

$$\log|\zeta(1/2+iT\tau+ix)| \approx \sum_{\ell=1}^{\log\log T} Y_\ell(x).$$

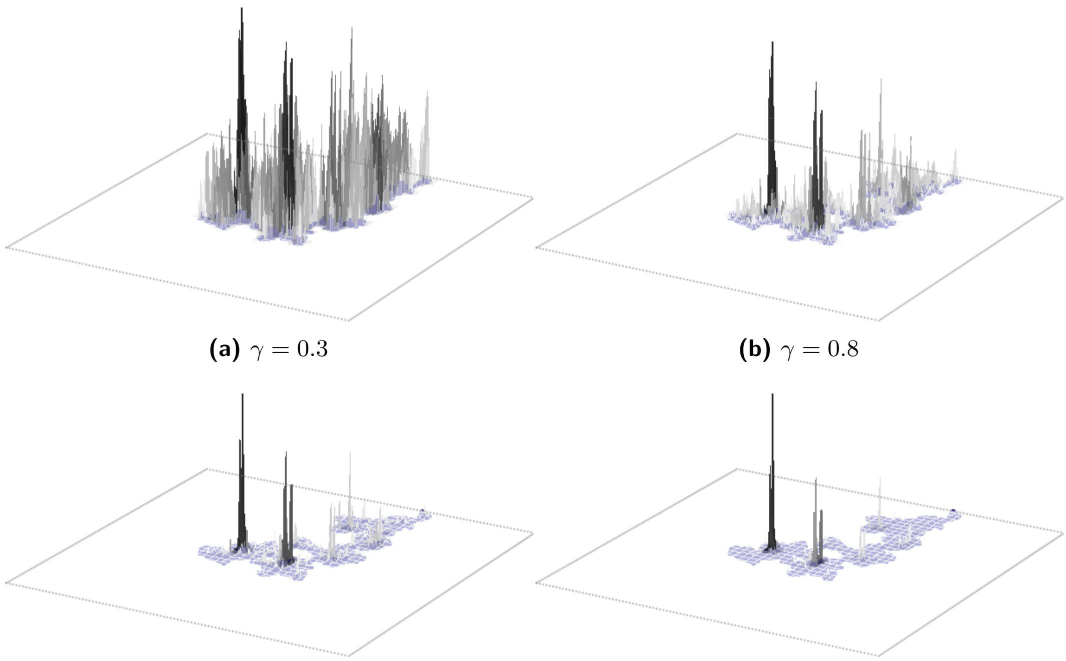
The claim now is that for all $\ell\neq\ell'$, Y_ℓ and $Y_{\ell'}$ are asymptotically (as $T\to\infty$) independent and for all ℓ, $Y_\ell(x)$ and $Y_\ell(y)$ are either

- strongly correlated if $|x-y|\ll e^{-\ell}$
- or strongly decorrelated if $|x-y|\gg e^{-\ell}$.

This correlation structure is very similar to the branching random walk picture, unveiling the log-correlations present in the Riemann zeta function. More details can be found in [BK21].

1.4 Brownian multiplicative chaos

This thesis investigates some connections between Gaussian multiplicative chaos and planar Brownian motion. This story can be seen as starting with isomorphism theorems which relate local times of random walk/Brownian motion to half of the Gaussian free field squared. Since the GFF is log-correlated in dimension 2, the square root of the local time L can be thought of as being a (non-Gaussian) log-correlated field and it is therefore sensible to try to make sense of $e^{\gamma\sqrt{L}}$. The resulting object has



(a) $\gamma = 0.3$ **(b)** $\gamma = 0.8$ **(c)** $\gamma = 1.3$ **(d)** $\gamma = 1.8$

Figure 1.6: Simulation of $\mu_\gamma = e^{\gamma\sqrt{L}}$ for $\gamma = 0.3$, 0.8, 1.3 and 1.8, for the same underlying sample of Brownian path which is drawn in blue. The domain D is a square and the starting point x_0 is its middle

been studied in [BBK94, AHS20, Jeg20a] and is now referred to as Brownian multiplicative chaos. See Figure 1.6 for a simulation. In the next section, we describe the construction of [Jeg20a].

1.4.1 Subcritical construction

Let $U \subset \mathbb{R}^2$ be a bounded simply connected domain in the plane and let $x_0 \in U$ be a starting point. Let $(B_t)_{0\leq t\leq \tau}$ be a Brownian motion which starts at x_0 and which is killed at the first time τ it exits U. Let L be the occupation field of B, i.e. for any Borel set A,

$$L(A) = \int_0^\tau \mathbf{1}_{\{B_t \in A\}} \mathrm{d}t.$$

Brownian multiplicative chaos measure is formally defined as $e^{\gamma\sqrt{L}}$ where $\gamma \in (0,2)$ is a parameter. As in the case of log-correlated Gaussian fields, the occupation field L is not well defined pointwise and one needs to work in order to define such an object. This has first been done in [BBK94] for a strict subset of the L^2-phase, i.e. for $\gamma \in (0,1)$. Recently, [AHS20] and simultaneously [Jeg20a] extended the construction of this object to the whole subcritical regime $\gamma \in (0,2)$. We now present the approach of [Jeg20a].

We will approximate the field L by looking at the local times of small circles: for every $x \in U$ and $\varepsilon > 0$, define

$$L_{x,\varepsilon} := \lim_{r\to 0^+} \frac{1}{2r} \int_0^\tau \mathbf{1}_{\{\varepsilon - r \leq |B_t - x| \leq \varepsilon + r\}} \mathrm{d}t.$$

These local times are well defined jointly in x and ε, so we can define for any Borel set $A \subset U$,

$$\mu_\gamma^\varepsilon(A) = \sqrt{|\log \varepsilon|}\varepsilon^{\gamma^2/2} \int_A e^{\gamma\sqrt{\frac{1}{\varepsilon}L_{x,\varepsilon}}} \mathrm{d}x.$$

Note that the normalisation is the same one as in the case of Gaussian fields, except for the multiplicative factor $\sqrt{|\log \varepsilon|}$ in front. [Jeg20a] shows that, as soon as $\gamma \in (0,2)$, μ_γ^ε converges in probability as $\varepsilon \to 0$ for the topology of weak convergence. The limiting measure is nondegenerate and is interpreted as $e^{\gamma\sqrt{L}}$.

1.4.2 Thick points of random walk

Brownian multiplicative chaos measures have proven to be useful in the study of exceptional points of planar random walk where the walk goes back unusually often. Such a study was initiated by Erdős and Taylor in [ET60] who made the following conjecture. Let U_N be some discrete approximation of a bounded simply connected domain U by a portion of the square lattice $\frac{1}{N}\mathbb{Z}^2$ with mesh size $1/N$ and let $(X_t)_{0 \le t \le \tau_N}$ be a (continuous-time) simple random walk on $\frac{1}{N}\mathbb{Z}^2$ stopped upon exiting for the first time U_N. Let ℓ_x^N be the local time at $x \in U_N$ defined by

$$\ell_x^N := \int_0^{\tau_N} \mathbf{1}_{\{X_t = x\}} \mathrm{d}t.$$

Then, Erdős and Taylor showed that

$$\frac{1}{\pi} \le \liminf_{N\to\infty} \frac{\sup_{x\in U_N} \ell_x^N}{(\log N)^2} \le \limsup_{N\to\infty} \frac{\sup_{x\in U_N} \ell_x^N}{(\log N)^2} \le \frac{4}{\pi}$$

and conjectured that the upper bound is sharp. This conjecture was proven forty years later in the landmark paper [DPRZ01]. They moreover showed that for all $a \in (0,2)$, the set of a-thick points

$$\mathcal{T}_N(a) := \left\{ x \in U_N : \ell_x^N \ge \frac{2}{\pi} a (\log N)^2 \right\}$$

contains asymptotically $N^{2-a+o(1)}$ points. These estimates on the “fractal dimension” of the set of thick points have then been streamlined in [Ros05, BR07, Jeg20b].

[Jeg19] went a step further by establishing the scaling limit of the set of thick points. In particular, it is shown that

$$\frac{\log N}{N^{2-a}} \# \mathcal{T}_N(a)$$

converges in distribution to a nondegenerate random variable. The limiting variable is nothing else but the total mass of the Brownian chaos measure in U with parameter $\gamma = \sqrt{2a}$.

1.4.3 Critical case: main result of Chapter 3

The purpose of Chapter 3 is to initiate the study of the critical case $\gamma = 2$. Using the notations of Section 1.4.1, the first result in this direction is that the subcritical normalisation leads to a vanishing measure at criticality, i.e. $\mu_{\gamma=2}^\varepsilon(U)$ converges in probability to zero. The subcritical normalisation is then boosted in two different ways using either the Seneta-Heyde normalisation or the derivative martingale (using multiplicative cascades phrases):

$$\sqrt{|\log \varepsilon|}\mu_{\gamma=2}^{\varepsilon} = |\log \varepsilon|\varepsilon^2 e^{2\sqrt{\frac{1}{\varepsilon}L_{x,\varepsilon}}}\mathrm{d}x$$

and

$$-\frac{\mathrm{d}\mu_{\gamma}^{\varepsilon}(dx)}{\mathrm{d}\gamma}\Big|_{\gamma=2} = \sqrt{|\log \varepsilon|}\varepsilon^2\left(2|\log \varepsilon| - \sqrt{\frac{1}{\varepsilon}L_{x,\varepsilon}}\right)e^{2\sqrt{\frac{1}{\varepsilon}L_{x,\varepsilon}}}\mathrm{d}x$$

are both shown to converge in probability for the topology of weak converge. The two resulting limiting measures are nondegenerate and agree up to a universal multiplicative constant. See Theorem 3.2. In Theorem 3.4, we also show that the critical measure can be obtained as limit of the subcritical measures.

In analogy with the Gaussian case and the branching random walk setting, it is natural to expect that the critical chaos measure encodes the scaling limit of the most extreme thick points of random walk. We state precisely such a conjecture in Section 3.1.2.

Brownian multiplicative chaos measures share striking similarities with Liouville measure, i.e. GMC measure associated to the 2D Gaussian free field. For instance, in both settings, the measures are conformally covariant and the explicit formulas of the first moment of the measure have very similar flavours (both involving conformal radii). However, they are far from being equal since, in the Brownian setting, the measure is supported by a Brownian trajectory. Chapter 4 will elucidate the connection between these measures by showing that one can recover Liouville measure (actually the hyperbolic cosine of the GFF) from Brownian multiplicative chaos measures. See Theorem 4.5.

1.5 Brownian loop soup: main results of Chapter 4

The last chapter of this thesis will see another character come into play: Brownian loop soup. Introduced by Lawler and Werner [LW04], Brownian loop soup is an infinite collection of Brownian-like loops distributed as a Poisson point process with intensity $\theta\mu_U^{\mathrm{loop}}$. Here $\theta > 0$ is an intensity parameter and μ_U^{loop} is a certain infinite measure on loops which remain in a given planar domain U. See Figure 1.7 for a simulation. Brownian loop soup is a fundamental object which is closely related to other conformally invariant processes such as the GFF, Schramm–Loewner Evolutions (SLE) and Conformal Loop Ensembles (CLE).

The behaviour of the Brownian loop soup depends very much on the value of the intensity parameter θ. For instance, Sheffield and Werner [SW12] proved that if $\theta \leq 1/2$, there are infinitely many clusters of overlapping loops. In that case, the outer boundaries of the outermost clusters form a family of non-intersecting and non-nested loops which turns out to be distributed as a CLE. When $\theta > 1/2$, there is only one “giant” cluster of loops. From this perspective, $\theta = 1/2$ plays the role of a critical intensity. At this special intensity, Le Jan [LJ11] shows that the occupation field of the Brownian loop soup has exactly the same distribution as half of the GFF squared.

Our goal in Chapter 4 is to sharpen our understanding of the relationship between the critical Brownian loop soup and the Gaussian free field. Specifically, let us couple a Brownian loop soup at

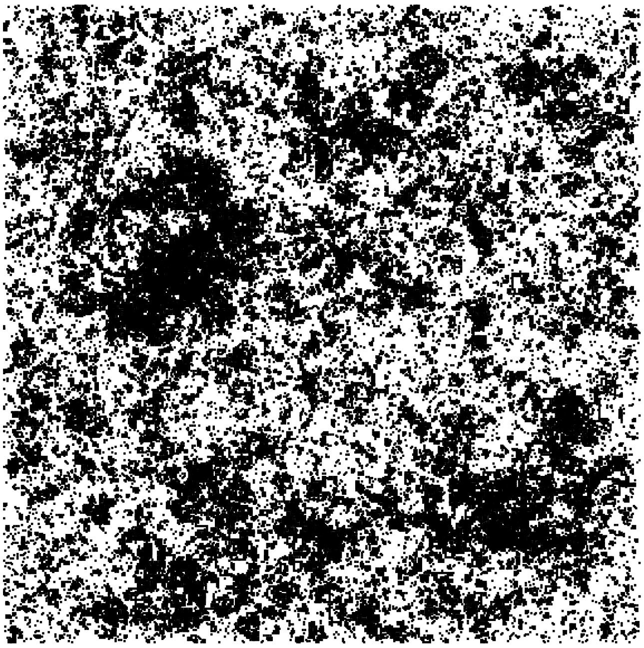


Figure 1.7: Simulation of Brownian loop soup made by S. Nacu and W. Werner. Only loops larger than a given threshold are displayed.

critical intensity $\theta = 1/2$ and a Gaussian free field in such a way that these two objects are related by Le Jan's identity. What does the Brownian loop soup look like in the vicinity of a point z sampled according to Liouville measure, i.e. Gaussian multiplicative chaos associated to the GFF? It is known that Liouville measure is supported on points where the GFF is atypically large (the so-called thick points). Via Le Jan's isomorphism, it is therefore natural to expect the occupation field of the Brownian loop soup to be atypically large at that point as well. How do loops combine to create such a thick local time? Does the thickness come from a single loop which visits z very often, or from an infinite number of loops that touch z, with each loop having a typical occupation field? We will show that the answer turns out to be an intermediate scenario. More precisely, we will show that Liouville-typical points are of infinite loop multiplicity, with the relative contribution of each loop to the overall thickness of the point being described by the Poisson–Dirichlet distribution $\theta = 1/2$. See Theorem 4.8.

In fact, our results are not restricted to the critical intensity $\theta = 1/2$ and hold without restrictions on $\theta > 0$. When $\theta \neq 1/2$, the occupation field of the Brownian loop soup is not distributed as half of the GFF squared and the corresponding multiplicative chaos does not agree with Liouville measure anymore. Therefore, our first task in Chapter 4 will be to construct the multiplicative chaos associated to the Brownian loop soup, i.e. we will make sense of a random measure formally defined as the exponential of γ times the square root of the occupation field. We will achieve this in two complementary ways.

From the continuum In Section 1.4, we recalled the definition of Brownian multiplicative chaos, a multiplicative chaos naturally associated to a single Brownian trajectory. Extending the definition to finitely many independent Brownian paths is routine (see [Jeg19]) and we can therefore try to construct the multiplicative chaos of the Brownian loop soup directly from these multi-trajectory

Brownian multiplicative chaos measures. To do so, we first thin the collection of loops we look at. Let K be a large positive real number, kill each loop independently of each other at rate K and consider the set $\mathcal{L}_U^\theta(K)$ of killed loops. As $K \to \infty$, $\mathcal{L}_U^\theta(K)$ increases to the whole Brownian loop soup $\mathcal{L}_U^\theta$. Although $\mathcal{L}_U^\theta(K)$ still contains infinitely many loops, it is of "finite density", so defining an associated Brownian chaos is not complicated: one can for instance truncate $\mathcal{L}_U^\theta(K)$ by looking at the n largest loops (according to the diameter say), consider the associated Brownian chaos $\mathcal{M}_a^{K,n}$ and then check that the increasing limit $\mathcal{M}_a^K = \lim_{n\to\infty} \mathcal{M}_a^{K,n}$ defines a random finite measure. At this stage, the measure $\mathcal{M}_a^K$ can be thought of as the exponential of $\gamma = \sqrt{2a}$ times the square root of the occupation field of $\mathcal{L}_U^\theta(K)$.

The multiplicative chaos of the Brownian loop soup is then built by renormalising $\mathcal{M}_a^K$ and letting $K \to \infty$: Theorem 4.1 states that

$$\frac{1}{(\log K)^\theta} \mathcal{M}_a^K \xrightarrow[K\to\infty]{} \mathcal{M}_a \quad \text{in probability,}$$

where the right hand side is defined by this convergence.

A characterisation of the joint law of $(\mathcal{L}_D^\theta, \mathcal{M}_a)$ which does not refer to the specific thinning we used is stated in Theorem 4.8.

From the discrete Random walk loop soups were introduced in [LTF07] and are discrete analogues of Brownian loop soup. In Theorem 4.12, we show that $\mathcal{M}_a$ can also be constructed by considering the uniform measure on thick points of the random walk loop soup and letting the mesh size tend to 0. Although the discrete approach to defining the multiplicative chaos $\mathcal{M}_a$ is easier to grasp, its proof is much more technical.

More precisely, let U_N be a discrete approximation of the domain U by a portion of the square lattice $\frac{1}{N}$ with mesh size $\frac{1}{N}$ and let $\mathcal{L}_{U_N}^\theta$ be a random walk loop soup in U_N. For any vertex $z \in U_N$ and any discrete path $(\wp(t))_{0\le t\le T(\wp)}$ parametrised by continuous time, we denote by $\ell_z(\wp)$ the local time of $\wp$ at z, i.e.

$$\ell_z(\wp) := N^2 \int_0^{T(\wp)} \mathbf{1}_{\{\wp(t)=z\}} \mathrm{d}t.$$

With our normalisation,

$$\mathbb{E}\Big[\sum_{\wp \in \mathcal{L}_{D_N}^\theta} \ell_z(\wp) \Big] \sim \frac{\theta}{2\pi} \log N \quad \text{as } N \to \infty$$

and we define the set of a-thick points by

$$\mathcal{T}_N(a) := \Big\{ z \in D_N : \sum_{\wp \in \mathcal{L}_{D_N}^\theta} \ell_z(\wp) \geq \frac{1}{2\pi} a (\log N)^2 \Big\}.$$

We encode this set in the following point measure: for all Borel set $A \subset \mathbb{C}$, define

$$\mathcal{M}_a^N(A) := \frac{(\log N)^{1-\theta}}{N^{2-a}} \sum_{z \in \mathcal{T}_N(a)} \mathbf{1}_{\{z\in A\}}.$$

Theorem 4.12 states that

$$(\mathcal{L}^\theta_{D_N}, \mathcal{M}^N_a) \xrightarrow[N \to \infty]{} (\mathcal{L}^\theta_D, c\mathcal{M}_a) \quad \text{in distribution,}$$

for some explicit constant c. $\mathcal{L}^\theta_D$ has the law of a Brownian loop soup with intensity θ and $\mathcal{M}_a$ is the associated multiplicative chaos. This result is close in spirit from the result of [Jeg19] we mentioned in Section 1.4.2 and, indeed, our proof crucially uses [Jeg19]. We nevertheless mention that a lot of work is required in order to prove Theorem 4.12.

Chapter 2

Density of imaginary multiplicative chaos via Malliavin calculus

We consider the imaginary Gaussian multiplicative chaos, i.e. the complex Wick exponential $\mu_\beta :=: e^{i\beta\Gamma(x)}:$ for a log-correlated Gaussian field Γ in $d \geq 1$ dimensions. We prove a basic density result, showing that for any nonzero continuous test function f, the complex-valued random variable $\mu_\beta(f)$ has a smooth density w.r.t. the Lebesgue measure on $\mathbb{C}$. As a corollary, we deduce that the negative moments of imaginary chaos on the unit circle do not correspond to the analytic continuation of the Fyodorov-Bouchaud formula, even when well-defined.

Somewhat surprisingly, basic density results are not easy to prove for imaginary chaos and one of the main contributions of the article is introducing Malliavin calculus to the study of (complex) multiplicative chaos. To apply Malliavin calculus to imaginary chaos, we develop a new decomposition theorem for non-degenerate log-correlated fields via a small detour to operator theory, and obtain small ball probabilities for Sobolev norms of imaginary chaos.

2.1 Introduction

In this paper we study imaginary Gaussian multiplicative chaos, formally written as $\mu_\beta :=: e^{i\beta\Gamma(x)}:$, where Γ is a log-correlated Gaussian field on a bounded domain $U \subset \mathbb{R}^d$ and β a real parameter. The study of imaginary chaos can be traced back to at least [DES93, Big92], in case of cascade fields to [BJM10], and to [LRV15b, JSW20] in a wider setting of log-correlated fields.

Imaginary multiplicative chaos distributions $:e^{i\beta\Gamma(x)}:$ can be rigorously defined as distributions in a Sobolev space of sufficiently negative index [JSW20]. In the case where Γ is the 2D continuum Gaussian free field (GFF), they are related to the sine-Gordon model [LRV19, JSW20] and the scaling limit of the spin-field of the critical XOR-Ising model is given by the real part of $:e^{i2^{-1/2}\Gamma(x)}:$ [JSW20]. Imaginary chaos has also played a role in the study of level sets of the GFF [SSV20], giving a connection to SLE-curves. In [CGPR21] it was shown using Wiener chaos methods that certain fields constructed using the Brownian Loop Soup converge to imaginary chaos. Recently, reconstruction theorems have been proved for both the continuum [AJ21] and the discrete version [GS20] of the imaginary chaos, showing that, somewhat surprisingly, when $d \geq 2$ it is possible to recover the underlying field from the information contained in the imaginary chaos in the whole subcritical phase $\beta \in (0, \sqrt{d})$.

In a wider context, real multiplicative chaos $:e^{\gamma\Gamma(x)}:$, with $\gamma \in \mathbb{R}$ has been the subject of a lot of recent progress (see e.g. reviews [RV14, Pow20b]). Complex and in particular imaginary multiplicative chaos appear then naturally, for example, as analytic extensions in γ. Complex variants of multiplicative chaos also come up when studying the statistics of zeros of the Riemann zeta function on the critical line [SW20].

The main result of this paper is the existence and smoothness of density for random variables of the type $\mu_\beta(f)$. The main contribution, however, is probably the technique used to prove the main result. Indeed, whereas in the case of imaginary multiplicative cascades [BM09] and real multiplicative chaos [RV10] rather direct Fourier methods give the existence of a density, this approach is problematic in the case of imaginary chaos. The main obstacle is the presence of cancellations that are difficult to control without an exact recursive independence structure or monotonicity. We circumvent these problems by turning to Malliavin calculus. Interestingly, in order to apply methods of Malliavin calculus we have to first obtain new decomposition theorems for log-correlated fields, and prove quite technical concentration estimates for tails of imaginary chaos.

2.1.1 The main result: existence of density

Let us now denote by μ_β the imaginary chaos with parameter $\beta \in (0, \sqrt{d})$ in d dimensions. In the appendix of [LSZ17b] and in [JSW20] the tails of this random variable were studied and it was shown that $\mathbb{P}[|\mu(f)| > t]$ behaves roughly like $\exp(-t^{2d/\beta^2})$ – this basically follows from the fact that using Onsager inequalities, one can obtain a very good control on the moments of imaginary chaos.

In the present article we are interested in the local properties of the law of $\mu(f)$ and our main result is that this random variable has a smooth density. The following slightly informal statement is made precise in Theorem 2.9.

Theorem. *Let Γ be a non-degenerate log-correlated field in an open domain U and let f be a nonzero continuous function with compact support in U. Then the law of $\mu_\beta(f)$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{C}$ and the density is a Schwartz function.*

Moreover, for any $\eta > 0$ the density is uniformly bounded from above for $\beta \in (\eta, \sqrt{d})$ and converges to zero pointwise as $\beta \to \sqrt{d}$.

Finally, the same holds in the case where μ_β is the imaginary chaos corresponding to the field $\hat{\Gamma}$ with covariance $\mathbb{E}[\hat{\Gamma}(x)\hat{\Gamma}(y)] = -\log|x-y|$ on the unit circle, with f being any nonzero continuous function defined on the circle.

Remark. The reason why the circle field is brought out separately is because it does not satisfy our definition of non-degenerate log-correlated fields, see Section 2.2, and requires a bit of extra work. With similar work other cases of degenerate log-correlated fields could be handled. However, a unified approach to handle a more general class of log-correlated fields is still lacking.

The requirement of compact support for f can also be dropped in many situations. For example, the theorem is also true in the case where Γ is the zero-boundary GFF on a bounded simply connected domain in $\mathbb{R}^2$ and $f \equiv 1$.

This theorem has already proved to be useful in further study of imaginary chaos[1], but we also expect the result and the method to be useful more generally in the study of complex chaos [LRV15b]

[1] A work in preparation studies the monofractal structure of imaginary chaos.

and in studying the integrability results related to multiplicative chaos [Rem20, KRV20] and the Sine-Gordon model. In a follow-up work, we will prove by independent methods that this density is in fact everywhere positive.

2.1.2 An application to the Fyodorov-Bouchaud formula

Let us mention here one direct application of our results, linking our studies to recent integrability results on the Gaussian multiplicative chaos stemming from Liouville conformal field theory [KRV20, Rem20]. Namely, in [Rem20] the author proved that for real $\gamma \in (0, \sqrt{2})$ the total mass of $: e^{\gamma \widehat{\Gamma}(x)} :$, where $\widehat{\Gamma}$ is the log-correlated Gaussian field on S^1 with covariance $C(x, y) = -\log|x-y|$, has an explicit density w.r.t. the Lebesgue measure; this was conjectured in [FB08] and proved by different methods in [CN19]. Moreover, in Theorem 1.1 of [Rem20] the author proves an explicit expression for the $p-$th moment of $Y_\gamma := \int_{S^1} : e^{\gamma \widehat{\Gamma}(x)} : dx$ with $-\infty < p < 4/\gamma^2$:

$$\mathbb{E}\left(Y_\gamma^p\right) = \frac{\Gamma(1 - p\gamma^2/2)}{\Gamma(1-\gamma^2/2)^p}, \tag{2.1}$$

where with a slight abuse of notation Γ is here the usual Γ-function.[2] Notice that for any p, the expression is analytic in γ (outside of isolated singularities) and in particular analytic in a neighbourhood around the imaginary axis. So naively one might think that at least as long as the moments are defined for $: e^{i\beta \widehat{\Gamma}(x)} :$, they would correspond to the expression given by (2.1) with $\gamma = i\beta$. And indeed, it is not hard to see that for $p \in \mathbb{N}$ this is the case. Our results however imply that this cannot be true in general, even in the case where the $p-$th moment is well-defined for the imaginary chaos. In other words, the analytic extension of the moment formulas is in general different from naively changing γ in the Wick exponential.

Corollary 2.1. *Let $\widehat{\mu}_\beta$ be the imaginary chaos corresponding to the log-correlated field $\widehat{\Gamma}$ on the unit circle. Then $\mathbb{E}\left(\widehat{\mu}_\beta(S^1)^{-1}\right)$ converges to zero as $\beta \to 1$. In particular, $\mathbb{E}\left(\widehat{\mu}_\beta(S^1)^{-1}\right)$ does not agree with the analytic continuation of Equation (2.1) for $\gamma \in (-i, i)$.*

Proof. From Theorem 2.9 it follows that

$$|\mathbb{E}\left(\widehat{\mu}_\beta(S^1)^{-1}\right)| \le \mathbb{E}\left(|\widehat{\mu}_\beta(S^1)|^{-1}\right) \to 0$$

as $\beta \to 1$. On the other hand a direct check shows that in Equation (2.1), the expression remains uniformly positive for $p = -1$, when we set $\gamma = i\beta$ and let $\beta \to 1$. □

2.1.3 Other results: a decomposition of log-correlated fields and Sobolev norms of imaginary chaos

As mentioned, our main tool in the proof of Theorem 2.9 is Malliavin calculus which is an infinite-dimensional differential calculus on the Wiener space introduced by Malliavin in the seventies [Mal78]. Whereas Malliavin calculus has been used to prove density results in various other settings [Nua06], we

[2] Notice that in that paper the author is using a different normalization of the field with local behaviour of $-2\log|x-y|$.

believe that it is a novel tool in the context of multiplicative chaos and could have further interesting applications. In order to apply Malliavin calculus, we need to derive some results that could be of independent interest.

First, we derive a new decomposition theorem for non-degenerate log-correlated fields. This statement is made precise in Theorem 2.16 and the proof has an operator-theoretic flavour.

Theorem. *Let Γ be a non-degenerate log-correlated Gaussian field on an open domain $U \subseteq \mathbb{R}^d$ with covariance kernel given by $-\log|x-y| + g(x,y)$ and g subject to some regularity conditions. Then, for every $V \Subset U$, there exists $\alpha > 0$ such that we may write (possibly in a larger probability space)*

$$\Gamma|_V = Y + Z,$$

where Y is an almost $\star$-scale invariant field and Z is a Hölder-regular field independent of Y, both defined on the whole of $\mathbb{R}^d$.

Second, we develop a way to study the small ball probabilities of $\|f\mu\|_{H^{-d/2}(\mathbb{R}^d)}$. The precise version of the following statement is given by Proposition 2.33.

Proposition. *Let $f \in C_c^\infty(U)$. Then for all $\beta \in (0, \sqrt{d})$ the probability $\mathbb{P}[\|f\mu\|_{H^{-d/2}(\mathbb{R}^d)} \leq \lambda]$ decays super-polynomially in λ.*

This result is closely related to small ball probabilities of the Malliavin determinant of $\mu_\beta(f)$. To prove it we establish concentration results on the tail of imaginary chaos.

2.1.4 Structure of the article

We have set up the article to highlight how the general theory of Malliavin calculus is applied to prove such a density result and what are the concrete estimates of imaginary chaos needed to apply it. After collecting some preliminaries in Section 2.2, we use Section 2.3 to walk the reader through the relevant notions and results of Malliavin calculus in the context of imaginary multiplicative chaos, thereby building up the backbone of the proof of the main theorem. In that section we state carefully the main result, and prove it up to technical estimates. The remaining proofs are then collected in Section 2.5 and in Section 2.6; the former contains some general lemmas of Malliavin calculus, and the latter deals with concentration results for imaginary chaos, including the proof of the Proposition 2.33 above. In Section 2.4 we prove the decomposition theorem stated above.

2.2 Basic notions and definitions

2.2.1 Log-correlated Gaussian fields and imaginary chaos

In this section we establish the formal setup for the log-correlated field Γ and of the imaginary chaos associated to Γ, often denoted by $:\exp(i\beta\Gamma):$ with $\beta \in \mathbb{R}$.

2.2.1.1 Log-correlated Gaussian fields

Let $U \subset \mathbb{R}^d$ be a bounded and simply connected domain and suppose we are given a kernel of the form

$$C(x,y) = \log \frac{1}{|x-y|} + g(x,y) \tag{2.2}$$

where g is bounded from above and satisfies $g(x,y) = g(y,x)$. Furthermore, we assume that $g \in H^{d+\varepsilon}_{\text{loc}}(U \times U) \cap L^2(U \times U)$ for some $\varepsilon > 0$. We may also extend $C(x,y)$ as 0 outside of $U \times U$. Then C defines a Hilbert–Schmidt operator on $L^2(\mathbb{R}^d)$, and hence C is self-adjoint and compact.

Assuming C is positive definite, by spectral theorem there exists a sequence of strictly positive eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots > 0$ and corresponding orthogonal eigenfunctions $(f_k)_{k\geq 1}$ spanning the subspace $L := (\operatorname{Ker} C)^\perp$ in $L^2(\mathbb{R}^d)$. We may now construct the log-correlated field Γ with covariance kernel $C(x,y)$ via its Karhunen–Loève expansion

$$\Gamma = \sum_{k\geq 1} A_k C^{1/2} f_k = \sum_{k \geq 1} A_k \sqrt{\lambda_k} f_k, \tag{2.3}$$

where $(A_k)_{k\geq 1}$ is an i.i.d. sequence of standard normal random variables. It has been shown in [JSW20, Proposition 2.3] that the above series converges in $H^{-\varepsilon}(\mathbb{R}^d)$ for any fixed $\varepsilon > 0$.

From the KL-expansion one can see that heuristically Γ is a standard Gaussian on the space $H_\Gamma := C^{1/2}L$. The space $H := H_\Gamma$ is called the Cameron–Martin space of Γ, and it becomes a Hilbert space by endowing it with the inner product $\langle f, g\rangle_H = \langle C^{-1/2} f, C^{-1/2} g\rangle_{L^2}$, where $C^{-1/2}f, C^{-1/2}g \in L$. This definition makes sense since $C^{1/2}$ is an injection on L. We will define the KL-basis $(e_k)_{k\geq 1}$ for H by setting $e_k := \sqrt{\lambda_k} f_k$, and we will also write $\langle \Gamma, h\rangle_H := \sum_{k=1}^\infty A_k \langle h, e_k\rangle_H$ for $h \in H$. The left hand side in the latter definition is purely formal since $\Gamma \notin H$ almost surely.

Let us finally define what we mean by a non-degenerate log-correlated field in all of this paper.

Definition 2.2 (Non-degenerate log-correlated field)**.** *Consider a kernel $C_\Gamma(x,y) = C(x,y)$ from (2.2) and the associated log-correlated field Γ, given by (2.3). We call the kernel C and the field Γ non-degenerate when C is an injective operator on $L^2(\mathbb{R}^d)$, i.e.* $\operatorname{Ker} C = \{0\}$.

Note that for covariance operators injectivity is equivalent to being strictly positive in the sense that $\langle C_\Gamma f, f\rangle > 0$ for all $f \in L^2(V)$, $f \neq 0$.[3]

The standard log-correlated field on the circle.

The only degenerate field we will work with in this paper is the standard log-correlated field on the circle. I.e. it is the field Γ on the unit circle which has the covariance $C_\Gamma(x,y) = \log \frac{1}{|x-y|}$, where one now thinks of x and y as being complex numbers of modulus 1. Equivalently, we may consider the

[3]On $\mathbb{R}^d$ one could also imagine a different definition of non-degenerate fields. Namely, a canonical way to define a log-correlated field Γ_d on $\mathbb{R}^d$ for any $d \geq 1$ is to take $H^{d/2}(\mathbb{R}^d)$ as the Cameron–Martin space of the field. It would then be natural to call any log-correlated field on $\mathbb{R}^d$ non-degenerate if its Cameron–Martin spaces is equivalent to $H^{d/2}(\mathbb{R}^d)$. We will basically see in Section 4 that very roughly our condition implies that the Cameron–Martin space of a suitable extension of the non-degenerate field Γ to the whole plane coincides up to an equivalent norm with $H^{d/2}(\mathbb{R}^d)$.

field on $[0,1]$ with the covariance

$$\mathbb{E}[\Gamma(e^{2\pi it})\Gamma(e^{2\pi is})] = \log \frac{1}{2|\sin(\pi(t-s))|},$$

in which case we may write

$$\Gamma(e^{2\pi it}) = \sqrt{2}\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}(A_k \cos(2\pi kt) + B_k \sin(2\pi kt))$$

where A_k and B_k are i.i.d. standard normal random variables.

This circle field is degenerate because it is conditioned to satisfy $\int_0^1 \Gamma(e^{2\pi i\theta})\, d\theta = 0$ and the operator C maps constant functions to zero. It is not hard to see that the restriction of the field $\Gamma(e^{2\pi i\cdot})$ to $I_0 := [-1/4, 1/4]$ is again non-degenerate.

2.2.1.2 Imaginary chaos

Let us now fix $\beta \in (0, \sqrt{d})$. For any $f \in L^\infty(U)$ we may define the imaginary chaos μ tested against f via the regularization and renormalisation procedure

$$\mu(f) := \lim_{\varepsilon\to 0} \int_U f(x) e^{i\beta\Gamma_\varepsilon(x) + \frac{\beta^2}{2}\mathbb{E}\Gamma_\varepsilon(x)^2}\, dx,$$

where Γ_ε is a convolution approximation of Γ against some smooth mollifier φ_ε. An easy computation shows that the convergence takes place in $L^2(\Omega)$. Importantly, the limiting random variable does not depend on the choice of mollifier. Again, one has to be careful however when defining $\mu(f)$ for uncountably many f simultaneously. Indeed, μ turns out to have a.s. infinite total variation, but it does define a random $H^s(\mathbb{R}^d)$-valued distribution when $s < -\beta^2/2$ [JSW20]. One may also (via a change of the base measure in the proofs of [JSW20]) fix $f \in L^\infty(\mathbb{R}^d)$ and consider $g \mapsto \mu(fg)$ as an element of $H^s(\mathbb{R}^d)$. Although μ is not defined pointwise, we will below freely use the notation $\int_U f(x)\mu(x)\, dx$ to refer to $\mu(f)$.

2.2.2 Malliavin calculus: basic definitions

In this subsection we will collect some very basic notions of Malliavin calculus: the Malliavin derivative and Malliavin smoothness. We will mainly follow [Nua06] in our definitions, making some straightforward adaptations for complex-valued random variables both here and in the following sections.

Let $C_p^\infty(\mathbb{R}^n;\mathbb{R})$ be the class of real-valued smooth functions defined on $\mathbb{R}^n$ such that f and all its partial derivatives grow at most polynomially.

Definition 2.3. *We say that F is a smooth (real) random variable if it is of the form*

$$F(\Gamma) = f(\langle \Gamma, h_1\rangle_H, \dots, \langle \Gamma, h_n\rangle_H)$$

for some $h_1, \dots, h_n \in H$ and $f \in C_p^\infty(\mathbb{R}^n;\mathbb{R})$, $n \geq 1$.

For such a variable F we define its Malliavin derivative DF by

$$DF = \sum_{k=1}^{n} \frac{\partial}{\partial_k} f(\langle \Gamma, h_1 \rangle_H, \dots, \langle \Gamma, h_n \rangle_H) h_k.$$

Thus we see that DF is an H-valued random variable and in fact, in the case of smooth variables, DF corresponds to the usual derivative map: for any $h \in H$, we have that

$$\langle DF(\Gamma), h \rangle_H = \lim_{\varepsilon \to 0} \frac{F(\Gamma + \varepsilon h) - F(\Gamma)}{\varepsilon}.$$

One may also define $D^m F$ as a $H^{\otimes m}$-valued random variable by setting

$$D^m F = \sum_{k_1, \dots, k_m = 1}^{n} \frac{\partial^m}{\partial_{k_1} \dots \partial_{k_m}} f(\langle \Gamma, h_1 \rangle_H, \dots, \langle \Gamma, h_n \rangle_H) h_{k_1} \otimes \dots \otimes h_{k_m}.$$

In our case H is a space of functions defined on U and hence $H^{\otimes m}$ can be seen as a space of functions defined on U^m. At times it will be convenient to write down the arguments of the function explicitly using subscripts, e.g. for all $t_1, \dots, t_m \in U$ we set

$$D^m_{t_1, \dots, t_m} F := D^m F(t_1, \dots, t_m),$$

with

$$D^m F(t_1, \dots, t_m) = \sum_{k_1, \dots, k_m = 1}^{n} \frac{\partial^m}{\partial k_1 \dots \partial k_m} f(\langle \Gamma, h_1 \rangle_H, \dots, \langle \Gamma, h_n \rangle_H) h_{k_1}(t_1) \dots h_{k_m}(t_m).$$

We extend the above definition in a natural way to complex smooth random variables by setting

$$D(F + iG) = DF + iDG$$

when F and G are real smooth random variables. Thus in general D will map complex random variables to the complexification of H, which we denote by $H_{\mathbb{C}}$. We will assume that the inner product $\langle \cdot, \cdot \rangle_{H_{\mathbb{C}}}$ is conjugate linear in the second variable. From here onwards we will use F for complex-valued Malliavin smooth random variables, unless otherwise stated.

To define D for a larger class of random variables one uses approximation by the smooth functions above. More precisely, we define for any non-negative integer k and real $p \geq 1$ the class of random variables $\mathbb{D}^{k,p}$ as the completion of (complex) smooth random variables with respect to the norm

$$\|F\|_{k,p}^p := \mathbb{E}|F|^p + \sum_{j=1}^{k} \mathbb{E}\|D^j F\|^p_{H_{\mathbb{C}}^{\otimes j}}.$$

The spaces $\mathbb{D}^{k,p}$ are decreasing with p and k, and we denote their intersection by $\mathbb{D}^\infty$.

Finally, viewing D as an unbounded operator on $L^2(\Omega; \mathbb{C})$ with values in $L^2(\Omega; H_{\mathbb{C}})$, we may define its adjoint δ which is also called the divergence operator. More specifically we have

$$\mathbb{E}[F\delta u] = \mathbb{E}\langle DF, u\rangle_{H_{\mathbb{C}}}$$

for any u such that $|\mathbb{E}\langle DF, u\rangle_{H_{\mathbb{C}}}|^2 \lesssim \mathbb{E}F^2$ for all $F \in \mathbb{D}^{1,2}$.

2.3 Density of imaginary chaos via Malliavin calculus

Let f be a continuous function of compact support in U. Our goal is to apply Malliavin calculus to show that the random variable $M := \mu(f)$ has a smooth density with respect to the Lebesgue measure on $\mathbb{C}$.

We start by walking through the basic results of Malliavin calculus that we want to apply and we then reduce the proof of Theorem 2.9 to concrete estimates on imaginary chaos. Some useful lemmas of Malliavin calculus are proven in Section 2.5 and the estimates on imaginary chaos are verified in Section 2.6, with input from Section 2.4.

Formally one can write the Malliavin derivative DM of $M = \mu(f)$ as

$$\begin{aligned} D_t M &= \int f(x) D_t :e^{i\beta \sum_{n=1}^{\infty} \langle \Gamma, e_n\rangle_H e_n(x)}: \, dx \\ &= \int f(x) \sum_{k=1}^{\infty} :e^{i\beta\Gamma(x)}: i\beta e_k(t) e_k(x)\, dx \\ &= i\beta \int f(x)\mu(x) C(t,x)\, dx. \end{aligned}$$

The content of the following proposition is to make the above computations rigorous by truncating the series $\sum_{n=1}^{\infty} \langle \Gamma, e_n\rangle_H e_n(x)$ to be able to work with Malliavin smooth random variables, as in Definition 2.3.

Proposition 2.4. *Let $f \in L^\infty(\mathbb{C})$. Then $M \in \mathbb{D}^\infty$ and*

$$D_t M = i\beta \int_U f(x)\mu(x) C(t,x)\, dx$$

for all $t \in U$.

The reason we are interested in showing that M belongs to $\mathbb{D}^\infty$ is the following classical result of Malliavin calculus, stating sufficient conditions for the existence of a smooth density. For convenience we state it here directly for complex valued random variables.

Proposition 2.5. *Let $F \in \mathbb{D}^\infty$ be a complex valued random variable and let*

$$\det \gamma_F := \frac{1}{4}(\|DF\|^4_{H_{\mathbb{C}}} - |\langle DF, D\overline{F}\rangle_{H_{\mathbb{C}}}|^2) \tag{2.4}$$

be the Malliavin determinant of F. If $\mathbb{E}|\det(\gamma_F)|^{-p} < \infty$ for all $p \geq 1$, then F has a density ρ w.r.t. the Lebesgue measure in $\mathbb{C}$ and ρ is a Schwartz function.

The proof follows rather directly from [Nua06, Proposition 2.1.5]:

Proof. Following [Nua06], the Malliavin matrix of a random vector $F = (F_1, \dots, F_n) \in \mathbb{R}^n$ is given by $\gamma_F := (\langle DF_j, DF_k \rangle_H)_{j,k}^n$. We will use Proposition 2.1.5 from [Nua06], which states that if $F_i \in \mathbb{D}^\infty$ and $\mathbb{E}|\det \gamma_F|^{-p} < \infty$ for all $p \geq 1$, then F has a density w.r.t. the Lebesgue measure on $\mathbb{R}^n$ which is a Schwartz function.

As $\operatorname{Re} F, \operatorname{Im} F \in \mathbb{D}^\infty$ by assumption, it is enough to check that $\det \gamma_F$ is equal to the given formula in the case $F = (\operatorname{Re} F, \operatorname{Im} F)$. This is easy to check by writing

$$\begin{aligned}\det \gamma_F &= \langle DF_1, DF_1 \rangle_H \langle DF_2, DF_2 \rangle_H - \langle DF_1, DF_2, \rangle_H^2 \\ &= \frac{1}{16} \|DF + D\overline{F}\|_{H_\mathbb{C}}^2 \|DF - D\overline{F}\|_{H_\mathbb{C}}^2 - \frac{1}{16} |\langle DF + D\overline{F}, DF - D\overline{F} \rangle_{H_\mathbb{C}}|^2\end{aligned}$$

and expanding the squares on the right hand side. We leave the details to the reader. □

Thus to show that F has a smooth and bounded density it will be enough to show that the negative moments of $\|DF\|_{H_\mathbb{C}}^4 - |\langle DF, D\overline{F} \rangle_{H_\mathbb{C}}|^2$ are all finite. In fact this quantity is not straightforward to control directly and to make calculations possible, we first apply the following projection bounds, whose proofs we postpone to Section 2.5:

Lemma 2.6 (Projection bounds)**.** *Let h be any function in $H_\mathbb{C}$. Then*

$$\frac{\det \gamma_F}{\|DF\|_{H_\mathbb{C}}^2} \geq \frac{1}{4} \frac{(|\langle DF, h \rangle_{H_\mathbb{C}}| - |\langle D\overline{F}, h \rangle_{H_\mathbb{C}}|)^2}{\|h\|_{H_\mathbb{C}}^2}. \tag{2.5}$$

and

$$\det \gamma_F \geq \frac{1}{4} \frac{(|\langle DF, h \rangle_{H_\mathbb{C}}| - |\langle D\overline{F}, h \rangle_{H_\mathbb{C}}|)^4}{\|h\|_{H_\mathbb{C}}^4}. \tag{2.6}$$

To further show that the density is uniformly bounded in β outside any interval surrounding the origin, we need to have some quantitative control on the densities. We will use the following simple adaption of Lemma 7.3.2 in [NN18] to the complex case to do this:

Lemma 2.7. *Let $p > 2$ and F be a complex Malliavin random variable in $\mathbb{D}^{2,\infty}$. Then there is a constant $c = c_p > 0$ such that the density ρ of F satisfies for all $x \in \mathbb{C}$*

$$\rho(x) \leq c_p (\mathbb{E}|\delta(A)|^p)^{m/p},$$

where the complex covering vector field A is defined by

$$A = \frac{\|DF\|_{H_\mathbb{C}}^2 DF - \langle DF, D\overline{F} \rangle_{H_\mathbb{C}} D\overline{F}}{\|DF\|_{H_\mathbb{C}}^4 - |\langle DF, D\overline{F} \rangle_{H_\mathbb{C}}|^2}.$$

Bounding $\delta(A)$ is again technically not straightforward, but the following general bound could possibly be of independent interest. It is again proved in Section 2.5.

Proposition 2.8. *Let F be a complex Malliavin random variable in $\mathbb{D}^{2,\infty}$. We have*

$$|\delta(A)| \lesssim \frac{\|DF\|_{H_\mathbb{C}}^2 (|\delta(DF)| + \|D^2 F\|_{H_\mathbb{C} \otimes H_\mathbb{C}})}{\|DF\|_{H_\mathbb{C}}^4 - |\langle DF, D\overline{F} \rangle_{H_\mathbb{C}}|^2}.$$

Using the above results on Malliavin calculus, we can now reduce Theorem 2.9 to concrete propositions on imaginary chaos. Proving the estimates needed for these propositions is basically the content of Section 2.6.

We start with a precise statement of the main theorem:

Theorem 2.9. *Let U be an open bounded domain and Γ a non-degenerate log-correlated field in U as in Definition 2.2 and f be a nonzero continuous function of compact support in U. We denote by μ_β the imaginary chaos associated to Γ. Then*

- *the law of $\mu_\beta(f)$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{C}$ and the density is a Schwartz function;*
- *for any $\eta > 0$ the density is uniformly bounded from above for $\beta \in (\eta, \sqrt{d})$ and converges to zero pointwise as $\beta \to \sqrt{d}$.*

Finally, the same holds in the case where Γ is defined on the unit circle with covariance $\mathbb{E}[\hat{\Gamma}(x)\hat{\Gamma}(y)] = -\log|x-y|$ and f is any nonzero continuous function on the circle.

There are basically two technical chaos estimates needed to deduce the theorem. First, super-polynomial bounds on small ball probabilities of the Mallian determinant are used both to prove that the density exists and is a Schwartz function, and to show uniformity:

Proposition 2.10. *Let Γ, f, $M = \mu(f)$ be as in the theorem above. Then we have the following bounds for the Malliavin determinant $\det \gamma_M$. For any $\nu > 0$, there exist absolute constants $C, c, a > 0$ such that for all $\varepsilon > 0$ sufficiently small and for all $\beta \in (\nu, \sqrt{d})$,*

$$\mathbb{P}\left(\det \gamma_M \geq (d-\beta^2)^{-4}\varepsilon\right) \geq 1 - C\exp\left(-a\varepsilon^{-c/2}\right). \tag{2.7}$$

and

$$\mathbb{P}\left(\frac{\det \gamma_M}{\|DM\|_{H_\mathbb{C}}^2} \geq (d-\beta^2)^{-2}\varepsilon\right) \geq 1 - C\exp\left(-a\varepsilon^{-c}\right). \tag{2.8}$$

Here the bound on $\frac{\|DM\|_{H_\mathbb{C}}^2}{\det \gamma_M}$ is necessary, when bounding the divergence of the covering field via Proposition 2.8. Second, in order to apply Lemma 2.7 we also need upper bounds on $|\delta(DM)|$ and $\|D^2M\|_{H_\mathbb{C}\otimes H_\mathbb{C}}$:

Proposition 2.11. *Let Γ, f, $M = \mu(f)$ be as in the theorem above. Then for all $N \geq 1$, there exists $C = C(N) > 0$ such that for all $\beta \in (0, \sqrt{d})$*

$$\mathbb{E}\left[|\delta(DM)|^{2N}\right] \leq C(d-\beta^2)^{-3N} \tag{2.9}$$

and

$$\mathbb{E}\left[\|D^2M\|_{H_\mathbb{C}\otimes H_\mathbb{C}}^{2N}\right] \leq C(d-\beta^2)^{-3N}. \tag{2.10}$$

We can now prove Theorem 2.9 modulo these propositions.

Proof of Theorem 2.9. To apply Proposition 2.5 to prove that $M = \mu(f)$ has a density w.r.t. Lebesgue measure, and that moreover this density is a Schwartz function, we need to verify two conditions:

- That $M \in \mathbb{D}^\infty$ – this is the content of Proposition 2.4;
- And that $\mathbb{E}|\det(\gamma_M)|^{-p} < \infty$ for all $p \geq 1$ – this follows directly from the bound (2.7) in Proposition 2.10.

Finally, it remains to argue that the density is uniformly bounded from above for $\beta \in (\eta, \sqrt{d})$ for some fixed $\eta > 0$, and converges to zero pointwise on $\mathbb{R}^d$ as $\beta \to \sqrt{d}$. This follows from Lemma 2.7, once we show that $\mathbb{E}|\delta(A)|^4$ is uniformly bounded in $\beta \in (\eta, \sqrt{d})$ and tends to zero as $\beta \to \sqrt{d}$. By Proposition 2.8

$$\mathbb{E}|\delta(A)|^4 \lesssim \mathbb{E}\Big|\frac{\|DM\|_{H_\mathbb{C}}^2(|\delta(DM)| + \|D^2M\|_{H_\mathbb{C}\otimes H_\mathbb{C}})}{\|DM\|_{H_\mathbb{C}}^4 - |\langle DM, D\overline{M}\rangle_{H_\mathbb{C}}|^2}\Big|^4.$$

By using the inequality $(x+y)^4 \lesssim x^4 + y^4$ and then Cauchy–Schwarz we have that

$$\mathbb{E}|\delta(A)|^4 \lesssim \sqrt{\mathbb{E}\Big|\frac{\|DM\|_{H_\mathbb{C}}^2}{\det \gamma_M}\Big|^8 \mathbb{E}|\delta(DM)|^8} + \sqrt{\mathbb{E}\Big|\frac{\|DM\|_{H_\mathbb{C}}^2}{\det \gamma_M}\Big|^8 \mathbb{E}|\|D^2M\|_{H_\mathbb{C}\otimes H_\mathbb{C}}|^8}.$$

We thus conclude from (2.8) in Proposition 2.10 and Proposition 2.11. □

The proofs of the above-mentioned chaos estimates appear in Section 2.6. More precisely,

- In Section 2.6.2 we prove that M is in $\mathbb{D}^\infty$, i.e. Proposition 2.4. This boils down to bounding moments of DM and is a rather standard calculation. Similar computations with small improvements on existing estimates allow to prove Proposition 2.11 in Section 2.6.3.
- In Section 2.6.4, we prove Proposition 2.10, which requires a novel approach. It is also in this subsection where we make use of the almost global decomposition theorem for non-degenerate log-correlated fields, proved in Section 2.4.

The missing general results of Malliavin calculus are proved in Section 2.5.

2.4 Almost global decompositions of non-degenerate log-correlated fields

It is often useful to try to decompose the log-correlated Gaussian field Γ on the open set $U \subset \mathbb{R}^d$ as a sum of two independent fields Y and Z, where Y is in some sense canonical and easy to calculate with, and Z is regular. In [JSW19] it was shown that such decompositions exist around every point $x_0 \in U$ when $g \in H^s_{\mathrm{loc}}(U \times U)$ for some $s > d$ and Y is taken to be a so-called almost $\star$-scale invariant field.

Our goal in this section is to establish a more general variant of this decomposition theorem which removes the need to restrict to small balls and works in any subdomain $V \Subset U$ (we write $A \Subset B$ to indicate that $\overline{A} \subset B$) by simply assuming that Γ is non-degenerate on V, meaning that C_Γ defines an injective integral operator on $L^2(V)$, as explained in Section 2.2.

In the context of the present article, the usefulness of this result is strongly interlinked with the following standard comparison result for Cameron–Martin spaces. In the case of Reproducing Kernel Hilbert spaces, this can be found for example in [Aro50].

Lemma 2.12. *Let Y and Z be two independent distribution-valued Gaussian fields and denote $\Gamma = Y + Z$. Let $(H_\Gamma, \|\cdot\|_{H_\Gamma})$ and $(H_Y, \|\cdot\|Y_{H_Y})$ be the Cameron–Martin spaces of Γ and Y respectively. Then $H_Y \subset H_\Gamma$ and moreover for every $h \in H_Y$, we have that $\|h\|_{H_Y} \geq \|h\|_{H_\Gamma}$.*

Basically, via this Lemma our decomposition allows to meaningfully transfer calculations on the initial field Γ to easier ones on the almost $\star$-scale invariant fields Y, where Fourier methods become available.

We will start by recalling the basic definitions related to $\star$-scale invariant and almost $\star$-scale invariant log-correlated fields. We then state the theorem and discuss heuristics, and finally prove the theorem in two last subsections. In this section all function spaces are the standard function spaces for real-valued functions, i.e. we don't need to consider their complexified counterparts.

2.4.1 Overview of $\star$-scale and almost $\star$-scale invariant log-correlated fields

To define $\star$-scale invariant and almost $\star$-scale invariant fields, we first need to pick a seed covariance k. For simplicity we will in what follows make the following assumptions on k:

Assumption 2.13. *The seed covariance $k\colon \mathbb{R}^d \to \mathbb{R}$ satisfies the following properties:*

- $k(x) \geq 0$ *for all* $x \in \mathbb{R}^d$ *and* $k(0) = 1$*;*
- $k(x) = k((|x|, 0, \dots, 0)) =: k(|x|)$ *is rotationally symmetric and* $\operatorname{supp} k \subset B(0,1)$*,*
- *There exists* $s > \frac{d+1}{2}$ *such that* $0 \leq \hat{k}(\xi) \lesssim (1 + |\xi|^2)^{-s}$ *for all* $\xi \in \mathbb{R}^d$*.*

The fact that k is supported in $B(0,1)$ yields the useful property that distant regions of the associated Gaussian field will be independent.

Definition 2.14. *Let $k\colon \mathbb{R}^d \to \mathbb{R}$ be as above. The $\star$-scale invariant covariance kernel C_X associated to k is given by*

$$C_X(x,y) := \int_0^\infty k(e^u(x-y))\,du.$$

Similarly, the related almost *$\star$-scale invariant covariance kernel $C_Y = C_{Y(\alpha)}$ associated to k and a parameter $\alpha > 0$ is given by*

$$C_Y(x,y) := \int_0^\infty k(e^u(x-y))(1 - e^{-\alpha u})\,du.$$

We often use approximations Y_δ of Y, which can be defined via the stochastic integrals

$$Y_\delta(x) = \int_{\mathbb{R}^d \times [0, \log\frac{1}{\delta}]} e^{du/2} \tilde{k}(e^u(t-x)) \sqrt{1 - e^{-\alpha u}} dW(t,u), \tag{2.11}$$

where W is the standard white noise on $\mathbb{R}^{d+1}$ and $\tilde{k}(x) = \mathcal{F}^{-1}\sqrt{\mathcal{F}k}(x)$ with $\mathcal{F}$ denoting the Fourier transform.

We also define the tail field $\hat{Y}_\delta := Y - Y_\delta$, which becomes independent on the length-scale δ. The following lemma then gives basic estimates on the covariance of this tail field. See Appendix 2.A for the proof.

Lemma 2.15. *There exists a constant $C > 0$ such that*

$$\mathbb{E}[\hat{Y}_\delta(x)\hat{Y}_\delta(y)] \leq \frac{\delta}{|x-y|}$$

and

$$\mathbb{E}[\hat{Y}_\delta(x)\hat{Y}_\delta(y)] \geq \frac{\delta}{|x-y|} - C.$$

Moreover $\mathbb{E}[\hat{Y}_\delta(x)\hat{Y}_\delta(y)] = 0$ whenever $|x-y| \geq \delta$.

2.4.2 Statement of the theorem and the high level argument

The main theorem of this section can be stated as follows.

Theorem 2.16. *Let Γ be a non-degenerate log-correlated Gaussian field on an open domain $U \subseteq \mathbb{R}^d$ as in Definition 2.2. Assume further that the covariance kernel given by (2.2) satisfies $g \in H^s_{\text{loc}}(U \times U)$ for some $s > d$.*

Then for every seed kernel k satisfying Assumption 2.13 and every $V \Subset U$, there exists $\alpha > 0$ such that we may write (possibly in a larger probability space)

$$\Gamma|_V = Y + Z,$$

where Y is an almost $\star$-scale invariant field with seed covariance k and parameter α and Z is a Hölder-regular field independent of Y, both defined on the whole of $\mathbb{R}^d$. Moreover, there exists $\varepsilon > 0$ such that the operator C_Z maps $H^s(\mathbb{R}^d) \to H^{s+d+\varepsilon}(\mathbb{R}^d)$ for all $s \in [-d, 0]$.

Notice that the 2D zero boundary Gaussian free field is a non-degenerate log-correlated field in the open disk. However, there is no hope to decompose it using an almost $\star$-scale invariant field on the whole of $\mathbb{D}$, so in that sense the above theorem is as global as you could hope.

Remark 2.17. In [JSW19, Theorem B] it was shown that even for a degenerate log-correlated field Γ, one can for any $x \in U$ find a ball $B(x, r(x))$, restricted to which Γ is non-degenerate and can be decomposed as an independent sum of an almost star-scale invariant field and a Hölder-regular field. In this sense one can see Theorem 2.16 as a generalization in the special case of non-degenerate fields.

Before going to the proof of Theorem 2.16, let us try to illustrate the high level argument in terms of the following toy problem on the unit circle $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$: Let Γ be a non-degenerate log-correlated field on $\mathbb{T}$ with covariance of the form $\log \frac{1}{|x-y|} + g(|x-y|)$, where now also the g term only depends on the distance between the two points. This means that we can write the covariance using the Fourier series

$$C_\Gamma(x,y) = \frac{g_0}{2} + \text{Re} \sum_{n=1}^{\infty} (\frac{1}{n} + g_n) x^n y^{-n},$$

where

$$g_n := \frac{1}{\pi} \int_{\mathbb{T}} g(|1-x|) x^{-n} |dx|,$$

with $|dx|$ denoting the arc-length measure. As Γ is assumed to be non-degenerate, we know that $\frac{1}{n} + g_n > 0$ for all $n \geq 1$.

The almost $\star$-scale field would correspond to a field with covariance of the form

$$C_Y(x, y) = \operatorname{Re} \sum_{n=1}^{\infty} (\frac{1}{n} - \frac{1}{n^{1+\alpha}}) x^n y^{-n},$$

and thus the difference between the tail and the two covariances would be

$$C_\Gamma(x, y) - C_Y(x, y) = \frac{g_0}{2} + \operatorname{Re} \sum_{n=1}^{\infty} (\frac{1}{n^{1+\alpha}} + g_n) x^n y^{-n}.$$

It is now easy to see that if $g_n = O(n^{-s})$ for some $s > 1 + \alpha$, the coefficients in the above difference are positive for all large enough n. By further reducing α, we can guarantee that $\frac{1}{n^{1+\alpha}} + g_n > 0$ for all $n \geq 1$, so that the difference $C_\Gamma - C_Y$ is again a positive definite kernel.

The main issue in implementing this strategy for general log-correlated covariances on domains in $\mathbb{R}^d$ is the fact that in general we do not have a canonical basis such that C_Γ and C_X would be simultaneously diagonalizable. To still be able to make useful calculations, we thus want to find some universal, non-basis dependent setting, where both can be studied. This is comfortably offered for example by the Fourier transform on spaces $L^2(\mathbb{R}^d)$ and $H^s(\mathbb{R}^d)$. Thus as a first step we will find a suitable extension of Γ to a log-correlated field on the whole of $\mathbb{R}^d$ with covariance of the form $C_X + R$ where C_X is the covariance of a $\star$-scale invariant field and R is the kernel of an integral operator which maps $L^2(\mathbb{R}^d)$ to $H^s(\mathbb{R}^d)$ for some $s > d$ (in particular it is in this sense more regular than C_X which maps $L^2(\mathbb{R}^d)$ to $H^d(\mathbb{R}^d)$). The second step is then to actually make the calculations work, and to do this in the general set-up we make use of some operator-theoretic methods.

2.4.3 Extension of log-correlated fields to the whole space

Let us begin by solving the aforementioned extension problem. In what follows we will denote by the same symbols both the integral operators and their kernels, and C_X (resp. $C_{Y(\alpha)}$) will always refer to the covariance operator of a $\star$-scale (resp. almost $\star$-scale) invariant field with a fixed seed covariance k (resp. and parameter α).

First of all, we note the existence of the following partition of unity consisting of squares of smooth functions.

Lemma 2.18. *Let $U \subset \mathbb{R}^d$ be an open domain and $V \Subset U$ an open subdomain. Then there exists an open set W with $V \Subset W \Subset U$ and non-negative functions $a, b \in C^\infty(\mathbb{R}^d)$ such that $a^2 + b^2 \equiv 1$, $b(x) = 0$ for all $x \in \overline{V}$, $b(x) > 0$ for all $x \in \mathbb{R}^d \setminus \overline{V}$ and $a(x) = 0$ for all $x \in \mathbb{R}^d \setminus W$.*

Proof. Pick any W with $V \Subset W \Subset U$. It is well-known that one can pick a function $u \in C^\infty(\mathbb{R}^d)$ which is 1 in V, 0 outside W and $0 \leq u(x) < 1$ for $x \in W \setminus \overline{V}$. The function $u(x)^2 + (1-u(x))^2 \geq \frac{1}{2}$ is everywhere strictly positive and therefore the function $v(x) := \sqrt{u(x)^2 + (1-u(x))^2}$ is smooth and strictly positive. Finally define $a(x) \coloneqq u(x)/v(x)$ and $b(x) \coloneqq (1-u(x))/v(x)$ to obtain the desired functions. □

Secondly we need the following estimates on the covariance operator C_X.

Lemma 2.19. *For any $s \in \mathbb{R}$ the operator C_X is a bounded invertible operator $H^s(\mathbb{R}^d) \to H^{s+d}(\mathbb{R}^d)$. The same holds for $C_{Y^{(\alpha)}}$ for any $\alpha > 0$. In particular the Cameron–Martin space of $Y^{(\alpha)}$ equals $H^{d/2}(\mathbb{R}^d)$ with an equivalent norm.*

Moreover the Fourier transform of the associated kernel

$$K(u) \coloneqq C_X(u,0) = \int_0^\infty k(e^s u)\, ds$$

is smooth and satisfies

$$|\nabla_\xi \hat{K}(\xi)| \lesssim (1+|\xi|^2)^{-\frac{d+1}{2}}.$$

Proof. We have $C_X f = K * f$, so it is enough to study the Fourier transform of K. We compute

$$\hat{K}(\xi) = \int_0^\infty e^{-du}\hat{k}(e^{-u}\xi)du = \int_0^1 v^{d-1}\hat{k}(v\xi)\, dv = |\xi|^{-d}\int_0^{|\xi|} v^{d-1}\hat{k}(v)\, dv.$$

Since $\hat{k}(0) > 0$ and also $\hat{k}(\xi) = O(|\xi|^{-\alpha})$ for some $\alpha > d+1$, we see that the above quantity is bounded from above and below by a constant multiple of $(1+|\xi|^2)^{-d/2}$, which implies the claim that C_X maps $H^s(\mathbb{R}^d)$ to $H^{s+d}(\mathbb{R}^d)$ continuously and bijectively.

Similarly $C_{Y^{(\alpha)}} f = K_\alpha * f$ with

$$\hat{K}_\alpha(\xi) = \int_0^1 v^{d-1}\hat{k}(v\xi)(1-v^\alpha)\, dv = |\xi|^{-d}\int_0^{|\xi|} v^{d-1}\hat{k}(v)(1-|\xi|^{-\alpha}v^\alpha)\, dv$$

and one again sees that this is bounded from above and below by a constant multiple of $(1+|\xi|^2)^{-d/2}$. In particular $H_{Y^{(\alpha)}} = C^{1/2}_{Y^{(\alpha)}} L^2(\mathbb{R}^d) = H^{d/2}(\mathbb{R}^d)$.

Next we note that since k is compactly supported, $\hat{k}$ is smooth and also $|\nabla \hat{k}(\xi)| = O(|\xi|^{-\alpha})$. Thus

$$\nabla \hat{K}(\xi) = \int_0^1 v^d \nabla \hat{k}(v\xi) dv = |\xi|^{-d-1}\int_0^{|\xi|} v^d \nabla \hat{k}(v)\, dv,$$

from which the second claim follows. □

As a corollary of the following lemma from [JSW19] we can rephrase (2.2) using a $\star$-scale invariant covariance instead of pure logarithm.

Lemma 2.20 ([JSW19, Proposition 4.1 (vi)])**.** *The covariance C_X of a $\star$-scale invariant field X satisfies $C_X(x,y) = \log\frac{1}{|x-y|} + g_0(x,y)$, where $g_0(x,y)$ belongs to $H^{s'}(\mathbb{R}^d)$ for some $s' > d$.*

Let us next prove the extension itself. We emphasise that the kernel R in the proposition below is not necessarily definite positive.

Proposition 2.21. *Let C_Γ be as in Theorem 2.16. Let $V \Subset U$ be an open subdomain. Let X be a $\star$-scale invariant log-correlated field with a seed covariance k satisfying Assumption 2.13.*

Then there exists a bounded integral operator $R\colon L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ *such that* $C_X + R$ *is strictly positive and the corresponding kernels satisfy*

$$C_\Gamma(x, y) = C_X(x, y) + R(x, y)$$

for all $x, y \in V$. *The kernel* R *is Hölder-continuous with some exponent* $\gamma > 0$ *and moreover, there exists* $\delta > 0$ *such that* R *defines a bounded operator* $H^r(\mathbb{R}^d) \to H^{r+d+2\delta}(\mathbb{R}^d)$ *for all* $r \in [-d, 0]$.

Proof. Let $V \Subset W \Subset U$ and $a, b \in C^\infty(\mathbb{R}^d)$ be as in Lemma 2.18 and consider the (distribution-valued) Gaussian field $Z = a\Gamma + bX$ defined on $\mathbb{R}^d$. Here Γ and X are independent and have covariance operators C_Γ and C_X respectively. By using Lemma 2.20 we can write $C_\Gamma(x, y) = C_X(x, y) + \tilde{g}(x, y)$ with $\tilde{g} \in H^{s'}_{\mathrm{loc}}(\mathbb{R}^d \times \mathbb{R}^d)$ for some $s' > d$. Thus we may write the kernel of the covariance operator of Z as

$$C_Z(x, y) = a(x)a(y)C_\Gamma(x, y) + b(x)b(y)C_X(x, y) = C_X(x, y) + R(x, y),$$

where

$$R(x, y) \coloneqq (a(x)a(y) + b(x)b(y) - 1)C_X(x, y) + a(x)a(y)\tilde{g}(x, y). \tag{2.12}$$

Note that $G(x, y) \coloneqq a(x)a(y)\tilde{g}(x, y)$ is an element of $H^{s'}(\mathbb{R}^d \times \mathbb{R}^d)$. For any $f \in H^r(\mathbb{R}^d)$ with $r \in [-s', 0]$ we have that the corresponding operator G satisfies

$$\begin{aligned}\|Gf\|^2_{H^{r+s'}(\mathbb{R}^d)} &= \int (1 + |\xi|^2)^{r+s'} \Big| \int \hat{G}(\xi, \zeta)\overline{\hat{f}(\zeta)}\, d\zeta\Big|^2 d\xi \\ &\lesssim \|G\|^2_{H^{s'}(\mathbb{R}^d \times \mathbb{R}^d)} \|f\|^2_{H^r(\mathbb{R}^d)}.\end{aligned}$$

We conclude that G is a bounded operator $H^r(\mathbb{R}^d) \to H^{r+s'}(\mathbb{R}^d)$.

Let us then consider the operator T with kernel

$$T(x, y) \coloneqq (a(x)a(y) + b(x)b(y) - 1)C_X(x, y)$$

corresponding to the first term in the definition of R. Again for $f \in L^2(\mathbb{R}^d)$ we have

$$\|Tf\|^2_{H^{d+1}(\mathbb{R}^d)} = \int (1 + |\xi|^2)^{d+1} \Big| \int \hat{T}(\xi, \zeta)\overline{\hat{f}(\zeta)}\, d\zeta\Big|^2 d\xi.$$

Note that since $a^2 + b^2 = 1$ we have

$$T(x, y) = (a(x)(a(y) - a(x)) + b(x)(b(y) - b(x)))C_X(x, y).$$

The maps $f \mapsto af$ and $f \mapsto bf = (b - 1)f + f$ are bounded operators $H^\alpha(\mathbb{R}^d) \to H^\alpha(\mathbb{R}^d)$ for any $\alpha \in \mathbb{R}$ since a and $b - 1$ are compactly supported and smooth. Thus it is enough to show that $A\colon f \mapsto [x \mapsto \int (a(y) - a(x))K(x - y)f(y)\, dy]$ and $B\colon f \mapsto [x \mapsto \int (b(y) - b(x))K(x - y)f(y)\, dy]$ are bounded operators $H^r(\mathbb{R}^d) \to H^{r+d+1}(\mathbb{R}^d)$, where $K(u) = C_X(u, 0)$.

We will show the claim for A – the same proof works for B as well since we only use the fact that a is smooth and has compact support and we can again reduce to this situation by replacing b with $b - 1$.

A small computation shows that we can write

$$\widehat{Af}(\xi) = \int \hat{a}(\xi - \zeta)(\hat{K}(\xi) - \hat{K}(\zeta))\hat{f}(\zeta)\, d\zeta$$

We can bound

$$\begin{aligned}\int \hat{a}(\xi - \zeta)(\hat{K}(\xi) - \hat{K}(\zeta))\hat{f}(\zeta)\, d\zeta \lesssim &\int_{\mathbb{R}^d \setminus B(\xi, |\xi|/2)} |\hat{a}(\xi - \zeta)||\hat{f}(\zeta)|\, d\zeta \\ &+ \int_{B(\xi,|\xi|/2)} |\hat{a}(\xi - \zeta)||\xi - \zeta| \sup_{z \in B(\xi,|\xi|/2)} |\nabla \hat{K}(z)||\hat{f}(\zeta)|\, d\zeta\end{aligned}$$

which by Lemma 2.19 is bounded by

$$\lesssim (1 + |\xi|^2)^{-d-1}\|f\|_{H^r(\mathbb{R}^d)} + (1 + |\xi|^2)^{-\frac{d+1}{2}} \int_{\mathbb{R}^d} |\hat{a}(\xi - \zeta)||\xi - \zeta||\hat{f}(\zeta)|\, d\zeta.$$

Now, for the first term we have that

$$\int_{\mathbb{R}^d} (1 + |\xi|^2)^{r+d+1}(1 + |\xi|^2)^{-2d-2}\|f\|^2_{H^r(\mathbb{R}^d)} < \infty.$$

For the second term we let $p(\xi) := |\xi||\hat{a}(\xi)|$ and note that since $|\hat{f}(\zeta)||\hat{f}(\zeta')| \leq (|\hat{f}(\zeta)|^2 + |\hat{f}(\zeta')|^2)/2$ we have

$$\begin{aligned}&\int_{\mathbb{R}^d} (1 + |\xi|^2)^{r+d+1}(1 + |\xi|^2)^{-d-1} \left(\int_{\mathbb{R}^d} p(\xi - \zeta)|\hat{f}(\zeta)|\, d\zeta\right)^2 d\xi \\ &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} (1 + |\xi|^2)^r p(\xi - \zeta)p(\xi - \zeta')|\hat{f}(\zeta)||\hat{f}(\zeta')|\, d\zeta\, d\zeta'\, d\xi \\ &\leq \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} (1 + |\xi|^2)^r p(\xi - \zeta)p(\xi - \zeta')|\hat{f}(\zeta)|^2\, d\zeta\, d\zeta'\, d\xi.\end{aligned}$$

Integrating over ζ' gives just $\|p\|_{L^1(\mathbb{R}^d)}$ and then by using the inequality $(1+|\xi|^2)^r \lesssim (1+|\zeta-\xi|^2)^{-r}(1+|\zeta|^2)^r$ we may also integrate over ξ and ζ separately to see that the above is bounded by a constant times

$$\|p\|_{L^1(\mathbb{R}^d)}\|(1 + |\cdot|)^{-r}p(\cdot)\|_{L^1(\mathbb{R}^d)}\|f\|^2_{H^r(\mathbb{R}^d)}.$$

Thus putting things together we obtain that

$$\|Af\|^2_{H^{r+d+1}(\mathbb{R}^d)} = \int (1 + |\xi|^2)^{r+d+1}|\widehat{Af}(\xi)|^2 \lesssim \|f\|^2_{H^r(\mathbb{R}^d)},$$

showing that R maps $H^r(\mathbb{R}^d) \to H^{r+d+2\delta}$ for $\delta > 0$ small enough.

Let us next show that R is Hölder-continuous. As $\tilde{g}$ belongs to $H^{s'}_{\mathrm{loc}}(\mathbb{R}^d \times \mathbb{R}^d)$ for some $s' > d$, it follows from the Sobolev embedding $H^{d+\delta}(\mathbb{R}^{2d}) \to C^\delta(\mathbb{R}^{2d})$ where $C^\delta(\mathbb{R}^{2d})$ is the space of δ-Hölder functions vanishing at infinity, that $\tilde{g}$ is γ-Hölder for some $\gamma > 0$. By (2.12) this implies that we only need to show that $(a(x)a(y) + b(x)b(y) - 1)C_X(x, y)$ is Hölder-continuous. As this term is compactly supported, we can add a smooth cutoff function ρ such that

$$(a(x)a(y) + b(x)b(y) - 1)C_X(x,y) = \rho(x)\rho(y)(a(x)(a(y) - a(x)) + b(x)(b(y) - b(x)))C_X(x,y)$$

for all $x, y \in \mathbb{R}^d$. Moreover, since $C_X(x,y) = \log \frac{1}{|x-y|} + g_0(x,y)$ with g_0 smooth, it is enough to show that

$$(a(y) - a(x))\rho(x)\rho(y) \log \frac{1}{|x-y|}$$

is Hölder-continuous (the term with $b(y) - b(x)$ can again be handled in a similar manner). Let us write the above as

$$\int_0^1 \nabla a(x + u(y-x))\, du \cdot (y-x)\rho(x)\rho(y) \log \frac{1}{|x-y|}.$$

As a is smooth, the map $(x,y) \mapsto \int_0^1 \nabla a(x + u(y-x))\, du$ is in particular a Hölder continuous map $\mathbb{R}^{2d} \to \mathbb{R}^d$. Thus it is enough to show that $(x,y) \mapsto (y-x) \log \frac{1}{|x-y|}$ is Hölder-continuous but this follows easily by checking that each component function $(y_j - x_j) \log \frac{1}{|x-y|}$ is Hölder continuous in each coordinate. The Hölder constants are also easily seen to be bounded for $x, y \in \operatorname{supp} \rho$.

Finally let us note that C_Z is strictly positive since if $f \in L^2(\mathbb{R}^d)$ is nonzero, then at least one of $f|_V$ or $f|_{\operatorname{supp} b}$ is nonzero. In the first case $\int a(x)a(y)C_\Gamma(x,y)f(x)f(y) > 0$ by the assumption that C_Γ was assumed to be injective in V, while in the second case $\int b(x)b(y)C_X(x,y)f(x)f(y) > 0$ since C_X is strictly positive on whole of $\mathbb{R}^d$. □

2.4.4 Deducing the decomposition theorem

Having obtained the desired extension, we are ready to prove the decomposition theorem. The second part of the proof consists in showing that we may subtract $C_{Y(\alpha)}$ from $C_X + R$ for some small enough $\alpha > 0$ and still obtain a positive operator.

To do this, we need to use the following classical stability property of strictly positive operators of the form $1 + K$ with K compact and self-adjoint that follows directly from the spectral theorem.

Lemma 2.22. *Let $\mathcal{H}$ be a Hilbert space and T a self-adjoint compact operator on $\mathcal{H}$ and suppose that $1 + T$ is strictly positive. Then there exists $\varepsilon > 0$ such that $1 + A + T$ is strictly positive for any self-adjoint A with $\|A\|_{\mathcal{H}\to\mathcal{H}} \le \varepsilon$.*

As a consequence of the above lemma and the smoothing properties of the map R obtained in Lemma 2.21 we first create a necessary lee-room. Notice that $C_X + R = C_X^{1/2}(I + C_X^{-1/2} R C_X^{-1/2}) C_X^{1/2}$ and hence

$$\langle (C_X + R)f, f\rangle_{L^2(\mathbb{R}^d)} = \langle (I + C_X^{-1/2} R C_X^{-1/2}) C_X^{1/2} f, C_X^{1/2} f\rangle_{L^2(\mathbb{R}^d)}.$$

The following statement is thus effectively saying that in fact $\langle (C_X + R)f, f\rangle_{L^2(\mathbb{R}^d)} > 0$ not only for $f \in L^2(\mathbb{R}^d)$, but also for $f \in H^{-d/2}(\mathbb{R}^d)$.

Lemma 2.23. *There is some $\varepsilon > 0$ such that $1 + A + C_X^{-1/2} R C_X^{-1/2}$ is a strictly positive operator on $L^2(\mathbb{R}^d)$ for any self-adjoint A with $\|A\|_{op} \le \varepsilon$.*

Proof. As R maps to functions supported in some compact domain D, by Rellich-type lemmas for fractional Sobolev spaces the operator $\tilde{R} = C_X^{-1/2} R C_X^{-1/2}$ is compact. As it is also self-adjoint on

$L^2(\mathbb{R}^d)$, there is an orthonormal basis of $L^2(\mathbb{R}^d)$ consisting of eigenfunctions of $\tilde{R}$. To show that $1+\tilde{R}$ is strictly positive it is enough to show that $\tilde{R}$ has no eigenfunctions with eigenvalues ≤ -1. Assume that f is an eigenfunction of $\tilde{R}$ with nonzero eigenvalue λ. Then by Lemma 2.21 we know that $\tilde{R}$ maps $H^s(\mathbb{R}^d) \to H^{s+2\delta}(\mathbb{R}^d)$ for any $s \in [0, d/2]$ and thus after applying $\tilde{R}$ to f roughly $1/\delta$ times we see that actually $f \in H^{d/2}(\mathbb{R}^d)$. Thus there exists some $g \in L^2(\mathbb{R}^d)$ such that $f = C_X^{1/2} g$, and we have that

$$(1+\lambda)\|f\|^2_{L^2(\mathbb{R}^d)} = \langle (1+\tilde{R})f, f\rangle_{L^2(\mathbb{R}^d)} = \langle (1+\tilde{R})C_X^{1/2}g, C_X^{1/2}g\rangle_{L^2(\mathbb{R}^d)} = \langle (C_X+R)g, g\rangle_{L^2(\mathbb{R}^d)} > 0$$

by the assumption on $C_X + R$, implying that $\lambda > -1$. Thus $1+\tilde{R}$ is strictly positive and the claim follows from Lemma 2.22. □

The final important technical ingredient is that for any $\alpha_0 > 0$,

$$(C_X - C_{Y^{(\alpha)}})^{-1/2} - C_X^{-1/2} \colon L^2(\mathbb{R}^d) \to H^{\frac{-d-\alpha_0}{2}}(\mathbb{R}^d)$$

converges pointwise to 0 when we let the parameter α of the almost $\star$-scale invariant field $Y^{(\alpha)}$ to 0.

Lemma 2.24. *For all $\alpha > 0$ set $U_\alpha \coloneqq C_X - C_{Y^{(\alpha)}}$ and let $U_0 = C_X$. Then $U_\alpha^{1/2}$ is a bounded bijection $H^s(\mathbb{R}^d) \to H^{s+\frac{d+\alpha}{2}}(\mathbb{R}^d)$ for all $s \in \mathbb{R}$, and for any $\alpha_0 > 0$, we have*

$$\sup_{\alpha_0 \geq \alpha > 0} \|U_\alpha^{-1/2}\|_{L^2(\mathbb{R}^d) \to H^{-\frac{d+\alpha_0}{2}}(\mathbb{R}^d)} < \infty.$$

Moreover, for any fixed $\alpha_0 > 0$ and $f \in L^2(\mathbb{R}^d)$ we have

$$\lim_{\alpha \to 0} \|(U_\alpha^{-1/2} - C_{Y^{(\alpha)}}^{-1/2})f\|_{H^{-\frac{d+\alpha_0}{2}}(\mathbb{R}^d)} = 0.$$

Before proving the lemma, let us see how it implies the theorem:

Proof of Theorem 2.16: We begin by writing

$$\langle (C_X - C_{Y^{(\alpha)}} + R)f, f\rangle_{L^2(\mathbb{R}^d)} = \langle (1+\tilde{R}_\alpha)U_\alpha^{1/2} f, U_\alpha^{1/2} f\rangle_{L^2(\mathbb{R}^d)},$$

where $U_\alpha = C_X - C_{Y^{(\alpha)}}$ and $\tilde{R}_\alpha = U_\alpha^{-1/2} R U_\alpha^{-1/2}$. It thus suffices to show that for some sufficiently small $\alpha > 0$ we have

$$\langle (1+\tilde{R}_\alpha)g, g\rangle_{L^2(\mathbb{R}^d)} > 0$$

for all nonzero $g \in L^2(\mathbb{R}^d)$. Indeed, this implies that $C_X - C_{Y^{(\alpha)}} + R$ is a positive integral operator on $L^2(\mathbb{R}^d)$, whose kernel by Lemma 2.21 and [JSW19, Proposition 4.1 (iii)] is Hölder-continuous, and thus the corresponding Gaussian process has an almost surely Hölder-continuous version (see e.g. [AT07, Theorem 1.3.5]). In addition by Lemma 2.21 and Lemma 2.24 we see that R and $C_X - C_{Y^\alpha}$ map $H^s(\mathbb{R}^d) \to H^{s+d+\varepsilon}(\mathbb{R}^d)$ for some $\varepsilon > 0$ and all $s \in [-d, 0]$.

To show that $1+\tilde{R}_\alpha$ is positive on $L^2(\mathbb{R}^d)$ on the other hand we may write $1+\tilde{R}_\alpha = 1+\tilde{R}+(\tilde{R}_\alpha - \tilde{R})$, where $\tilde{R} = C_X^{-1/2} R C_X^{-1/2}$. By Lemma 2.23 it is enough to show that $\|\tilde{R}_\alpha - \tilde{R}\|_{L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)}$ can be made as small as we wish by choosing α small.

As $\tilde{R}_\alpha - \tilde{R}$ is self-adjoint we have

$$\|\tilde{R}_\alpha - \tilde{R}\|_{L^2(\mathbb{R}^d)\to L^2(\mathbb{R}^d)} = \sup_{u\in L^2(\mathbb{R}^d),||u||_2=1} |\langle(\tilde{R}_\alpha - \tilde{R})u, u\rangle|_{L^2(\mathbb{R}^d)}.$$

By linearity and self-adjointness of $C_X^{-1/2}$, R and $U_\alpha^{-1/2}$, we can write $\langle(\tilde{R}_\alpha - \tilde{R})u, u\rangle_{L^2(\mathbb{R}^d)}$ as

$$\langle(U_\alpha^{-1/2} - C_X^{-1/2})RC_X^{-1/2}u, u\rangle_{L^2(\mathbb{R}^d)} + \langle(U_\alpha^{-1/2} - C_X^{-1/2})RU_\alpha^{-1/2}u, u\rangle_{L^2(\mathbb{R}^d)}.$$

Now choose $\alpha_0 = \delta$ in Lemma 2.24 and observe that then for all $\alpha < \alpha_0$, the unit ball of $L^2(\mathbb{R}^d)$ under $RU_\alpha^{-1/2}$ and $RC_X^{-1/2}$ is contained in a fixed compact set of $H^{\frac{d+\delta}{2}}(\mathbb{R}^d)$. As Lemma 2.24 establishes uniform boundedness as well as pointwise convergence, we have that $U_\alpha^{-1/2} \to C_X^{-1/2}$ uniformly on this set and thus conclude the theorem. □

We finally prove the lemma:

Proof of Lemma 2.24. Note that U_α is a Fourier multiplier operator with the symbol

$$\hat{u}_\alpha(\xi) = \int_0^1 v^{d-1+\alpha}\hat{k}(v\xi)\,dv = |\xi|^{-d-\alpha}\int_0^{|\xi|} v^{d-1+\alpha}\hat{k}(v)\,dv.$$

As by assumption $\hat{k}$ is non-negative and decays faster than any polynomial, we have that

$$(1+|\xi|^2)^{-\frac{d+\alpha}{2}} \lesssim \hat{u}_\alpha(\xi) \lesssim (1+|\xi|^2)^{-\frac{d+\alpha}{2}}$$

where the hidden constant does not depend on α. In particular for every $\alpha < \alpha_0$, we have $(1+|\xi|^2)^{-\frac{d+\alpha_0}{2}} \lesssim \hat{u}_\alpha(\xi)$.

Let us now fix α_0 and consider for $\alpha < \alpha_0$ the self-adjoint operator $T_\alpha = U_\alpha^{-1/2} - C_Y^{-1/2}$ which maps $L^2(\mathbb{R}^d)$ to $H^{-\frac{d+\alpha}{2}}(\mathbb{R}^d) \subseteq H^{-\frac{d+\alpha_0}{2}}(\mathbb{R}^d)$. For any fixed $f \in L^2(\mathbb{R}^d)$ we have

$$\|T_\alpha f\|_{H^{-\frac{d+\alpha_0}{2}}(\mathbb{R}^d)} = \int (1+|\xi|^2)^{-\frac{d+\alpha_0}{2}}|\hat{u}_\alpha(\xi)^{-1/2} - \hat{K}(\xi)^{-1/2}|^2|\hat{f}(\xi)|^2\,d\xi.$$

For any fixed ξ the integrand tends to 0 as $\alpha \to 0$. Thus, as $\hat{u}_\alpha(\xi) \gtrsim (1+|\xi|^2)^{-\frac{d+\alpha_0}{2}}$ for all $\alpha < \alpha_0$, we can apply the dominated convergence theorem to deduce that $T_\alpha f \to 0$ in $H^{-\frac{d+\alpha_0}{2}}(\mathbb{R}^d)$. □

2.5 General bounds on $\det_\gamma M$ and $\delta(A)$

In this section we prove two non-standard lemmas for Malliavin calculus, that we believe could be of independent interest. Firstly, we prove a certain projection bound for the determinant of complex Malliavin variables. Second, we obtain an estimate on the complex covering fields that is again a much easier starting point for further calculations.

2.5.1 Proof of the projection bound – Proposition 2.6

Proof of Proposition 2.6. Let us first expand

$$
\begin{aligned}
&\|DF\|^2_{H_\mathbb{C}}\Big\|DF - \frac{\langle DF, D\overline{F}\rangle_{H_\mathbb{C}}}{\|DF\|^2_{H_\mathbb{C}}} D\overline{F}\Big\|^2_{H_\mathbb{C}} \\
&= \|DF\|^2_{H_\mathbb{C}}\Big(\|DF\|^2_{H_\mathbb{C}} - \frac{\overline{\left\langle DF, D\overline{F}\right\rangle_{H_\mathbb{C}}}}{\|DF\|^2_{H_\mathbb{C}}}\left\langle DF, D\overline{F}\right\rangle_{H_\mathbb{C}} \\
&\quad - \frac{\left\langle DF, D\overline{F}\right\rangle_{H_\mathbb{C}}}{\|DF\|^2_{H_\mathbb{C}}}\left\langle D\overline{F}, DF\right\rangle_{H_\mathbb{C}} + \frac{|\left\langle DF, D\overline{F}\right\rangle_{H_\mathbb{C}}|^2}{\|DF\|^4_{H_\mathbb{C}}}\|D\overline{F}\|^2_{H_\mathbb{C}}\Big) \\
&= \|DF\|^4_{H_\mathbb{C}} - |\left\langle DF, D\overline{F}\right\rangle_{H_\mathbb{C}}|^2.
\end{aligned}
$$

By (2.4), we deduce that

$$
\det \gamma_F = \frac{1}{4}\|DF\|^2_{H_\mathbb{C}}\Big\|DF - \frac{\langle DF, D\overline{F}\rangle_{H_\mathbb{C}}}{\|DF\|^2_{H_\mathbb{C}}} D\overline{F}\Big\|^2_{H_\mathbb{C}}.
$$

As we have the following projection inequality

$$
\|DF\|_{H_\mathbb{C}} \geq \Big\|DF - \frac{\langle DF, D\overline{F}\rangle_{H_\mathbb{C}}}{\|DF\|^2_{H_\mathbb{C}}} D\overline{F}\Big\|_{H_\mathbb{C}},
$$

the result follows, once we show that for any $h \in H_\mathbb{C}$,

$$
\Big\|DF - \frac{\langle DF, D\overline{F}\rangle_{H_\mathbb{C}}}{\|DF\|^2_{H_\mathbb{C}}} D\overline{F}\Big\|_{H_\mathbb{C}} \geq \frac{\big||\langle DF, h\rangle_{H_\mathbb{C}}| - |\langle D\overline{F}, h\rangle_{H_\mathbb{C}}|\big|}{\|h\|_{H_\mathbb{C}}}. \tag{2.13}
$$

By Cauchy–Schwarz inequality and the triangle inequality we have

$$
\begin{aligned}
\Big\|DF - \frac{\langle DF, D\overline{F}\rangle_{H_\mathbb{C}}}{\|DF\|^2_{H_\mathbb{C}}} D\overline{F}\Big\|_{H_\mathbb{C}} &\geq \frac{|\langle DF - \frac{\langle DF, D\overline{F}\rangle_{H_\mathbb{C}}}{\|DF\|^2_{H_\mathbb{C}}} D\overline{F}, h\rangle_{H_\mathbb{C}}|}{\|h\|_{H_\mathbb{C}}} \\
&\geq \frac{|\langle DF, h\rangle_{H_\mathbb{C}}| - \frac{|\langle DF, D\overline{F}\rangle_{H_\mathbb{C}}|}{\|DF\|^2_{H_\mathbb{C}}}|\langle D\overline{F}, h\rangle_{H_\mathbb{C}}|}{\|h\|_{H_\mathbb{C}}} \\
&\geq \frac{|\langle DF, h\rangle_{H_\mathbb{C}}| - |\langle D\overline{F}, h\rangle_{H_\mathbb{C}}|}{\|h\|_{H_\mathbb{C}}}.
\end{aligned}
$$

By now repeating the bound with $\overline{h}$ in place of h we obtain (2.13) which finishes the proof. □

2.5.2 Bounding $\delta(A)$ via derivatives in independent Gaussian directions – Proposition 2.8

For a succinct write-up, it is helpful to use directional derivatives in independent random directions, although the proposition could also be proved by first proving a version for smooth random variables and then taking limits.

Now, recall that for smooth random variables F, and $h \in H_{\mathbb{C}}$ we could write

$$\langle DF(\Gamma), h\rangle_H = \frac{d}{dt}\Big|_{t=0} F(\Gamma + th). \tag{2.14}$$

We consider directional derivatives in independent random directions, with the law of Γ. More precisely, let $X \sim \Gamma$ be an independent Gaussian field defined on a new probability space $(\Omega_X, \mathcal{F}_X, \mathbb{P}_X)$ whose expectation we denote by $\mathbb{E}_X$. For a Malliavin variable $F \in \mathbb{D}^{2,\infty}$, as $DF \in H_{\mathbb{C}}$ and X is independent of Γ, one can define

$$\mathcal{D}_X F \coloneqq \langle X, DF(\Gamma)\rangle_H. \tag{2.15}$$

The usefulness of this definition comes from the following simple lemma, which would be true on any manifold, when we would consider the directional derivatives $\mathcal{D}_X$ in directions given by the standard Gaussian on the tangent space with the norm given by the metric:

Lemma 2.25. *Let $X, Y \sim \Gamma$ be independent and $F, G \in \mathbb{D}^{1,\infty}$. We then have that $\mathbb{E}_X[\mathcal{D}_X F \cdot \overline{\mathcal{D}_X G}] = \langle DF, DG\rangle_{H_{\mathbb{C}}}$.*

We are now ready to prove Proposition 2.8.

Proof of Proposition 2.8. Write $\Delta := 4 \det \gamma_F = \|DF\|_{H_{\mathbb{C}}}^4 - |\langle DF, D\overline{F}\rangle_{H_{\mathbb{C}}}|^2$. Then by the integration by parts rule for the divergence operator δ (e.g. [Nua06, Proposition 1.3.3]), $\delta(A)$ equals

$$\frac{\|DF\|_{H_{\mathbb{C}}}^2 \delta(DF) - \langle DF, D\overline{F}\rangle_{H_{\mathbb{C}}} \delta(D\overline{F})}{\Delta} - \langle D\frac{\|DF\|_{H_{\mathbb{C}}}^2}{\Delta}, D\overline{F}\rangle_{H_{\mathbb{C}}} + \langle D\frac{\langle DF, D\overline{F}\rangle_{H_{\mathbb{C}}}}{\Delta}, DF\rangle_{H_{\mathbb{C}}}.$$

The first term is $\lesssim \Delta^{-1}\|DF\|_{H_{\mathbb{C}}}^2 |\delta(DF)|$ in absolute value, so it is enough to consider the other two terms. By the product rule for Malliavin derivatives, we may write

$$\langle D\frac{\langle DF, D\overline{F}\rangle_{H_{\mathbb{C}}}}{\Delta}, DF\rangle_{H_{\mathbb{C}}} - \langle D\frac{\|DF\|_{H_{\mathbb{C}}}^2}{\Delta}, D\overline{F}\rangle_{H_{\mathbb{C}}}$$

as

$$\begin{aligned} &= \Delta^{-1}\left(\langle D\langle DF, D\overline{F}\rangle_{H_{\mathbb{C}}}, DF\rangle_{H_{\mathbb{C}}} - \langle D\|DF\|_{H_{\mathbb{C}}}^2, D\overline{F}\rangle_{H_{\mathbb{C}}}\right) - \\ &- \Delta^{-2}\left(\langle DF, D\overline{F}\rangle_{H_{\mathbb{C}}}\langle D\Delta, DF\rangle_{H_{\mathbb{C}}} - \|DF\|_{H_{\mathbb{C}}}^2\langle D\Delta, D\overline{F}\rangle_{H_{\mathbb{C}}}\right) \end{aligned}$$

To bound the first term, we first notice that by Cauchy–Schwarz

$$\langle D\langle DF, D\overline{F}\rangle_{H_{\mathbb{C}}}, DF\rangle_{H_{\mathbb{C}}} \le \|D\langle DF, D\overline{F}\rangle_{H_{\mathbb{C}}}\|\|DF\|_{H_{\mathbb{C}}}.$$

For the first term, it is now helpful to use the averaging for a quick bound. We write

$$\|D\langle DF, D\overline{F}\rangle_{H_{\mathbb{C}}}\|_{H_{\mathbb{C}}} = 2|\mathbb{E}_{X,Y}\mathcal{D}_Y F \cdot \mathcal{D}_X\mathcal{D}_Y F|.$$

By Cauchy–Schwarz this can be bounded by

$$2\sqrt{\mathbb{E}_{X,Y}|\mathcal{D}_Y F|^2}\sqrt{\mathbb{E}_{X,Y}|\mathcal{D}_X\mathcal{D}_Y F|^2} = 2\|DF\|_{H_{\mathbb{C}}}\|D^2 F\|_{H_{\mathbb{C}}\otimes H_{\mathbb{C}}}.$$

Similarly, one can bound

$$\langle D\|DF\|^2_{H_\mathbb{C}}, D\overline{F}\rangle_{H_\mathbb{C}} \leq 2\|DF\|_{H_\mathbb{C}}\|D^2F\|_{H_\mathbb{C}\otimes H_\mathbb{C}},$$

and thus

$$\Delta^{-1}\left(\langle D\langle DF, D\overline{F}\rangle_{H_\mathbb{C}}, DF\rangle_{H_\mathbb{C}} - \langle D\|DF\|^2_{H_\mathbb{C}}, D\overline{F}\rangle_{H_\mathbb{C}}\right) \leq 4\frac{\|DF\|^2_{H_\mathbb{C}}\|D^2F\|_{H_\mathbb{C}\otimes H_\mathbb{C}}}{\Delta}.$$

It remains to handle

$$\Delta^{-2}\left(\langle DF, D\overline{F}\rangle_{H_\mathbb{C}}\langle D\Delta, DF\rangle_{H_\mathbb{C}} - \|DF\|^2_{H_\mathbb{C}}\langle D\Delta, D\overline{F}\rangle_{H_\mathbb{C}}\right),$$

which we can rewrite as

$$\Delta^{-2}\langle D\Delta, \langle D\overline{F}, DF\rangle_{H_\mathbb{C}}DF - \|DF\|^2_{H_\mathbb{C}}D\overline{F}\rangle_{H_\mathbb{C}}.$$

Notice that

$$\|DF\|^2_{H_\mathbb{C}}\Delta = \|\langle D\overline{F}, DF\rangle_{H_\mathbb{C}}DF - \|DF\|^2_{H_\mathbb{C}}D\overline{F}\|^2_{H_\mathbb{C}}, \tag{2.16}$$

and thus by Cauchy–Schwarz we can bound the expression just above by

$$\Delta^{-3/2}\|D\Delta\|_{H_\mathbb{C}}\|DF\|_{H_\mathbb{C}}.$$

Thus the proposition follows from the following claim:

Claim 2.26. *We have that* $\|D\Delta\|_{H_\mathbb{C}} \lesssim \Delta^{1/2}\|DF\|_{H_\mathbb{C}}\|D^2F\|_{H_\mathbb{C}\otimes H_\mathbb{C}}$.

Proof of claim. Maybe the nicest way to prove this claim is to use derivatives in random directions as above. First, observe that using averaging we can write a neat analogue of Equation (2.16) :

$$\Delta = \frac{1}{2}\mathbb{E}_{Z,W}|\mathcal{D}_ZF\cdot\mathcal{D}_W\overline{F} - \mathcal{D}_Z\overline{F}\cdot\mathcal{D}_WF|^2.$$

Thus we have

$$\mathcal{D}_X\Delta = \operatorname{Re}\mathbb{E}_{Z,W}(\mathcal{D}_ZF\cdot\mathcal{D}_W\overline{F} - \mathcal{D}_Z\overline{F}\cdot\mathcal{D}_WF)\mathcal{D}_X(\mathcal{D}_ZF\cdot\mathcal{D}_W\overline{F} - \mathcal{D}_Z\overline{F}\cdot\mathcal{D}_WF).$$

By triangle inequality and Cauchy–Schwarz we obtain

$$|\mathcal{D}_X\Delta|^2 \lesssim \Delta\mathbb{E}_{Z,W}|\mathcal{D}_X(\mathcal{D}_ZF\cdot\mathcal{D}_W\overline{F})|^2$$

and hence

$$\|D\Delta\|^2_{H_\mathbb{C}} = \mathbb{E}_X|\mathcal{D}_X\Delta|^2 \lesssim \Delta\|DF\|^2_{H_\mathbb{C}}\|D^2F\|^2_{H_\mathbb{C}\otimes H_\mathbb{C}},$$

from which the claim follows. □

□

2.6 Estimates for Malliavin variables in the case of imaginary chaos

The aim of this section is to prove the probabilistic bounds needed to apply the tools of Malliavin calculus to $M = \mu(f)$. We start by going through some old and new Onsager inequalities and related integral bounds. In Section 2.6.2, we prove by a rather standard argument that M is in $\mathbb{D}^\infty$, i.e. Proposition 2.4. In Section 2.6.3 we derive bounds on $|\delta(DM)|$ and $\|D^2 M\|_{H_\mathbb{C} \otimes H_\mathbb{C}}$ and deduce Proposition 2.11 by a quite similar argument.

Finally, in Section 2.6.4 we prove bounds on the Malliavin determinant of M and this is the main technical input of the paper. Here things get quite interesting – we rely both on the decomposition theorem, Theorem 2.16, and projection bounds for Mallivan determinants from Section 2.5, but also need to find ways to get a good grip on the concentration of $M = \mu(f)$, and on Sobolev norms of the imaginary chaos μ itself.

2.6.1 Onsager inequalities and related bounds

In this section, we collect a few Onsager inequalities and related bounds. To this end, we define for any Gaussian field Γ and $\mathbf{x} = (x_1, \dots, x_N), \mathbf{y} = (y_1, \dots, y_M)$ the quantity

$$\mathcal{E}(\Gamma; \mathbf{x}; \mathbf{y}) = -\sum_{1 \le j < k \le N} \mathbb{E}\Gamma(x_j)\Gamma(x_k) - \sum_{1 \le j < k \le M} \mathbb{E}\Gamma(y_j)\Gamma(y_k) + \sum_{\substack{1 \le j \le N \\ 1 \le k \le M}} \mathbb{E}\Gamma(x_j)\Gamma(y_k).$$

Also, we let $\Gamma_\delta = \Gamma * \varphi_\delta$ be a mollification of Γ where $\varphi_\delta = \delta^{-d}\varphi(\cdot/\delta)$ and φ is a smooth non-negative function with compact support that satisfies $\int_{\mathbb{R}^d} \varphi = 1$.

The following is a restatement of a standard Onsager inequality from [JSW20].[4]

Lemma 2.27 (Proposition 3.6(ii) of [JSW20]). *Let K be a compact subset of U or the circle $K = S^1$. There exists $C = C(K) > 0$ such that the following holds true: Let $N \ge 1, \delta > 0$ and for all $i = 1 \dots N$ let $x_i, y_i \in K$ be such that $D(x_i, \delta)$ and $D(y_i, \delta)$ are included in K. For all $i = 1 \dots N$, denote $z_i := x_i$ and $z_{N+i} := y_i$ and set $d_j := \min_{k \ne j} |z_k - z_j|$. Then*

$$\mathcal{E}(\Gamma_\delta; \mathbf{x}; \mathbf{y}) \le \frac{1}{2} \sum_{j=1}^{2N} \log \frac{1}{d_j} + CN^2. \tag{2.17}$$

Moreover, the same holds for the field Γ itself.

We will also need stronger Onsager inequalities for (almost) $\star$-scale invariant fields, whose rather standard proof is pushed to the appendix 2.A.

Lemma 2.28. *Let Y_ε and $\hat{Y}_\varepsilon$ be defined as in Section 2.4.1 and let $\mathbf{x} = (x_1, \dots, x_N)$ and $\mathbf{y} = (y_1, \dots, y_N)$ be two N-tuples of points in U. For all $j = 1, \dots, N$, denote $z_j := x_j$ and $z_{N+j} = y_j$ and set $d_j := \min_{k \ne j} |z_k - z_j|$. Then*

$$\mathcal{E}(Y_\varepsilon; \mathbf{x}; \mathbf{y}) \le \frac{1}{2} \sum_{j=1}^{2N} \log \frac{1}{d_j \vee \varepsilon}$$

[4] In fact, the cited result does not contain the case of the circle, however essentially the same proof works.

and

$$\mathcal{E}(\hat{Y}_\varepsilon(\varepsilon\cdot);\mathbf{x};\mathbf{y}) \leq \frac{1}{2}\sum_{j=1}^{2N}\log\frac{1}{d_j}. \tag{2.18}$$

Moreover, if R is a Gaussian field such that $M := \sup_{x\in U}\mathbb{E}[R(x)^2] < \infty$, then

$$\mathcal{E}(R;\mathbf{x};\mathbf{y}) \leq NM. \tag{2.19}$$

Both of these Onsager inequalities are used in conjunction with the following bounds:

Lemma 2.29. *For $N \geq 2$, there exists $C > 0$ such that*

- *for all $\beta \in (0,\sqrt{d})$,*

$$\int_{B(0,1)^N}\prod_{i=1}^N\left(\min_{j\neq i}|z_i - z_j|\right)^{-\beta^2/2} dz_1\dots dz_N \leq C^N(d-\beta^2)^{-\lfloor N/2\rfloor}N^{\frac{N\beta^2}{2d}}; \tag{2.20}$$

- *for all $\beta \in (0,\sqrt{d})$,*

$$\int_{B(0,1)^N}\prod_{i=1}^N\left|\log\min_{j\neq i}|z_i - z_j|\right|^{1/2}\left(\min_{j\neq i}|z_i - z_j|\right)^{-\beta^2/2} dz_1\dots dz_N \leq C^N(d-\beta^2)^{-2\lfloor N/2\rfloor}N^N; \tag{2.21}$$

- *for all $\beta \in (0,\sqrt{d})$,*

$$\int_{B(0,1)^N}\prod_{i=1}^N\left|\log\min_{j\neq i}|z_i - z_j|\right|\left(\min_{j\neq i}|z_i - z_j|\right)^{-\beta^2/2} dz_1\dots dz_N \leq C^N(d-\beta^2)^{-3\lfloor N/2\rfloor}N^N; \tag{2.22}$$

- *for all $\beta > 0$,*

$$\int_{B(0,1)^N}\left(\prod_{i=1}^N\min_{j\neq i}\max(\delta,|z_i - z_j|)\right)^{-\beta^2/2} dz_1\dots dz_N \leq C^N N^N(\log\frac{1}{\delta})^{N/2}\delta^{-\max(0,\beta^2-d)N/2}; \tag{2.23}$$

Proof. We only sketch the proof, as all the main ideas can be found in proof of [JSW20, Lemma 3.10].

Let us start with showing (2.20). By carefully following the proof of [JSW20, Lemma 3.10] which shows that (2.20) is less than $c^{2\lfloor N/2\rfloor}N^{\frac{N\beta^2}{2d}}$, one can actually see that the constant c there can be taken to be equal to $c'(d-\beta^2)^{-1/2}$ for some constant $c' > 0$ independent of β (at one point in the proof there is a term of order $(d-\beta^2)^{-k}$ coming from $\Gamma(1-\frac{d}{\beta^2})^k$ where $k \leq \lfloor N/2\rfloor$).

We will next show (2.23). By mimicking the beginning of the proof of [JSW20, Lemma 3.10], we can bound the left hand side of (2.23) by

$$C^N\sum_{k=1}^{\lfloor N/2\rfloor}\sum_F\int_{B(0,1)^N}\prod_{i=1}^k(\delta\vee|u_{2i-1}|)^{-\beta^2}\prod_{i=2k+1}^N(\delta\vee|u_i|)^{-\beta^2/2}du_1\dots du_N$$

where $C > 0$ and the second sum runs over all nearest neighbour configurations F such that the induced graph with vertices $\{1, \dots, N\}$ and edges $(i, F(i))$ has k components. Of course, the domain on which we integrate is actually much smaller than $B(0,1)$, but integrating over this larger domain will be enough for our purposes. After integration, we obtain that the left hand side of (2.23) is at most

$$C^N \sum_{k=1}^{\lfloor N/2 \rfloor} \sum_F A_{\beta^2}^k A_{\beta^2/2}^{N-2k} \le C^N N^N \sum_{k=1}^{\lfloor N/2 \rfloor} A_{\beta^2}^k A_{\beta^2/2}^{N-2k},$$

where

$$A_{\beta^2} := \int_0^1 r^{d-1} (\delta \vee r)^{-\beta^2} \, dr.$$

Now, by Jensen's inequality $A_{\beta^2/2}^2 \le d^{-1} A_{\beta^2}$, giving us the bound $C^N N^N A_{\beta^2}^{N/2}$. Noting that

$$A_{\beta^2} \lesssim \log \frac{1}{\delta} \delta^{-\max(0, \beta^2 - d)}$$

concludes the proof of (2.23).

We finally turn to the proof of (2.21) and (2.22). By again mimicking the beginning of the proof of [JSW20, Lemma 3.10], we can bound the left hand side of (2.21) by

$$\begin{aligned} &C^N \sum_{k=1}^{\lfloor N/2 \rfloor} M_k \int_{B(0,1)^N} \prod_{i=1}^k |u_{2i-1}|^{-\beta^2} |\log |u_{2i-1}|| \prod_{i=2k+1}^N |u_i|^{-\beta^2/2} |\log |u_i||^{1/2} \\ &\le C^N \sum_{k=1}^{\lfloor N/2 \rfloor} M_k \left(\int_0^1 r^{-\beta^2+d-1} |\log r| dr \right)^k \le C^N \sum_{k=1}^{\lfloor N/2 \rfloor} M_k (d-\beta^2)^{-2k} \le C^N (d-\beta^2)^{-2\lfloor N/2 \rfloor} N^N, \end{aligned}$$

where M_k is the number of nearest neighbour functions $\{1, \dots, N\} \to \{1, \dots, N\}$ with k components and C is some large enough constant. This concludes the proof of (2.21); the proof of (2.22) is similar. □

2.6.2 M belongs to $\mathbb{D}^\infty$ – proof of Proposition 2.4

The purpose of this section is to prove Proposition 2.4. Before doing so, we collect two auxiliary lemmas from Malliavin calculus.

Lemma 2.30 ([Nua06, Lemma 1.2.3])**.** *Let $(F_n, n \ge 1)$ be a sequence of (complex) random variables in $\mathbb{D}^{1,2}$ that converges to F in $L^2(\Omega)$ and such that $\sup_n \mathbb{E}\left[\|DF_n\|_{H_\mathbb{C}}^2 \right] < \infty$. Then F belongs to $\mathbb{D}^{1,2}$ and the sequence of derivatives $(DF_n, n \ge 1)$ converges to DF in the weak topology of $L^2(\Omega; H_\mathbb{C})$.*

Second, we need a rather direct consequence of [Nua06, Lemma 1.5.3]:

Lemma 2.31. *Let $p > 1$, $k \ge 1$ and let $(F_n, n \ge 1)$ be a sequence of (complex) random variables converging to F in $L^p(\Omega)$. Suppose that $\sup_n \|F_n\|_{k,p} < \infty$. Then F belongs to $\mathbb{D}^{k,p}$ and $\|F\|_{k,p} \le C_{k,p} \limsup_n \|F_n\|_{k,p}$ for some $C_{k,p} > 0$.*

Proof of Lemma 2.31. See Appendix 2.A. □

We now have the ingredients needed to prove Proposition 2.4. The proof of this result is rather standard, but needs a bit of care as the most convenient way of obtaining Malliavin smooth random variables is truncating the Karhunen–Loève expansion of Γ. Doing so we face the issue that there is no Onsager inequality available for this approximation of the field that we are aware of. We will bypass this difficulty by considering a further convolution of this truncated version of Γ against a smooth mollifier φ and then use the Onsager inequality (2.17) for convolution approximations.

Proof of Proposition 2.4. Here, we sketch the proof and give full details in the Appendix 2.B. We start by showing that M belongs to $\mathbb{D}^\infty$. Let $n \geq 1, \delta > 0, j \geq 0$ and $p \geq 1$. In the following, we will denote

$$\Gamma_\delta = \Gamma * \varphi_\delta, \quad \Gamma_{n,\delta} = \sum_{k=1}^n A_k e_k * \varphi_\delta, \quad M_\delta = \int_{\mathbb{C}} f(x) e^{i\beta\Gamma_\delta(x) + \frac{\beta^2}{2}\mathbb{E}[\Gamma_\delta(x)^2]} dx$$

and

$$M_{n,\delta} = \int_{\mathbb{C}} f(x) e^{i\beta\Gamma_{n,\delta}(x) + \frac{\beta^2}{2}\mathbb{E}[\Gamma_{n,\delta}(x)^2]} dx.$$

$M_{n,\delta}$ is a smooth random variable (in the sense of Definition 2.3) and $D^j M_{n,\delta}$ is equal to

$$(i\beta)^j \int_{\mathbb{C}} dx f(x) e^{i\beta\Gamma_{n,\delta}(x) + \frac{\beta^2}{2}\mathbb{E}[\Gamma_{n,\delta}(x)^2]} \sum_{k_1,\dots,k_j=1}^n (e_{k_1} * \varphi_\delta)(x) \dots (e_{k_j} * \varphi_\delta)(x) e_{k_1} \otimes \dots \otimes e_{k_j}.$$

Combining Onsager inequalities, (2.20) and Lemma 2.31, one can show by taking the limit $n \to \infty$ that for all $k \geq 1$, $M_\delta \in \mathbb{D}^{k,2p}$ and that

$$\sup_{\delta > 0} \|M_\delta\|_{k,2p} < \infty.$$

Details of this are in the appendix. Now, because $(M_\delta, \delta > 0)$ converges in L^{2p} towards M, Lemma 2.31 then implies that for all $k \geq 1$, $M \in \mathbb{D}^{k,2p}$. This concludes the proof that $M \in \mathbb{D}^\infty$.

The proof of the formula for DM now follows via a series of approximation arguments. From the first part by taking $n \to \infty$, one can rather quickly deduce that

$$DM_\delta = i\beta \int_{\mathbb{C}} dx f(x) e^{i\beta\Gamma_\delta(x) + \frac{\beta^2}{2}\mathbb{E}[\Gamma_\delta(x)^2]} \sum_{k=1}^\infty (e_k * \varphi_\delta)(x) e_k.$$

Next, one argues that $(DM_\delta, \delta > 0)$ converges in $L^2(\Omega; H)$ towards

$$i\beta \int_{\mathbb{C}} dx f(x) \mu(x) C(x, \cdot)$$

and concludes that it necessarily corresponds to DM by Lemma 2.30. Here one again uses Onsager inequalities and dominated convergence. The full details are found in the appendix. □

2.6.3 Bounds on $|\delta(DM)|$ and $\|D^2M\|_{H_{\mathbb{C}}\otimes H_{\mathbb{C}}}$ – proof of Proposition 2.11

The goal of this section is to control the tails of $|\delta(DM)|$ and $\|D^2M\|_{H_{\mathbb{C}}\otimes H_{\mathbb{C}}}$. We first note that these two random variables can be written explicitly in terms of imaginary chaos.

Lemma 2.32. *Let $f \in L^\infty(\mathbb{C})$. Then*

$$\delta(DM) = \beta \int_{\mathbb{C}} f(x) \frac{d}{d\beta}\mu(x)dx, \tag{2.24}$$

$$\|D^2M\|^2_{H_{\mathbb{C}}\otimes H_{\mathbb{C}}} = \beta^4 \operatorname{Re} \int_{\mathbb{C}\times\mathbb{C}} f(x)f(y)\mu(x)\overline{\mu(y)}C(x,y)^2 dxdy, \tag{2.25}$$

where the expression $\frac{d}{d\beta}\mu(x)$ is given sense by $\lim_{\delta\to 0}\left(i\Gamma_\delta(x) + \beta\mathbb{E}\Gamma_\delta^2(x)\right) : \exp(i\beta\Gamma_\delta(x)) :$ with the limit, say, in $H^{-d}(U)$ and in probability.

The proof of (2.25) is very similar to the proof of the formula of DM and we omit the details. The origin of (2.24) can be explained by the following formal computation, that can be turned into a rigorous proof in a very similar manner as what we did in the proof of Proposition 2.4 when we obtained the explicit expression of DM – one needs to use smooth approximations both for the field Γ, and smooth Malliavin variables.

'Formal' proof of Lemma 2.32. By Proposition 2.4, and then by integration by parts for δ (Proposition 1.3.3 of [Nua06]), we have

$$\begin{aligned}\delta(DM) &= i\beta \int_{\mathbb{C}} f(x)\delta(\mu(x)C(x,\cdot))dx \\ &= i\beta \int_{\mathbb{C}} f(x)\left(\mu(x)\delta(C(x,\cdot)) - \langle D\mu(x), C(x,\cdot)\rangle_{H_{\mathbb{C}}}\right)dx.\end{aligned}$$

Noticing that $\delta(C(x,\cdot)) = \Gamma(x)$ (see (1.44) of [Nua06]) and that by Proposition 2.4 $\langle D\mu(x), C(x,\cdot)\rangle_{H_{\mathbb{C}}} = i\beta\mu(x)C(x,x)$, we obtain

$$\delta(DM) = \beta \int_{\mathbb{C}} f(x)\mu(x)(i\Gamma(x) + \beta C(x,x))dx = \beta \int_{\mathbb{C}} f(x)\frac{d}{d\beta}\mu(x)dx.$$

This shows (2.24). □

Proof of Proposition 2.11. We will only write the details for the variable $\delta(DM)$ since bounding the moments of $\|D^2M\|_{H_{\mathbb{C}}\otimes H_{\mathbb{C}}}$ is very similar to bounding the moments of imaginary chaos itself (with the use of (2.22) instead of (2.20)).

Let $N \geq 1$ and let $K \Subset U$ be the support of f. By Lemma 2.32 we have

$$\mathbb{E}[|\delta(DM)|^{2N}] \leq \|f\|_\infty^{2N}\beta^{2N} \int_{K^{2N}} \left|\mathbb{E}\Big[\prod_{j=1}^N \frac{d}{d\beta}\mu(x_j)\frac{d}{d\beta}\overline{\mu(y_j)}\Big]\right| dx_1\dots dx_N dy_1 \dots dy_N.$$

By a limiting argument, one can justify the formal identity:

$$\mathbb{E}\Big[\prod_{j=1}^N \frac{d}{d\beta}\mu(x_j)\frac{d}{d\beta}\overline{\mu(y_j)}\Big] = \Big[\prod_{\ell=1}^N \frac{d}{d\beta_\ell}\frac{d}{d\gamma_\ell}E((\beta_j)_{j=1}^N, (\gamma_j)_{j=1}^N)\Big]_{\beta_1=\dots=\beta_N=\gamma_1=\dots\gamma_N=\beta}.$$

where

$$E((\beta_j)_{j=1}^N, (\gamma_j)_{j=1}^N) := e^{-\sum_{j<k} \beta_j \beta_k C(x_j,x_k) - \sum_{j<k} \gamma_j \gamma_k C(y_j,y_k) + \sum_{j,k} \beta_j \gamma_k C(x_j,y_k)}.$$

Let $(z_1, \dots, z_{2N}) := (x_1, \dots, x_N, y_1, \dots, y_N)$. By induction one sees that after differentiating w.r.t. the first k of the variables $\beta_1, \dots, \beta_N, \gamma_1, \dots, \gamma_N$ and expanding one is left with a finite number of terms of the form

$$\pm \prod_{j=1}^N \beta_j^{n_j} \gamma_j^{m_j} \prod_{j=1}^{\ell} C(z_{a_j}, z_{b_j}) E((\beta_j)_{j=1}^N, (\gamma_j)_{j=1}^N),$$

where $0 \le n_j, m_j, \ell \le k$, $1 \le a_1 < a_2 < \dots < a_\ell \le k$ and $1 \le b_1, \dots, b_\ell \le 2N$ with $a_j \ne b_j$ for all j. Hence we have

$$\mathbb{E}[|\delta(DM)|^{2N}] \le C_N \sum_{\ell=1}^{2N} \sum_{1 \le a_1 < \dots < a_\ell \le 2N} \sum_{b_1,\dots,b_\ell=1}^{2N} \int_{K^{2N}} \prod_{j=1}^{\ell} \mathbf{1}_{a_j \ne b_j} |C(z_{a_j}, z_{b_j})| e^{\mathcal{E}(\Gamma;\mathbf{x};\mathbf{y})} \, dx_j dy_j.$$

Note that $|C(z_{a_j}, z_{b_j}| \le C \log \frac{4R}{|z_{a_j} - z_{b_j}|}$ for large enough $C > 0$ and R so large that $K \subset B(0,R)$. Thus applying Lemma 2.27 to each summand, we can bound the whole sum by

$$C_N \int_{K^{2N}} \prod_{j=1}^{2N} \log \frac{4R}{\min_{k \ne j} |z_j - z_k|} (\min_{k \ne j} |z_j - z_k|)^{-\beta^2/2} \, dz_1 \dots dz_{2N}.$$

By scaling this is less than

$$C_N \int_{B(0,1/4)^{2N}} \prod_{j=1}^{2N} \log \frac{1}{\min_{k \ne j} |z_j - z_k|} (\min_{k \ne j} |z_j - z_k|)^{-\beta^2/2} \, dz_1 \dots dz_{2N},$$

which by Lemma 2.29 is less than $C_N (d - \beta^2)^{3N}$. □

2.6.4 Small ball probabilities for the Malliavin determinant of M – proof of Proposition 2.10

This section contains the main probabilistic input to Theorem 2.9 – the proof of Proposition 2.10. Roughly, the content of this proposition is to establish super-polynomial decay of $\mathbb{P}(\det \gamma_M < \varepsilon)$ as $\varepsilon \to 0$, where $\det \gamma_M := (\|DM\|_{H_{\mathbb{C}}}^4 - |\langle DM, D\overline{M} \rangle_{H_{\mathbb{C}}}|^2)/4$ is the Malliavin determinant of $M = \mu(f)$.

We will start by presenting a toy model explaining the strategy; then we explain the proof setup and prove the proposition modulo some technical chaos lemmas. The section finishes by proving the technical estimates.

2.6.4.1 A toy model: small ball probabilities for $\| : \exp(i\beta \mathrm{GFF}) : \|_{H^{-1}(\mathbb{R}^2)}$

To explain the strategy of our proof, we consider a toy problem asking about the small ball probabilities for norms of imaginary chaos. For concreteness, let us do it here with the 2D Gaussian free field; see Proposition 2.33 at the end of this section for a more general statement.

Consider the 2D zero boundary GFF on $K = [0,1]^2$ and the imaginary chaos μ_β. We know that as a generalized function $\mu_\beta \in H^{-1}(K)$ for all $\beta \in (0, \sqrt{2})$. Can we prove super-polynomial bounds for $\mathbb{P}\left(\|\mu\|_{H^{-1}(K)} < \varepsilon\right)$? Moreover, can we obtain bounds that are tight as $\beta \to \sqrt{2}$?

Writing out the norm squared, we have that

$$\|\mu\|^2_{H^{-1}(K)} = \int_{K^2} \mu(x)G(x,y)\overline{\mu(y)}\, dx\, dy > 0,$$

where G is the Dirichlet Green's function on K. Now, the expectation $\mathbb{E}\|\mu\|^2_{H^{-1}(K)}$ is easy to calculate and it is bounded. As all moments exist, one could imagine proving bounds near zero by using concentration results on μ. However, these concentration results do not see the special role of zero and would not suffice for good enough bounds for asymptotics near 0.

The idea is then to use only the decorrelated high-frequency part of Γ to stay away from zero. To make this more precise, denote by Γ_δ the part of the GFF containing only frequencies less than δ^{-1} and let $\hat{\Gamma}_\delta$ denote the tail of the GFF. Consider now the projection bound $\|f_\delta\|_{H^{-1}(K)}\|\mu\|_{H^{-1}(K)} \geq \langle \mu, f_\delta\rangle_{H^{-1}(K)}$. Setting $f_\delta(x) = \Delta(: e^{i\beta\Gamma_\delta(x)} :)$, we get that

$$\|\mu\|_{H^{-1}(K)} \geq \frac{\int_K : e^{-i\beta\hat{\Gamma}_\delta(x)} :\, dx}{\|f_\delta\|_{H^{-1}(K)}}.$$

A small calculation shows that $\|f_\delta\|_{H^{-1}(K)} = \| : e^{i\beta\Gamma_\delta(y)} : \|_{H^1(K)}$. It is further believable that we should have $\| : e^{i\beta\Gamma_\delta(y)} : \|_{H^1(K)} \asymp \delta^{-\beta^2/2}\|\Gamma_\delta\|_{H^1(K)}$, and that this expression admits Gaussian concentration. As in the concrete case $\mathbb{E}\|\Gamma_\delta\|_{H^1(K)} \asymp \delta^{-1}$, we can conclude that the denominator is of order $\delta^{-1-\beta^2/2}$ with super-polynomial concentration on fluctuations.

In the numerator, the term of the form $\int_K : e^{-i\beta\hat{\Gamma}_\delta(x)} : dx$ remains. Such a tail chaos is very highly concentrated around 1, with fluctuations of unit order having a super-polynomial cost in δ. Thus the whole ratio will concentrate around

$$C\frac{\int_K : e^{-i\beta\hat{\Gamma}_\delta(x)} : dx}{\delta^{-1-\beta^2/2}} \sim C\delta^{1+\beta^2/2},$$

with super-polynomial cost for fluctuations on the same scale. Thus setting $\varepsilon = \delta^{1+\beta^2/2}$ we obtain super-polynomial decay for $\mathbb{P}\left(\|\mu\|_{H^{-1}(K)} < \varepsilon\right)$.

Whereas this is good enough for any fixed β, observe that as $\beta \to \sqrt{2}$, we have $\mathbb{E}\|\mu\|^2_{H^{-1}(K)} = O((2-\beta^2)^{-2})$, but $\mathbb{E}|\langle \mu, f_\delta\rangle_{H^{-1}(K)}|^2 = \mathbb{E}|\int : e^{-i\beta\hat{\Gamma}_\delta(x)} : |^2 = O((2-\beta^2)^{-1})$. As further $\|f_\delta\|_{H^{-1}(K)} \asymp \delta^{-\beta^2/2}\|\Gamma_\delta\|_{H^1(K)}$ and $\|\Gamma_\delta\|_{H^1(K)}$ does not depend on β, we see that we are losing in terms of $\beta^2 - 2$.

Illustratively, we are losing in high frequencies because we are replacing

$$\int \mu(x)G(x,y)\overline{\mu(y)} \qquad \text{by} \qquad \int : e^{i\beta\hat{\Gamma}_\delta(x)} :: e^{-i\beta\hat{\Gamma}_\delta(y)} : .$$

After taking expectation, in terms of near-diagonal contributions, as $G(x,y) \sim -\log|x-y|$ near the diagonal, this basically translates to replacing $-\int |x|^{-\beta^2/2}\log|x|$ with $\int |x|^{-\beta^2/2}$ and results in the loss of a factor of $2-\beta^2$ as $\beta^2 \to 2$. Thus we have to tweak our test function f_δ further to at the same time guarantee sufficient concentration and not to lose too much on tails.

We will see later on that this strategy gives us more generally the following result.

Proposition 2.33. *Let $f \in C_c^\infty(U)$. Then for each $\nu \in (0, \sqrt{d})$, there exist constants $c_1, c_2, c_3 > 0$ such that*

$$\mathbb{P}[\|f\mu\|_{H^{-d/2}(\mathbb{R}^d)} \leq (d-\beta^2)^{-2}\lambda] \leq c_1 e^{-c_2\lambda^{-c_3}}$$

for all $\lambda > 0$ and all $\beta \in (\nu, \sqrt{d})$.

The same strategy for the determinant requires some extra input, yet the key ideas are present already in this toy model: the projection bound corresponds to the analogue of Malliavin determinants given by Lemma 2.6, the concentration of the numerator to Lemma 2.34 and that of the denominator to Lemma 2.35. The only new technical ingredient will enter as Lemma 2.36.

2.6.4.2 Proof setup and proof of Proposition 2.10 modulo technical lemmas

Let f be a bounded continuous function whose support is a compact subset of U and set $M = \mu(f)$. Our goal in this section is to obtain lower bounds on $\mathbb{P}[\det \gamma_M \geq \lambda]$, where $\det \gamma_M$ is the Malliavin determinant (2.4).

As in the toy problem, it is not so clear how to obtain sharp bounds directly and the idea is to use the projection bound from Lemma 2.6, which says that

$$\mathbb{P}[\det \gamma_M \geq \lambda] \geq \mathbb{P}\Big[\frac{(|\langle DM, h\rangle_{H_\mathbb{C}}| - |\langle D\overline{M}, h\rangle_{H_\mathbb{C}}|)^4}{\|h\|_{H_\mathbb{C}}^4} \geq 4\lambda\Big] \tag{2.26}$$

for any $h \in H_\mathbb{C}$. A key step is the specific choice of $h(x)$, which needs to at the same time give a precise enough bound and allow for chaos computations. Moreover, we have to ensure that it also belongs to the Cameron–Martin space. Here, one of the technical difficulties is that in general we do not have a good understanding of the Cameron–Martin space of Γ. To deal with that, we will use the decomposition theorem, Theorem 2.16 to be able to work with almost $\star$-scale invariant fields.

More precisely, let us fix an open set V with $\overline{V}$ a compact subset of U such that $\operatorname{supp} f \subset V$. Then by Theorem 2.16 one can write $\Gamma|_V = Y + Z =: X$ where Y is an almost $\star$-scale invariant field with smooth and compactly supported seed covariance k and parameter α, and Z is an independent Hölder-continuous field. Recall further the approximations Y_ε of Y of such a field from Section 2.4.1 and the notation for its tail field $\hat{Y}_\varepsilon := Y - Y_\varepsilon$.

Now, notice that

$$\det \gamma_M = \frac{\beta^4}{4}\Big(\Big|\int f(x)f(y)\mu(x)\overline{\mu(y)}C(x,y)\,dx\,dy\Big|^2 - \Big|\int f(x)f(y)\mu(x)\mu(y)C(x,y)\,dx\,dy\Big|^2\Big),$$

where the right hand side only depends on μ, and thus on Γ, restricted to V. Thus, to obtain bounds on $\det \gamma_M$, we can instead of working with the (complexified) Cameron–Martin space $H_\mathbb{C} = H_{\Gamma,\mathbb{C}}$, just as well work with the Cameron–Martin space of $Y + Z$, which is defined on the whole plane. Apologising for the abuse of notation, we still denote it by $H_\mathbb{C}$. This small trick allows us to use the independence structure of the field Y, and also puts Fourier techniques in our hand.

Definition of h. Whereas the decomposition theorem and the change of Cameron–Martin space make the computations potentially doable, they become practically doable only with a very careful choice of the test function h. Namely, we set

$$h(x) = h_\delta(x) = e^{i\beta Y_\delta(x) - \frac{\beta^2}{2}\mathbb{E}[Y_\delta(x)^2]} \int f(y) : e^{i\beta Z(y)} :: e^{i\beta \hat{Y}_\delta(y)} : R_\delta(x,y)\, dy,$$

where $R_\delta(x,y) = g_\delta(x) g_\delta(y) \mathbb{E}[\hat{Y}_\delta(x)\hat{Y}_\delta(y)]$ is defined using a smooth indicator g_δ of δ-separated squares and the parameter δ will be chosen in a suitable way according to λ.

More precisely, let $\mathcal{Q}_\delta$ be the collection of cubes of the form

$$[4k_1\delta, (4k_1+1)\delta] \times \cdots \times [4k_d\delta, (4k_d+1)\delta],$$

where $k_1, \ldots, k_d \in \mathbb{Z}$. Note in particular that the cubes are δ-separated and hence the restrictions of $\hat{Y}_\delta$ to two distinct cubes in $\mathcal{Q}_\delta$ are independent. We then set

$$g_\delta = \varphi_\delta * \mathbf{1}_{\bigcup \mathcal{Q}_\delta \cap V}, \tag{2.27}$$

where φ is a smooth mollifier supported in the unit ball and $\varphi_\delta(x) = \delta^{-d}\varphi(x/\delta)$.

We note that h is indeed almost surely an element of $H_\mathbb{C}$, since the Malliavin derivative of $(i\beta)^{-1}\int f(y) : e^{i\beta Z(y)} : g_\delta(y) : e^{i\beta\hat{Y}_\delta(y)} : dy$ with respect to the field $\hat{Y}_\delta$ equals

$$x \mapsto \int f(y) : e^{i\beta Z(y)} : g_\delta(y) : e^{i\beta \hat{Y}_\delta(y)} : \mathbb{E}[\hat{Y}_\delta(x)\hat{Y}_\delta(y)]\, dy$$

and lies in $H_{\hat{Y}_\delta,\mathbb{C}}$. In particular, since $Y = Y_\delta + \hat{Y}_\delta$ is an independent sum, it lies in $H_{Y,\mathbb{C}}$ as well and, by Lemma 2.19, this as a set of functions coincides with $H^{d/2}_\mathbb{C}(\mathbb{R}^d)$. Moreover, the map $x \mapsto g_\delta(x) e^{i\beta Y_\delta(x) - \frac{\beta^2}{2}\mathbb{E}[Y_\delta(x)^2]}$ is almost surely smooth so multiplying by it shows that

$$x \mapsto g_\delta(x) e^{i\beta Y_\delta(x) - \frac{\beta^2}{2}\mathbb{E}[Y_\delta(x)^2]} \int f(y) : e^{i\beta Z(y)} : g_\delta(y) : e^{i\beta\hat{Y}_\delta(y)} : \mathbb{E}[\hat{Y}_\delta(x)\hat{Y}_\delta(y)]\, dy \in H^{d/2}_\mathbb{C}(\mathbb{R}^d).$$

Finally, as $Y + Z$ is an independent sum, Lemma 2.12 implies that $H^{d/2}_\mathbb{C}(\mathbb{R}^d) \subset H_\mathbb{C}$ as desired.

Proof of Proposition 2.10 In order to derive bounds on $\mathbb{P}[\det \gamma_M < \lambda]$ and $\mathbb{P}(\frac{\det \gamma_M}{\|DM\|^2_{H_\mathbb{C}}} < \lambda)$ for $\lambda > 0$ small, we will look at the three terms $|\langle DM, h_\delta\rangle_{H_\mathbb{C}}|$, $|\langle D\overline{M}, h_\delta\rangle_{H_\mathbb{C}}|$ and $\|h_\delta\|_{H_\mathbb{C}}$ appearing in (2.26) separately and collect the results in the following lemmas.

Lemma 2.34. *For every $\nu > 0$, there exists a constant $c_2 > 0$ such that for all $c > 0$ small enough*

$$\mathbb{P}[|\langle DM, h_\delta\rangle_{H_\mathbb{C}}| \le c(d-\beta^2)^{-2}\delta^d] \le \exp\left(-c_2\delta^{-d\wedge 2}\right)$$

for all small enough $\delta > 0$ and all $\beta \in (\nu, \sqrt{d})$.

Lemma 2.35. *For all $\eta > 0$ small enough, we can choose $C > 0$ such that*

$$\|h_\delta\|^2_{H_\mathbb{C}} \le C\delta^{\beta^2 - 2d - 2\eta} W^2 |\langle DM, h_\delta\rangle_{H_\mathbb{C}}|,$$

where W is a Y_δ-measurable positive random variable. Moreover, we can pick $c_1, c_2 > 0$ such that for *all* $\delta \in (0,1)$ *and* $t \geq c_1 \delta^{-2-\eta}$ *we have*

$$\mathbb{P}(W > t) \leq \exp(-c_2 \delta^{\eta} t^{\frac{2}{d}}).$$

Lemma 2.36. *For every* $\nu > 0$*, there exists a constant* $c_1 > 0$ *such that the following holds. For every* $c > 0$*, we can choose* $c_2 > 0$ *such that*

$$\mathbb{P}[|\langle \overline{DM}, h_\delta \rangle_{H_\mathbb{C}}| \geq c(d-\beta^2)^{-2}\delta^d] \leq \exp(-c_2 \delta^{-c_1})$$

for all small enough $\delta > 0$ *and all* $\beta \in (\nu, \sqrt{d})$*.*

We now explain how we deduce Proposition 2.10 from these lemmas, and then in the next subsections turn to their proofs.

Proof of Proposition 2.10. By Lemma 2.6, we have that

$$\mathbb{P}\left(\frac{\det \gamma_M}{\|DM\|^2_{H_\mathbb{C}}} \geq \varepsilon/4\right) \geq \mathbb{P}\left(\frac{(|\langle DM, h_\delta\rangle_{H_\mathbb{C}}| - |\langle D\overline{M}, h_\delta\rangle_{H_\mathbb{C}}|)^2}{\|h_\delta\|^2_{H_\mathbb{C}}} \geq \varepsilon\right)$$

and

$$\mathbb{P}\left(\det \gamma_M \geq \varepsilon/4\right) \geq \mathbb{P}\left(\frac{(|\langle DM, h_\delta\rangle_{H_\mathbb{C}}| - |\langle D\overline{M}, h_\delta\rangle_{H_\mathbb{C}}|)^2}{\|h_\delta\|^2_{H_\mathbb{C}}} \geq \sqrt{\varepsilon}\right),$$

so it suffices to bound $\mathbb{P}\left(\frac{(|\langle DM, h_\delta\rangle_{H_\mathbb{C}}| - |\langle D\overline{M}, h_\delta\rangle_{H_\mathbb{C}}|)^2}{\|h_\delta\|^2_{H_\mathbb{C}}} \leq \varepsilon\right)$ from above. Here h_δ is as above and we will choose δ depending on ε.

Using Lemma 2.35, we first bound for some $\eta > 0$

$$\frac{(|\langle DM, h_\delta\rangle_{H_\mathbb{C}}| - |\langle D\overline{M}, h_\delta\rangle_{H_\mathbb{C}}|)^2}{\|h_\delta\|^2_{H_\mathbb{C}}} \geq C^{-1}\delta^{-\beta^2+2d+2\eta} W^{-2}(|\langle DM, h_\delta\rangle_{H_\mathbb{C}}| - 2|\langle D\overline{M}, h_\delta\rangle_{H_\mathbb{C}}|).$$

Hence, taking c to be the constant from Lemma 2.34 we can bound

$$\mathbb{P}\Big(\frac{(|\langle DM, h_\delta\rangle_{H_\mathbb{C}}| - |\langle D\overline{M}, h_\delta\rangle_{H_\mathbb{C}}|)^2}{\|h_\delta\|^2_{H_\mathbb{C}}} \leq (d-\beta^2)^{-2}\delta^{3d+5}\Big)$$

by

$$\mathbb{P}\left(|\langle DM, h_\delta\rangle_{H_\mathbb{C}}| - 2|\langle D\overline{M}, h_\delta\rangle_{H_\mathbb{C}}| \leq \frac{c}{2}(d-\beta^2)^{-2}\delta^d\right) + \mathbb{P}\left(C\delta^{\beta^2-2d-2\eta}W^2 > \frac{c}{2}\delta^{-2d-5}\right).$$

The second term can be bounded using Lemma 2.35 loosely by $\exp(-c_1\delta^{-c_1})$ for some $c_1 > 0$.

For the first term, Lemma 2.34 gives that

$$\mathbb{P}(|\langle DM, h_\delta\rangle_{H_\mathbb{C}}| \leq c(d-\beta^2)^{-2}\delta^d) \leq \exp(-c_2\delta^{-d\wedge 2})$$

and Lemma 2.36 gives constants $c_3 > 0$

$$\mathbb{P}(2|\langle \overline{DM}, h_\delta\rangle_{H_\mathbb{C}}| \geq \frac{c}{2}(d-\beta^2)^{-2}\delta^d) \leq \exp(-\delta^{-c_3}),$$

and we thus obtain the proposition.

The case of the standard log-correlated field on circle needs extra attention, and is treated in Section 2.6.4.6. □

One can see that a simplified version of the above proof can also be used to prove Proposition 2.33.

Proof of Proposition 2.33. Recall that on the support of f, we can write $\Gamma_{|V} = Y + Z = X$, where Y is almost $\star-$scale invariant and Z is Holder regular, both defined on the whole space. Note that by Lemma 2.19 and Theorem 2.16 the operators C_Y and C_Z are bounded from $H^{-d/2}(\mathbb{R}^d)$ to $H^{d/2}(\mathbb{R}^d)$ and hence so is C_X. Thus for any $\varphi \in H^{-d/2}(\mathbb{R}^d)$ we have

$$\langle C_X \varphi, \varphi \rangle_{L^2(\mathbb{R}^d)} \leq \|C_X \varphi\|_{H^{d/2}(\mathbb{R}^d)} \|\varphi\|_{H^{-d/2}(\mathbb{R}^d)} \leq \|C_X\|_{H^{-d/2}(\mathbb{R}^d) \to H^{d/2}(\mathbb{R}^d)} \|\varphi\|^2_{H^{-d/2}(\mathbb{R}^d)}$$

so that in particular

$$\|f\mu\|^2_{H^{-d/2}(\mathbb{R}^d)} \gtrsim \langle C_X(f\mu), f\mu \rangle_{L^2(\mathbb{R}^d)} = \beta^{-2} \|DM\|^2_{H_\mathbb{C}} \geq \beta^{-2} \frac{|\langle DM, h_\delta \rangle_{H_\mathbb{C}}|^2}{\|h_\delta\|^2_{H_\mathbb{C}}}.$$

Using this inequality one can proceed as in the proof of Proposition 2.10 except one does not need to take care of the term $\langle D\overline{M}, h_\delta \rangle$. □

The rest of this subsection is dedicated to the proofs of Lemmas 2.34, 2.35 and 2.36, and sketching the extension to the case of the circle.

2.6.4.3 Proof of Lemma 2.34

Proof of Lemma 2.34. Let us fix some $\nu > 0$ small. Note that $\langle DM, h_\delta \rangle_{H_\mathbb{C}}$ is equal to

$$\begin{aligned} i\beta \int f(x) : e^{i\beta X(x)} : \overline{h_\delta(x)}\, dx &= i\beta \int f(x) f(y) : e^{i\beta(\hat{Y}_\delta(x) + Z(x))} :: e^{-i\beta(\hat{Y}_\delta(y) + Z(y))} : R_\delta(x, y) \\ &= i\beta \sum_{Q \in \mathcal{Q}_\delta} \int_{Q \times Q} f(x) f(y) : e^{i\beta(\hat{Y}_\delta(x) + Z(x))} :: e^{-i\beta(\hat{Y}_\delta(y) + Z(y))} : R_\delta(x, y)\, dx\, dy \end{aligned}$$

since $R_\delta(x, y) = 0$ if x and y are not in the same square in $\mathcal{Q}_\delta$. Moreover the summands are mutually independent, when we condition on the field Z, and by scaling each term agrees in law with

$$\delta^{2d} J_Q := \delta^{2d} \int_{\delta^{-1}Q \times \delta^{-1}Q} f(\delta x) : e^{i\beta Z(\delta x)} :: e^{-i\beta Z(\delta y)} : f(\delta y) : e^{i\beta \hat{Y}_\delta(\delta x)} :: e^{i\beta \hat{Y}_\delta(\delta y)} : R_\delta(\delta x, \delta y)\, dx\, dy.$$

We can write

$$\mathbb{E}[J_Q | Z] = \int_{\delta^{-1}Q \times \delta^{-1}Q} f(\delta x) f(\delta y) : e^{i\beta Z(\delta x)} :: e^{-i\beta Z(\delta y)} : e^{\beta^2 \mathbb{E}[\hat{Y}_\delta(\delta x) \hat{Y}_\delta(\delta y)]} R_\delta(\delta x, \delta y)\, dx\, dy.$$

Whenever Q is such that $f(x) \geq \|f\|_\infty / 2$ for all $x \in Q$ (or similarly if $f(x) \leq -\|f\|_\infty / 2$), and the event $E_Q := \{\sup_{x,y \in Q} |Z(x) - Z(y)| \leq \pi / (4\beta)\}$ holds, a basic calculation that uses Lemma 2.15 shows that

- $\mathbb{E}[J_Q | Z, E_Q] \geq C(d - \beta^2)^{-2}$, for some constant $C > 0$ that is uniform over $\beta \in (\nu, d)$ and depends only on $\|f\|_\infty$

- $\mathbb{E}[J_Q^2|Z, E_Q] \leq c(d-\beta^2)^{-4}$ for some constant $c > 0$ that is again uniform over $\beta \in (\nu, d)$ and depends solely on $\|f\|_\infty$.

In particular, by the Paley-Zygmund inequality for any such square Q it holds that $\mathbb{P}[J_Q \geq \lambda(d-\beta^2)^{-2}|Z, E_Q] \geq p$, where $\lambda = C/2$ and $p > 0$ is some constant. In the following, we denote by $\tilde{\mathcal{Q}}_\delta$ the collection of those squares in which f is larger than $\|f\|_\infty/2$ (again, we may consider $-f$ instead of f if needed).

Now, recall that Z is a Hölder continuous Gaussian field, and thus by local chaining inequalities (e.g. Proposition 5.35 in [vH]), we have that for some universal constant $C > 0$

$$\mathbb{P}\left(\sup_{|x-y|\leq 2\delta} |Z(x) - Z(y)| > \pi/(4\beta)\right) \leq C\exp(-C\delta^{-2}).$$

Thus denoting $E = \{\sup_{|x-y|\leq 2\delta} |Z(x) - Z(y)| \leq \pi/(4\beta)\}$, we can bound

$$\mathbb{P}[|\langle DM, h_\delta\rangle_{H_\mathbb{C}}| \leq c(d-\beta^2)^{-2}\delta^d] \leq P(E^c) + \mathbb{P}\Big[|\langle DM, h_\delta\rangle_{H_\mathbb{C}}| \leq c(d-\beta^2)^{-2}\delta^d|E\Big].$$

As $\mathbb{P}(E^c) \leq C\exp(-C\delta^{-2})$ and $E \subseteq \bigcap_Q E_Q$, it remains to only take care of the second term working under the assumption that the event E_Q holds for all Q. For any $t > 0$ to be chosen later, we have

$$\begin{aligned}\mathbb{P}\Big[|\langle DM, h_\delta\rangle_H| \leq (d-\beta^2)^{-2}t|E\Big] &\leq \mathbb{P}\Big[J_Q \geq (d-\beta^2)^{-2}\lambda \text{ for at most } t/(\beta\lambda\delta^{2d}) \text{ distinct } Q \in \tilde{\mathcal{Q}}_\delta|E\Big] \\ &\leq \mathbb{P}[\mathrm{Bin}(|\tilde{\mathcal{Q}}_\delta|, p) \leq t/(\beta\lambda\delta^{2d})] \\ &\leq e^{-2|\tilde{\mathcal{Q}}_\delta|\left(p-\left\lceil\frac{t}{\beta\lambda\delta^{2d}}\right\rceil|\tilde{\mathcal{Q}}_\delta|^{-1}\right)^2}\end{aligned}$$

where $\mathrm{Bin}(n,p)$ denotes the Binomial distribution. In the second line we used the conditional independence of J_Q given Z and the conditional probability obtained above; on the last line we used the Hoeffding's inequality

$$\mathbb{P}[\mathrm{Bin}(n,p) \leq m] \leq e^{-2n(p-\frac{m}{n})^2}.$$

Noting that $c_1\delta^{-d} \leq |\tilde{\mathcal{Q}}_\delta| \leq c_2\delta^{-d}$ for some $c_1, c_2 > 0$, we see that by choosing $t = p\beta\lambda\delta^d/(2c_2)$ we get

$$\mathbb{P}\Big[|\langle DM, h_\delta\rangle_H| \leq (d-\beta^2)^{-2}t|E\Big] \leq e^{-2c_1\frac{p}{3}\delta^{-d}}$$

for small enough $\delta > 0$ and the lemma follows. □

2.6.4.4 Proof of Lemma 2.35

Proof of Lemma 2.35. We start with some immediate bounds that allow the usage of inequalities on Sobolev spaces $H^s_\mathbb{C}(\mathbb{R}^d)$. First, by Lemma 2.19 we have

$$C^{-1}\|\cdot\|_{H^{d/2}_\mathbb{C}(\mathbb{R}^d)} \leq \|\cdot\|_{H_{Y,\mathbb{C}}} \leq C\|\cdot\|_{H^{d/2}_\mathbb{C}(\mathbb{R}^d)}$$

for some $C > 0$. On the other hand, by Lemma 2.12, we have that

$$\|\cdot\|_{H_{\mathbb{C}}} \leq \|\cdot\|_{H_{Y,\mathbb{C}}} \leq \|\cdot\|_{H_{\hat{Y}_\delta,\mathbb{C}}}.$$

Now let $\psi \in C_c^\infty(\mathbb{R}^d)$ be a non-negative function which equals 1 in the support of g_δ (recall that g_δ is defined in (2.27)). Set

$$F(x) := e^{i\beta Y_\delta(x) - \frac{\beta^2}{2}\mathbb{E}[Y_\delta(x)^2]}\psi(x)$$

and

$$G(x) := \int f(y) : e^{i\beta Z(y)} :: e^{i\beta \hat{Y}_\delta(y)} : g_\delta(y)\mathbb{E}[\hat{Y}_\delta(x)\hat{Y}_\delta(y)]\, dx\, dy$$

so that $g_\delta(x)F(x)G(x) = h_\delta(x)$. Using the above norm bounds in conjunction with the classical inequality $\|FG\|_{H^{d/2}(\mathbb{R}^d)} \lesssim \|F\|_{H_{\mathbb{C}}^{d/2+\varepsilon}(\mathbb{R}^d)}\|G\|_{H_{\mathbb{C}}^{d/2}(\mathbb{R}^d)}$ for any $\varepsilon > 0$ (see e.g. Theorem 5.1 in [BH15]), we can bound $\|h_\delta\|_{H_{\mathbb{C}}}$ by some constant times

$$\|gFG\|_{H_{\mathbb{C}}^{d/2}(\mathbb{R}^d)} \lesssim \|g_\delta\|_{H_{\mathbb{C}}^{d/2+\varepsilon}(\mathbb{R}^d)}\|F\|_{H_{\mathbb{C}}^{d/2+\varepsilon}(\mathbb{R}^d)}\|G\|_{H_{\mathbb{C}}^{d/2}(\mathbb{R}^d)} \lesssim \|g_\delta\|_{H_{\mathbb{C}}^{d/2+\varepsilon}(\mathbb{R}^d)}\|F\|_{H_{\mathbb{C}}^{d/2+\varepsilon}(\mathbb{R}^d)}\|G\|_{H_{\hat{Y}_\delta,\mathbb{C}}}.$$

We can bound $\|g_\delta\|_{H_{\mathbb{C}}^{d/2+\varepsilon}(\mathbb{R}^d)} \lesssim \delta^{-d-\varepsilon}$ by scaling and triangle inequality. Further, by definition we have that $\|G\|^2_{H_{\hat{Y}_\delta,\mathbb{C}}} = |\langle DM, h_\delta\rangle_{H_{\mathbb{C}}}|$. Thus it remains to deal with $\|F\|_{H_{\mathbb{C}}^{d/2+\varepsilon}(\mathbb{R}^d)}$. To do this, we will use Gaussian concentration inequalities.

Namely, by Theorem 4.5.7 in [Bog98], if X is isonormal on a Hilbert space H', and any $T : H' \to \mathbb{R}$ is $L-$Lipschitz w.r.t $\|\cdot\|_{H'}$, then for all $t > 0$

$$\mathbb{P}(T(X) - \mathbb{E}T(X) > t) \leq \exp(-\frac{t^2}{2L^2}).$$

We will make use of this concentration in the case $T = \|\cdot\|_{H^{d/2+\varepsilon}(\mathbb{R}^d)}$ to bound $W := T(F)$. We first apply Theorem A in [AF92], which gives that for $f \in H^{d/2+\varepsilon}(\mathbb{R}^d)$ we have $\|\exp(if)\psi\|_{H_{\mathbb{C}}^{d/2+\varepsilon}} \lesssim \|f\|_{H^{d/2+\varepsilon}(\mathbb{R}^d)} + \|f\|^{d/2+\varepsilon}_{H^{d/2+\varepsilon}(\mathbb{R}^d)}$.[5] This together with the fact that $\mathbb{E}[Y_\delta(x)^2]$ is constant in x gives us that $\|F\|_{H_{\mathbb{C}}^{d/2+\varepsilon}(\mathbb{R}^d)} \leq c\delta^{\beta^2/2}(\|Y_\delta\tilde{\psi}\|_{H^{d/2+\varepsilon}(\mathbb{R}^d)} + \|Y_\delta\tilde{\psi}\|^{d/2+\varepsilon}_{H^{d/2+\varepsilon}(\mathbb{R}^d)})$ for some $c > 0$. Here $\tilde{\psi} \in C_c^\infty(\mathbb{R}^d)$ is some function which is 1 in the support of ψ. Further, we have the following bounds:

Claim 2.37. *It holds that*

1. $\|\cdot\|_{H^{d/2+\varepsilon}(\mathbb{R}^d)}$ *is* $O(\delta^{-2\varepsilon})-$*Lipschitz with respect to* $\|\cdot\|_{H_{Y_\delta}}$.
2. $(\mathbb{E}\|\tilde{\psi}Y_\delta\|_{H^{d/2+\varepsilon}(\mathbb{R}^d)}|)^2 \leq \mathbb{E}\|\tilde{\psi}Y_\delta\|^2_{H^{d/2+\varepsilon}(\mathbb{R}^d)} \lesssim \delta^{-d-4\varepsilon}$.

Proof of Claim 2.37. Recall from the proof of Lemma 2.19 that the operator C_{Y_δ} is a Fourier multiplier operator with the symbol

$$\hat{K}_\delta(\xi) := \int_\delta^1 v^{d-1}(1 - v^\alpha)\hat{k}(v\xi)dv$$

[5] In [AF92] the authors consider compositions with real-valued functions; in our case one can apply it directly to the real and imaginary part. Note that by the theorem the first operator in the chain $f \mapsto e^{if} - 1 \mapsto (e^{if} - 1)\psi \mapsto e^{if}\psi$ is bounded and the other two are bounded since ψ is smooth.

and k is by assumption smooth. Moreover,

$$\|f\|_{H_{Y_\delta}}^2 = \int \hat{K}_\delta(\xi)^{-1}|\hat{f}(\xi)|^2\, d\xi$$

and

$$\mathbb{E}[\|\tilde{\psi}Y_\delta\|^2_{H^{d/2+\varepsilon}(\mathbb{R}^d)} = \int (1+|\xi|^2)^{d/2+\varepsilon} \int |\hat{\tilde{\psi}}(\zeta)|^2 \hat{K}_\delta(\xi-\zeta)\, d\zeta\, d\xi.$$

The two claims thus directly follow from bounding $\hat{K}_\delta$ respectively by

$$\hat{K}_\delta(\xi) \lesssim \delta^{-2\varepsilon}(1+|\xi|^2)^{-d/2-\varepsilon}, \tag{2.28}$$

$$\text{and} \quad \hat{K}_\delta(\xi) \lesssim \delta^{-d-4\varepsilon}(1+|\xi|^2)^{-d-2\varepsilon}, \tag{2.29}$$

where the underlying constants do not depend on δ. These inequalities are clear when $|\xi| \le 1$, and follow by integrating the bounds $\hat{k}(v\xi) \le C|v\xi|^{-d-2\varepsilon}$ and $\hat{k}(v\xi) \le C|v\xi|^{-2d-4\varepsilon}$ for $|\xi| > 1$. □

We can finally apply the Gaussian concentration to deduce that for all $\varepsilon \in (0, d/2)$, there are some $c, C' > 0$, such that for all $t > c\delta^{-d-4\varepsilon}$

$$\mathbb{P}(\|\tilde{\psi}Y_\delta\|_{H^{d/2+\varepsilon}(\mathbb{R}^d)} > t) \le \exp\left(-C'\delta^\varepsilon t^2\right),$$

and thus for some $c', C'' > 0$ and for all $t > c'\delta^{-2-4\varepsilon}$

$$\mathbb{P}(\|\tilde{\psi}Y_\delta\|_{H^{d/2+\varepsilon}(\mathbb{R}^d)} + \|\tilde{\psi}Y_\delta\|^{d/2+\varepsilon}_{H^{d/2+\varepsilon}(\mathbb{R}^d)} > t) \le \exp\left(-C'\delta^\varepsilon t^{\frac{2}{d}}\right),$$

implying the lemma. □

2.6.4.5 Proof of Lemma 2.36

Proof. We have

$$\left\langle DM, \overline{h_\delta}\right\rangle = i\beta \int f(x)f(y)e^{-2\beta^2\mathbb{E}[X_\delta(x)^2]} : e^{i2\beta X_\delta(x)} :: e^{i\beta \hat{Y}_\delta(x)} :: e^{i\beta\hat{Y}_\delta(y)} : R_\delta(x,y)dxdy,$$

which we can write as a sum

$$i\beta \sum_{Q\in\mathcal{Q}_\delta} \int_{Q\times Q} f(x)f(y)e^{-2\beta^2\mathbb{E}[X_\delta(x)^2]} : e^{i2\beta X_\delta(x)} :: e^{i\beta\hat{Y}_\delta(x)} :: e^{i\beta\hat{Y}_\delta(y)} : R_\delta(x,y)dxdy =: i\beta \sum_{Q\in\mathcal{Q}_\delta} L_Q.$$

We can then first bound

$$\mathbb{E}|\left\langle \overline{DM}, h_\delta\right\rangle|^{2N} \le \beta^{2N}\mathbb{E}|\sum_{Q\in\mathcal{Q}_\delta} L_Q|^{2N}.$$

If we expand the $2N$-th moment of such a sum, we obtain terms of the form

$$\beta^{2N}\mathbb{E}\Big[L_{Q_1}\dots L_{Q_N}\overline{L_{Q_1'}\dots L_{Q_N'}}\Big].$$

Before taking expectation in each such term we separate the field $Y_\delta = Y_{\sqrt{\delta}} + \widetilde{Y}_\delta$, with $\widetilde{Y}_\delta := Y_\delta - Y_{\sqrt{\delta}}$ being independent of $Y_{\sqrt{\delta}}$. We can then write each term as

$$= \beta^{2N} \int \prod_{j=1}^{N} f(x_j) f(y_j) f(x'_j) f(y'_j) R_\delta(x_j, y_j) R_\delta(x'_j, y'_j) e^{4\beta^2 \mathcal{E}(Y_{\sqrt{\delta}}; \mathbf{x}; \mathbf{x}')} e^{\beta^2 \mathcal{E}(\hat{Y}_\delta; \mathbf{x}, \mathbf{y}; \mathbf{x}', \mathbf{y}')}$$
$$\times e^{-2\beta^2 \sum_{j=1}^{N} (\mathbb{E}[X_\delta(x_j)^2] + \mathbb{E}[X_\delta(x'_j)^2])} \mathbb{E}\left(\prod_{j=1}^{N} : e^{i2\beta(Z(x_j) + \widetilde{Y}_\delta(x_j))} :: e^{-i2\beta(Z(x'_j) + \widetilde{Y}_\delta(x'_j))} : \right),$$

where the integration is over $x_j, y_j \in Q_j$ and $x'_j, y'_j \in Q'_j$. We bound the expectation by

$$\mathbb{E} \left| \prod_{j=1}^{N} : e^{i2\beta(Z(x_j) + \widetilde{Y}_\delta(x_j))} :: e^{-i2\beta(Z(x'_j) + \widetilde{Y}_\delta(x'_j))} : \right| \le C^N \delta^{-2N\beta^2},$$

since $\mathbb{E}[\widetilde{Y}_\delta(x)^2] = \frac{1}{2} \log \frac{1}{\delta} + O(1)$. Now, there is some $c > 0$ such that $\mathcal{E}(Y_{\delta^{1/2}}; \mathbf{x}; \mathbf{x}') \ge \mathcal{E}(Y_\delta^{1/2}, \mathbf{q}; \mathbf{q}') - c\sqrt{\delta} N^2$, where $\mathbf{q}$ and $\mathbf{q}'$ denote the vectors of midpoints for the ordered squares Q_j and Q'_j. This can be seen by noting that since the seed covariance k is Lipschitz, we have

$$|\mathbb{E}[Y_{\sqrt{\delta}}(x) Y_{\sqrt{\delta}}(x')] - \mathbb{E}[Y_{\sqrt{\delta}}(q) Y_{\sqrt{\delta}}(q')]| \lesssim \int_0^{\frac{1}{2}\log\frac{1}{\delta}} e^u ||x - x'| - |q - q'|| (1 - e^{-\alpha u}) \, du \lesssim \sqrt{\delta}$$

when $|x - q|, |x' - q'| \lesssim \delta$. Thus we obtain the upper bound

$$\|f\|_\infty^{4N} \beta^{2N} \delta^{2\beta^2 N} e^{c\sqrt{\delta} N^2} e^{4\beta^2 \mathcal{E}(Y_{\delta^{1/2}}; \mathbf{q}_1; \mathbf{q}_2)} \mathbb{E}[J_{Q_1} \dots J_{Q_N} \overline{J_{Q'_1} \dots J_{Q'_N}}],$$

where now

$$J_Q = \int_{Q \times Q} : e^{i\beta \hat{Y}_\delta(x)} :: e^{i\beta \hat{Y}_\delta(y)} : R_\delta(x, y) dx dy.$$

By Hölder's inequality we can bound

$$\mathbb{E}[J_{Q_1} \dots J_{Q_N} \overline{J_{Q'_1} \dots J_{Q'_N}}] \le \mathbb{E}|J_{Q_1}|^{2N}.$$

By scaling the right hand side equals

$$\delta^{4Nd} \int_{[0,1]^{4Nd}} \prod_{j=1}^{N} R_\delta(\delta x_j, \delta y_j) R_\delta(\delta x'_j, \delta y'_j) e^{\beta^2 \mathcal{E}(Y^{(\delta)}; \mathbf{x}, \mathbf{y}; \mathbf{x}', \mathbf{y}')}$$
$$\le \delta^{4Nd} \int_{[0,1]^{4Nd}} \prod_{j=1}^{N} \sqrt{\log \frac{C}{|x_j - \pi(x_j)|} \log \frac{C}{|y_j - \pi(y_j)|} \log \frac{C}{|x'_j - \pi(x'_j)|} \log \frac{C}{|y'_j - \pi(y'_j)|}} e^{\beta^2 \mathcal{E}(Y^{(\delta)}; \mathbf{x}, \mathbf{y}; \mathbf{x}', \mathbf{y}')},$$

where we have used Lemma 2.15 and $\pi(x)$ denotes the closest point to point x in the set

$$\{x_1, \dots, x_N, y_1, \dots, y_N, x'_1, \dots, x'_N, y'_1, \dots, y'_N\} \setminus \{x\}.$$

By relabeling the points as $z_1, \dots, z_{4N}$ and using Lemma 2.28 we then have the upper bound

$$\delta^{4Nd} \int_{[0,1]^{4Nd}} \prod_{j=1}^{4N} \sqrt{\log \frac{C}{|z_j - z_{F(j)}|}} \frac{1}{|z_j - z_{F(j)}|^{\beta^2/2}},$$

which by Lemma 2.29 is bounded by

$$C^N (d-\beta^2)^{-4N} \delta^{4Nd} N^{4N}$$

for some constant $C > 0$. Hence we can bound $\mathbb{E}| \left\langle \overline{DM}, h_\delta \right\rangle |^{2N}$ by

$$C^N (d-\beta^2)^{-4N} \delta^{4Nd} N^{4N} \beta^{2N} \delta^{2\beta^2 N} e^{2c\sqrt{\delta}N^2} \delta^{-2Nd} \int_{K^{2N}} \exp\left(4\beta^2 \mathcal{E}(Y_{\delta^{1/2}}; \mathbf{x}; \mathbf{x}')\right),$$

where for convenience we have turned $\mathbf{q}, \mathbf{q}'$ back to $\mathbf{x}, \mathbf{x}'$ by paying the same price. The latter integral is the $2N$-th moment of the 2β chaos of field $Y_{\delta^{1/2}}$, which by Lemma 2.28 and (2.23) is bounded by $C^N N^{2N} \left(\log \frac{1}{\delta}\right)^N \delta^{-N\max(2\beta^2 - \frac{d}{2}, 0)}$, giving

$$\mathbb{E}| \left\langle \overline{DM}, h_\delta \right\rangle |^{2N} \leq C^N e^{c\sqrt{\delta}N^2} (d-\beta^2)^{-4N} \left(\log \frac{1}{\delta}\right)^N \delta^{N(2d + \min(\frac{d}{2}, 2\beta^2))} N^{6N}.$$

Note that for any fixed $b, C, \nu > 0$ we have $2b^{-1} C \log \frac{1}{\delta} < \delta^{-\nu}$ and δ small enough. One thus sees that

$$\mathbb{P}[|\langle \overline{DM}, h_\delta \rangle_H| \geq b(d-\beta^2)^{-2}\delta^d] \leq 2^{-N} e^{c\sqrt{\delta}N^2} \delta^{-\nu N} \delta^{N\min(\frac{d}{2}, 2\beta^2)} N^{6N}$$

yields the desired upper bound by choosing e.g. $N = \delta^{-\beta^2/(24d)}$. □

2.6.4.6 Special case: the standard log-correlated field on the circle

In this section we will briefly explain how to extend the proof of Proposition 2.10 to the case where we are interested in the total mass of the imaginary chaos defined using the field Γ on the unit circle which has the covariance $\log \frac{1}{|x-y|}$, where one now thinks of x and y as being complex numbers of modulus 1. See Section 2.2 for the precise definitions.

Recall, that the extra complication in this case is that the field is degenerate in the sense that it is conditioned to satisfy $\int_0^1 \Gamma(e^{2\pi i\theta})\, d\theta = 0$. In terms of the proof of Proposition 2.10 this creates some annoyance, as the function h_δ we used in the projection bounds does not anymore belong to the Cameron–Martin space $H_{\mathbb{C}}$ of Γ, and we will instead need to look at the function $\tilde{h}_\delta = h_\delta - \int h_\delta(y)\, dy$.

As the field $\Gamma(e^{2\pi i \cdot})$ is non-degenerate when restricted to $I_0 := [-1/4, 1/4]$ (see again Section 2.2), it is also beneficial to introduce a smooth bump function ψ supported in $I_0 := [-1/4, 1/4]$, and thus set

$$h_\delta(x) = \psi(x) e^{i\beta Y_\delta(x) - \frac{\beta^2}{2}\mathbb{E}[Y_\delta(x)^2]} \int \psi(y) : e^{i\beta(\hat{Y}_\delta(y) + Z(y))} : R_\delta(x,y)\, dy.$$

This will let us still use the decomposition $X = Y + Z$ where $\Gamma_{|I_0} = X_{|I_0}$ and streamline most of the proof.

In the case of Lemmas 2.34 and 2.36, i.e. in terms $\langle DM, \tilde{h}_\delta \rangle_{H_{\mathbb{C}}}$ and $\langle \overline{DM}, \tilde{h}_\delta \rangle_{H_{\mathbb{C}}}$, this subtraction of the mean introduces the extra term $i\beta M \int h_\delta(y)\,dy$. In the case of Lemma 2.35, we have an extra term of the form $|\int h_\delta(y)|$. The next lemma guarantees that both terms are negligible.

Lemma 2.38. *For all $c > 0$ there is some $c_1 > 0$ such that we have*

$$\mathbb{P}[|\int h_\delta(y)\,dy| > c\delta(1-\beta^2)^{-1/2}] \leq e^{-c_1\delta^{-1}c^{\frac{2}{\beta^2}}}$$

and

$$\mathbb{P}[|M\int h_\delta(y)\,dy| > c\delta(1-\beta^2)^{-1}] \leq e^{-c_1\delta^{-1/2}c^{\frac{1}{\beta^2}}}$$

for all δ small enough.

Proof. We will bound the N–th moment of $|M\int h_\delta(y)|$, use the Chebyshev inequality and optimize over N. Note that by the Cauchy–Schwarz inequality we have

$$\mathbb{E}\left[\left|M\int h_\delta(y)\,dy\right|^N\right] \leq \mathbb{E}[|M|^{2N}]^{1/2}\mathbb{E}\left[\left|\int h_\delta(y)\,dy\right|^{2N}\right]^{1/2}$$

and by [JSW20, Theorem 1.3] we know that (recall that we are currently in a one-dimensional setting)

$$\mathbb{E}[|M|^{2N}] \leq C^N(d-\beta^2)^{-N}N^{\beta^2 N}$$

for some $C > 0$. We mention that, in the article [JSW20], the dependence of the above constant in terms of β was not stated but follows from their approach (see (2.20)). To bound $\mathbb{E}[|\int_0^1 h_\delta(y)\,dy|^{2N}]$, we note that by Jensen's inequality we have

$$\mathbb{E}\Big[\Big|\int_0^1 h_\delta(y)\,dy\Big|^{2N}\Big] \leq \mathbb{E}\Big[\Big(\int_0^1 |h_\delta(y)|^2\,dy\Big)^N\Big],$$

where the right hand side equals

$$\mathbb{E}\Big[\Big(\int_0^1 |\psi(x)|^2 e^{-\beta^2\mathbb{E}[Y_\delta(x)^2]}\Big|\int \psi(y) : e^{i\beta(\hat{Y}_\delta(y)+Z(y))} : R_\delta(x,y)\,dy\Big|^2\,dx\Big)^N\Big].$$

We bound $|\psi(x)|^2 e^{-\beta^2\mathbb{E}[Y_\delta(x)^2]}$ by $C\delta^{\beta^2}$ and since $R_\delta(x,y) = 0$ whenever x, y do not belong to the same square, we can bound the above expression by

$$C^N\delta^{N\beta^2}\delta^{-N}\sum_{Q\in\mathcal{Q}_\delta}\mathbb{E}\Big[\Big(\int_{Q^3}\psi(y)\psi(z) : e^{i\beta(\hat{Y}_\delta(y)+Z(y))} : R_\delta(x,y)R_\delta(x,z) : e^{-i\beta(\hat{Y}_\delta(z)+Z(z))} :\,dz\,dx\,dy\Big)^N\Big].$$

By developing the expectation into a multiple integral, using an Onsager inequality associated to the smooth field Z (see (2.19)) and then rewriting the multiple integrals as an expectation, we see that we can get rid of the field Z in the above expectation by only paying a multiplicative price C^N.

Thus it remains to bound

$$C^N\delta^{N\beta^2}\delta^{-N}\sum_{Q\in\mathcal{Q}_\delta}\mathbb{E}\Big[\Big(\int_{Q^3}\psi(y)\psi(z) : e^{i\beta\hat{Y}_\delta(y)} : R_\delta(x,y)R_\delta(x,z) : e^{-i\beta\hat{Y}_\delta(z)} :\,dz\,dx\,dy\Big)^N\Big].$$

By scaling we see that each term in the sum is equal in law to

$$\delta^{3N} J_Q := \delta^{3N}\mathbb{E}\Big[\Big(\int_{\delta^{-1}Q\times\delta^{-1}Q\times\delta^{-1}Q}\psi(\delta y)\psi(\delta z) : e^{i\beta\hat{Y}_\delta(\delta y)} : R_\delta(\delta x,\delta y)R_\delta(\delta x,\delta z) : e^{-i\beta\hat{Y}_\delta(\delta z)} : dz\,dx\,dy\Big)^N\Big].$$

To bound this expectation, we expand the product and obtain a multiple integral over x_i, y_i, z_i, $i = 1\dots N$. The expectation of the product of $: e^{i\beta\hat{Y}_\delta(\delta y)} :$ and $: e^{-i\beta\hat{Y}_\delta(\delta z)} :$ leads to $\mathcal{E}(\hat{Y}_\delta(\delta\cdot);\mathbf{y};\mathbf{z})$ that we bound using the Onsager inequality (2.18). Since for any fixed y and z,

$$\psi(\delta y)\psi(\delta z)\int_{\delta^{-1}Q} R_\delta(\delta x,\delta y)R_\delta(\delta x,\delta z)\,dx < C,$$

we can first integrate the variables x_i and control the remaining integral over y_i and z_i, $i = 1\dots N$ with (2.20). Overall, J_Q is bounded by $(d-\beta^2)^{-N}N^{\beta^2 N}$.

Altogether we obtain that

$$\mathbb{E}\Big[\Big|\int_0^1 h_\delta(y)\,dy\Big|^{2N}\Big] \le C^N(d-\beta^2)^{-N}\delta^{(\beta^2+2)N}N^{\beta^2 N}$$

and hence

$$\mathbb{E}\Big[\Big|M\int h_\delta(y)\,dy\Big|^{N}\Big] \le C^N(d-\beta^2)^{-N}\delta^{(\frac{\beta^2}{2}+1)N}N^{\beta^2 N},$$

which gives us the tail estimates

$$\mathbb{P}\Big[\Big|\int h_\delta(y)\,dy\Big| \ge \lambda(d-\beta^2)^{-1/2}\Big] \le \frac{C^N\delta^{(\frac{\beta^2}{2}+1)N}N^{\frac{\beta^2}{2}N}}{\lambda^N}.$$

and

$$\mathbb{P}\Big[\Big|M\int h_\delta(y)\,dy\Big| \ge \lambda(d-\beta^2)^{-1}\Big] \le \frac{C^N\delta^{(\frac{\beta^2}{2}+1)N}N^{\beta^2 N}}{\lambda^N}.$$

Optimising over N now concludes. □

Appendix 2.A Some standard proofs

Proof of Lemma 2.15. It is calculationally somewhat easier to work with the rescaled field $Y^{(\epsilon)}(x) = \hat{Y}_\epsilon(\delta x)$, which can be expressed using white noise as:

$$Y^{(\delta)}(x) := \int_{\mathbb{R}^d\times[0,\infty)} e^{du/2}\tilde{k}(e^u(t-x))\sqrt{1-\delta^\alpha e^{-\alpha u}}dW(t,u).$$

The first inequality then follows directly:

$$\mathbb{E}[Y^{(\delta)}(x)Y^{(\delta)}(y)] = \int_0^\infty k(e^u(x-y))(1-\delta^\alpha e^{-\alpha u})\,du \le \int_0^\infty k(e^u(x-y))\,du \le \log\frac{1}{|x-y|}$$

by the fact that k is supported in $B(0,1)$ and $k(t)\le 1$ for all t.

For the second inequality we compute

$$\begin{aligned}\int_0^\infty k(e^u(x-y))(1-\delta^\alpha e^{-\alpha u})\,du &\geq \int_0^\infty k(e^u(x-y))(1-e^{-\alpha u})\,du \\ &\geq \int_0^{\log\frac{1}{|x-y|}} k(e^u(x-y))\,du - \int_0^\infty e^{-\alpha u}\,du \\ &\geq \log\frac{1}{|x-y|} + \int_0^{\log\frac{1}{|x-y|}} (k(e^u(x-y))-1)\,du - \frac{1}{\alpha}\end{aligned}$$

Note that by Taylor's theorem we have for all $t \in \mathbb{R}$ the inequality

$$k(t) \geq 1 + k'(0)t - ct^2$$

for some constant $c > 0$, and in fact since k is smooth and symmetric we have $k'(0) = 0$. Hence

$$\int_0^{\log\frac{1}{|x-y|}} (k(e^u(x-y))-1)\,du \geq -c\int_0^{\log\frac{1}{|x-y|}} e^{2u}|x-y|^2 = -c(\frac{1}{2|x-y|^2}|x-y|^2 - \frac{|x-y|^2}{2}) \geq -\frac{c}{2},$$

from which the claim follows.

Finally, the independence comes from the fact that k is supported in $B(0,1)$ □

Proof of Lemma 2.28. Let us begin with the field Y_ε. Set $q_j = 1$ for $1 \leq j \leq N$ and $q_j = -1$ for $N+1 \leq j \leq 2N$ and note that

$$\mathcal{E}(Y_\varepsilon;\mathbf{x};\mathbf{y}) = -\frac{1}{2}\mathbb{E}\left[\Big(\sum_{j=1}^{2N} q_j Y_{d_j\wedge\varepsilon}(z_j)\Big)^2\right] + \frac{1}{2}\sum_{j=1}^{2N}\mathbb{E}[Y_{d_j\wedge\varepsilon}(z_j)^2] \leq \frac{1}{2}\sum_{j=1}^{2N}\log\frac{1}{d_j\wedge\varepsilon}$$

since $\mathbb{E}[Y_\varepsilon(x)Y_\varepsilon(y)] = \mathbb{E}[Y_s(x)Y_t(y)]$ for all $s,t \leq \varepsilon\wedge|x-y|$ and $\mathbb{E}[Y_\delta(x)^2] \leq \log\frac{1}{\delta}$ for all $\delta \in (0,1)$.

As the field $\hat{Y}_\varepsilon(\varepsilon x)$ has the same distribution as the field $Y^{(\varepsilon)}(x)$ from the proof of Lemma 2.15, we have

$$\mathcal{E}(\hat{Y}_\varepsilon(\varepsilon\cdot);\mathbf{x};\mathbf{y}) = -\frac{1}{2}\mathbb{E}\left[\Big(\sum_{j=1}^{2N} q_j \hat{Y}^{(\varepsilon)}_{d_j}(z_j)\Big)\right] + \frac{1}{2}\sum_{j=1}^{2N}\mathbb{E}[\hat{Y}^{(\varepsilon)}_{d_j}(z_j)^2] \leq \frac{1}{2}\sum_{j=1}^{2N}\log\frac{1}{d_j}.$$

Finally, if R is a regular field then

$$\mathcal{E}(R;\mathbf{x};\mathbf{y}) = -\frac{1}{2}\mathbb{E}\left[\Big(\sum_{j=1}^{2N} q_j R(z_j)\Big)\right] + \frac{1}{2}\sum_{j=1}^{2N}\mathbb{E}[R(z_j)^2] \leq N\sup_{1\leq j\leq 2N}\mathbb{E}[R(z_j)^2].$$ □

Proof of Lemma 2.31. We prove this lemma in the context of real-valued random variables. The extension to complex-valued random variables follows immediately.

In page 58 of [Nua06], an operator L on the set of variables with finite second moment is introduced and used to define the norm $|||F|||_{k,p} := \mathbb{E}\left[((I-L)^{k/2}F)^p\right]^{1/p}$. The norms $|||\cdot|||_{k,p}$ and $\|\cdot\|_{k,p}$ are equivalent (see [Nua06] page 77). Hence $\sup_n \mathbb{E}\left[((I-L)^{k/2}F_n)^p\right] < \infty$. By weak compactness of balls in $L^p(\Omega)$, we can extract a subsequence $(n(i), i \geq 1)$ such that $((I-L)^{k/2}F_{n(i)}, i \geq 1)$ converges weakly towards some element G. Since the L^p-norm is weakly lower-semicontinuous, we moreover have

$$\mathbb{E}\left[G^p\right] \leq \liminf_i \mathbb{E}\left[((I-L)^{k/2}F_{n(i)})^p\right] \leq \limsup_n \mathbb{E}\left[((I-L)^{k/2}F_n)^p\right].$$

In the proof of [Nua06, Lemma 1.5.3], D. Nualart shows that $F=(I-L)^{-k/2}G$. This implies that

$$\|F\|_{k,p} \leq C_{k,p} \,|||F|||_{k,p} = C_{k,p}\mathbb{E}\left[G^p\right]^{1/p} \leq C_{k,p} \limsup_n |||F_n|||_{k,p} \leq C'_{k,p} \limsup_n \|F_n\|_{k,p}.$$

This concludes the proof. □

Appendix 2.B Proof of Proposition 2.4

Proof of Proposition 2.4. We start by showing that M belongs to $\mathbb{D}^\infty$. Let $n \geq 1, \delta > 0, j \geq 0$ and $p \geq 1$. In the following, we will denote

$$\Gamma_\delta = \Gamma * \varphi_\delta, \quad \Gamma_{n,\delta} = \sum_{k=1}^n A_k e_k * \varphi_\delta, \quad M_\delta = \int_{\mathbb{C}} f(x) e^{i\beta\Gamma_\delta(x)+\frac{\beta^2}{2}\mathbb{E}[\Gamma_\delta(x)^2]}dx$$

and

$$M_{n,\delta} = \int_{\mathbb{C}} f(x) e^{i\beta\Gamma_{n,\delta}(x)+\frac{\beta^2}{2}\mathbb{E}[\Gamma_{n,\delta}(x)^2]}dx.$$

$M_{n,\delta}$ is a smooth random variable and $D^j M_{n,\delta}$ is equal to

$$(i\beta)^j \int_{\mathbb{C}} dx f(x) e^{i\beta\Gamma_{n,\delta}(x)+\frac{\beta^2}{2}\mathbb{E}[\Gamma_{n,\delta}(x)^2]} \sum_{k_1,\dots,k_j=1}^n (e_{k_1} * \varphi_\delta)(x)\dots(e_{k_j} * \varphi_\delta)(x) e_{k_1} \otimes \dots \otimes e_{k_j}. \tag{2.30}$$

Since $(e_{k_1} \otimes \dots \otimes e_{k_j}, k_1,\dots,k_j = 1\dots n)$ is an orthonormal family of $H^{\otimes j}$, we deduce that

$$\left\|D^j M_{n,\delta}\right\|^2_{H_{\mathbb{C}}^{\otimes j}} = \beta^{2j} \int_{\mathbb{C}^2} f(x)f(y) e^{i\beta\Gamma_{n,\delta}(x)-i\beta\Gamma_{n,\delta}(y)+\frac{\beta^2}{2}\mathbb{E}[\Gamma_{n,\delta}(x)^2]+\frac{\beta^2}{2}\mathbb{E}[\Gamma_{n,\delta}(y)^2]}$$
$$\times \left(\sum_{k=1}^n (e_k * \varphi_\delta)(x)(e_k * \varphi_\delta)(y)\right)^j dxdy.$$

Thanks to the convolution, all the integrated terms are uniformly bounded in n and $x_1\dots x_p$, $y_1\dots y_p$. By dominated convergence theorem and then by using (2.17) which provides an Onsager inequality for convolution approximations, we deduce that

$$\limsup_{n\to\infty} \mathbb{E}\left[\left\|D^j M_{n,\delta}\right\|^{2p}_{H_{\mathbb{C}}^{\otimes j}}\right]$$
$$\leq \beta^{2jp} \int_{\mathbb{C}^{2p}} dx_1\dots dx_p dy_1\dots dy_p \prod_{l=1}^p f(x_l)f(y_l)\left(C * (\varphi_\delta \otimes \varphi_\delta)(x_l,y_l)\right)^j e^{\beta^2 \mathcal{E}(\Gamma_\delta;\mathbf{x};\mathbf{y})}$$
$$\leq C_{j,p} \|f\|^{2p}_\infty \int_{K^{2p}} dz_1\dots dz_{2p} \prod_{l=1}^{2p} \left(\min_{l'\neq l}|z_l - z_{l'}|\right)^{-\beta^2/2} \left(\max_{l'\neq l} C * (\varphi_\delta \otimes \varphi_\delta)(z_l, z_{l'})\right)^{j/2}$$

where K is the support of f. Importantly, the above constant $C_{j,p}$ does not depend on δ. Notice that

$$C * (\varphi_\delta \otimes \varphi_\delta)(x,y) \leq C \log \frac{c}{|x-y| \vee \delta}.$$

Hence, if we let $\varepsilon > 0$ be such that $\beta^2/2 + \varepsilon < d/2$, there exists $C'_{j,p} > 0$ independent of δ such that

$$\limsup_{n\to\infty} \mathbb{E}\left[\left\| D^j M_{n,\delta} \right\|^{2p}_{H_{\mathbb{C}}^{\otimes j}} \right] \leq C'_{j,p} \int_{K^{2p}} dz_1 \dots dz_{2p} \prod_{l=1}^{2p} \left(\min_{l' \neq l} |z_l - z_{l'}| \right)^{-\beta^2/2-\varepsilon} \leq C''_{j,p} \tag{2.31}$$

by (2.20). Since $(M_{n,\delta}, n \geq 1)$ converges in L^{2p} towards M_δ, Lemma 2.31 and (2.31) imply that for all $k \geq 1$, $M_\delta \in \mathbb{D}^{k,2p}$ and that

$$\sup_{\delta > 0} \|M_\delta\|_{k,2p} < \infty. \tag{2.32}$$

Now, because $(M_\delta, \delta > 0)$ converges in L^{2p} towards M, Lemma 2.31 implies that for all $k \geq 1$, $M \in \mathbb{D}^{k,2p}$. This concludes the proof that $M \in \mathbb{D}^\infty$.

We now turn to the proof of the formula for DM. On the one hand, (2.30) gives

$$DM_{n,\delta} = i\beta \int_{\mathbb{C}} dx f(x) e^{i\beta\Gamma_{n,\delta}(x) + \frac{\beta^2}{2}\mathbb{E}[\Gamma_{n,\delta}(x)^2]} \sum_{k=1}^{n} (e_k * \varphi_\delta)(x) e_k.$$

One can then show that $(DM_{n,\delta}, n \geq 1)$ converges in $L^2(\Omega; H)$ towards

$$i\beta \int_{\mathbb{C}} dx f(x) e^{i\beta\Gamma_\delta(x) + \frac{\beta^2}{2}\mathbb{E}[\Gamma_\delta(x)^2]} \sum_{k=1}^{\infty} (e_k * \varphi_\delta)(x) e_k.$$

On the other hand, the first part of the proof showed that $\sup_n \mathbb{E}\left[\|DM_{n,\delta}\|^2_{H_{\mathbb{C}}} \right] < \infty$ and Lemma 2.30 implies that $(DM_{n,\delta}, n \geq 1)$ converges to DM_δ in the weak topology of $L^2(\Omega; H)$. Hence

$$DM_\delta = i\beta \int_{\mathbb{C}} dx f(x) e^{i\beta\Gamma_\delta(x) + \frac{\beta^2}{2}\mathbb{E}[\Gamma_\delta(x)^2]} \sum_{k=1}^{\infty} (e_k * \varphi_\delta)(x) e_k.$$

Let us now show that $(DM_\delta, \delta > 0)$ converges in $L^2(\Omega; H)$ towards

$$i\beta \int_{\mathbb{C}} dx f(x) \mu(x) C(x, \cdot).$$

Firstly, since

$$C(x, \cdot) = \sum_{k \geq 1} e_k(x) e_k(\cdot)$$

and the $e_k, k \geq 1$, form an orthonormal family of H, we have

$$\mathbb{E}\left[\left\|\int_{\mathbb{C}} dx f(x)\mu(x)C(x,\cdot) - \int_{\mathbb{C}} dx f(x)e^{i\beta\Gamma_\delta(x)+\frac{\beta^2}{2}\mathbb{E}[\Gamma_\delta(x)^2]}C(x,\cdot)\right\|_{H_{\mathbb{C}}}^2\right] \tag{2.33}$$
$$= \sum_{k\geq 1}\mathbb{E}\left[\left(\int_{\mathbb{C}} f(x)\mu(x)e_k(x)dx - \int_{\mathbb{C}} f(x)e^{i\beta\Gamma_\delta(x)+\frac{\beta^2}{2}\mathbb{E}[\Gamma_\delta(x)^2]}e_k(x)dx\right)^2\right].$$

Each single term in the above sum goes to zero as $\delta \to 0$. Moreover, using Onsager inequality for convolution approximations (2.17), one can obtain a domination in a similar manner as what we did in the first part of the proof. By the dominated convergence theorem, it implies that (2.33) goes to zero as $\delta \to 0$. Secondly,

$$\mathbb{E}\left[\left\|\int_{\mathbb{C}} dx f(x)e^{i\beta\Gamma_\delta(x)+\frac{\beta^2}{2}\mathbb{E}[\Gamma_\delta(x)^2]}\sum_{k\geq 1}(e_k*\varphi_\delta)(x)e_k - \int_{\mathbb{C}} dx f(x)e^{i\beta\Gamma_\delta(x)+\frac{\beta^2}{2}\mathbb{E}[\Gamma_\delta(x)^2]}C(x,\cdot)\right\|_{H_{\mathbb{C}}}^2\right] \tag{2.34}$$
$$= \sum_{k\geq 1}\mathbb{E}\left[\left(\int_{\mathbb{C}} f(x)e^{i\beta\Gamma_\delta(x)+\frac{\beta^2}{2}\mathbb{E}[\Gamma_\delta(x)^2]}((e_k*\varphi_\delta)(x) - e_k(x))dx\right)^2\right]$$
$$\leq C\|f\|_\infty^2\int_{K^2}|x-y|^{-\beta^2}\left|\sum_{k\geq 1}((e_k*\varphi_\delta)(x)-e_k(x))((e_k*\varphi_\delta)(y)-e_k(y))\right|dxdy$$

where K is as before the support of f. The above integrand is dominated by the integrable function $C|x-y|^{-\beta^2}\log(c/|x-y|)$. Dominated convergence theorem thus implies that (2.34) goes to zero as $\delta \to 0$. Putting things together, we have shown the aforementioned convergence: $(DM_\delta, \delta > 0)$ converges in $L^2(\Omega; H)$ towards

$$i\beta\int_{\mathbb{C}} dx f(x)\mu(x)C(x,\cdot).$$

With (2.32), we notice that $\sup_\delta \mathbb{E}\left[\|DM_\delta\|_{H_{\mathbb{C}}}^2\right] < \infty$ and Lemma 2.30 also shows that $(DM_\delta, \delta > 0)$ converges to DM in the weak topology of $L^2(\Omega; H)$. This yields

$$DM = i\beta\int_{\mathbb{C}} dx f(x)\mu(x)C(x,\cdot). \qquad \square$$

Chapter 3

Critical Brownian multiplicative chaos

Brownian multiplicative chaos measures, introduced in [Jeg20a, AHS20, BBK94], are random Borel measures that can be formally defined by exponentiating γ times the square root of the local times of planar Brownian motion. So far, only the subcritical measures where the parameter γ is less than 2 were studied. This article considers the critical case where $\gamma = 2$, using three different approximation procedures which all lead to the same universal measure. On the one hand, we exponentiate the square root of the local times of small circles and show convergence in the Seneta–Heyde normalisation as well as in the derivative martingale normalisation. On the other hand, we construct the critical measure as a limit of subcritical measures. This is the first example of a non-Gaussian critical multiplicative chaos.

We are inspired by methods coming from critical Gaussian multiplicative chaos, but there are essential differences, the main one being the lack of Gaussianity which prevents the use of Kahane's inequality and hence a priori controls. Instead, a continuity lemma is proved which makes it possible to use tools from stochastic calculus as an effective substitute.

3.1 Introduction

Thick points of planar Brownian motion/random walk are points that have been visited unusually often by the trajectory. The study of these points has a long history going back to the famous conjecture of Erdős and Taylor [ET60] on the leading order of the number of times a planar simple random walk visits the most visited site during the first n steps. Since then, the understanding of these thick points has considerably improved. On the random walk side, [DPRZ01] settled Erdős–Taylor conjecture and computed the number of thick points at the level of exponent, for random walk having symmetric increments with finite moments of all order. [Ros05, BR07], and more recently [Jeg20b], streamlined the proof and extended these results to a wide class of planar random walk. On the Brownian motion side, [BBK94] constructed random measures supported on the set of thick points. Their results concern only a partial range $\{a \in (0, 1/2)\}$ of the thickness parameter a[1]. [AHS20] and [Jeg20a] extended simultaneously the results of [BBK94] by building these random measures for the whole subcritical range $\{a \in (0, 2)\}$. [Jeg19] gave an axiomatic characterisation of these measures and showed that they describe the scaling limit of thick points of planar simple random walk for any fixed $a < 2$. All these aforementioned works are subcritical results. The aim of this paper is to extend the theory

[1] a is related to the parameter γ in Gaussian multiplicative chaos theory by $a = \gamma^2/2$, so $a < 1/2$ corresponds to $\gamma < 1$.

to the critical point $a = 2$ by constructing a random measure supported by the thickest points of a planar Brownian trajectory. This enables us to formulate a precise conjecture on the convergence in distribution of the supremum of local times of planar random walk.

Our construction is inspired by Gaussian multiplicative chaos theory (GMC), i.e. the study of random measures formally defined as the exponential of γ times a log-correlated Gaussian field, such as the two-dimensional Gaussian free field (GFF), where $\gamma \geq 0$ is a parameter. Since such a field is not defined pointwise but is rather a random generalised function, making sense of such a measure requires some nontrivial work. The theory was introduced by Kahane [Kah85] and has expanded significantly in recent years. By now it is relatively well understood, at least in the subcritical case where $\gamma < \sqrt{2d}$ [RV10, DS11, RV11, Sha16, Ber17] and even in the critical case $\gamma = \sqrt{2d}$ [DRSV14b, DRSV14a, JS17, JSW19, Pow18, APS19, APS20]. In this article, the log-correlated field we have in mind is the (square root of) the local time process of a planar Brownian motion, appropriately stopped. The main interest of our construction from GMC point of view is that this field is non-Gaussian, so that our results give the first example of a critical chaos for a truly non-Gaussian field.[2]

3.1.1 Main results

Let $\mathbb{P}_x$ be the law under which $(B_t)_{t\geq 0}$ is a planar Brownian motion starting from $x \in \mathbb{R}^2$. Let $D \subset \mathbb{R}^2$ be an open bounded simply connected domain, $x_0 \in D$ be a starting point and τ be the first exit time of D:

$$\tau := \inf\{t \geq 0 : B_t \notin D\}.$$

For all $x \in \mathbb{R}^2, t > 0, \varepsilon > 0$, define the local time $L_{x,\varepsilon}(t)$ of $(|B_s - x|, s \geq 0)$ at ε up to time t (here $|\cdot|$ stands for the Euclidean norm):

$$L_{x,\varepsilon}(t) := \lim_{r \to 0^+} \frac{1}{2r} \int_0^t \mathbf{1}_{\{\varepsilon - r \leq |B_s - x| \leq \varepsilon + r\}} ds. \tag{3.1}$$

[Jeg20a, Proposition 1.1] shows that we can make sense of the local times $L_{x,\varepsilon}(\tau)$ simultaneously for all x and ε with the convention that $L_{x,\varepsilon}(\tau) = 0$ if the circle $\partial D(x, \varepsilon)$ is not entirely included in D. We can thus define for any thickness parameter $\gamma \in (0, 2]$ and any Borel set A,

$$m_\varepsilon^\gamma(A) := \sqrt{|\log \varepsilon|} \varepsilon^{\gamma^2/2} \int_A e^{\gamma \sqrt{\frac{1}{\varepsilon} L_{x,\varepsilon}(\tau)}} dx. \tag{3.2}$$

We recall:

Theorem A (Theorem 1.1 of [Jeg20a])**.** *Let $\gamma \in (0, 2)$. The sequence of random measures m_ε^γ converges as $\varepsilon \to 0$ in probability for the topology of weak convergence on D towards a Borel measure m^γ called Brownian multiplicative chaos.*

See [AHS20] for a different construction of the subcritical Brownian multiplicative chaos, as well as [BBK94] for partial results. See also [Jeg19] for more properties on these measures.

[2] We point out the work of [SW20] on the Riemann zeta function where the limiting field is Gaussian, but not the approximation. See also [FK14, Web15, NSW18, LOS18, BWW18, Jun18] for subcritical results.

Our first result towards extending the theory to the critical point $\gamma = 2$ is the fact that the subcritical normalisation yields a vanishing measure in the critical case:

Proposition 3.1. *$m_\varepsilon^{\gamma=2}(D)$ converges in $\mathbb{P}_{x_0}$-probability to zero.*

To obtain a non-trivial object we thus need to renormalise the measure slightly differently. Firstly, we consider the Seneta–Heyde normalisation: for all Borel set A, define

$$m_\varepsilon(A) := \sqrt{|\log \varepsilon|} m_\varepsilon^{\gamma=2}(A) = |\log \varepsilon| \varepsilon^2 \int_A e^{2\sqrt{\frac{1}{\varepsilon} L_{x,\varepsilon}(\tau)}} dx. \tag{3.3}$$

Secondly, we consider the derivative martingale normalisation which formally corresponds to (minus) the derivative of m_ε^γ with respect to γ evaluated at $\gamma = 2$: for all Borel set A, define

$$\mu_\varepsilon(A) := -\frac{\mathrm{d} m_\varepsilon^\gamma(A)}{\mathrm{d}\gamma}\Big|_{\gamma=2} = \sqrt{|\log \varepsilon|} \varepsilon^2 \int_A \left(-\sqrt{\frac{1}{\varepsilon} L_{x,\varepsilon}(\tau)} + 2 \log \frac{1}{\varepsilon} \right) e^{2\sqrt{\frac{1}{\varepsilon} L_{x,\varepsilon}(\tau)}} dx. \tag{3.4}$$

Theorem 3.2. *The sequences of random positive measures $(m_\varepsilon)_{\varepsilon>0}$ and random signed measures $(\mu_\varepsilon)_{\varepsilon>0}$ converge in $\mathbb{P}_{x_0}$-probability for the topology of weak convergence towards random Borel measures m and μ. Moreover, the limiting measures satisfy:*

1. *$m = \sqrt{\frac{2}{\pi}} \mu$ $\mathbb{P}_{x_0}$-a.s. In particular, μ is a random positive measure.*
2. *Nondegeneracy: $\mu(D) \in (0, \infty)$ $\mathbb{P}_{x_0}$-a.s.*
3. *First moment: $\mathbb{E}_{x_0} [\mu(D)] = \infty$.*
4. *Nonatomicity: $\mathbb{P}_{x_0}$-a.s. simultaneously for all $x \in D$, $\mu(\{x\}) = 0$.*

Our next main result is the construction of critical Brownian multiplicative chaos as a limit of subcritical measures. Before stating such a result, we need to ensure that we can make sense of the subcritical measures simultaneously for all $\gamma \in (0, 2)$.

Proposition 3.3. *Let $\mathcal{M}$ be the set of finite Borel measures on $\mathbb{R}^2$. The process $\gamma \in (0, 2) \mapsto m^\gamma \in \mathcal{M}$ of subcritical Brownian multiplicative chaos measures possesses a modification such that for all continuous nonnegative function f, $\gamma \in (0, 2) \mapsto \int f dm^\gamma \in \mathbb{R}$ is lower semi-continuous.*

Theorem 3.4. *Let $\gamma \in (0, 2) \mapsto m^\gamma$ be the process of subcritical Brownian multiplicative chaos measures from Proposition 3.3. Then, $(2-\gamma)^{-1} m^\gamma$ converges towards 2μ as $\gamma \to 2^-$ in probability for the topology of weak convergence of measures.*

Remark 3.5. In Proposition 3.3, we do not obtain continuity of the process in γ. The main difficulty here is that, in order to use Kolmogorov's continuity theorem, one has to consider moments of order larger than 1. When $\gamma \geq \sqrt{2}$, the second moment blows up and we have to deal with non-integer moments which are difficult to estimate without the use of Kahane's convexity inequalities but this tool is restricted to the Gaussian setting. To bypass this difficulty, we apply Kolmogorov's criterion to versions of the measures that are restricted to specific 'good' events allowing us to make L^2-computations. The drawback is that it does not yield continuity of the process but only lower semi-continuity. See Appendix 3.B.

We mention that the construction of the critical measure as a limit of subcritical measures is only partially known in the GMC realm. Such a result has first been proved to hold in the specific case of the two-dimensional GFF [APS19] exploiting on the one hand the construction of Liouville measures as multiplicative cascades [APS20] and on the other hand the strategy of Madaule [Mad16] who proves a result analogous to Theorem 3.4 in the case of multiplicative cascades/branching random walk. It has then been extended to a wide class of log-correlated Gaussian fields in dimension two by comparing them to the GFF [JSW19]. In other dimensions, a natural reference log-correlated Gaussian field is lacking and the result is so far unknown. We believe that the approach we use in this paper to prove Theorem 3.4 can be adapted in order to show that critical GMC measures can be built from their subcritical versions in *any* dimension.[3]

Theorem 3.4 can be seen as exchanging the limit in ε and the derivative with respect to γ. Surprisingly, a factor of 2 pops up when one exchanges the two:

$$\lim_{\gamma \to 2^-} \lim_{\varepsilon \to 0} \frac{(m_\varepsilon^\gamma - m_\varepsilon^2)}{2-\gamma} = \lim_{\gamma \to 2^-} \frac{1}{2-\gamma} m^\gamma = 2 \lim_{\varepsilon \to 0} \mu_\varepsilon = 2 \lim_{\varepsilon \to 0} \lim_{\gamma \to 2^-} \frac{(m_\varepsilon^\gamma - m_\varepsilon^2)}{2-\gamma}.$$

This factor of 2 is present as well in the context of GMC [APS19, JSW19] and cascades [Mad16].

Theorem 3.4 is important because it hints at the universal nature of the measure μ, in the following sense. First, recall that the article [Jeg19] gives an axiomatic characterisation of the subcritical measures m^γ implying their universality in the sense that different approximations yield the same limiting measures. Thus, Theorem 3.4 can be seen as showing a form of universality for μ as well. Furthermore, the subcritical measures m^γ are known to be conformally covariant [Jeg20a, AHS20] and Theorem 3.4 allows us to extend this conformal covariance to the critical measures.

Corollary 3.6. *Let $\phi : D \to D'$ be a conformal map between two bounded simply connected domains. Let $x_0 \in D$ and denote by μ^D and $\mu^{D'}$ the critical Brownian multiplicative chaos measures built in Theorem 3.2 for the domains (D, x_0) and $(D', \phi(x_0))$ respectively. Then we have*

$$(\mu^D \circ \phi^{-1})(dx) \overset{\text{law}}{=} \left| (\phi^{-1})'(x) \right|^4 \mu^{D'}(dx).$$

Proof. Let $\gamma \in (0,2)$ and denote by $m^{\gamma,D}$ and $m^{\gamma,D'}$ the subcritical measures built in Theorem A for the domains (D, x_0) and $(D', \phi(x_0))$ respectively. By [Jeg20a, Corollary 1.4 *(iv)*], it is known that

$$(m^{\gamma,D} \circ \phi^{-1})(dx) \overset{\text{law}}{=} \left| (\phi^{-1})'(x) \right|^{2+\gamma^2/2} m^{\gamma,D'}(dx). \tag{3.5}$$

By Theorem 3.4, we obtain the desired result by dividing both sides of the above equality by $2(2-\gamma)$ and then by letting $\gamma \to 2$.

Let us note that in [Jeg20a] the conformal covariance (3.5) of the subcritical measures is stated between domains that are assumed to have a boundary composed of a finite number of analytic curves. This extra assumption was made to match the framework of [AHS20] but we emphasise that it is useless in our context. Proposition 6.2 of [Jeg20a] only requires the domain to be bounded and simply

[3] After the first version of the current paper was finished, the fact that the critical GMC measure can be obtained as a limit of the subcritical measures has been established in any dimension in [Pow20b].

connected. This proposition characterises the law of $m^{\gamma,D}$ together with the Brownian motion from which it has been built. The conformal covariance then follows from this proposition as it is written in Section 5 of [AHS20]. □

Note that we could not hope to apply directly the approach used in the subcritical case to prove conformal covariance at criticality. Indeed, in the subcritical regime, this is based on a characterisation of the law of the couple formed by the measure together with the Brownian motion from which it has been built. This characterisation is in turn based on L^1 computations that are infinite at criticality (Theorem 3.2, point 3).

3.1.2 Conjecture on the supremum of local times of random walk

In recent years, much effort has been put in the study of the supremum of log-correlated fields, the ultimate goal being the convergence in distribution of the supremum properly centred. In many examples, the limiting law is a Gumbel distribution randomly shifted by the log of the total mass of an associated critical chaos. This has been established for example in the following instances: branching random walk [Aï13], local times of random walk on regular trees [Abe18], cover time of binary trees [CLS18, DRZ19], discrete GFF [BDZ16], log-correlated Gaussian field [Mad15, DRZ17]. See [Arg17, Shi15] and [BL16, Section 2] for more references. By analogy with these results, it is natural to make the following conjecture that we present in the more natural setting of random walk.

For $x \in \mathbb{Z}^2$ and $N \geq 1$, let ℓ_x^N be the total number of times a planar simple random walk starting from the origin has visited the vertex x before exiting the square $[-N,N]^2$. Define a random Borel measure μ_N on $\mathbb{R}^2 \times \mathbb{R}$ by setting for all Borel sets $A \subset \mathbb{R}^2$ and $T \subset \mathbb{R}$,

$$\mu_N(A \times T) := \sum_{x \in \mathbb{Z}^2} \mathbf{1}_{\{x/N \in A\}} \mathbf{1}_{\left\{ \sqrt{\ell_x^N} - 2\pi^{-1/2} \log N + \pi^{-1/2} \log \log N \in T \right\}}.$$

Conjecture 3.7. *There exist constants $c_1, c_2 > 0$ such that $(\mu_N, N \geq 1)$ converges in distribution for the topology of vague convergence on $\mathbb{R}^2 \times (\mathbb{R} \cup \{+\infty\})$ towards the Poisson point process*

$$\mathrm{PPP}(c_1 \mu \otimes c_2 e^{-c_2 t} dt)$$

where μ is the critical Brownian multiplicative chaos in the domain $[-1,1]^2$ with the origin as a starting point. In particular, for all $t \in \mathbb{R}$,

$$\mathbb{P}\left(\sup_{x \in \mathbb{Z}^2} \sqrt{\ell_x^N} \leq \frac{2}{\sqrt{\pi}} \log N - \frac{1}{\sqrt{\pi}} \log \log N + t \right) \xrightarrow[N \to \infty]{} \mathbb{E}\left[\exp\left(-c_1 \mu([-1,1]^2) e^{-c_2 t} \right) \right].$$

The leading order term $2\pi^{-1/2} \log N$ has been conjectured by Erdős and Taylor [ET60] and proven by [DPRZ01]. See also [Ros05, BR07, Jeg20b]. We expect $-\pi^{-1/2} \log \log N$ to be the second order term since, with this choice of constant, the expectation of $\mu_N(\mathbb{R}^2 \times (0,\infty))$ blows up like $\log N$. Indeed, in analogy with the case of the 2D discrete GFF (see [BL16]), this should be the correct way of scaling the point measure to get a nondegenerate limit.

Let us compare this conjecture with the case of the 2D discrete GFF $(\phi_N(x))_{x \in \mathbb{Z}^2}$, that is the centred Gaussian vector whose covariance is given by $\mathbb{E}[\phi_N(x)\phi_N(y)] = \mathbb{E}_x\left[\ell_y^N\right]$. [BDZ16] (see [BL20] for the link with Liouville measure) showed that for all $t \in \mathbb{R}$,

$$\mathbb{P}\left(\sup_{x\in\mathbb{Z}^2} \frac{1}{\sqrt{2}}\phi_N(x) \leq \frac{2}{\sqrt{\pi}}\log N - \frac{3}{4\sqrt{\pi}}\log\log N + t\right) \xrightarrow[N\to\infty]{} \mathbb{E}\left[\exp\left(-c_1\mu^L([-1,1]^2)e^{-c_2 t}\right)\right]$$

where $c_1, c_2 > 0$ are some constants and μ^L is the Liouville measure in $[-1,1]^2$. Despite strong links between local times and half of the GFF squared (see lecture notes [Ros14] for an overview of the topic), Conjecture 3.7 would show that the supremum of the former is slightly smaller than the supremum of the latter, enhancing subtle differences between the two fields (see [Jeg19, Corollary 1.1] and [Jeg20a, Corollary 1.1] for results in this direction).

Let us mention that [Jeg20b] shows results analogous to Conjecture 3.7 in dimensions larger or equal to three and that [Jeg19] establishes the subcritical analogue of Conjecture 3.7 in dimension two. A first step towards solving Conjecture 3.7 might be to give a characterisation of the law of critical Brownian multiplicative chaos analogous to the subcritical characterisation of [Jeg19]. Since the first moment blows up, fixing the normalisation of the measure is one of the main challenges in this regard.

3.1.3 Proof outline

We now explain the main ideas and difficulties of the proof of Theorems 3.2 and 3.4.

We start by recalling that, as noticed in [Jeg20a], if the domain D is a disc $D = D(x,\eta)$ centred at x, then the local times $L_{x,r}(\tau), r > 0$, exhibit the following Markovian structure: for all $\eta' \in (0,\eta)$ and all $z \in D(0,\eta)\backslash D(0,\eta')$, under $\mathbb{P}_z$ and conditioned on $L_{x,\eta'}(\tau)$,

$$\left(\sqrt{\frac{1}{r}L_{x,r}(\tau)}, r = \eta' e^{-s}, s \geq 0\right) \overset{\text{law}}{=} (X_s, s \geq 0) \tag{3.6}$$

with $(X_s, s \geq 0)$ being a zero-dimensional Bessel process starting from $\sqrt{L_{x,\eta'}(\tau)/\eta'}$. This is an easy consequence of rotational invariance of Brownian motion and second Ray-Knight isomorphism for local times of one-dimensional Brownian motion. In order to exploit this relation, we will very often stop the Brownian trajectory at the first exit time $\tau_{x,R}$ of the disc $D(x,R)$, R being the diameter of the domain D.

What makes the critical case so special is that the approximating measures are not normalised by the first moment any more (otherwise we would get a vanishing measure as shown in Proposition 3.1). We thus need to introduce good events before being able to even make L^1-computations. Defining the right events and showing that they do not change the measures with high probability is one of the crucial steps of this paper that we are about to explain. We first explain the most natural events to consider and we then explain why we will actually consider different events.

Naive definition of good events In analogy with the case of log-correlated Gaussian fields, it is natural to consider the following events to make the measures bounded in L^1: let $\beta > 0$ be large and for all $x \in D$ and $\varepsilon > 0$, define

$$G_\varepsilon(x) := \left\{ \forall \delta \in [\varepsilon, 1], \sqrt{\frac{1}{\delta} L_{x,\delta}(\tau_{x,R})} \leq 2 \log \frac{1}{\delta} + \beta \right\}.$$

Here, we stop the Brownian path at time $\tau_{x,R}$ to be able to use (3.6). One would expect that as $\beta \to \infty$, $\mathbb{P}_{x_0}\left(\bigcap_{x \in D} \bigcap_{\varepsilon > 0} G_\varepsilon(x)\right) \to 1$ since, by analogy with the Gaussian case (see [Pow18, Corollary 2.4] for instance), the following should hold true:

$$\sup_{x \in D} \sup_{\varepsilon > 0} \left(\sqrt{\frac{1}{\varepsilon} L_{x,\varepsilon}(\tau_{x,R})} - 2 \log \frac{1}{\varepsilon} \right) < \infty \qquad \mathbb{P}_{x_0} - \text{a.s.} \tag{3.7}$$

Because of the lack of self-similarity and Gaussianity of our model, showing (3.7) turns out to be far from easy (see the introduction of Section 3.4 for more about this). We thus take a detour to justify that the introduction of the events $G_\varepsilon(x)$ is harmless. We first control the supremum of the more regular local times of small annuli allowing us to introduce good events associated to these local times. Crucially, these good events will be enough to make the measures bounded in L^1. Using *repulsion estimates* associated to zero-dimensional Bessel process X, we will finally be able to transfer the restrictions on the local times of annuli (requiring for all $k \geq 0$, $\min_{[k,k+1]} X \leq 2k + 2\log(k) + \beta/2$) over to restrictions on the local times of circles (requiring for all $s \geq 0$, $X_s \leq 2s + \beta$). This is the content of Section 3.4.

Other repulsion estimates with a similar flavour will tell us that, once we restrict ourselves to the events $G_\varepsilon(x)$, we will be able to restrict further the measures to the good events

$$G'_\varepsilon(x) := \left\{ \forall \delta \in [\varepsilon, 1], \sqrt{\frac{1}{\delta} L_{x,\delta}(\tau_{x,R})} \leq 2 \log \frac{1}{\delta} + \beta - \frac{\sqrt{|\log \delta|}}{M \log(2 + |\log \delta|)^2} \right\}$$

for some large $M > 0$. This is the content of Lemma 3.15. This second layer of good event will make the measures bounded in L^2 (Proposition 3.16). We will conclude the proof by showing that the measures restricted to the second layer of good events converge in L^2 (Proposition 3.17).

Actual definition of good events We now explain why we actually define different good events. This paper extensively uses the relation (3.6) between local times and zero-dimensional Bessel process. When making L^1-computations, we will bound from above the local times $L_{x,\varepsilon}(\tau)$ by $L_{x,\varepsilon}(\tau_{x,R})$ and we will use directly (3.6). Difficulties arise when we start to make L^2-computations since we need to consider local times at two different centres. We will resolve this issue with the following reasoning. Consider a Brownian excursion from $\partial D(x, 1)$ to $\partial D(x, 2)$ and condition on the initial and final points of the excursion (this will be important to keep track of the number of excursions). Because of this conditioning, rotational symmetry is broken and the law of the local times $(L_{x,\delta}(\tau_{x,2}), \delta \leq 1)$ is no longer given by a zero-dimensional Bessel process. But if we condition further on the fact that the excursion went deep inside $D(x, 1)$, then it will have forgotten its starting position and the law of $(L_{x,\delta}(\tau_{x,2}), \delta \leq 1)$ will be very close to the one given in (3.6). This is the content of the **continuity lemma** (Lemma 3.20) which is a much more precise version of [Jeg20a, Lemma 5.1] giving a quantitative estimate of the error in the aforementioned approximation. Importantly, this approximation cannot be true if we look at the local times $L_{x,\delta}(\tau_{x,2})$ for *all* radii $\delta \leq 1$. Instead, we must restrict ourselves to dyadic radii $\delta \in \{e^{-n}, n \geq 0\}$ so that the Brownian path has enough space to forget its initial position. See Remark 3.21. This is one reason why we cannot define the good events $G_\varepsilon(x)$ and $G'_\varepsilon(x)$ using this continuum of radii. Another reason is that it would prevent us from decoupling the two-point estimates needed in the proof of Proposition 3.17 (see especially (3.64)).

Moreover, we will not define the good events using only local times at dyadic radii neither. Indeed, doing so would then require us to estimate probabilities associated to zero-dimensional Bessel process evaluated at discrete times. These probabilities are much harder to estimate than their continuous time counterpart and our approach cannot afford to lose too much on these estimates (especially in the identifications of the different limiting measures). We will resolve this using the following surprising trick: we will consider a field $(h_{x,\delta}, x \in D, \delta \in (0,1])$ that interpolates the local times $\sqrt{\frac{1}{\delta} L_{x,\delta}(\tau_{x,R})}$ between dyadic radii by zero-dimensional Bessel bridges that have a very small range of dependence (see Lemma 3.13). In this way, the one-point estimates will be the same as if we considered local times at all radii but we will be able to decouple things to make the two-point computations. We believe this new idea will be useful in subsequent studies.

Paper outline The rest of the paper is organised as follows. Section 3.2 proves Theorems 3.2 and 3.4 subject to the intermediate results Proposition 3.14, Lemma 3.15 and Propositions 3.16 and 3.17. Section 3.3 collects preliminary results that will be used throughout the paper. In particular, it states and proves the continuity lemma and contains results on Bessel processes and barrier estimates associated to 1D Brownian motion. Section 3.4 proves Proposition 3.14 and Lemma 3.15 showing that we can safely add the two layers of good events. Section 3.5 is dedicated to the L^2 estimates needed to prove Proposition 3.16 and 3.17. Appendix 3.A justifies the existence of the field $(h_{x,\delta}, x \in D, \delta \in (0,1])$ interpolating local times with zero-dimensional Bessel bridges. Finally, Appendix 3.B sketches the proof of Proposition 3.3.

We end this introduction with some notations that will be used throughout the paper. We will denote:

Notation 3.8. *For $x > 0$ and $d \geq 0$, $\mathbb{P}_x^d$ and $\mathbb{E}_x^d$ the law and the expectation under which $(X_t)_{t\geq 0}$ is a d-dimensional Bessel process starting from x at time 0. $\mathbb{P}_x$ and $\mathbb{E}_x$ will denote the law and the expectation of 1D Brownian motion starting at x. Note that under $\mathbb{P}_x$, the process X takes negative and positive values, whereas the process stays nonnegative under $\mathbb{P}_x^1$.*

Notation 3.9. *For $x \in D$, k_x the smallest nonnegative integer such that $e^{-k_x} \leq |x - x_0|$;*

Notation 3.10. *R the diameter of the domain D and for $x \in D$ and $r > 0$, $\tau_{x,r}$ the first hitting time of $\partial D(x,r)$;*

Notation 3.11. *For $a_\varepsilon \in \mathbb{R}, b_\varepsilon > 0, \varepsilon > 0$, we will denote $a_\varepsilon \lesssim b_\varepsilon$ (resp. $a_\varepsilon = O(b_\varepsilon)$, resp. $a_\varepsilon = o(b_\varepsilon)$) if there exists some constant $C > 0$ such that for all $\varepsilon > 0$, $a_\varepsilon \leq C b_\varepsilon$ (resp. $|a_\varepsilon| \leq C b_\varepsilon$, resp. $a_\varepsilon / b_\varepsilon \to 0$ as $\varepsilon \to 0$). Sometimes we will emphasise the dependency on some parameter η by writing for instance $a_\varepsilon = o_\eta(b_\varepsilon)$;*

Notation 3.12. *For $x \in \mathbb{R}$, $(x)_+ = \max(x, 0)$.*

In this paper, C, c, etc. will denote generic constants that may vary from line to line.

3.2 High level proof of Theorems 3.2 and 3.4

To ease notations, we will prove the convergences stated in Theorem 3.2 along the radii $\varepsilon \in \{e^{-k}, k \geq 0\}$. The proof extends naturally to all radii $\varepsilon \in (0,1]$. In particular, in what follows we will write $\sup_{\varepsilon>0}$, $\limsup_{\varepsilon>0}$, etc. but we actually mean $\sup_{\varepsilon \in \{e^{-k}, k \geq 0\}}$, $\limsup_{\varepsilon \in \{e^{-k}, k \geq 0\}}$, etc.

We start off by defining the field $(h_{x,\delta}, x \in D, \delta \in (0,1])$ mentioned in Section 3.1.3. Recall Notation 3.10. We will also denote for any $x = (x_1, x_2) \in \mathbb{R}^2$, $\lfloor x \rfloor = (\lfloor x_1 \rfloor, \lfloor x_2 \rfloor)$.

Lemma 3.13. *By enlarging the probability space we are working on if necessary, we can construct a random field $(h_{x,\delta}, x \in D, \delta \in (0,1])$ such that*

- *for all $x \in D$, and $n \geq 0$, conditionally on $\{L_{x,\delta}(\tau_{x,R}), \delta = e^{-n}, e^{-n-1}\}$, $(h_{x,e^{-t}}, t \in [n, n+1])$ has the law of a zero-dimensional Bessel bridge from $\sqrt{e^n L_{x,e^{-n}}(\tau_{x,R})}$ to $\sqrt{e^{n+1} L_{x,e^{-n-1}}(\tau_{x,R})}$ that is independent of $(B_t, t \geq 0)$ and $(h_{y,\delta}, y \in D, \delta \notin [e^{-n-1}, e^{-n}])$;*
- *for all $n_0 \geq 0$ and $x, y \in D$, conditionally on $\{L_{z,\delta}(\tau_{z,R}), z = x, y, \delta = e^{-n}, n \geq n_0\}$, $(h_{x,\delta}, \delta \leq e^{-n_0})$ and $(h_{y,\delta}, \delta \leq e^{-n_0})$ are independent as soon as $|x - y| \geq 2e^{-n_0}$;*
- *for all $n \geq 0$ and $z \in e^{-n-10}\mathbb{Z}^2 \cap D$, $(h_{x,\delta}, x \in D, \lfloor e^{n+10} x \rfloor = e^{n+10} z, e^{-n-1} \leq \delta \leq e^{-n})$ is continuous.*

See Appendix 3.A for a proof of the existence of such a process. Note that by (3.6), for all $n_0 \geq 0$ and for all $x \in D$, conditionally on $L_{x,e^{-n_0}}(\tau_{x,R})$, $(h_{x,e^{-s-n_0}}, s \geq 0)$ has the law of a zero-dimensional Bessel process starting from $\sqrt{e^{n_0} L_{x,e^{-n_0}}(\tau_{x,R})}$.

We now introduce the good events that we will work with: let $\beta, M > 0$ be large and define for all $x \in D$ and $\varepsilon \leq |x - x_0|$, $\varepsilon = e^{-k}$,

$$G_\varepsilon(x) := \{\forall s \in [k_x, k], h_{x,e^{-s}} \leq 2s + \beta\}$$

and

$$G'_\varepsilon(x) := \left\{\forall s \in [k_x, k], h_{x,e^{-s}} \leq 2s + \beta - \frac{\sqrt{s}}{M \log(2+s)^2}\right\}.$$

If $|x - x_0| < \varepsilon$, the above good events do not impose anything by convention. Let us mention that if $\varepsilon = e^{-k-t_0}$ for some $k \geq 0$ and $t_0 \in (0,1)$, one would need to consider the process

$$s \mapsto \begin{cases} h_{x,e^{-s}} & \text{if } s \in [k_x, k], \\ \sqrt{e^s L_{x,e^{-s}}(\tau_{x,R})} & \text{if } s \in [k, k+t_0] \end{cases}$$

instead of $s \mapsto h_{x,e^{-s}}$ to define the good events when $\varepsilon \notin \{e^{-k}, k \geq 0\}$. Again, in what follows we will restrict ourselves to $\varepsilon \in \{e^{-k}, k \geq 0\}$ to ease notations.

We now consider modified versions of the measures $m_\varepsilon^\gamma, \gamma \in (0,2)$, and m_ε defined respectively in (3.2) and (3.3):

$$\hat{m}_\varepsilon^\gamma(dx) := \mathbf{1}_{G_\varepsilon(x)} m_\varepsilon^\gamma(dx), \quad \hat{\hat{m}}_\varepsilon^\gamma(dx) := \mathbf{1}_{G'_\varepsilon(x)} \mathbf{1}_{\{|x-x_0|\geq 1/M\}} \hat{m}_\varepsilon^\gamma(dx) \tag{3.8}$$

and

$$\hat{m}_\varepsilon(dx) := \mathbf{1}_{G_\varepsilon(x)} m_\varepsilon(dx), \quad \hat{\hat{m}}_\varepsilon(dx) := \mathbf{1}_{G'_\varepsilon(x)} \mathbf{1}_{\{|x-x_0|\geq 1/M\}} \hat{m}_\varepsilon(dx). \tag{3.9}$$

We also consider modified versions of the measure μ_ε defined in (3.4): for all Borel set A, set

$$\hat{\mu}_\varepsilon(A) := \sqrt{|\log \varepsilon|} \varepsilon^2 \int_A \left(-\sqrt{\frac{1}{\varepsilon} L_{x,\varepsilon}(\tau_{x,R})} + 2 \log \frac{1}{\varepsilon} + \beta \right) e^{2\sqrt{\frac{1}{\varepsilon} L_{x,\varepsilon}(\tau)}} \mathbf{1}_{G_\varepsilon(x)} dx \tag{3.10}$$

and we decompose further

$$\hat{\hat{\mu}}_\varepsilon(dx) := \mathbf{1}_{G'_\varepsilon(x)} \mathbf{1}_{\{|x-x_0|\geq 1/M\}} \hat{\mu}_\varepsilon(dx).$$

We emphasise that in (3.10) the local times are stopped at time τ or $\tau_{x,R}$ depending on whether the local time is in the exponential or not.

A first step towards the proof of Theorem 3.2 consists in showing that these changes of measures are harmless:

Proposition 3.14. *Let A be a Borel set. The following three limits hold in $\mathbb{P}_{x_0}$-probability:*

$$\limsup_{\beta \to \infty} \limsup_{\varepsilon \to 0} |\hat{m}_\varepsilon(A) - m_\varepsilon(A)| = 0, \tag{3.11}$$

$$\limsup_{\beta \to \infty} \limsup_{\varepsilon \to 0} |\hat{\mu}_\varepsilon(A) - \mu_\varepsilon(A)| = 0, \tag{3.12}$$

$$\limsup_{\beta \to \infty} \limsup_{\gamma \to 2^-} (2-\gamma)^{-1} \limsup_{\varepsilon \to 0} |\hat{m}_\varepsilon^\gamma(A) - m_\varepsilon^\gamma(A)| = 0. \tag{3.13}$$

Once the good events $G_\varepsilon(x)$ are introduced, we can perform L^1 computations. Next, we will show:

Lemma 3.15. *Let A be a Borel set and fix $\beta > 0$. We have*

$$\limsup_{M \to \infty} \limsup_{\varepsilon \to 0} \mathbb{E}_{x_0} \left[\hat{m}_\varepsilon(A) - \hat{\hat{m}}_\varepsilon(A) \right] = 0, \tag{3.14}$$

$$\limsup_{M \to \infty} \limsup_{\varepsilon \to 0} \mathbb{E}_{x_0} \left[\hat{\mu}_\varepsilon(A) - \hat{\hat{\mu}}_\varepsilon(A) \right] = 0, \tag{3.15}$$

$$\limsup_{M \to \infty} \limsup_{\gamma \to 2} (2-\gamma)^{-1} \limsup_{\varepsilon \to 0} \mathbb{E}_{x_0} \left[\hat{m}_\varepsilon^\gamma(A) - \hat{\hat{m}}_\varepsilon^\gamma(A) \right] = 0. \tag{3.16}$$

The second layer of good events makes the sequences $(\hat{\hat{m}}_\varepsilon(D), \varepsilon > 0)$, $(\hat{\hat{\mu}}_\varepsilon(D), \varepsilon > 0)$ and $((2-\gamma)^{-1} \hat{\hat{m}}_\varepsilon^\gamma(D), \gamma \in [1,2), \varepsilon < \varepsilon_\gamma)$ bounded in L^2. Here

$$\varepsilon_\gamma := \exp\left(-\exp(2/(2-\gamma))\right) \tag{3.17}$$

goes to zero very rapidly as $\gamma \to 2$. We recall that a sequence $(\nu_n, n \geq 1)$ of random Borel measures on D is tight for the topology of weak convergence on D if, and only if, the sequence $(\nu_n(D), n \geq 1)$ of real-valued random variables is tight (see [Bis20, Exercise 3.8] for instance).

Proposition 3.16. *Fix $\beta > 0$ and $M > 0$. We have*

$$\int_{D\times D} \sup_{\varepsilon>0} \mathbb{E}_{x_0}\left[\hat{\hat{m}}_\varepsilon(dx)\hat{\hat{m}}_\varepsilon(dy)\right] < \infty, \tag{3.18}$$

$$\int_{D\times D} \sup_{\varepsilon>0} \mathbb{E}_{x_0}\left[\hat{\hat{\mu}}_\varepsilon(dx)\hat{\hat{\mu}}_\varepsilon(dy)\right] < \infty, \tag{3.19}$$

$$\int_{D\times D} \sup_{\gamma\in[1,2)} (2-\gamma)^{-2} \sup_{\varepsilon<\varepsilon_\gamma} \mathbb{E}_{x_0}\left[\hat{\hat{m}}_\varepsilon^\gamma(dx)\hat{\hat{m}}_\varepsilon^\gamma(dy)\right] < \infty. \tag{3.20}$$

In particular, $\sup_{\varepsilon>0} \mathbb{E}_{x_0}\left[\hat{\hat{\mu}}_\varepsilon(D)^2\right] < \infty$ and $(\hat{\hat{\mu}}_\varepsilon, \varepsilon > 0)$ is tight for the topology of weak convergence on D. Moreover, any subsequential limit $\hat{\hat{\mu}}$ of $(\hat{\hat{\mu}}_\varepsilon, \varepsilon > 0)$ satisfies: $\mathbb{P}_{x_0}$-a.s. simultaneously for all $x \in D$, $\hat{\hat{\mu}}(\{x\}) = 0$.

Finally, we will show:

Proposition 3.17. *Fix $\beta > 0$ and $M > 0$ and let A be a Borel set. Let $(\gamma_n, n \geq 1) \in [1,2)^{\mathbb{N}}$ be a sequence converging to 2.*

1. *$(\hat{\hat{m}}_\varepsilon(A), \varepsilon > 0)$, $(\hat{\hat{\mu}}_\varepsilon(A), \varepsilon > 0)$ and for all $n \geq 1$, $(\hat{\hat{m}}_\varepsilon^{\gamma_n}(A), \varepsilon < \varepsilon_{\gamma_n})$ are Cauchy sequences in L^2. Let $\hat{\hat{m}}(A)$, $\hat{\hat{\mu}}(A)$ and $\hat{\hat{m}}^{\gamma_n}(A), n \geq 1$, be the limiting random variables.*
2. *$\hat{\hat{m}}(A) = \sqrt{2/\pi}\hat{\hat{\mu}}(A) \quad \mathbb{P}_{x_0}$-a.s.*
3. *$(2-\gamma_n)^{-1}\hat{\hat{m}}^{\gamma_n}(A)$ converges in L^2 towards $2\hat{\hat{\mu}}(A)$ as $n \to \infty$.*

We now have all the ingredients to prove Theorems 3.2 and 3.4.

Proof of Theorems 3.2 and 3.4. Let A be a Borel set. Let $\beta > 0$. For all $M > 0$, we have

$$\begin{aligned} &\limsup_{\varepsilon,\delta\to 0} \mathbb{E}_{x_0}\left[|\hat{\mu}_\varepsilon(A) - \hat{\mu}_\delta(A)|\right] \\ &\leq 2\limsup_{\varepsilon\to 0} \mathbb{E}_{x_0}\left[|\hat{\mu}_\varepsilon(A) - \hat{\hat{\mu}}_\varepsilon(A)|\right] + \limsup_{\varepsilon,\delta\to 0} \mathbb{E}_{x_0}\left[|\hat{\hat{\mu}}_\varepsilon(A) - \hat{\hat{\mu}}_\delta(A)|^2\right]^{1/2}. \end{aligned}$$

By Proposition 3.17, the second right hand side term vanishes whereas by Lemma 3.15 the first right hand side term goes to zero as $M \to \infty$. The left hand side term being independent of M, it has to vanish. In other words, $(\hat{\mu}_\varepsilon(A), \varepsilon > 0)$ converges in L^1 towards some $\hat{\mu}(A,\beta)$ (we keep track of the dependence in β here). Let $\hat{\mu}(A,\infty)$ be the almost sure limit of the nondecreasing sequence $\hat{\mu}(A,\beta)$ as $\beta \to \infty$. We now have for any small $\rho > 0$ and large $\beta > 0$,

$$\begin{aligned} &\limsup_{\varepsilon\to 0} \mathbb{P}_{x_0}\left(|\mu_\varepsilon(A) - \hat{\mu}(A,\infty)| > \rho\right) \leq \limsup_{\varepsilon\to 0} \mathbb{P}_{x_0}\left(|\mu_\varepsilon(A) - \hat{\mu}_\varepsilon(A,\beta)| > \rho/3\right) \\ &+ \limsup_{\varepsilon\to 0} \mathbb{P}_{x_0}\left(|\hat{\mu}_\varepsilon(A,\beta) - \hat{\mu}(A,\beta)| > \rho/3\right) + \mathbb{P}_{x_0}\left(|\hat{\mu}(A,\beta) - \hat{\mu}(A,\infty)| > \rho/3\right). \end{aligned}$$

The second right hand side term vanishes since $(\hat{\mu}_\varepsilon(A,\beta), \varepsilon > 0)$ converges (in L^1) towards $\hat{\mu}(A,\beta)$. The third term goes to zero as $\beta \to \infty$ since $(\hat{\mu}(A,\beta), \beta > 0)$ converges (almost surely) to $\hat{\mu}(A,\infty)$.

The first term goes to zero as $\beta \to \infty$ by Proposition 3.14. We have thus obtained the convergence in $\mathbb{P}_{x_0}$-probability of $(\mu_\varepsilon(A), \varepsilon > 0)$.

Let $(\gamma_n, n \geq 1) \in [1,2)^{\mathbb{N}}$ be a sequence converging to 2. By mimicking the above lines, Proposition 3.14, Lemma 3.15 and Proposition 3.17 imply that

$$\lim_{\varepsilon \to 0} \left(m_\varepsilon(A) - \sqrt{\frac{2}{\pi}} \mu_\varepsilon(A) \right) = 0 \quad \text{and} \quad \lim_{n \to \infty} \lim_{\varepsilon \to 0} \left(\frac{1}{2 - \gamma_n} m_\varepsilon^{\gamma_n}(A) - 2\mu_\varepsilon(A) \right) = 0$$

in $\mathbb{P}_{x_0}$-probability. By [Jeg20a], we already know that $(m_\varepsilon^{\gamma_n}(A), \varepsilon > 0)$ converges to $m^{\gamma_n}(A)$ in probability. We have thus obtained the convergence in probability of $(m_\varepsilon(A), \varepsilon > 0), (\mu_\varepsilon(A), \varepsilon > 0)$ and $((2-\gamma_n)^{-1} m^{\gamma_n}(A), n \geq 1)$ and the limits satisfy

$$\lim_{\varepsilon \to 0} m_\varepsilon(A) = \sqrt{\frac{2}{\pi}} \lim_{\varepsilon \to 0} \mu_\varepsilon(A) \quad \text{and} \quad \lim_{n \to \infty} \frac{1}{2 - \gamma_n} m^{\gamma_n}(A) = 2 \lim_{\varepsilon \to 0} \mu_\varepsilon(A).$$

Obtaining the convergence of the measures and the identification of the limiting measures as stated in Theorems 3.2 and 3.4 is now routine.

The only points that remained to be checked are points 2-4 of Theorem 3.2. Point 4 follows from the fact that any subsequential limit $\hat{\hat{\mu}}$ of $(\hat{\hat{\mu}}_\varepsilon, \varepsilon > 0)$ are non-atomic (see Proposition 3.16) and that $\mu(D) - \hat{\hat{\mu}}(D)$ is as small as desired (in probability, by tuning the parameters β and M) by Proposition 3.14 and Lemma 3.15. We now turn to Point 3. Since $(\hat{m}_\varepsilon(D), \varepsilon > 0)$ converges in L^1 towards $\hat{m}(D)$, $\mathbb{E}_{x_0}[\hat{m}(D)] = \lim_{\varepsilon \to 0} \mathbb{E}_{x_0}[\hat{m}_\varepsilon(D)]$. Now, by monotonicity, $\mathbb{E}_{x_0}[m(D)] \geq \lim_{\beta \to \infty} \lim_{\varepsilon \to 0} \mathbb{E}_{x_0}[\hat{m}_\varepsilon(D)]$ which is infinite by (3.49).

Finally, let us prove Point 2 of Theorem 3.2. The fact that $\mu(D)$ is finite $\mathbb{P}_{x_0}$-a.s. follows directly from Proposition 3.14 and Lemma 3.28. We now want to show that it is positive $\mathbb{P}_{x_0}$-a.s. By Point 3 of Theorem 3.2, we already know that it is positive with a positive probability. We are going to bootstrap this to obtain a probability equal to 1. Let $p \geq 1$ and consider the sequence of stopping times defined by $\sigma_0^{(2)} = 0$ and for all $i \geq 1$,

$$\sigma_i^{(1)} := \inf\{t > \sigma_{i-1}^{(2)}, |B_t - x_{i-1}| = 2^{-p}\}, \quad \sigma_i^{(2)} := \inf\{t > \sigma_i^{(1)}, |B_t - x_0| = 2^{-p+1} i\}$$

and $x_i := B_{\sigma_i^{(2)}}$. For $i \geq 0$, let μ_i be the critical Brownian multiplicative chaos in the domain $(D(x_i, 2^{-p}), x_i)$ between the times $\sigma_i^{(2)}$ and $\sigma_{i+1}^{(1)}$. Let $I := \lfloor \mathrm{d}(x_0, \partial D) 2^p / 10 \rfloor$. Since $\mu \leq \sum_{i=0}^{I} \mu_i$, we have

$$\mathbb{P}_{x_0}(\mu(D) = 0) \leq \mathbb{P}_{x_0}\left(\forall i = 0 \ldots I, \mu_i(D(x_i, 2^{-p})) = 0\right).$$

By Markov property and translation invariance, the probability on the right hand side is equal to

$$\mathbb{P}_{x_0}\left(\mu_0(D(x_0, 2^{-p})) = 0\right)^{I+1}.$$

By scaling of critical Brownian multiplicative chaos coming from Corollary 3.6, the probability $\mathbb{P}_{x_0}(\mu_0(D(x_0, 2^{-p})) = 0)$ does not depend on p. Moreover, thanks to Theorem 3.2, Point 3, it is strictly less than one. By letting $p \to \infty$, we thus deduce that $\mathbb{P}_{x_0}(\mu(D) = 0) = 0$ concluding the proof. □

Proposition 3.1 now follows:

Proof of Proposition 3.1. Recall that $m_\varepsilon^{\gamma=2}(D) = m_\varepsilon(D)/\sqrt{|\log \varepsilon|}$. By Theorem 3.2, $(m_\varepsilon(D), \varepsilon > 0)$ converges in $\mathbb{P}_{x_0}$-probability towards a nondegenerate random variable. Hence $(m_\varepsilon^{\gamma=2}(D), \varepsilon > 0)$ converges in $\mathbb{P}_{x_0}$-probability to zero as desired. □

The remaining of the paper is devoted to the proof of the above intermediate statements.

3.3 Preliminaries

3.3.1 Local times as exponential random variables

In this short section we recall some results of [Jeg20a] that allow us to approximate local times of circles by exponential random variables. We start by recalling the behaviour of the Green function.

Lemma 3.18 ([Jeg20a], Lemma 2.1)**.** *For all $x \in \mathbb{C}$, $r > \varepsilon > 0$ and $y \in \partial D(x, \varepsilon)$, we have:*

$$\mathbb{E}_y \left[L_{x,\varepsilon}(\tau_{\partial D(x,r)}) \right] = 2\varepsilon \log \frac{r}{\varepsilon}. \tag{3.21}$$

In the following lemma, we denote by $\mathrm{CR}(x, D)$ the conformal radius of D seen from x and by G_D the Green function of D with Dirichlet boundary conditions normalised so that $G_D(x, y) \sim -\log|x - y|$ as $x \to y$. Recall also Notation 3.10.

Lemma 3.19. *Let $\eta > 0$, $x \in D$ and $\varepsilon > 0$ such that the disc $D(x, \varepsilon)$ is included in D and is at distance at least η from ∂D. Let $y \in \partial D(x, \varepsilon)$. Then $L_{x,\varepsilon}(\tau)$ under $\mathbb{P}_y$ stochastically dominates and is stochastically dominated by exponential variables with mean*

$$2\varepsilon \log \frac{\mathrm{CR}(x, D)}{\varepsilon} + o_\eta(\varepsilon).$$

In particular,

$$\mathbb{E}_y \left[e^{2\sqrt{\frac{1}{\varepsilon} L_{x,\varepsilon}(\tau)}} \right] = (1 + o_\eta(1)) 2\sqrt{2\pi}\, \mathrm{CR}(x, D)^2 \sqrt{|\log \varepsilon|} \varepsilon^{-2}. \tag{3.22}$$

Moreover, if $x_0 \notin D(x, \varepsilon)$,

$$\mathbb{P}_{x_0} (\tau_{x,\varepsilon} < \tau) = (1 + o_\eta(1)) \frac{G_D(x_0, x)}{|\log \varepsilon|}. \tag{3.23}$$

Proof. (3.23) is part of [Jeg20a, Lemma 2.2]. The claim about the stochastic dominations is a consequence of [Jeg20a, Section 2] as explained at the beginning of the proof of [Jeg20a, Proposition 3.1]. (3.22) is then an easy computation with exponential variables. □

3.3.2 Continuity lemma

We now state a refinement of Lemma 5.1 of [Jeg20a]. We indeed need a quantitative estimate on the error that we make when we forget about the exit point of the excursion.

Lemma 3.20. *Let $k, k', n \geq 0$ with $k' \geq k+1$ and $n \geq k'-k$. Denote $\eta = e^{-k}$, $\eta' = e^{-k'}$ and for all $i = 1 \dots k'-k$, $r_i = \eta e^{-i}$. Consider $0 < r_n < \dots < r_{k'-k+1} < r_{k'-k} = \eta'$ and for $i = 1 \dots n$, $T_i \in \mathcal{B}([0,\infty))$. For any $y \in \partial D(0, \eta/e)$, we have*

$$1 - p(\eta'/\eta) \leq \frac{\mathbb{P}_y \left(\forall i = 1 \dots n, L_{0,r_i}(\tau_{0,\eta}) \in T_i | \tau_{0,\eta'} < \tau_{0,\eta}, B_{\tau_{0,\eta}} \right)}{\mathbb{P}_y \left(\forall i = 1 \dots n, L_{0,r_i}(\tau_{0,\eta}) \in T_i | \tau_{0,\eta'} < \tau_{0,\eta} \right)} \leq 1 + p(\eta'/\eta) \tag{3.24}$$

with $p(u) \leq \frac{1}{c} \exp\left(-c |\log u|^{1/2}\right)$ for some universal constant $c > 0$.

Remark 3.21. It is crucial that we consider dyadic radii $r \in \{\eta e^{-i}, i = 1 \dots k'-k\}$ between η' and η/e since there is no hope to obtain such a result if we were looking at the local times $L_{0,r}(\tau_{0,\eta})$ for all $r \leq \eta/e$. Indeed, if we condition the Brownian motion to spend very little time in the disc $D(0,\eta/e)$ before hitting $\partial D(0,\eta)$ (which is a function of $L_{0,r}(\tau_{0,\eta}), r \leq \eta/e$), $B_{\tau_{0,\eta}}$ will favour points on $\partial D(0,\eta)$ close to the starting position y, even if we condition further the trajectory to visit $D(0,\eta')$ before exiting $D(0,\eta)$.

Proof of Lemma 3.20. The proof is inspired from the one of [Jeg20a, Lemma 5.1]. In this proof, we will write $u = \pm v$ when we mean $-v \leq u \leq v$. To ease notations, we will denote $\tau_\eta := \tau_{0,\eta}, \tau_{\eta'} := \tau_{0,\eta'}$ and for all $i = 1 \dots n$, $L_{r_i} := L_{0,r_i}(\tau_{0,\eta})$. Take $C \in \mathcal{B}(\partial D(0,\eta))$. We will denote $\mathrm{Leb}(C)$ for the Lebesgue measure on $\partial D(0,\eta)$ of C. It is enough to show that

$$\begin{aligned} &\mathbb{P}_y \left(B_{\tau_\eta} \in C, \tau_{\eta'} < \tau_\eta, \forall i = 1 \dots n, L_{r_i} \in T_i \right) \\ &\quad = \left(1 \pm \frac{1}{c} \exp\left(-c \left| \log \frac{\eta'}{\eta} \right|^{1/3} \right) \right) \frac{\mathbb{P}_y \left(B_{\tau_\eta} \in C, \tau_{\eta'} < \tau_\eta \right)}{\mathbb{P}_y \left(\tau_{\eta'} < \tau_\eta \right)} \mathbb{P}_y \left(\tau_{\eta'} < \tau_\eta, \forall i = 1 \dots n, L_{r_i} \in T_i \right). \end{aligned} \tag{3.25}$$

Moreover, establishing (3.25) can be reduced to show that

$$\begin{aligned} &\mathbb{P}_y \left(B_{\tau_\eta} \in C, \tau_{\eta'} < \tau_\eta, \forall i = 1 \dots n, L_{r_i} \in T_i \right) \\ &\quad = \left(1 \pm \frac{1}{c} \exp\left(-c \left| \log \frac{\eta'}{\eta} \right|^{1/3} \right) \right) \frac{\mathrm{Leb}(C)}{2\pi\eta} \mathbb{P}_y \left(\tau_{\eta'} < \tau_\eta, \forall i = 1 \dots n, L_{r_i} \in T_i \right). \end{aligned} \tag{3.26}$$

Indeed, applying (3.26) to $T_i = [0,\infty)$ for all i gives

$$\mathbb{P}_y \left(B_{\tau_\eta} \in C, \tau_{\eta'} < \tau_\eta \right) = \left(1 \pm \frac{1}{c} \exp\left(-c \left| \log \frac{\eta'}{\eta} \right|^{1/3} \right) \right) \mathbb{P}_y \left(\tau_{\eta'} < \tau_\eta \right) \frac{\mathrm{Leb}(C)}{2\pi\eta},$$

which combined with (3.26) leads to (3.25) with slightly different constants. Finally, after reformulation of (3.26), to finish the proof we only need to prove that

$$\mathbb{P}_y \left(B_{\tau_\eta} \in C | \tau_{\eta'} < \tau_\eta, \forall i = 1 \dots n, L_{r_i} \in T_i \right) = \left(1 \pm \frac{1}{c} \exp\left(-c \left| \log \frac{\eta'}{\eta} \right|^{1/3} \right) \right) \frac{\mathrm{Leb}(C)}{2\pi\eta}. \tag{3.27}$$

The skew-product decomposition of Brownian motion (see [Kal02], Corollary 16.7 for instance) tells us that we can write

$$(B_t, t \geq 0) \overset{\text{(d)}}{=} (|B_t| \, e^{i\theta_t}, t \geq 0) \text{ with } (\theta_t, t \geq 0) = (w_{\sigma_t}, t \geq 0)$$

where $(w_t, t \geq 0)$ is a one-dimensional Brownian motion independent of the radial part $(|B_t|, t \geq 0)$ and $(\sigma_t, t \geq 0)$ is a time-change that is adapted to the filtration generated by $(|B_t|, t \geq 0)$:

$$\sigma_t = \int_0^t \frac{1}{|B_s|^2} ds.$$

In particular, under $\mathbb{P}_y$, we have the following equality in law

$$(\tau_\eta, |B_t|, t < \tau_\eta, B_{\tau_\eta}) \overset{\text{(d)}}{=} \left(\tau_\eta, |B_t|, t < \tau_\eta, \eta e^{i\theta_0 + i\varsigma\mathcal{N}} \right) \tag{3.28}$$

where θ_0 is the argument of y, $\mathcal{N}$ is a standard normal random variable independent of the radial part $(|B_t|, t \geq 0)$ and

$$\varsigma = \sqrt{\int_0^{\tau_\eta} \frac{1}{|B_s|^2} ds}.$$

We now investigate a bit the distribution of $e^{i\theta_0 + it\mathcal{N}}$ for some $t > 0$. More precisely, we want to give a quantitative description of the fact that if t is large, the previous distribution should approximate the uniform distribution on the unit circle. Using the probability density function of $\mathcal{N}$ and then using Poisson summation formula, we find that the probability density function $f_t(\theta)$ of $e^{i\theta_0 + it\mathcal{N}}$ at a given angle θ is given by

$$\begin{aligned} f_t(\theta) &= \frac{1}{\sqrt{2\pi}t} \sum_{n \in \mathbb{Z}} e^{-(\theta - \theta_0 + 2\pi n)^2/(2t^2)} = \frac{1}{2\pi} \sum_{p \in \mathbb{Z}} e^{ip(\theta - \theta_0)} e^{-p^2 t^2/2} \\ &= \frac{1}{2\pi} \left(1 + 2 \sum_{p=1}^{\infty} \cos(p(\theta - \theta_0)) e^{-p^2 t^2/2} \right). \end{aligned}$$

In particular, we can control the error in the approximation mentioned above by: for all $\theta \in [0, 2\pi]$,

$$\left| f_t(\theta) - \frac{1}{2\pi} \right| \leq \frac{1}{\pi} \sum_{p=1}^{\infty} e^{-p^2 t^2/2} \leq C_1 \max\left(1, \frac{1}{t}\right) e^{-t^2/2}$$

for some universal constant $C_1 > 0$.

We now come back to the objective (3.27). Using the identity (3.28) and because the local times

L_{r_i} are measurable with respect to the radial part of Brownian motion, we have by triangle inequality

$$\begin{aligned}
&\left|\mathbb{P}_y\left(B_{\tau_\eta} \in C | \tau_{\eta'} < \tau_\eta, \forall i = 1 \dots n, L_{r_i} \in T_i\right) - \frac{\mathrm{Leb}(C)}{2\pi\eta}\right| \\
&\leq \mathbb{E}_y\left[\int_0^{2\pi}\left|f_\varsigma(\theta) - \frac{1}{2\pi}\right| \mathbf{1}_{\{\eta e^{i\theta}\in C\}} d\theta \middle| \tau_{\eta'} < \tau_\eta, \forall i = 1 \dots n, L_{r_i} \in T_i\right] \\
&\leq C_1 \frac{\mathrm{Leb}(C)}{\eta} \mathbb{E}_y\left[\max\left(1, \frac{1}{\varsigma}\right) e^{-\varsigma^2/2} \middle| \tau_{\eta'} < \tau_\eta, \forall i = 1 \dots n, L_{r_i} \in T_i\right] \\
&\leq C_1 \frac{\mathrm{Leb}(C)}{\eta} \mathbb{E}_y\left[\max\left(1, \frac{1}{\varsigma'}\right) e^{-(\varsigma')^2/2} \middle| \tau_{\eta'} < \tau_\eta, \forall i = 1 \dots n, L_{r_i} \in T_i\right]
\end{aligned}$$

where

$$\varsigma' := \sqrt{\int_{\tau_{r_n}}^{\tau_\eta} \frac{1}{|B_s|^2} ds}.$$

To conclude the proof, we want to show that

$$\mathbb{E}_y\left[\max\left(1, \frac{1}{\varsigma'}\right) e^{-(\varsigma')^2/2} \middle| \tau_{\eta'} < \tau_\eta, \forall i = 1 \dots n, L_{r_i} \in T_i\right] \leq \frac{1}{c} \exp\left(-c\left|\log \frac{\eta'}{\eta}\right|^{1/2}\right).$$

By conditioning on the trajectory up to $\tau_{\eta'}$, it is enough to show that for any $T_i' \in \mathcal{B}([0,\infty)), i = 1 \dots n$, for any $z \in \partial D(0, \eta')$,

$$\mathbb{E}_z\left[\max\left(1, \frac{1}{\varsigma'}\right) e^{-(\varsigma')^2/2} \middle| \forall i = 1 \dots n, L_{r_i} \in T_i'\right] \leq \frac{1}{c} \exp\left(-c\left|\log \frac{\eta'}{\eta}\right|^{1/2}\right). \tag{3.29}$$

In the following, we fix such T_i' and such a z.

Consider the sequence of stopping times defined by: $\sigma_0^{(2)} := 0$ and for all $i = 1 \dots k' + k$,

$$\sigma_i^{(1)} := \inf\left\{t > \sigma_{i-1}^{(2)} : |B_t| = \eta' e^{i-1/2}\right\} \text{ and } \sigma_i^{(2)} := \inf\left\{t > \sigma_i^{(1)} : |B_t| \in \{\eta' e^i, \eta' e^{i-1}\}\right\}.$$

We only keep track of the portions of trajectories during the intervals $\left[\sigma_i^{(1)}, \sigma_i^{(2)}\right]$ by bounding from below ς' by

$$(\varsigma')^2 \geq \sum_{i=1}^{k'-k} \frac{\sigma_i^{(2)} - \sigma_i^{(1)}}{(\eta' e^i)^2}.$$

Notice that by Markov property, conditioning on $\{\forall i = 1 \dots n, L_{r_i} \in T_i'\}$ impacts the variables $\sigma_i^{(2)} - \sigma_i^{(1)}$ only through $\left|B_{\sigma_i^{(2)}}\right|$. Since

$$v \mapsto \max\left(1, \frac{1}{v^{1/2}}\right) e^{-v/2}$$

is convex, we deduce by Jensen's inequality that

$$\begin{aligned}
&\mathbb{E}_z\left[\max\left(1, \frac{1}{\varsigma'}\right) e^{-(\varsigma')^2/2} \middle| \forall i = 1 \dots n, L_{r_i} \in T_i'\right] \\
&\leq \frac{1}{k'-k} \sum_{i=1}^{k'-k} \mathbb{E}_z\left[\max\left(1, \frac{1}{k'-k} \frac{(\eta' e^i)^2}{\sigma_i^{(2)} - \sigma_i^{(1)}}\right)^{1/2} \exp\left(-\frac{k'-k}{2} \frac{\sigma_i^{(2)} - \sigma_i^{(1)}}{(\eta' e_i)^2}\right) \middle| \left|B_{\sigma_i^{(2)}}\right|\right].
\end{aligned}$$

By Markov property and Brownian scaling, we have obtained

$$\begin{aligned}&\mathbb{E}_z\left[\max\left(1,\frac{1}{\varsigma'}\right)e^{-(\varsigma')^2/2}\middle|\forall i=1\dots n, L_{r_i}\in T_i'\right]\\&\leq\max_{r=1,e^{-1}}\mathbb{E}_{e^{-1/2}}\left[\max\left(1,\frac{1}{(k'-k)\sigma_*}\right)^{1/2}\exp\left(-\frac{(k'-k)\sigma_*}{2}\right)\middle||B_{\sigma_*}|=r\right].\end{aligned}$$

where $\sigma_* := \inf\{t>0: |B_t|\in\{1,e^{-1}\}\}$. Now, one can show (see [Doo55, Section 14] for instance) that there exists a universal constant $c>0$ such that for all $s\geq 1$,

$$\mathbb{E}_{e^{-1/2}}\left[e^{-s\sigma_*}\right]\leq e^{-c\sqrt{s}}.$$

Since $\min_{r=1,e^{-1}}\mathbb{P}_{e^{-1/2}}(|B_{\sigma_*}|=r|)\geq c$ for some universal constant $c>0$, we also have

$$\max_{r=1,e^{-1}}\mathbb{E}_{e^{-1/2}}\left[e^{-s\sigma_*}\middle||B_{\sigma_*}|=r\right]\leq Ce^{-c\sqrt{s}}.$$

From this, we deduce that

$$\max_{r=1,e^{-1}}\mathbb{E}_{e^{-1/2}}\left[\max\left(1,\frac{1}{(k'-k)\sigma_*}\right)\middle||B_{\sigma_*}|=r\right]\leq C$$

and therefore, by Cauchy–Schwarz, we obtain that

$$\max_{r=1,e^{-1}}\mathbb{E}_{e^{-1/2}}\left[\max\left(1,\frac{1}{(k'-k)\sigma_*}\right)^{1/2}\exp\left(-\frac{(k'-k)\sigma_*}{2}\right)\middle||B_{\sigma_*}|=r\right]\leq Ce^{-c\sqrt{k'-k}}.$$

Recalling that $k'-k=\log\eta'/\eta$, this shows (3.29) which finishes the proof of Lemma 3.20. □

3.3.3 Bessel process

The purpose of this section is to collect properties of Bessel processes that will be needed in this paper. Recall Notation 3.8.

We start off by recalling the following result that can be found for instance in the lecture notes [Law18], Proposition 2.2.

Lemma B. *For each $x,t>0$ and $d\geq 0$, the measures $\mathbb{P}_x$ and $\mathbb{P}_x^d$, considered as measures on paths $\{X_s, s\leq t\}$, restricted to the event $\{\forall s\leq t, X_s>0\}$ are mutually absolutely continuous with Radon-Nikodym derivative*

$$\frac{d\mathbb{P}_x^d}{d\mathbb{P}_x}=\left(\frac{X_t}{x}\right)^a\exp\left(-\frac{a(a-1)}{2}\int_0^t\frac{ds}{X_s^2}\right)$$

where $a=(d-1)/2$.

We now state a consequence of Lemma B and Girsanov's theorem that will allow us to transfer computations on zero-dimensional Bessel process over to 1D Brownian motion and 3D Bessel process. Let us mention that since 0 is absorbing for the zero-dimensional Bessel process X, we will very often write $\mathbf{1}_{\{X_t>0\}}$ instead of $\mathbf{1}_{\{\forall s\leq t, X_s>0\}}$ for this specific process.

Lemma 3.22. *Let $\gamma \in (0,2]$, $t > 0$, $r > 0$ and let $f : \mathcal{C}([0,t],[0,\infty)) \to [0,\infty)$ be a nonnegative measurable function. Then*

$$\begin{aligned} &\sqrt{t}e^{-\frac{\gamma^2}{2}t}\mathbb{E}_r^0\left[e^{\gamma X_t}\mathbf{1}_{\{X_t>0\}}f(X_s, s\leq t)\right] \\ &= \sqrt{r}e^{\gamma r}\mathbb{E}_r\left[\left(\frac{t}{X_t+\gamma t}\right)^{1/2}\exp\left(-\frac{3}{8}\int_0^t\frac{ds}{(X_s+\gamma s)^2}\right)\mathbf{1}_{\{\forall s\leq t, X_s+\gamma s>0\}}f(X_s+\gamma s, s\leq t)\right]. \end{aligned} \tag{3.30}$$

In particular,

$$\sqrt{t}e^{-\frac{\gamma^2}{2}t}\mathbb{E}_r^0\left[e^{\gamma X_t}\mathbf{1}_{\{X_t>0\}}f(X_s, s\leq t)\right] \leq \sqrt{r}e^{\gamma r}\mathbb{E}_r\left[\left(\frac{t}{X_t+\gamma t}\right)_+^{1/2}f(X_s+\gamma s, s\leq t)\right]. \tag{3.31}$$

Moreover,

$$\begin{aligned} &\sqrt{t}e^{-2t}\mathbb{E}_r^0\left[e^{2X_t}\mathbf{1}_{\{X_t>0\}}\mathbf{1}_{\{\forall s\leq t, X_s<2s+\beta\}}f\left(X_s, s\leq t\right)\right] \\ &= 2^{-1/2}\sqrt{r}e^{2r}(\beta-r)\mathbb{E}_{\beta-r}^3\left[\frac{1}{X_t}\left(1-\frac{X_t-\beta}{2t}\right)^{-1/2}\mathbf{1}_{\{\forall s\leq t, 2s-X_s+\beta>0\}}\right. \\ &\qquad\qquad \left.\times f(2s-X_s+\beta, s\leq t)\exp\left(-\frac{3}{8}\int_0^t\frac{ds}{(2s-X_s+\beta)^2}\right)\right] \end{aligned} \tag{3.32}$$

and

$$\begin{aligned} &\sqrt{t}e^{-2t}\mathbb{E}_r^0\left[e^{2X_t}\mathbf{1}_{\{X_t>0\}}\mathbf{1}_{\{\forall s\leq t, X_s<2s+\beta\}}f(X_s, s\leq t)\right] \\ &\leq \sqrt{r}e^{2r}(\beta-r)\mathbb{E}_{\beta-r}^3\left[\frac{1}{X_t}\left(\frac{t}{2t+\beta-X_t}\right)_+^{1/2}f(\beta+2s-X_s, s\leq t)\right]. \end{aligned} \tag{3.33}$$

Finally,

$$\lim_{\beta\to\infty}\lim_{t\to\infty} te^{-2t}\mathbb{E}_r^0\left[e^{2X_t}\mathbf{1}_{\{\forall s\leq t, X_s<2s+\beta\}}\right] = \infty. \tag{3.34}$$

Proof of Lemma 3.22. By Lemma B, the left hand side of (3.30) is equal to

$$\sqrt{r}\sqrt{t}e^{-\frac{\gamma^2}{2}t}\mathbb{E}_r\left[X_t^{-1/2}\exp\left(-\frac{3}{8}\int_0^t\frac{ds}{X_s^2}\right)e^{\gamma X_t}\mathbf{1}_{\{\forall s\leq t, X_s>0\}}f(X_s, s\leq t)\right].$$

Girsanov's theorem concludes the proof of (3.30). (3.31) follows directly from (3.30). Now, by (3.30), the left hand side of (3.32) is equal to

$$\sqrt{r}e^{2r}\mathbb{E}_r\Bigg[\left(\frac{t}{X_t+2t}\right)_+^{1/2}\exp\left(-\frac{3}{8}\int_0^t\frac{ds}{(X_s+2s)^2}\right)\mathbf{1}_{\{\forall s\le t, X_s+2s>0\}}$$
$$\times\,\mathbf{1}_{\{\forall s\le t, X_s<\beta\}}f(X_s+2s, s\le t)\Bigg]$$
$$=\sqrt{r}e^{2r}\mathbb{E}_{\beta-r}\Bigg[\left(\frac{t}{2t+\beta-X_t}\right)_+^{1/2}\exp\left(-\frac{3}{8}\int_0^t\frac{ds}{(2s+\beta-X_s)^2}\right)\mathbf{1}_{\{\forall s\le t, 2s+\beta-X_s>0\}}$$
$$\times\,\mathbf{1}_{\{\forall s\le t, X_s>0\}}f(2s+\beta-X_s, s\le t)\Bigg].$$

By Lemma B, this is in turn equal to the right hand side of (3.32). (3.33) is an easy consequence of (3.32) and we now turn to the proof of (3.34). We use (3.32) and we add the stronger constraint that $\{\forall s\le t, 2s-X_s+\beta>r/2+s\}$ in order to have a lower bound. On this event, we can bound

$$\exp\left(-\frac{3}{8}\int_0^t\frac{ds}{(2s-X_s+\beta)^2}\right)\ge\exp\left(-\frac{3}{8}\int_0^\infty\frac{ds}{(r/2+s)^2}\right)=c_r.$$

Moreover, we simply bound

$$\left(1-\frac{X_t-\beta}{2t}\right)^{-1/2}\ge\left(1+\frac{\beta}{2t}\right)^{-1/2},$$

which overall shows that

$$te^{-2t}\mathbb{E}_r^0\left[e^{2X_t}\mathbf{1}_{\{\forall s\le t, X_s<2s+\beta\}}\right]\ge c_r(\beta-r)\left(1+\frac{\beta}{2t}\right)^{-1/2}\sqrt{t}\mathbb{E}^3_{\beta-r}\left[\frac{1}{X_t}\mathbf{1}_{\{\forall s\le t, 2s-X_s+\beta>r/2+s\}}\right].$$

Since X_t under $\mathbb{P}^3_{\beta-r}(\cdot|\forall s\le t, 2s-X_s+\beta>r/2+s)$ is stochastically dominated by X_t under $\mathbb{P}^3_{\beta-r}$, we can further bound

$$\mathbb{E}^3_{\beta-r}\left[\frac{1}{X_t}\mathbf{1}_{\{\forall s\le t, 2s-X_s+\beta>r/2+s\}}\right]\ge\mathbb{E}^3_{\beta-r}\left[\frac{1}{X_t}\right]\mathbb{P}^3_{\beta-r}\left(\forall s\le t, 2s-X_s+\beta>r/2+s\right).$$

Lemma 3.23, Point 2, shows that $\mathbb{E}^3_{\beta-r}\left[\frac{\sqrt{t}}{X_t}\right]\to\sqrt{2/\pi}$ as $t\to\infty$. Therefore

$$\liminf_{t\to\infty} te^{-2t}\mathbb{E}_r^0\left[e^{2X_t}\mathbf{1}_{\{\forall s\le t, X_s<2s+\beta\}}\right]\ge c_r(\beta-r)\mathbb{P}^3_{\beta-r}\left(\forall s\ge 0, 2s-X_s+\beta>r/2+s\right).$$

To see that the above probability remains bounded away from zero as $\beta\to\infty$, we can for instance notice that a three-dimensional Bessel process which starts at $\beta-r$ is stochastically dominated by the sum of three independent one-dimensional Bessel processes $X^{(i)}$, $i=1,2,3$, starting at the origin, plus $\beta-r$ (this follows by bounding $\sqrt{a^2+b^2+c^2}\le|a|+|b|+|c|$). Therefore

$$\mathbb{P}^3_{\beta-r}\left(\forall s\ge 0, 2s-X_s+\beta>r/2+s\right)\ge\mathbb{P}\left(\forall s\ge 0, \sum_{i=1}^3 X_s^{(i)}<r/2+s\right)>0.$$

This concludes the proof of (3.34). □

We now collect some properties of three-dimensional Bessel process.

Lemma 3.23. *Let $K > 0$.*

1. *Uniformly over $r \in [0, K]$,*

$$\mathbb{P}_r^3 \left(\forall t \geq 0, X_t \geq \frac{\sqrt{t}}{M \log(2+t)^2} \right) \to 1$$

as $M \to \infty$.

2. $\mathbb{E}_r^3 \left[\frac{1}{X_t} \right] = \sqrt{\frac{2}{\pi t}} + o\left(\frac{1}{\sqrt{t}} \right)$ *as $t \to \infty$, where the error is uniform over $r \in [0, K]$.*

3. *For any $q \in (0, 3)$, $\sup_{t \geq 1} \sup_{r > 0} \mathbb{E}_r^3 \left[\frac{t^{q/2}}{X_t^q} \right]$ is finite.*

4. *For any $q \in (0, 1)$, $\sup_{t \geq 1} \sup_{K \geq 0} \sup_{r \in [0,K]} \mathbb{E}_r^3 \left[\left(1 - \frac{X_t - K}{2t} \right)_+^{-q} \right]$ is finite.*

Proof of Lemma 3.23. Points 1-2 are part of [Pow18, Lemma 2.9]. To verify Point 3, notice that X_t under $\mathbb{P}_0^3$ is stochastically dominated by X_t under $\mathbb{P}_r^3$ for any $r > 0$. By scaling, we deduce that

$$\sup_{t \geq 1} \sup_{r > 0} \mathbb{E}_r^3 \left[\frac{t^{q/2}}{X_t^q} \right] \leq \sup_{t \geq 1} \mathbb{E}_0^3 \left[\frac{t^{q/2}}{X_t^q} \right] = \mathbb{E}_0^3 \left[\frac{1}{X_1^q} \right].$$

The density of X_1 under $\mathbb{P}_0^3$ is explicit (see [Law18, Proposition 2.5] for instance) and is given by

$$\sqrt{\frac{2}{\pi}} y^2 e^{-y^2/2} dy.$$

We can therefore directly check that $\mathbb{E}_0^3 \left[X_1^{-q} \right]$ is finite as soon as $q < 3$. This concludes the proof of Point 3. Point 4 follows from a similar direct computation. □

We conclude this section on Bessel processes with estimates that will be used repeatedly in the paper.

Lemma 3.24. *There exists a universal constant $C > 0$ such that the following estimates hold true. For all $K \geq 1, r \in [0, K]$ and $t \geq 1$,*

$$t e^{-2t} \mathbb{E}_r^0 \left[e^{2X_t} \mathbf{1}_{\{\forall s \leq t, X_s \leq 2s + K\}} \mathbf{1}_{\{X_t > 0\}} \right] \leq C \sqrt{r} (K - r) e^{2r} \tag{3.35}$$

and

$$\sqrt{t} e^{-2t} \mathbb{E}_r^0 \left[(-X_t + 2t + K) e^{2X_t} \mathbf{1}_{\{\forall s \leq t, X_s \leq 2s + K\}} \mathbf{1}_{\{X_t > 0\}} \right] \leq C \sqrt{r} (K - r) e^{2r}. \tag{3.36}$$

Moreover, for all $K \geq 1, r \in [0, K]$, $\gamma \in (1, 2)$ and $t \geq \exp(1/(2 - \gamma))$,

$$\frac{1}{2 - \gamma} \sqrt{t} e^{-\gamma^2 t/2} \mathbb{E}_r^0 \left[e^{\gamma X_t} \mathbf{1}_{\{\forall s \leq t, X_s \leq 2s + K\}} \mathbf{1}_{\{X_t > 0\}} \right] \leq C \sqrt{r} (K - r) e^{\gamma r}. \tag{3.37}$$

Proof of Lemma 3.24. By (3.33), the left hand side of (3.36) is at most

$$2^{-1/2}\sqrt{r}(K-r)e^{2r}\mathbb{E}^3_{K-r}\left[\left(1-\frac{X_t-K}{2t}\right)_+^{-1/2}\right].$$

The expectation with respect to the three-dimensional Bessel process is bounded uniformly in $r \in [0,K], K>0, t\geq 1$ by Lemma 3.23, point 4. This concludes the proof of (3.36). Now, by (3.33) and then by Cauchy–Schwarz inequality, the left hand side of (3.35) is at most

$$\begin{aligned}&2^{-1/2}\sqrt{r}(K-r)e^{2r}\mathbb{E}^3_{K-r}\left[\frac{\sqrt{t}}{X_t}\left(1-\frac{X_t-K}{2t}\right)_+^{-1/2}\right]\\ &\leq 2^{-1/2}\sqrt{r}(K-r)e^{2r}\mathbb{E}^3_{K-r}\left[\frac{t}{X_t^2}\right]^{1/2}\mathbb{E}^3_{K-r}\left[\left(1-\frac{X_t-K}{2t}\right)_+^{-1}\right]^{1/2}.\end{aligned}$$

Lemma 3.23, points 3 and 4, then concludes the proof of (3.35). We now turn to the proof of (3.37). By (3.31), the left hand side of (3.37) is at most

$$\begin{aligned}&\frac{1}{2-\gamma}\sqrt{r}e^{\gamma r}\mathbb{E}_r\left[\left(\frac{t}{X_t+\gamma t}\right)_+^{1/2}\mathbf{1}_{\{\forall s\leq t, X_s\leq(2-\gamma)s+K\}}\right]\\ &\leq \frac{1}{2-\gamma}\sqrt{r}e^{\gamma r}\mathbb{E}_r\left[\left(\frac{t}{X_t+\gamma t}\right)_+^{1/2}\mathbf{1}_{\{X_t\leq -\gamma t/2\}}\right]\\ &+\sqrt{\frac{2}{\gamma}}\frac{1}{2-\gamma}\sqrt{r}e^{\gamma r}\mathbb{P}_r\left(\forall s\leq t, X_s\leq (2-\gamma)s+K\right).\end{aligned}$$

By Hölder's inequality and an analogue of Lemma 3.23, Point 4, for Brownian motion rather than 3D Bessel process, we see that the last expectation above is at most

$$\mathbb{E}_0\left[\left(1+\frac{X_t+r}{\gamma t}\right)_+^{-2/3}\right]^{3/4}\mathbb{P}_0\left(X_t\leq -r-\gamma t/2\right)^{1/4}\lesssim e^{-\gamma^2 t/32}\leq 2-\gamma$$

by recalling that $t\geq \exp(1/(2-\gamma))$. On the other hand (see [Res92, Proposition 6.8.1] for instance),

$$\mathbb{P}_0\left(\forall s\geq 0, X_s<(2-\gamma)s+K-r\right)=1-e^{-2(K-r)(2-\gamma)}\leq 2(K-r)(2-\gamma).$$

Since

$$\mathbb{P}_0\left(\exists s\geq t, X_s\geq (2-\gamma)s+K-r\right)\lesssim e^{-\frac{(2-\gamma)^2}{16}t}\leq 2-\gamma,$$

it implies that

$$\mathbb{P}_0\left(\forall s\leq t, X_s<(2-\gamma)s+K-r\right)\lesssim (K-r)(2-\gamma).$$

Putting things together yields (3.37). This concludes the proof. □

3.3.4 Barrier estimates for 1D Brownian motion

The purpose of this section is to prove the following lemma.

Lemma 3.25. *There exists $C > 0$ such that the following claims hold true. For all $K, H \geq 1$ and all integer $n \geq 1$,*

$$\mathbb{P}_0 \left(\forall k = 0 \dots n-1, \min_{[k,k+1]} X \leq 2 \log(1+k) + K \right) \leq \frac{CK^2}{\sqrt{n}} \tag{3.38}$$

and

$$\mathbb{P}_0 \left(\forall k = 0 \dots n-1, \min_{[k,k+1]} X \leq 2 \log(k+1) + K, \exists s \in [0,n], X_s > K + H \right) \leq \frac{CK^2 e^{-H/64}}{\sqrt{n}}. \tag{3.39}$$

Moreover, for all $K, H \geq 1$, $\gamma \in [1,2)$ and all integer $n \geq (2-\gamma)^{-4}$,

$$\mathbb{P}_0 \left(\forall k = 0 \dots n-1, \min_{[k,k+1]} X \leq (2-\gamma)k + 2\log(1+k) + K \right) \leq CK^2(2-\gamma) \tag{3.40}$$

and

$$\begin{aligned} \mathbb{P}_0 \Big(\forall k = 0 \dots n-1, \min_{[k,k+1]} X \leq (2-\gamma)k + 2\log(1+k) + K, \\ \exists s \leq n, X_s \geq (2-\gamma)s + K + H \Big) \leq CK^2 e^{-H/64}(2-\gamma). \end{aligned} \tag{3.41}$$

We start off with the following intermediate result.

Lemma 3.26. *Let $c > 0$. There exists $C > 0$ such that the following estimates hold. For all $n \geq 1$ and $K \geq 1$,*

$$\mathbb{P}_0 \left(\forall s \leq n, X_s \leq c \log(1+s) + K \right) \leq CK^2/\sqrt{n}. \tag{3.42}$$

Moreover, for all $\gamma \in [1,2)$, for all $n \geq (2-\gamma)^{-4}$ and $K \geq 1$,

$$\mathbb{P}_0 \left(\forall s \leq n, X_s \leq (2-\gamma)s + c\log(1+s) + K \right) \leq CK^2(2-\gamma). \tag{3.43}$$

Proof. We start by proving (3.42). If $K > n^{1/4}$, then the result is clear by bounding the probability by one. In the rest of the proof we thus assume that $K \leq n^{1/4}$. Let us denote $K_n = c\log(1+n) + K$. By the reflection principle,

$$\mathbb{P}_0 \left(\forall s \leq n, X_s \leq K_n, X_n \geq -n^{1/4} \right) = \int_{-n^{1/4}}^{K_n} \frac{1}{\sqrt{2\pi n}} \left(e^{-\frac{x^2}{2n}} - e^{-\frac{(2K_n - x)^2}{2n}} \right) dx$$

For all $x \in [-n^{1/4}, K_n]$, we can bound

$$e^{-\frac{x^2}{2n}} - e^{-\frac{(2K_n-x)^2}{2n}} = e^{-\frac{x^2}{2n}} \left(1 - e^{-2\frac{(K_n^2 - K_n x)}{n}} \right) \lesssim e^{-\frac{x^2}{2n}} \frac{K_n^2 - K_n x}{n} \lesssim e^{-\frac{x^2}{2n}} \frac{K_n n^{1/4}}{n},$$

implying that

$$\mathbb{P}_0 \left(\forall s \leq n, X_s \leq K_n, X_n \geq -n^{1/4} \right) \lesssim \frac{K_n n^{1/4}}{n} \int_{-n^{1/4}}^{K_n} \frac{1}{\sqrt{2\pi n}} e^{-\frac{x^2}{2n}} dx \lesssim \frac{K_n}{n}.$$

Another similar consequence of the reflection principle is that

$$\mathbb{P}_0\left(\forall s \leq n, X_s \leq K_n\right) \gtrsim K_n/\sqrt{n}.$$

Therefore

$$\begin{aligned}&\mathbb{P}_0\left(X_n \geq -n^{1/4} | \forall s \leq n, X_s \leq c\log(1+s) + K\right)\\&\leq \mathbb{P}_0\left(X_n \geq -n^{1/4} | \forall s \leq n, X_s \leq c\log(1+n) + K\right) \lesssim 1/\sqrt{n}\end{aligned}$$

and

$$\begin{aligned}&\mathbb{P}_0\left(\forall s \leq n, X_s \leq c\log(1+s) + K\right)\\&\lesssim n^{-1/2}\mathbb{P}_0\left(\forall s \leq n, X_s \leq c\log(1+s) + K\right) + \mathbb{P}_0\left(\forall s \leq n, X_s \leq c\log(1+s) + K, X_n \leq -n^{1/4}\right)\\&\lesssim n^{-1/2}\mathbb{P}_0\left(\forall s \leq n, X_s \leq c\log(1+s) + K\right) + \mathbb{P}_0\left(X_n \leq -n^{1/4}, \exists s \in [n, n+n^{1/4}], X_s > K\right)\\&+ \mathbb{P}_0\left(\forall s \leq n+n^{1/4}, X_s \leq c'(s \wedge (n+n^{1/4}-s))^{1/20} + K\right).\end{aligned}$$

By equation (25) of [BDZ16], the last right hand side term is at most $CK^2/\sqrt{n}$. The second right hand side term being at most

$$\mathbb{P}_0\left(\max_{[0,n^{1/4}]} X \geq K + n^{1/4}\right) \lesssim e^{-cn^{1/4}},$$

we deduce that

$$\begin{aligned}&\mathbb{P}_0\left(\forall s \leq n, X_s \leq c\log(1+s) + K\right)\\&\lesssim n^{-1/2}\mathbb{P}_0\left(\forall s \leq n, X_s \leq c\log(1+s) + K\right) + K^2/\sqrt{n}\end{aligned}$$

which concludes the proof of (3.42).

We now turn to the proof of (3.43). Since $n \geq (2-\gamma)^{-4}$,

$$\begin{aligned}\mathbb{P}_0\left(\exists s \geq n, X_s > (2-\gamma)s\right) \leq \mathbb{P}_0\left(\exists s \geq n, X_s > s^{3/4}\right) &\leq \sum_{k \geq n} \mathbb{P}_0\left(\max_{[k,k+1]} X > k^{3/4}\right)\\&\leq \sum_{k \geq n} e^{-c\sqrt{k}} \lesssim e^{-c\sqrt{n}} \lesssim 2-\gamma.\end{aligned}$$

Hence

$$\begin{aligned}&\mathbb{P}_0\left(\forall s \leq n, X_s \leq (2-\gamma)s + c\log(1+s) + K\right)\\&\lesssim 2-\gamma + \mathbb{P}_0\left(\forall s \leq 2n, X_s \leq (2-\gamma)s + C(s \wedge (2n-s))^{1/20} + K\right)\\&= 2-\gamma + e^{-(2-\gamma)^2 n}\mathbb{E}_0\left[e^{-(2-\gamma)X_{2n}}\mathbf{1}_{\left\{\forall s \leq 2n, X_s \leq C(s \wedge (2n-s))^{1/20} + K\right\}}\right]\end{aligned}$$

by Girsanov's theorem. Now, by equation (25) of [BDZ16], we conclude that

$$\begin{aligned}
&\mathbb{P}_0\left(\forall s \le n, X_s \le (2-\gamma)s + 3\log(1+s) + K\right)\\
&\lesssim 2-\gamma + Ke^{-(2-\gamma)^2 n}\int_{-\infty}^{K}(K-x)n^{-3/2}e^{-x^2/(4n)}e^{-(2-\gamma)x}dx\\
&= 2-\gamma + K\int_{-\infty}^{K+2(2-\gamma)n}(K+2(2-\gamma)n-y)n^{-3/2}e^{-y^2/(4n)}dy\\
&\lesssim K^2(2-\gamma).
\end{aligned}$$

This finishes the proof of (3.43). □

Proof of Lemma 3.25. We start by proving (3.38). By Lemma 3.26, there exists some universal constant $C_1 > 0$ such that for all $t \ge 1$,

$$\mathbb{P}_0\left(\forall s \in [1,t], X_s \le 3\log(1+s) + 2K\right) \le C_1 K^2/\sqrt{t+1}. \tag{3.44}$$

We thus aim to take care of the minima in (3.38). Let $n \ge 1$ and define

$$p_n := \sup_{0 \le t_0 < 1} \mathbb{P}_0\left(\forall k \le n-1, \min_{[k+t_0,k+1+t_0]} X \le 2\log(k+1) + K\right).$$

Let $0 \le t_0 < 1$. Set $\tau := \inf\{s > t_0 : X_s \ge 3\log(1+s) + 2K\}$. We are going to decompose the above probability according to the value of τ. Let $k \ge 1$. Notice that on the event $\{k + t_0 \le \tau < k+1+t_0, \min_{[k-1+t_0,k+t_0]} X \le 2\log k + K\}$, we have $\max_{u,v\in[k-1+t_0,\tau]} |X_u - X_v| \ge \log(k+1) + K$. If $k = 0$, on the event $\{t_0 \le \tau < 1+t_0\}$, we simply have $\max_{u\in[0,\tau]} |X_u - X_0| \ge K$ when X starts at 0. Hence

$$\begin{aligned}
&\mathbb{P}_0\left(\forall k \le n-1, \min_{[k+t_0,k+1+t_0]} X \le 2\log(k+1) + K\right)\\
&\le \mathbb{P}_0(\tau \ge n+t_0) + \sum_{k=0}^{n-1}\mathbb{P}_0\Big(k+t_0 \le \tau < k+1+t_0, \max_{u,v\in[k-1+t_0,\tau]}|X_u - X_v| \ge \log(k+1)+K,\\
&\qquad \forall j = k+1\dots n, \min_{[j+t_0,j+1+t_0]} X \le 2\log(j+1)+K\Big).
\end{aligned}$$

By applying Markov's property to the stopping time τ, and by writing $\tilde{X}$ a Brownian motion independent of τ, we see that the last probability written above is equal to

$$\begin{aligned}
&\mathbb{E}_0\Bigg[\mathbf{1}_{\{k+t_0\le\tau<k+1+t_0,\max_{u,v\in[k-1+t_0,\tau]}|X_u-X_v|\ge\log(k+1)+K\}} \\
&\quad\times \mathbb{P}_0\left(\forall j=k+1\dots n,\min_{[j-\tau+t_0,j+1-\tau+t_0]}\tilde{X}\le 2\log(j+1)-3\log(1+\tau)-K\Big|\tau\right)\Bigg] \\
&\le \mathbb{P}_0\left(k+t_0\le\tau<k+1+t_0,\max_{u,v\in[k-1+t_0,\tau]}|X_u-X_v|\ge\log(k+1)+K\right) \\
&\quad\times\sup_{0\le t_0'<1}\mathbb{P}_0\left(\forall j=0\dots n-k-1,\min_{[j+t_0',j+1+t_0']}X\le 2\log(1+j)+K\right) \\
&\le \mathbb{P}_0\left(\max_{u,v\in[0,2]}|X_u-X_v|\ge\log(k+1)+K\right)p_{n-k-1} \\
&\le \mathbb{P}_0\left(2\max_{[0,2]}|X|\ge\log(k+1)+K\right)p_{n-k-1}\le e^{-(\log(k+1)+K)^2/16}p_{n-k-1}.
\end{aligned}$$

Moreover, by (3.44),

$$\sup_{0\le t_0<1}\mathbb{P}_0(\tau\ge n+t_0)\le\mathbb{P}_0\left(\forall s\in[1,n],X_s\le 3\log(1+s)+2K\right)\le C_1K^2/\sqrt{n+1}.$$

We have thus proven that

$$p_n\le\frac{C_1K^2}{\sqrt{n+1}}+\sum_{k=0}^{n-1}e^{-(\log(k+1)+K)^2/16}p_{n-k-1}. \tag{3.45}$$

This recursive relation allows us to conclude the proof of (3.38). We detail the arguments. Define

$$C_2:=\sup_{n\ge1}\sqrt{n+1}\sum_{k=0}^{n-1}e^{-(\log(k+1))^2/16}\frac{1}{\sqrt{n-k}}<\infty$$

and assume that K is large enough so that we can define

$$C_K=C_1/(1-e^{-K^2/16}C_2).$$

We clearly have $p_0\le1\le C_KK^2/\sqrt{1+0}$. Let $n\ge1$ and assume now that for all $k\le n-1$, $p_k\le C_KK^2/\sqrt{k+1}$. By (3.45), we have

$$p_n\le\frac{C_1K^2}{\sqrt{n+1}}+\sum_{k=0}^{n-1}e^{-(\log(k+1)+K)^2/16}\frac{C_KK^2}{\sqrt{n-k}}\le\frac{K^2}{\sqrt{n+1}}\left(C_1+e^{-K^2/16}C_2C_K\right)=\frac{C_KK^2}{\sqrt{n+1}}.$$

This concludes the proof by induction of the fact that $p_n\le C_KK^2/\sqrt{n+1}$ for all $n\ge1$. Since C_K does not grow with K, this concludes the proof of (3.38).

We now turn to the proof of (3.39). We are first going to show that

$$\mathbb{P}_0\Big(\forall k = 0\dots n-1, \min_{[k,k+1]} X \le 2\log(1+k) + K, \\ \exists s \le n, X_s \ge 3\log(1+s) + H + K\Big) \le Ce^{-H^2/16}K^2/\sqrt{n}. \tag{3.46}$$

By considering the stopping time

$$\inf\left\{s > 0, X_s \ge 3\log(1+s) + H + K\right\},$$

and by following almost the same arguments as above, one can show that the probability in (3.46) is at most

$$\sum_{k=0}^{n-1} e^{-(\log(k+1)+H)^2/16} \sup_{0\le t_0<1} \mathbb{P}_0\left(\forall j \le n-1-k, \min_{[k+t_0,k+1+t_0]} X \le 2\log(k+1) + K\right)$$
$$\le \sum_{k=0}^{n-1} e^{-(\log(k+1)+H)^2/16} p_{n-k} \lesssim \sum_{k=0}^{n-1} e^{-(\log(k+1)+H)^2/16} \frac{K^2}{\sqrt{n-k}} \lesssim e^{-H^2/16}\frac{K^2}{\sqrt{n}}$$

thanks to the estimates on p_n. This shows (3.46). Now, it implies that

$$\mathbb{P}_0\left(\forall k = 0\dots n-1, \min_{[k,k+1]} X \le 2\log(k+1) + K, \exists s \in [0,n], X_s > K + H\right)$$
$$\le Ce^{-H^2/64}K^2/\sqrt{n} + \mathbb{P}_0\left(\forall s \in [0,n], X_s \le 3\log(s+1) + K + H/2, \exists s \in [0,n], X_s > K + H\right).$$

If H is larger than $6\log(n+1)$, then the probability on the right hand side vanishes and we directly obtain (3.39). Let us now assume that $H \le 6\log(n+1)$ and denote $k_0 := \left\lfloor e^{\frac{H}{6}} - 1\right\rfloor \le n$ and consider the stopping time $\tau = \inf\{s > 0 : X_s > K + H\}$. By Markov property, the last probability written above is at most equal to

$$\sum_{k=k_0}^{n-1} \mathbb{P}_0\left(k \le \tau < k+1, \forall s \in (\tau, n), X_s \le (c+1)\log(s+1) + K + H/2\right)$$
$$\le \sum_{k=k_0}^{n-1} \mathbb{P}_0\left(k \le \tau < k+1\right)$$
$$\times \mathbb{P}_{K+H}\left(\forall s \in [0, n-k-1], X_s \le 3\log(s+1) + 2\log(k+1) + K + H/2\right)$$
$$\lesssim \sum_{k=k_0}^{n-1} \mathbb{P}_0\left(k \le \tau < k+1\right)\log(k+1)^2/\sqrt{n-k}$$

by Lemma 3.26. Now, using the explicit density of τ (which is a consequence of the reflection principle), we see that

$$\mathbb{P}_0\left(k \le \tau < k+1\right) = \int_k^{k+1} \frac{K+H}{\sqrt{2\pi t^3}} \exp\left(-\frac{(K+H)^2}{2t}\right) \lesssim \frac{K+H}{(k+1)^{3/2}}.$$

Hence,

$$\mathbb{P}_0\left(\forall k=0\dots n-1, \min_{[k,k+1]} X \leq 2\log(k+1)+K, \exists s\in[0,n], X_s > K+H\right)$$
$$\lesssim e^{-H^2/64}K^2\frac{1}{\sqrt{n}}+(K+H)\sum_{k=k_0}^{n-1}\frac{\log(k+1)^2}{\sqrt{k^3(n-k)}}.$$

The behaviour of the above sum is given by

$$\sum_{k=k_0}^{n-1}\frac{\log(k+1)^2}{\sqrt{k^3(n-k)}}\lesssim\frac{1}{\sqrt{n}}\sum_{k=k_0}^{\lfloor n/2\rfloor}\frac{\log(k)^2}{k^{3/2}}+\frac{\log(n)^2}{n^{3/2}}\sum_{k=\lfloor n/2\rfloor+1}^{n-1}\frac{1}{\sqrt{n-k}}\lesssim\frac{1}{\sqrt{n}}\left(\frac{\log(k_0)^2}{\sqrt{k_0}}+\frac{\log(n)^2}{\sqrt{n}}\right).$$

By recalling that $k_0=\left\lfloor e^{\frac{H}{6}}-1\right\rfloor\leq n$, we have therefore obtained that

$$\mathbb{P}_0\left(\forall k=0\dots n-1, \min_{[k,k+1]} X \leq 2\log(k+1)+K, \exists s\in[0,n], X_s > K+H\right)$$
$$\lesssim e^{-H^2/64}K^2\frac{1}{\sqrt{n}}+(K+H)H^2e^{-H/12}\frac{1}{\sqrt{n}}.$$

This concludes the proof of (3.39).

We now turn to the proof of (3.40). This time we define for $n\geq 1$,

$$q_n := \sup_{0\leq t_0<1}\mathbb{P}_0\left(\forall k\leq n-1, \min_{[k+t_0,k+1+t_0]} X\leq(2-\gamma)k+2\log(1+k)+K\right).$$

By considering for $0\leq t_0<1$, the stopping time

$$\inf\left\{s>t_0 : X_s>(2-\gamma)s+3\log(1+s)+2K\right\},$$

we can show using a reasoning very similar to the one above that

$$q_n\leq\mathbb{P}_0\left(\forall s\in[1,n], X_s\leq(2-\gamma)s+3\log(1+s)+2K\right)+\sum_{k=0}^{n-1}e^{-(\log(1+k)+K)^2/16}q_{n-k-1}.$$

Take $n\geq(2-\gamma)^{-4}$. By (3.43), the first right hand side term above is at most $CK^2(2-\gamma)$. Moreover, for all $k\in[n/2,n]$,

$$q_k-q_n\leq\mathbb{P}_0\left(\exists s\geq n/2, X_s>(2-\gamma)s\right)\leq\mathbb{P}_0\left(\exists s\geq n/2, X_s>2^{-1/4}s^{3/4}\right)\lesssim e^{-\sqrt{n}/4}\lesssim 2-\gamma.$$

Therefore,

$$q_n\lesssim K^2(2-\gamma)+\sum_{k=0}^{\lfloor n/2\rfloor}e^{-(\log(k+1)+K)^2/16}q_n+\sum_{k=\lfloor n/2\rfloor+1}^{n-1}e^{-(\log(k+1)+K)^2/16}$$
$$\lesssim K^2(2-\gamma)+e^{-K^2/16}q_n$$

which shows that $q_n \lesssim K^2(2-\gamma)$ as soon as K is large enough. This finishes the proof of (3.40). (3.41) follows from (3.40) in a similar manner that (3.39) follows from (3.38). This concludes the proof. □

3.4 Adding good events: proof of Proposition 3.14 and Lemma 3.15

The purpose of this section is to prove Proposition 3.14 and Lemma 3.15. We start by discussing Proposition 3.14. As mentioned in Section 3.1.3, it is natural to expect the introduction of the good events $G_\varepsilon(x)$ to be harmless. Indeed, in analogy with the case of log-correlated Gaussian fields (see [Pow18, Corollary 2.4] for instance), the following should hold true:

$$\sup_{x \in D} \sup_{\varepsilon > 0} \left(\sqrt{\frac{1}{\varepsilon} L_{x,\varepsilon}(\tau_{x,R})} - 2 \log \frac{1}{\varepsilon} \right) < \infty \qquad \mathbb{P}_{x_0} - \text{a.s.} \tag{3.47}$$

which would imply (forgetting about the Bessel bridges) that $\mathbb{P}_{x_0}\left(\bigcap_{x \in D} \bigcap_{\varepsilon > 0} G_\varepsilon(x)\right) \to 1$ as $\beta \to \infty$. We have not been able to prove such a statement because of the following two main reasons.

1) For a fixed radius ε, we would like to be able to compare

$$\sup_{x \in D} \sqrt{\frac{1}{\varepsilon} L_{x,\varepsilon}(\tau_{x,R})} \quad \text{and} \quad \sup_{x \in \varepsilon \mathbb{Z}^2 \cap D} \sqrt{\frac{1}{\varepsilon} L_{x,\varepsilon}(\tau_{x,R})}, \tag{3.48}$$

the latter supremum being a supremum over a finite number of elements. To do so, we would need to be able to precisely control the way the local times vary with respect to the centre of the circle. Obtaining estimates precise enough turns out to be difficult to achieve (the estimates of Section C of [Jeg20a] leading to the continuity of the local time process $(x, \varepsilon) \mapsto L_{x,\varepsilon}(\tau)$ are too rough). We resolve this problem by first considering local time of annuli rather than circles. Indeed, comparing local times of annuli is much easier since if an annulus is included in another one, then the local time of the former is not larger than the local time of the latter.

2) Assuming that we are able to make the comparison (3.48), the next step would be to be able to bound from above

$$\mathbb{P}_{x_0}\left(\sup_{x \in \varepsilon \mathbb{Z}^2 \cap D} \sqrt{\frac{1}{\varepsilon} L_{x,\varepsilon}(\tau_{x,R})} \geq 2 \log \frac{1}{\varepsilon} \right).$$

If the bound is good enough, Borel-Cantelli lemma would allow us to conclude the proof of (3.47), at least along dyadic radii ε. Estimating accurately this probability is again challenging (a union bound is not good enough for instance). In the case of log-correlated Gaussian fields, the estimation of such probabilities is heavily based on the Gaussianity of the process. For instance, in [DRSV14a], Kahane's convexity inequalities allow the authors to import computations from cascades (Theorem 1.6 of [HS09]). We resolve this problem by asking the local times to stay under $2 \log \frac{1}{\varepsilon} + 2 \log \log \frac{1}{\varepsilon}$ instead of $2 \log \frac{1}{\varepsilon}$. Indeed, here we can do very naive computations using for instance union bounds. Importantly, this restriction is enough to turn the variables that we consider bounded in L^1. We can then make L^1 computations and use repulsion estimates to get rid of the extra $2 \log \log \frac{1}{\varepsilon}$ term.

3.4.1 Supremum of local times of annuli

Lemma 3.27. *For $x \in D$ and $\varepsilon > 0$, let*

$$\ell_{x,\varepsilon}(\tau_{x,R}) := \int_0^{\tau_{x,R}} \mathbf{1}_{\{\varepsilon \le |B_t - x| \le e\varepsilon\}} dt = \int_\varepsilon^{e\varepsilon} L_{x,r}(\tau_{x,R}) dr$$

be the amount of time the Brownian trajectory has spent in the annulus $D(x, e\varepsilon) \backslash D(x,\varepsilon)$ before hitting $\partial D(x,R)$. Then,

$$\sup_{\varepsilon \in \{e^{-n}, n \ge 1\}} \sup_{\substack{x \in D \\ |x - x_0| \ge e\varepsilon}} \sqrt{\frac{2}{(e^2-1)\varepsilon^2} \ell_{x,\varepsilon}(\tau_{x,R})} - 2 \log \frac{1}{\varepsilon} - 2 \log \log \frac{1}{\varepsilon} < \infty \qquad \mathbb{P}_{x_0} - \text{a.s.}$$

Proof of Lemma 3.27. For $x \in D$ and $\varepsilon > 0$, define

$$\ell_{x,\varepsilon} := \int_0^{\tau_{x,eR}} \mathbf{1}_{\left\{\varepsilon - \frac{\varepsilon}{|\log \varepsilon|} \le |B_t - x| \le e\varepsilon + \frac{\varepsilon}{|\log\varepsilon|}\right\}} dt$$

and notice that if $|x-y| \le \varepsilon / |\log \varepsilon|$, then $\ell_{x,\varepsilon}(\tau_{x,R}) \le \ell_{y,\varepsilon}$ $\mathbb{P}_{x_0}$-a.s. Hence

$$\begin{aligned} &\sup_{\varepsilon \in \{e^{-n}, n \ge 1\}} \sup_{\substack{x \in D \\ |x - x_0| \ge e\varepsilon}} \sqrt{\frac{2}{(e^2-1)\varepsilon^2} \ell_{x,\varepsilon}(\tau_{x,R})} - 2 \log \frac{1}{\varepsilon} - 2 \log \log \frac{1}{\varepsilon} \\ \le &\sup_{\varepsilon \in \{e^{-n}, n \ge 1\}} \sup_{\substack{x \in \frac{\varepsilon}{|\log\varepsilon|}\mathbb{Z}^2 \cap D \\ |x - x_0| \ge e\varepsilon + \varepsilon/|\log\varepsilon|}} \sqrt{\frac{2}{(e^2-1)\varepsilon^2} \ell_{x,\varepsilon}} - 2 \log \frac{1}{\varepsilon} - 2 \log \log \frac{1}{\varepsilon} \end{aligned}$$

$\mathbb{P}_{x_0}$-a.s. By Borel-Cantelli lemma, to conclude the proof it is now enough to show that

$$\sum_{\varepsilon \in \{e^{-n}, n \ge 1\}} \mathbb{P}_{x_0} \left(\sup_{\substack{x \in \frac{\varepsilon}{|\log\varepsilon|}\mathbb{Z}^2 \cap D \\ |x - x_0| \ge e\varepsilon + \varepsilon/|\log\varepsilon|}} \sqrt{\frac{2}{(e^2-1)\varepsilon^2} \ell_{x,\varepsilon}} - 2 \log \frac{1}{\varepsilon} - 2 \log \log \frac{1}{\varepsilon} \ge 0 \right) < \infty.$$

After a union bound, we want to estimate

$$\mathbb{P}_{x_0} \left(\sqrt{\frac{2}{(e^2-1)\varepsilon^2} \ell_{x,\varepsilon}} - 2 \log \frac{1}{\varepsilon} - 2 \log \log \frac{1}{\varepsilon} \ge 0 \right)$$

for a given $\varepsilon \in \{e^{-n}, n \ge 1\}$ and $x \in \frac{\varepsilon}{|\log\varepsilon|}\mathbb{Z}^2 \cap D$ such that $|x - x_0| \ge e\varepsilon + \varepsilon/|\log\varepsilon|$. Let $z \in \partial D(x, e\varepsilon + \varepsilon/|\log\varepsilon|)$. By (3.6), starting from z and conditioned on

$$\ell := \frac{1}{e\varepsilon + \varepsilon/|\log\varepsilon|} L_{x, e\varepsilon + \varepsilon/|\log\varepsilon|}(\tau_{x,eR}),$$

$$\ell_{x,\varepsilon} = \int_{\varepsilon - \varepsilon/|\log\varepsilon|}^{e\varepsilon + \varepsilon/|\log\varepsilon|} L_{x,r}(\tau_{x,eR}) dr \overset{(d)}{=} \left(e\varepsilon + \frac{\varepsilon}{|\log\varepsilon|} \right)^2 \int_0^{\log \frac{e\varepsilon + \varepsilon/|\log\varepsilon|}{\varepsilon - \varepsilon/|\log\varepsilon|}} e^{-2s} X_s^2 ds$$

where X_s is a zero-dimensional Bessel process starting at $\sqrt{\ell}$. By bounding

$$\log \frac{e\varepsilon + \varepsilon/|\log\varepsilon|}{\varepsilon - \varepsilon/|\log\varepsilon|} \leq 1 + \frac{3}{|\log\varepsilon|}, \quad \left(e\varepsilon + \frac{\varepsilon}{|\log\varepsilon|}\right)^2 \leq e^2\varepsilon^2\left(1 + \frac{1}{|\log\varepsilon|}\right)$$

(if ε is small enough) and

$$\frac{2}{1-e^{-2}}\left(1 + \frac{1}{|\log\varepsilon|}\right)\int_0^{1+3/|\log\varepsilon|} e^{-2s}X_s^2 ds \leq \left(1 + \frac{2}{|\log\varepsilon|}\right)\max_{s\leq 1+3/|\log\varepsilon|} X_s^2$$

we deduce that

$$\begin{aligned}
&\mathbb{P}_z\left(\sqrt{\frac{2}{(e^2-1)\varepsilon^2}\ell_{x,\varepsilon}} - 2\log\frac{1}{\varepsilon} - 2\log\log\frac{1}{\varepsilon} \geq 0\right) \\
&\leq \mathbb{E}_z\left[\mathsf{P}^0_{\sqrt{\ell}}\left(\left(1 + \frac{2}{|\log\varepsilon|}\right)\max_{s\leq 1+3/|\log\varepsilon|} X_s^2 \geq \left(2\log\frac{1}{\varepsilon} + 2\log\log\frac{1}{\varepsilon}\right)^2\right)\right].
\end{aligned}$$

Since $(X_s, s\geq 0)$ under $\mathsf{P}^0_{\sqrt{\ell}}$ is stochastically dominated by $(X_s, s\geq 0)$ under $\mathsf{P}_{\sqrt{\ell}}$ (zero-dimensional Bessel process has a negative drift), we obtain that

$$\begin{aligned}
&\mathsf{P}^0_{\sqrt{\ell}}\left(\left(1 + \frac{2}{|\log\varepsilon|}\right)\max_{s\leq 1+3/|\log\varepsilon|} X_s^2 \geq \left(2\log\frac{1}{\varepsilon} + 2\log\log\frac{1}{\varepsilon}\right)^2\right) \\
&\leq \mathsf{P}^0_{\sqrt{\ell}}\left(\max_{s\leq 1+3/|\log\varepsilon|} X_s \geq 2\log\frac{1}{\varepsilon} + 2\log\log\frac{1}{\varepsilon} - 3\right) \\
&\leq \mathsf{P}_0\left(\max_{s\leq 1+3/|\log\varepsilon|} X_s \geq 2\log\frac{1}{\varepsilon} + 2\log\log\frac{1}{\varepsilon} - 3 - \sqrt{\ell}\right) \\
&\leq \mathbf{1}_{\left\{\sqrt{\ell}\geq 2\log\frac{1}{\varepsilon}+2\log\log\frac{1}{\varepsilon}-3\right\}} \\
&+ 2\times\mathbf{1}_{\left\{\sqrt{\ell}<2\log\frac{1}{\varepsilon}+2\log\log\frac{1}{\varepsilon}-3\right\}}\exp\left(-\frac{1}{2(1+3/|\log\varepsilon|)}\left(2\log\frac{1}{\varepsilon} + 2\log\log\frac{1}{\varepsilon} - 3 - \sqrt{\ell}\right)^2\right).
\end{aligned}$$

We used reflection principle in the last inequality. Recalling that under $\mathbb{P}_z$ ℓ is an exponential variable with mean equal to $2|\log\varepsilon| + O(1)$ (see (3.21)), we see that

$$\mathbb{P}_z\left(\sqrt{\ell} \geq 2\log\frac{1}{\varepsilon} + 2\log\log\frac{1}{\varepsilon} - 3\right) \lesssim \varepsilon^2|\log\varepsilon|^{-4}.$$

Moreover, by denoting $A := \frac{2|\log\varepsilon|+2\log|\log\varepsilon|-3}{\sqrt{\mathbb{E}_z[\ell]}}$ and $\lambda = \frac{\mathbb{E}_z[\ell]}{2(1+3/|\log\varepsilon|)}$, we have

$$\begin{aligned}
&\mathbb{E}_z\left[\mathbf{1}_{\left\{\sqrt{\ell}<2\log\frac{1}{\varepsilon}+2\log\log\frac{1}{\varepsilon}-3\right\}}\exp\left(-\frac{1}{2(1+3/|\log\varepsilon|)}\left(2\log\frac{1}{\varepsilon} + 2\log\log\frac{1}{\varepsilon} - 3 - \sqrt{\ell}\right)^2\right)\right] \\
&\leq \int_0^{A^2} e^{-\lambda(A-\sqrt{t})^2}e^{-t}dt = 2e^{-\lambda A^2/(\lambda+1)}\int_{-\infty}^{A/\sqrt{\lambda+1}} e^{-u^2}\max\left(0, \frac{u}{\sqrt{\lambda+1}} + \frac{\lambda}{\lambda+1}A\right)du \\
&\lesssim Ae^{-\lambda A^2/(\lambda+1)} \lesssim \sqrt{|\log\varepsilon|}\varepsilon^2|\log\varepsilon|^{-4}.
\end{aligned}$$

Wrapping things up, we have proven that

$$\mathbb{P}_{x_0}\left(\sqrt{\frac{2}{(e^2-1)\varepsilon^2}\ell_{x,\varepsilon}} - 2\log\frac{1}{\varepsilon} - 2\log\log\frac{1}{\varepsilon} \geq 0\right) \lesssim |\log\varepsilon|^{-7/2}\varepsilon^2$$

and summing over $x \in \frac{\varepsilon}{|\log\varepsilon|}\mathbb{Z}^2 \cap D$, $|x-x_0| \geq e\varepsilon + \varepsilon/|\log\varepsilon|$,

$$\mathbb{P}_{x_0}\left(\sup_{\substack{x\in\frac{\varepsilon}{|\log\varepsilon|}\mathbb{Z}^2\cap D\\ |x-x_0|\geq e\varepsilon+\varepsilon/|\log\varepsilon|}} \sqrt{\frac{2}{(e^2-1)\varepsilon^2}\ell_{x,\varepsilon}} - 2\log\frac{1}{\varepsilon} - 2\log\log\frac{1}{\varepsilon} \geq 0\right) \lesssim |\log\varepsilon|^{-3/2}.$$

This is summable over $\varepsilon \in \{e^{-n}, n\geq 1\}$ as required. It concludes the proof. □

3.4.2 First layer of good events: proof of Proposition 3.14

We now have all the ingredients to prove Proposition 3.14. During the course of the proof, we will obtain intermediate results that we gather in the following lemma. Recall the definition (3.17) of ε_γ.

Lemma 3.28. *Firstly,*

$$\lim_{\beta\to\infty}\lim_{\varepsilon\to 0}\mathbb{E}_{x_0}\left[\hat{m}_\varepsilon(D)\right] = \infty. \tag{3.49}$$

Secondly, we have for $\beta>0$ fixed,

$$\sup_{\varepsilon>0}\mathbb{E}_{x_0}\left[\hat{m}_\varepsilon(D)\right] \leq \int_D \sup_{\varepsilon>0}\mathbb{E}_{x_0}\left[\hat{m}_\varepsilon(dx)\right] < \infty, \tag{3.50}$$

$$\sup_{\varepsilon>0}\mathbb{E}_{x_0}\left[\hat{\mu}_\varepsilon(D)\right] \leq \int_D \sup_{\varepsilon>0}\mathbb{E}_{x_0}\left[\hat{\mu}_\varepsilon(dx)\right] < \infty \tag{3.51}$$

and

$$\sup_{\gamma\in[1,2)}(2-\gamma)^{-1}\sup_{\varepsilon<\varepsilon_\gamma}\mathbb{E}_{x_0}\left[\hat{m}_\varepsilon^\gamma(D)\right] \leq \int_D \sup_{\gamma\in[1,2)}(2-\gamma)^{-1}\sup_{\varepsilon<\varepsilon_\gamma}\mathbb{E}_{x_0}\left[\hat{m}_\varepsilon^\gamma(dx)\right] < \infty. \tag{3.52}$$

Proof of Proposition 3.14 and Lemma 3.28. Let $\beta'>0$ be large. In light of Lemma 3.27 we introduce for all $\varepsilon = e^{-k} > 0$ and $x\in D$ at distance at least $e\varepsilon$ from x_0, the good event

$$H_\varepsilon(x) := \left\{\forall n = k_x+1\dots k, \sqrt{\frac{2}{(e^2-1)}e^{2n}\ell_{x,e^{-n}}(\tau_{x,R})} - 2n - 2\log n \leq \beta'\right\}$$

and set

$$H := \bigcap_{x\in D}\bigcap_{\varepsilon>0} H_\varepsilon(x).$$

Lemma 3.27 asserts that $\mathbb{P}_{x_0}(H)\to 1$ as $\beta'\to\infty$.

Seneta–Heyde norming. We are first going to show that for a fixed $\beta'>0$,

$$\int_A \sup_{\varepsilon>0}|\log\varepsilon|\varepsilon^2\mathbb{E}_{x_0}\left[e^{2\sqrt{\frac{1}{\varepsilon}L_{x,\varepsilon}(\tau)}}\mathbf{1}_{H_\varepsilon(x)}\right]dx < \infty. \tag{3.53}$$

First of all, if $|x-x_0|<1/|\log\varepsilon|$, then we simply bound

$$|\log\varepsilon|\varepsilon^2\mathbb{E}_{x_0}\left[e^{2\sqrt{\frac{1}{\varepsilon}L_{x,\varepsilon}(\tau)}}\mathbf{1}_{H_\varepsilon(x)}\right]\leq|\log\varepsilon|\varepsilon^2\mathbb{E}_{x_0}\left[e^{2\sqrt{\frac{1}{\varepsilon}L_{x,\varepsilon}(\tau_{x,R})}}\right]\lesssim|\log\varepsilon|^{3/2}$$

by (3.22). Take now $x\in D$ at distance at least $1/|\log\varepsilon|$ from x_0. We again bound $L_{x,\varepsilon}(\tau)$ by $L_{x,\varepsilon}(\tau_{x,R})$ to be able to use the link (3.6) between local times and zero-dimensional Bessel process:

$$|\log\varepsilon|\varepsilon^2\mathbb{E}_{x_0}\left[e^{2\sqrt{\frac{1}{\varepsilon}L_{x,\varepsilon}(\tau)}}\mathbf{1}_{H_\varepsilon(x)}\right]\leq|\log\varepsilon|\varepsilon^2\mathbb{E}_{x_0}\left[e^{2\sqrt{\frac{1}{\varepsilon}L_{x,\varepsilon}(\tau_{x,R})}}\mathbf{1}_{H_\varepsilon(x)}\right].$$

Denote by $r_x:=\sqrt{e^{k_x}L_{x,e^{-k_x}}(\tau_{x,R})}$. (3.6) tells us that, conditionally on r_x, the process

$$X_s:=\sqrt{e^{k_x+s}L_{x,e^{-k_x-s}}(\tau_{x,R})},\quad s\geq 0,$$

is a zero-dimensional Bessel process starting at r_x. The event $H_\varepsilon(x)$ requires

$$\begin{aligned}\min_{u\in[s-1,s]}X_u&\leq\left(\frac{2}{e^2-1}\int_{s-1}^{s}e^{2s-2u}X_u^2du\right)^{1/2}\\&=\left(\frac{2}{e^2-1}e^{2(k_x+s)}\int_{s-1}^{s}e^{-k_x-u}L_{x,e^{-k_x-u}}(\tau_{x,R})du\right)^{1/2}\\&=\left(\frac{2}{e^2-1}e^{2(k_x+s)}\int_{e^{-k_x-s}}^{e^{-k_x-s+1}}L_{x,\delta}(\tau_{x,R})d\delta\right)^{1/2}\\&\leq 2s+2k_x+2\log(s+k_x)+\beta'\leq 2s+2\log s+\beta'+4k_x\end{aligned}$$

for all $s=1\dots k-k_x$. Hence

$$\begin{aligned}&|\log\varepsilon|\varepsilon^2\mathbb{E}_{x_0}\left[e^{2\sqrt{\frac{1}{\varepsilon}L_{x,\varepsilon}(\tau_{x,R})}}\mathbf{1}_{H_\varepsilon(x)}\right]\\&\leq|\log\varepsilon|\varepsilon^2\mathbb{E}_{x_0}\left[\mathsf{E}^0_{r_x}\left[e^{2X_{k-k_x}}\mathbf{1}_{\left\{\forall s=1\dots k-k_x,\min_{u\in[s-1,s]}X_u\leq 2s+2\log s+\beta'+4k_x\right\}}\right]\right].\end{aligned}$$

Now, with (3.30), we have

$$\begin{aligned}&|\log\varepsilon|\varepsilon^2\mathbb{E}_{x_0}\left[e^{2\sqrt{\frac{1}{\varepsilon}L_{x,\varepsilon}(\tau_{x,R})}}\mathbf{1}_{H_\varepsilon(x)}\right]\leq|\log\varepsilon|\varepsilon^2+\frac{k}{\sqrt{k-k_x}}e^{-2k_x}\\&\times\mathbb{E}_{x_0}\left[e^{2r_x}\sqrt{r_x}\mathsf{E}_{r_x}\left[\left(\frac{k-k_x}{X_{k-k_x}+2(k-k_x)}\right)_+^{1/2}\mathbf{1}_{\left\{\forall s=1\dots k-k_x,\min_{u\in[s-1,s]}X_u\leq 2\log s+\beta'+4k_x+2\right\}}\right]\right].\end{aligned}$$

We now bound

$$\begin{aligned}
&\mathbb{E}_{r_x}\left[\left(\frac{k-k_x}{X_{k-k_x}+2(k-k_x)}\right)_+^{1/2}\mathbf{1}_{\{\forall s=1\ldots k-k_x,\min_{u\in[s-1,s]}X_u\le 2\log s+\beta'+4k_x+2\}}\right]\\
&\le \mathbb{P}_{r_x}\left(\forall s=1\ldots k-k_x,\ \min_{u\in[s-1,s]}X_u\le 2\log s+\beta'+4k_x+2\right)\\
&+\mathbb{E}_{r_x}\left[\left(\frac{k-k_x}{X_{k-k_x}+2(k-k_x)}\right)_+^{1/2}\mathbf{1}_{\{X_{k-k_x}\le -(k-k_x)\}}\right].
\end{aligned}$$

By (3.38), the first right hand side term is at most $C(k_x)^2k^{-1/2}$. The second right hand side term decays much faster and we have obtained

$$|\log\varepsilon|\varepsilon^2\mathbb{E}_{x_0}\left[e^{2\sqrt{\frac{1}{\varepsilon}L_{x,\varepsilon}(\tau_{x,R})}}\mathbf{1}_{H_\varepsilon(x)}\right]\lesssim|\log\varepsilon|\varepsilon^2+(k_x)^2e^{-2k_x}\mathbb{E}_{x_0}\left[e^{2r_x}\sqrt{r_x}\right]\lesssim(k_x)^3$$

where we have used (3.22) in the last inequality (or more precisely, the stochastic domination stated in Lemma 3.19 in order to also handle $\sqrt{r_x}$). To wrap things up, we have proven that

$$|\log\varepsilon|\varepsilon^2\mathbb{E}_{x_0}\left[e^{2\sqrt{\frac{1}{\varepsilon}L_{x,\varepsilon}(\tau_{x,R})}}\mathbf{1}_{H_\varepsilon(x)}\right]\lesssim\begin{cases}|\log\varepsilon|^{3/2} & \text{if } |x-x_0|\le 1/|\log\varepsilon|\\ |\log|x-x_0||^3 & \text{if } |x-x_0|\ge 1/|\log\varepsilon|\end{cases}$$

which concludes the proof of (3.53). Very few arguments need to be changed in order to show (3.50). The only difference is that, compared to the event $H_\varepsilon(x)$, the event $G_\varepsilon(x)$ ensures (in particular) the Bessel process X to stay below $s\mapsto 2s+\beta+2k_x$ at every integer s. This is more restrictive than asking $\min_{[s,s+1]}X$ to be not larger than $2s+2\log s+\beta+4k_x$, we can thus conclude using the reasoning above.

We now turn to the proof of (3.11). Fix $\beta'>0$. We are going to show that

$$\sup_{\varepsilon>0}|\log\varepsilon|\varepsilon^2\int_A\mathbb{E}_{x_0}\left[e^{2\sqrt{\frac{1}{\varepsilon}L_{x,\varepsilon}(\tau)}}\mathbf{1}_{H_\varepsilon(x)}\mathbf{1}_{G_\varepsilon(x)^c}\right]dx \tag{3.54}$$

goes to zero as $\beta\to\infty$. Let $\eta_0>0$ be small. By (3.53),

$$\sup_{\varepsilon>0}|\log\varepsilon|\varepsilon^2\int_A\mathbf{1}_{\{|x-x_0|\le\eta_0\}}\mathbb{E}_{x_0}\left[e^{2\sqrt{\frac{1}{\varepsilon}L_{x,\varepsilon}(\tau)}}\mathbf{1}_{H_\varepsilon(x)}\right]dx=o_{\eta_0\to 0}(1).$$

Fix now $\eta_0>0$. In what follows the constants underlying the bounds may depend on η_0. Recall the definition of $h_{x,\delta}$ constructed in Lemma 3.13. By a reasoning very similar to what we did above and using (3.39), one can show that

$$\sup_{\substack{\varepsilon=e^{-k}\\k\ge1}}|\log\varepsilon|\varepsilon^2\int_A\mathbf{1}_{\{|x-x_0|>\eta_0\}}\mathbb{E}_{x_0}\left[e^{2\sqrt{\frac{1}{\varepsilon}L_{x,\varepsilon}(\tau_{x,R})}}\mathbf{1}_{H_\varepsilon(x)}\mathbf{1}_{\{\exists s\in\{k_x,\ldots,k\},h_{x,e^{-s}}\ge 2s+\beta/2\}}\right]dx$$

goes to zero as $\beta \to \infty$. We are thus left to control

$$|\log \varepsilon|\varepsilon^2 \mathbb{E}_{x_0} \left[e^{2\sqrt{\frac{1}{\varepsilon}L_{x,\varepsilon}(\tau_{x,R})}} \mathbf{1}_{\{\forall s \in \{k_x,\dots,k\}, h_{x,e^{-s}} < 2s+\beta/2\}} \mathbf{1}_{G_\varepsilon(x)^c} \right]$$

for some $x \in D$ at distance at least η_0 from x_0. Denote $r_x = \sqrt{e^{-k_x} L_{x,e^{-k_x}}(\tau_{x,R})}$. By (3.6) and then by (3.31), this is equal to

$$\begin{aligned}
&|\log \varepsilon|\varepsilon^2 \mathbb{E}_{x_0} \left[\mathsf{E}^0_{r_x} \left[e^{2X_t} \mathbf{1}_{\{\forall s=0\dots k-k_x, X_s < 2s+\beta/2+2k_x\}} \mathbf{1}_{\{\exists s \leq k-k_x, X_s \geq 2s+\beta+2k_x\}} \right]\right] \\
&\lesssim \sqrt{k} \mathbb{E}_{x_0} \left[\sqrt{r_x} e^{2r_x} \mathsf{E}_{r_x} \left[\left(\frac{k-k_x}{X_{k-k_x}+2(k-k_x)} \right)_+^{1/2} \right.\right. \\
&\left.\left. \times \mathbf{1}_{\{\forall s=0\dots k-k_x, X_s < \beta/2+2k_x\}} \mathbf{1}_{\{\exists s \leq k-k_x, X_s \geq \beta+2k_x\}} \right]\right] \\
&\lesssim \sqrt{k} \mathbb{E}_{x_0} \left[\sqrt{r_x} e^{2r_x} \mathsf{P}_r \left(\forall s = 0\dots k-k_x, X_s < \beta/2+2k_x, \exists s \leq k-k_x, X_s \geq \beta+2k_x \right) \right] \\
&\lesssim \beta^2 e^{-\beta/256}
\end{aligned}$$

by (3.39). This concludes the proof of (3.54). We now have for any small $\rho > 0$,

$$\begin{aligned}
&\limsup_{\varepsilon \to 0} \mathbb{P}_{x_0} \left(|m_\varepsilon(A) - \hat{m}_\varepsilon(A)| \geq \rho \right) \\
&\leq \mathbb{P}_{x_0}(H^c) + \frac{1}{\rho} \limsup_{\varepsilon \to 0} |\log \varepsilon|\varepsilon^2 \int_A \mathbb{E}_{x_0} \left[e^{2\sqrt{\frac{1}{\varepsilon}L_{x,\varepsilon}(\tau)}} \mathbf{1}_{H_\varepsilon(x)} \mathbf{1}_{G_\varepsilon(x)^c} \right] dx.
\end{aligned}$$

By letting $\beta \to \infty$ and then $\beta' \to \infty$, we see that

$$\limsup_{\beta \to \infty} \limsup_{\varepsilon \to 0} \mathbb{P}_{x_0} \left(|m_\varepsilon(A) - \hat{m}_\varepsilon(A)| \geq \rho \right) = 0$$

as desired in (3.11).

To show (3.49), take $r > 0$ small enough so that $\{x \in D : D(x,r) \subset D\}$ has positive Lebesgue measure and notice that

$$\mathbb{E}_{x_0}\left[\hat{m}_\varepsilon(D)\right] \geq |\log \varepsilon|\varepsilon^2 \int_D \mathbf{1}_{\{D(x,r) \subset D\}} \mathbb{E}_{x_0} \left[e^{2\sqrt{\frac{1}{\varepsilon}L_{x,\varepsilon}(\tau_{x,r})}} \mathbf{1}_{\left\{\forall s \in [k_x,k], h^r_{x,e^{-s}} \leq 2s+\beta\right\}} \right] dx$$

where h^r is defined in a similar manner as h expect that we consider local times up to time $\tau_{x,r}$ rather than $\tau_{x,R}$. Using (3.6), we see that (3.49) is a direct consequence of (3.34) and Fatou's lemma.

Subcritical measures We have finished the part of the proof concerning the Seneta–Heyde normalisation and we now turn to the justification of (3.13) and (3.52). This is very similar to what we have just done. The only difference is that after using the link (3.6) between local times and zero-dimensional

Bessel process and the relation (3.30) to transfer computations to 1D Brownian motion, we have

$$\begin{aligned}
&\frac{1}{2-\gamma}\sqrt{|\log\varepsilon|}\varepsilon^{\gamma^2/2}\mathbb{E}_{x_0}\left[e^{2\sqrt{\frac{1}{\varepsilon}L_{x,\varepsilon}(\tau_{x,R})}}\mathbf{1}_{H_\varepsilon(x)}\right] \\
&\le \frac{1}{2-\gamma}\sqrt{|\log\varepsilon|}\varepsilon^{\gamma^2/2} + \frac{1}{2-\gamma}e^{-\gamma^2 k_x/2}\mathbb{E}_{x_0}\Bigg[\sqrt{r_x}e^{\gamma r_x} \\
&\times \mathbb{E}_{r_x}\left[\left(\frac{k-k_x}{X_{k-k_x}+\gamma(k-k_x)}\right)^{1/2}\mathbf{1}_{\left\{\forall s=1\ldots k-k_x, \min_{u\in[s-1,s]}X_u\le(2-\gamma)s+2\log s+\beta'+4k_x+2\right\}}\right]\Bigg].
\end{aligned}$$

We conclude as before by using (3.40) and (3.41) (note here that $k-k_x\ge(2-\gamma)^{-4}$ since ε_γ has been chosen small enough) instead of (3.38) and (3.39).

Derivative martingale We finish with the justification of (3.12) and (3.51). Recall that in the modified measure $\hat{\mu}_\varepsilon$, the Brownian motion is stopped either at time τ or at time $\tau_{x,R}$ depending on whether the local time $L_{x,\varepsilon}$ is in the exponential or not. Part of (3.12) consists in saying that, in the limit, this modification does not change the measure with high probability. We thus start by proving that

$$\limsup_{\varepsilon\to 0}\int_D\sqrt{|\log\varepsilon|}\varepsilon^2\mathbb{E}_{x_0}\left[\left(\sqrt{\frac{1}{\varepsilon}L_{x,\varepsilon}(\tau_{x,R})}-\sqrt{\frac{1}{\varepsilon}L_{x,\varepsilon}(\tau)}\right)e^{2\sqrt{\frac{1}{\varepsilon}L_{x,\varepsilon}(\tau)}}\right]dx=0. \tag{3.55}$$

Let $x\in D$. By applying Markov property to the first exit time τ of D, the integrand in (3.55) is at most equal to

$$\sqrt{|\log\varepsilon|}\varepsilon^2\mathbb{E}_{x_0}\left[e^{2\sqrt{\frac{1}{\varepsilon}L_{x,\varepsilon}(\tau)}}\sup_{z\in\partial D}\mathbb{E}_z\left[\sqrt{\frac{1}{\varepsilon}L_{x,\varepsilon}(\tau_{x,R})+\frac{1}{\varepsilon}L_{x,\varepsilon}(\tau)}-\sqrt{\frac{1}{\varepsilon}L_{x,\varepsilon}(\tau)}\,\middle|\,L_{x,\varepsilon}(\tau)\right]\right].$$

We decompose this expectation in two parts, the first one integrating on the event that $\sqrt{\frac{1}{\varepsilon}L_{x,\varepsilon}(\tau)}<|\log\varepsilon|/2$ and the second one integrating on the complement event. The first part decays quickly to zero and we explain how to deal with the second part. Recall that starting from any point of $\partial D(x,\varepsilon)$, $L_{x,\varepsilon}(\tau_{x,R})$ is a random variable with mean $2\varepsilon\log(R/\varepsilon)$ (see Lemma 3.18). By Cauchy–Schwarz inequality and then by bounding $\sqrt{a+b}-\sqrt{a}\le Cb/\sqrt{a}$ for $a>2b>1$, and using (3.23), we thus obtain that on the event that $\sqrt{\frac{1}{\varepsilon}L_{x,\varepsilon}(\tau)}\ge|\log\varepsilon|/2$,

$$\begin{aligned}
&\sup_{z\in\partial D}\mathbb{E}_z\left[\sqrt{\frac{1}{\varepsilon}L_{x,\varepsilon}(\tau_{x,R})+\frac{1}{\varepsilon}L_{x,\varepsilon}(\tau)}-\sqrt{\frac{1}{\varepsilon}L_{x,\varepsilon}(\tau)}\,\middle|\,L_{x,\varepsilon}(\tau)\right] \\
&\le \sup_{z\in\partial D}\mathbb{P}_z\left(\tau_{x,\varepsilon}<\tau_{x,R}\right)\left(\sqrt{\mathbb{E}_z\left[\frac{1}{\varepsilon}L_{x,\varepsilon}(\tau_{x,R})\,\middle|\,\tau_{x,\varepsilon}<\tau\right]+\frac{1}{\varepsilon}L_{x,\varepsilon}(\tau)}-\sqrt{\frac{1}{\varepsilon}L_{x,\varepsilon}(\tau)}\right) \\
&\lesssim \frac{|\log \mathrm{d}(x,\partial D)|}{|\log\varepsilon|}\left(\sqrt{\frac{1}{\varepsilon}L_{x,\varepsilon}(\tau)+2\log\frac{R}{\varepsilon}}-\sqrt{\frac{1}{\varepsilon}L_{x,\varepsilon}(\tau)}\right) \\
&\lesssim |\log \mathrm{d}(x,\partial D)|\left(\frac{1}{\varepsilon}L_{x,\varepsilon}(\tau)\right)^{-1/2}\lesssim\frac{|\log \mathrm{d}(x,\partial D)|}{|\log\varepsilon|}.
\end{aligned}$$

The integrand in (3.55) is therefore at most

$$o(1) + O(1)\frac{|\log \mathrm{d}(x,\partial D)|}{\sqrt{|\log \varepsilon|}}\varepsilon^2 \mathbb{E}_{x_0}\left[e^{2\sqrt{\frac{1}{\varepsilon}L_{x,\varepsilon}(\tau)}}\right] \leq o(1) + O(1)\frac{|\log \mathrm{d}(x,\partial D)||\log|x-x_0||}{|\log \varepsilon|}$$

by (3.22) and (3.23). This concludes the proof of (3.55).

Now, let $\beta > 0$. For any small $\rho > 0$ and large $\beta' > 0$, we have

$$\begin{aligned}
&\limsup_{\varepsilon \to 0} \mathbb{P}_{x_0}\left(|\mu_\varepsilon(A) - \hat{\mu}_\varepsilon(A)| > \rho\right) \leq \mathbb{P}_{x_0}\left(H^c\right) \\
&+ \frac{3}{\rho}\limsup_{\varepsilon \to 0} \int_D \sqrt{|\log \varepsilon|}\varepsilon^2 \mathbb{E}_{x_0}\left[\left(\sqrt{\frac{1}{\varepsilon}L_{x,\varepsilon}(\tau_{x,R})} - \sqrt{\frac{1}{\varepsilon}L_{x,\varepsilon}(\tau)}\right) e^{2\sqrt{\frac{1}{\varepsilon}L_{x,\varepsilon}(\tau)}}\right] dx \\
&+ \frac{3}{\rho}\limsup_{\varepsilon \to 0} \int_D \left(3\log\log\frac{1}{\varepsilon} + \beta\right)\sqrt{|\log \varepsilon|}\varepsilon^2 \mathbb{E}_{x_0}\left[e^{2\sqrt{\frac{1}{\varepsilon}L_{x,\varepsilon}(\tau)}}\mathbf{1}_{H_\varepsilon(x)}\right] dx \\
&+ \frac{3}{\rho}\limsup_{\varepsilon \to 0} \int_D \sqrt{|\log \varepsilon|}\varepsilon^2 \mathbb{E}_{x_0}\left[\left|-\sqrt{\frac{1}{\varepsilon}L_{x,\varepsilon}(\tau_{x,R})} + 2\log\frac{1}{\varepsilon}\right| e^{2\sqrt{\frac{1}{\varepsilon}L_{x,\varepsilon}(\tau)}}\mathbf{1}_{H_\varepsilon(x)}\mathbf{1}_{G_\varepsilon(x)^c}\right] dx.
\end{aligned}$$

(3.55) and (3.53) tell us that the second and respectively third right hand side terms vanish. When $\beta' > 0$ and $\rho > 0$ are fixed, one can show using a method very similar to what we did with the Seneta–Heyde normalisation that the last right hand side term goes to zero as $\beta \to \infty$. Hence

$$\limsup_{\beta \to \infty}\limsup_{\varepsilon \to 0} \mathbb{P}_{x_0}\left(|\mu_\varepsilon(A) - \hat{\mu}_\varepsilon(A)| > \rho\right) \leq \mathbb{P}_{x_0}\left(H^c\right).$$

The left hand side term is independent of β' whereas the right hand side term goes to zero as $\beta' \to 0$. Therefore, for any small $\rho > 0$,

$$\limsup_{\beta \to \infty}\limsup_{\varepsilon \to 0} \mathbb{P}_{x_0}\left(|\mu_\varepsilon(A) - \hat{\mu}_\varepsilon(A)| > \rho\right) = 0$$

as desired in (3.12). The proof of (3.51) is very similar to that of (3.50). We omit the details and it concludes the proof. □

3.4.3 Second layer of good events: proof of Lemma 3.15

Proof of Lemma 3.15. We start by proving (3.15). Let $\eta_0 > 0$. By Lemma 3.28, it is enough to show that

$$\int_D \sqrt{|\log \varepsilon|}\varepsilon^2 \mathbb{E}_{x_0}\left[\left(-\sqrt{\frac{1}{\varepsilon}L_{x,\varepsilon}(\tau_{x,R})} + 2\log\frac{1}{\varepsilon} + \beta\right) e^{2\sqrt{\frac{1}{\varepsilon}L_{x,\varepsilon}(\tau)}}\mathbf{1}_{G_\varepsilon(x)}\mathbf{1}_{G'_\varepsilon(x)^c}\right]\mathbf{1}_{\{|x-x_0|>\eta_0\}} dx \quad (3.56)$$

goes to zero as $\varepsilon \to 0$ and then $M \to \infty$. The constants underlying the following estimates may depend on η_0. We start off by bounding $L_{x,\varepsilon}(\tau)$ by $L_{x,\varepsilon}(\tau_{x,R})$ in the exponential above. By letting $t = k - k_x$,

$\beta_x = \beta + 2k_x$ and $r = \sqrt{e^{k_x} L_{x,e^{-k_x}}(\tau_{x,R})}$ and by using (3.6), we are left to estimate

$$\sqrt{t}e^{-2t}\mathbb{E}_{x_0}\left[\mathbb{E}^0_r\left[(-X_t+2t+\beta_x)e^{2X_t}\mathbf{1}_{\left\{\forall s\leq t, X_s<2s+\beta_x, \exists s\leq t, X_s\geq 2s+\beta_x-\frac{\sqrt{s}}{M\log(2+s)^2}\right\}}\right]\right].$$

By (3.33) and then by Cauchy–Schwarz inequality, this is at most

$$\begin{aligned}&\mathbb{E}_{x_0}\left[\sqrt{r}(\beta_x-r)e^{2r}\mathbb{E}^3_{\beta_x-r}\left[\left(\frac{t}{2t+\beta_x-X_t}\right)_+^{1/2}\mathbf{1}_{\left\{\exists s\leq t, X_s\leq\frac{\sqrt{s}}{M\log(2+s)^2}\right\}}\right]\right]\\&\lesssim\mathbb{E}_{x_0}\left[\sqrt{r}(\beta_x-r)e^{2r}\mathbb{E}^3_{\beta_x-r}\left[\left(\frac{t}{2t+\beta_x-X_t}\right)_+\right]^{1/2}\mathbb{P}^3_{\beta_x-r}\left(\exists s\geq 0, X_s\leq\frac{\sqrt{s}}{M\log(2+s)^2}\right)^{1/2}\right]\end{aligned}$$

which goes to zero as $M\to\infty$ uniformly in t by Lemma 3.23, Points 1 and 4. We have thus proven that the contribution of points at distance at least η_0 from x_0 to the integral (3.56) goes to zero as $\varepsilon\to 0$ and then $M\to 0$. This concludes the proof of (3.15).

The proof of (3.14) is very similar: the presence of an extra $\sqrt{|\log\varepsilon|}$ in the normalisation as well as the absence of the derivative term $(-X_t+2t+\beta)$ makes an extra multiplicative term $\sqrt{t}/X_t$ popping up in the expectation with respect to the 3D Bessel process. We conclude as before using Cauchy–Schwarz inequality and Lemma 3.23, Point 3.

We finish with the proof of (3.16). With the same notations as above, it is again enough to estimate

$$(2-\gamma)^{-1}\sqrt{t}e^{-\frac{\gamma^2}{2}t}\mathbb{E}_{x_0}\left[\mathbb{E}^0_r\left[e^{\gamma X_t}\mathbf{1}_{\left\{\forall s\leq t, X_s<2s+\beta_x, \exists s\leq t, X_s\geq 2s+\beta_x-\frac{\sqrt{s}}{M\log(2+s)^2}\right\}}\right]\right].$$

By (3.31), this is at most

$$\begin{aligned}&(2-\gamma)^{-1}\mathbb{E}_{x_0}\left[\sqrt{r}e^{\gamma r}\mathbb{E}_r\left[\left(\frac{t}{X_t+\gamma t}\right)_+^{1/2}\mathbf{1}_{\left\{\forall s\leq t, X_s<(2-\gamma)s+\beta_x, \exists s\leq t, X_s\geq(2-\gamma)s+\beta_x-\frac{\sqrt{s}}{M\log(2+s)^2}\right\}}\right]\right]\\&\lesssim o_{t\to\infty}(1)+(2-\gamma)^{-1}\mathbb{E}_{x_0}\Bigg[\sqrt{r}e^{\gamma\sqrt{r}}\\&\qquad\mathbb{P}_r\left(\forall s\leq t, X_s<(2-\gamma)s+\beta_x, \exists s\leq t, X_s\geq(2-\gamma)s+\beta_x-\frac{\sqrt{s}}{M\log(2+s)^2}\right)\Bigg]\end{aligned}\tag{3.57}$$

where we obtained the above estimate by decomposing the expectation according to whether $X_t\leq -\gamma t/2$ or not. By Girsanov's theorem and then by Lemma B, the above probability with respect to the one-dimensional Brownian motion is equal to

$$\begin{aligned}&e^{-(2-\gamma)^2t/2}\mathbb{E}_0\left[e^{-(2-\gamma)X_t}\mathbf{1}_{\left\{\forall s\leq t, X_s<\beta_x-r, \exists s\leq t, X_s\geq\beta_x-r-\frac{\sqrt{s}}{M\log(2+s)^2}\right\}}\right]\\&=e^{-(2-\gamma)^2t/2}\mathbb{E}^3_{\beta_x-r}\left[\frac{\beta_x-r}{X_t}e^{-(2-\gamma)(\beta_x-r-X_t)}\mathbf{1}_{\left\{\exists s\leq t, X_s\leq\frac{\sqrt{s}}{M\log(2+s)^2}\right\}}\right].\end{aligned}$$

By decomposing the above expectation according to whether $X_t \geq (2-\gamma)t/4$ or not, we see that it is at most, up to a multiplicative constant,

$$e^{-(2-\gamma)^2 t/4} + e^{-(2-\gamma)^2 t/2} \mathbb{E}^3_{\beta_x - r} \left[\frac{\beta_x - r}{(2-\gamma)t} e^{(2-\gamma)X_t} \mathbf{1}_{\left\{ \exists s \leq t, X_s \leq \frac{\sqrt{s}}{M \log(2+s)^2} \right\}} \right].$$

Now, by Lemma 3.23 point 1 and because X_t under $\mathbb{P}^3_{\beta_x - r}\left(\cdot \middle| \exists s \leq t, X_s \leq \frac{\sqrt{s}}{M \log(2+s)^2} \right)$ is stochastically dominated by X_t under $\mathbb{P}^3_{\beta_x - r}$, we see that the probability in (3.57) is at most, up to a multiplicative constant,

$$e^{-(2-\gamma)^2 t/4} + o_{M \to \infty}(1) e^{-(2-\gamma)^2 t/2} \mathbb{E}^3_{\beta_x - r} \left[\frac{\beta_x - r}{(2-\gamma)t} e^{(2-\gamma)X_t} \right].$$

By a similar procedure as above we can reintroduce $\frac{\beta_x - r}{X_t}$ in the expectation above in place of $\frac{\beta_x - r}{(2-\gamma)t}$ and reverse the computations using Lemma B and then Girsanov's theorem to obtain that

$$\begin{aligned} & e^{-(2-\gamma)^2 t/2} \mathbb{E}^3_{\beta_x - r} \left[\frac{\beta_x - r}{(2-\gamma)t} e^{(2-\gamma)X_t} \right] \\ & \lesssim e^{-(2-\gamma)^2 t/4} + \mathbb{P}_r \left(\forall s \leq t, X_s < (2-\gamma)s + \beta_x \right) \lesssim 2 - \gamma \end{aligned}$$

by (3.40). Wrapping things up, we have obtained that the probability in (3.57) is at most

$$o_{M \to \infty}(1)(2-\gamma)$$

as desired. This concludes the proof. □

3.5 L^2-estimates

3.5.1 Uniform integrability: proof of Proposition 3.16

This section is devoted to the proof of Proposition 3.16. We first state the following result for ease of reference.

Lemma 3.29. *Let I be a finite set of indices, $(r_i, i \in I) \in [0, \infty)^I$ and let $(X^{(i)}, i \in I) \sim \otimes_{i \in I} \mathbb{P}^0_{r_i}$ be independent zero-dimensional Bessel processes starting at r_i. Define the process $(X_s, s \geq 0)$ as follows: for all $n \geq 0$, let $X_n = \sqrt{\sum_{i \in I} (X_n^{(i)})^2}$ and conditionally on $(X_n^{(i)}, n \geq 1, i \in I)$, let $(X_s, s \in (n, n+1)), n \geq 0$, be independent zero-dimensional Bessel bridges between X_n and X_{n+1}. Then $X \sim \mathbb{P}^0_r$ with $r = \sqrt{\sum_{i \in I} r_i^2}$.*

Proof. This is a direct consequence of the fact that the sum of independent zero-dimensional squared Bessel processes is again distributed as a zero-dimensional squared Bessel process. □

Proof of Proposition 3.16. The constants underlying this proof may depend on β and M. We start by proving (3.19). We will then see that very few arguments need to be modified to obtain (3.18) and

(3.20). Let ε' be the only real number in $\{e^{-n}, n \geq 1\}$ be such that

$$1 \leq \frac{\varepsilon'}{e^4 M \varepsilon \exp\left((\log|\log\varepsilon|)^6\right)} < e. \tag{3.58}$$

We are first going to control the contribution of points $x, y \in D$ at distance at least $1/M$ from x_0 such that $|x-y| \leq \varepsilon'$. Let x and y be such points. On $G'_\varepsilon(y)$,

$$\sqrt{\frac{1}{\varepsilon} L_{y,\varepsilon}(\tau)} \leq 2\log\frac{1}{\varepsilon} - \frac{\sqrt{|\log\varepsilon|}}{M\log(2+|\log\varepsilon|)^2} + \beta.$$

We thus have

$$\begin{aligned} (\varepsilon')^2 \mathbb{E}_{x_0}\left[\hat{\hat{\mu}}_\varepsilon(x)\hat{\hat{\mu}}_\varepsilon(y)\right] &\lesssim (\varepsilon')^2 |\log\varepsilon|^3 \exp\left(-\frac{2\sqrt{|\log\varepsilon|}}{M\log(2+|\log\varepsilon|)^2}\right) \mathbb{E}_{x_0}\left[e^{2\sqrt{\frac{1}{\varepsilon}L_{x,\varepsilon}(\tau)}}\right] \\ &\lesssim \left(\frac{\varepsilon'}{\varepsilon}\right)^2 |\log\varepsilon|^{7/2} \exp\left(-\frac{2\sqrt{|\log\varepsilon|}}{M\log(2+|\log\varepsilon|)^2}\right) \end{aligned}$$

using (3.22) in the last inequality. This shows that

$$\int_{D\times D} \sup_{\varepsilon>0} \mathbb{E}_{x_0}\left[\hat{\hat{\mu}}_\varepsilon(dx)\hat{\hat{\mu}}_\varepsilon(dy)\right] \mathbf{1}_{\{|x-y|\leq\varepsilon'\}} < \infty.$$

We now focus on the remaining contribution. Let $x, y \in D$ at distance at least $1/M$ from x_0 be such that $|x-y| \geq \varepsilon'$. Without loss of generality, assume that the diameter of D is at most 1 so that we can define $\alpha = e^{-k_\alpha}, \eta = e^{-k_\eta} \in \{e^{-n}, n \geq 1 + \lfloor \log M \rfloor\}$ to be the only real numbers satisfying

$$\frac{1}{e^2 M} \leq \frac{\alpha}{|x-y|} < \frac{1}{eM} \quad \text{and} \quad \frac{1}{e^4 M} \leq \frac{\eta}{|x-y|\exp\left(-(\log|\log|x-y||)^6\right)} < \frac{1}{e^3 M}. \tag{3.59}$$

Notice that $D(x,\alpha) \cap D(y,\alpha) = \varnothing$ (as soon as M is at least $2/e$), that $\eta \geq \varepsilon$ because $|x-y| \geq \varepsilon'$, that $k_\eta \geq 1 + \log M \geq k_x$ and that $\eta < \alpha/e$. Define

$$G_{\eta,\varepsilon}(x) := \left\{\forall s \in [k_\eta, k], h_{x,e^{-s}} \leq 2s + \beta\right\}.$$

Importantly, the event $G_{\eta,\varepsilon}(x)$ is contained in $G_\varepsilon(x)$ and only cares about what happens inside the disc $D(x,\alpha/e)$. We similarly define $G_{\eta,\varepsilon}(y)$. We can bound $\mathbb{E}_{x_0}\left[\hat{\hat{\mu}}_\varepsilon(x)\hat{\hat{\mu}}_\varepsilon(y)\right]$ by

$$\begin{aligned} &|\log\varepsilon|\varepsilon^4 \mathbb{E}_{x_0}\Bigg[\left(-\sqrt{\frac{1}{\varepsilon}L_{x,\varepsilon}(\tau_{x,R})} + 2\log\frac{1}{\varepsilon} + \beta\right)\left(-\sqrt{\frac{1}{\varepsilon}L_{y,\varepsilon}(\tau_{y,R})} + 2\log\frac{1}{\varepsilon} + \beta\right) \\ &e^{2\sqrt{\frac{1}{\varepsilon}L_{x,\varepsilon}(\tau_{x,R})}} e^{2\sqrt{\frac{1}{\varepsilon}L_{y,\varepsilon}(\tau_{y,R})}} \mathbf{1}_{G_{\eta,\varepsilon}(x)} \mathbf{1}_{G_{\eta,\varepsilon}(y)} \mathbf{1}_{\left\{\sqrt{\frac{1}{\eta}L_{y,\eta}(\tau_{y,R})} \leq 2\log\frac{1}{\eta} + \beta - \frac{\sqrt{|\log\eta|}}{M\log(2+|\log\eta|)^2}\right\}}\Bigg]. \end{aligned} \tag{3.60}$$

In broad terms, our strategy now is to condition on $L_{x,\eta}(\tau_{x,R})$ and $L_{y,\eta}(\tau_{y,R})$ and integrate everything else. Let N_x be the number of excursions from $\partial D(x,\alpha/e)$ to $\partial D(x,\alpha)$ before hitting $\partial D(x,R)$. For $i = 1 \ldots N_x$ and $\delta \leq \alpha/e$, let $L^i_{x,\delta}$ be the local time of $\partial D(x,\delta)$ accumulated during the i-th excursion. We also write $r^i_{x,\eta} := \sqrt{\frac{1}{\eta} L^i_{x,\eta}}$ and $r_{x,\eta} := \sqrt{\frac{1}{\eta} L_{x,\eta}(\tau_{x,R})}$. Let I_x be the subset of $\{1, \dots, N_x\}$ corresponding to the above excursions that hit $\partial D(x,\eta)$. Define similar notations with x replaced by y et let $\mathcal{F}_{x,y}$ be the sigma algebra generated by N_x, N_y, I_x, I_y and the successive initial and final positions of the above-mentioned excursions (around both x and y).

Conditionally on the initial and final positions of the above excursions,

$$\left(L^i_{x,\delta}, i = 1 \dots N_x, \delta \leq \alpha/e \right) \quad \text{and} \quad \left(L^i_{y,\delta}, i = 1 \dots N_y, \delta \leq \alpha/e \right)$$

are independent. Moreover, for all $i = 1 \dots N_x$, conditioned on $\{i \in I_x\}$, $\left(L^i_{x,e^{-n}}, n \geq k_\alpha + 1 \right)$ is close to be independent of the initial and final positions of the given excursion: this is the content of the continuity Lemma 3.20. The Bessel bridges that we use to interpolate the local times between dyadic radii smaller than α around x and y do not create any further dependence since $D(x,\alpha) \cap D(y,\alpha) = \varnothing$. Hence, recalling (3.6) and Lemma 3.29, we see that by paying a multiplicative price $(1 + p(\frac{\eta}{\alpha}))^{|I_x|+|I_y|}$ and conditionally on $\mathcal{F}_{x,y}$, we can approximate the joint law of $(h_{x,\eta e^{-s}}, s \geq 0)$ and $(h_{y,\eta e^{-s}}, s \geq 0)$ by $\mathbb{P}^0_{r_{x,\eta}} \otimes \mathbb{P}^0_{r_{y,\eta}}$. Letting $t = \log \frac{\eta}{\varepsilon} = k - k_\eta$ and $\beta' := \beta + 2k_\eta$, we deduce that

$$\begin{aligned}
&|\log \varepsilon| \varepsilon^4 \mathbb{E}_{x_0} \Bigg[\left(-\sqrt{\frac{1}{\varepsilon} L_{x,\varepsilon}(\tau_{x,R})} + 2 \log \frac{1}{\varepsilon} + \beta \right) \left(-\sqrt{\frac{1}{\varepsilon} L_{y,\varepsilon}(\tau_{y,R})} + 2 \log \frac{1}{\varepsilon} + \beta \right) \\
&e^{2\sqrt{\frac{1}{\varepsilon} L_{x,\varepsilon}(\tau_{x,R})}} e^{2\sqrt{\frac{1}{\varepsilon} L_{y,\varepsilon}(\tau_{y,R})}} \mathbf{1}_{G_{\eta,\varepsilon}(x)} \mathbf{1}_{G_{\eta,\varepsilon}(y)} \mathbf{1}_{\left\{ \sqrt{\frac{1}{\eta} L_{y,\eta}(\tau_{y,R})} \leq 2 \log \frac{1}{\eta} + \beta - \frac{\sqrt{|\log \eta|}}{M \log(2 + |\log \eta|)^2} \right\}} \Bigg| \mathcal{F}_{x,y} \Bigg] \\
&\leq \left(1 + p\left(\frac{\eta}{\alpha} \right) \right)^{|I_x| + |I_y|} |\log \varepsilon| \varepsilon^4 \mathbb{E}_{x_0} \Bigg[\mathbf{1}_{\left\{ \sqrt{\frac{1}{\eta} L_{y,\eta}(\tau_{y,R})} \leq 2 \log \frac{1}{\eta} + \beta - \frac{\sqrt{|\log \eta|}}{M \log(2 + |\log \eta|)^2} \right\}} \\
&\times \mathbb{E}^0_{r_{x,\eta}} \left[\left(-X_t + 2t + \beta' \right) e^{2X_t} \mathbf{1}_{\{\forall s \leq t, X_s \leq 2s + \beta'\}} \right] \\
&\times \mathbb{E}^0_{r_{y,\eta}} \left[\left(-X_t + 2t + \beta' \right) e^{2X_t} \mathbf{1}_{\{\forall s \leq t, X_s \leq 2s + \beta'\}} \right] \Bigg| \mathcal{F}_{x,y} \Bigg].
\end{aligned}$$

Now, by (3.36),

$$\begin{aligned}
&\sqrt{|\log \varepsilon|} \varepsilon^2 \mathbb{E}^0_{r_{x,\eta}} \left[\left(-X_t + 2t + \beta' \right) e^{2X_t} \mathbf{1}_{\{\forall s \leq t, X_s \leq 2s + \beta'\}} \right] \\
&\lesssim \frac{\sqrt{|\log \varepsilon|} \varepsilon^2}{\sqrt{t} e^{-2t}} |\log \eta|^{3/2} e^{2r_{x,\eta}} \lesssim |\log \eta|^{3/2} \eta^2 e^{2\sqrt{\frac{1}{\eta} L_{x,\eta}(\tau_{x,R})}}.
\end{aligned} \tag{3.61}$$

We have a similar estimate for the expectation around the point y and we further bound

$$\begin{aligned}
&\sqrt{|\log \varepsilon|} \varepsilon^2 \mathbb{E}^0_{r_{y,\eta}} \left[\left(-X_t + 2t + \beta' \right) e^{2X_t} \mathbf{1}_{\{\forall s \leq t, X_s < 2s + \beta'\}} \right] \\
&\times \mathbf{1}_{\left\{ \sqrt{\frac{1}{\eta} L_{y,\eta}(\tau_{x,R})} \leq 2 \log \frac{1}{\eta} + \beta - \frac{\sqrt{|\log \eta|}}{M \log(2 + |\log \eta|)^2} \right\}} \\
&\lesssim |\log \eta|^{3/2} \eta^{-2} \exp \left(- \frac{2\sqrt{|\log \eta|}}{M \log(2 + |\log \eta|)^2} \right).
\end{aligned}$$

To wrap things up, we have proven that

$$\begin{aligned} &\mathbb{E}_{x_0}\left[\hat{\hat{\mu}}_\varepsilon(x)\hat{\hat{\mu}}_\varepsilon(y) \middle| \, |I_x| + |I_y|\right] \\ &\lesssim \left(1 + p\left(\frac{\eta}{\alpha}\right)\right)^{|I_x|+|I_y|} |\log \eta|^3 \exp\left(-\frac{2\sqrt{|\log \eta|}}{M \log(2+|\log \eta|)^2}\right) \mathbb{E}_{x_0}\left[e^{2\sqrt{\frac{1}{\eta}L_{x,\eta}(\tau_{x,R})}} \middle| \, |I_x| + |I_y| \right]. \end{aligned} \tag{3.62}$$

By the continuity Lemma 3.20 and recalling (3.59), there exists $c_* > 0$ such that

$$\log\left(1 + p\left(\frac{\eta}{\alpha}\right)\right) \lesssim \exp\left(-c_*\left(\log\frac{\alpha}{\eta}\right)^{1/3}\right) \lesssim \exp\left(-c_*\left(\log|\log|x-y||\right)^2\right).$$

If we take N to be equal to $\exp\left(c_*\left(\log|\log|x-y||\right)^2/2\right)$, we thus have

$$\left(1 + p\left(\frac{\eta}{\alpha}\right)\right)^N \lesssim 1$$

and (3.62) together with (3.22) yield

$$\begin{aligned} &\mathbb{E}_{x_0}\left[\hat{\hat{\mu}}_\varepsilon(x)\hat{\hat{\mu}}_\varepsilon(y)\mathbf{1}_{\{|I_x|+|I_y|\le N\}}\right] \\ &\lesssim |\log \eta|^3 \exp\left(-\frac{2\sqrt{|\log \eta|}}{M \log(2+|\log \eta|)^2}\right) \mathbb{E}_{x_0}\left[e^{2\sqrt{\frac{1}{\eta}L_{x,\eta}(\tau_{x,R})}} \right] \\ &\lesssim |\log \eta|^{7/2}\eta^{-2} \exp\left(-\frac{2\sqrt{|\log \eta|}}{M \log(2+|\log \eta|)^2}\right) \\ &\lesssim |\log|x-y||^{7/2}|x-y|^{-2}\exp\left(2(\log|\log|x-y||)^6\right)\exp\left(-\frac{2\sqrt{|\log|x-y||}}{M\log(2+|\log|x-y||)^2}\right) \\ &\lesssim |x-y|^{-2}\exp\left(-\frac{\sqrt{|\log|x-y||}}{M\log(2+|\log|x-y||)^2}\right). \end{aligned}$$

We now explain how to bound $\mathbb{E}_{x_0}\left[\hat{\hat{\mu}}_\varepsilon(x)\hat{\hat{\mu}}_\varepsilon(y)\mathbf{1}_{\{|I_x|+|I_y|>N\}}\right]$. $|I_x|$ is smaller than the number of excursions from $\partial D(x,\alpha/e)$ to $\partial D(x,\eta)$ before hitting $\partial D(x,R)$ and the probability for a Brownian trajectory starting at $\partial D(x,\alpha/e)$ to hit $\partial D(x,\eta)$ before hitting $\partial D(x,R)$ is given by

$$\frac{\log(\alpha/eR)}{\log(\eta/R)}.$$

By strong Markov property, we then obtain that for all $M > 0$,

$$\mathbb{P}_{x_0}\left(|I_x| > M\right) \le \left(\frac{\log(\alpha/eR)}{\log(\eta/R)}\right)^M \le \exp\left(-c\frac{(\log|\log|x-y||)^6}{|\log|x-y||}M\right).$$

Using (3.62), Cauchy–Schwarz and (3.22), we deduce that

$$\begin{aligned}
&\mathbb{E}_{x_0}\left[\hat{\mu}_\varepsilon(x)\hat{\mu}_\varepsilon(y)\mathbf{1}_{\{|I_x|+|I_y|>N\}}\right] \\
&\leq \sum_{p\geq\lfloor\log_2 N\rfloor} \mathbb{E}_{x_0}\left[\hat{\mu}_\varepsilon(x)\hat{\mu}_\varepsilon(y)\mathbf{1}_{\{2^p\leq|I_x|+|I_y|<2^{p+1}\}}\right] \\
&\lesssim |\log\eta|^3 \exp\left(-\frac{2\sqrt{|\log\eta|}}{M\log(2+|\log\eta|)^2}\right)\mathbb{E}_{x_0}\left[e^{4\sqrt{\frac{1}{\eta}L_{x,\eta}(\tau_{x,R})}}\right]^{1/2} \\
&\times \sum_{p\geq\lfloor\log_2 N\rfloor}\left(1+p\left(\frac{\eta}{\alpha}\right)\right)^{2^{p+1}}\left(\mathbb{P}_{x_0}\left(|I_x|\geq 2^{p-1}\right)+\mathbb{P}_{x_0}\left(|I_y|\geq 2^{p-1}\right)\right)^{1/2} \\
&\lesssim |x-y|^{-4}\exp\left(-c\frac{(\log|\log|x-y||)^6}{|\log|x-y||}N\right) \qquad (3.63)\\
&\leq |x-y|^{-4}\exp\left(-c\exp\left(\frac{c_*}{4}(\log|\log|x-y||)^2\right)\right)\lesssim 1.
\end{aligned}$$

This concludes the proof of (3.19).

Let $\hat{\mu}$ be any subsequential limit of $(\hat{\mu}_\varepsilon, \varepsilon>0)$. The claim about the non-atomicity of $\hat{\mu}$ follows from the following energy estimate which is a consequence of what we did before:

$$\begin{aligned}
&\mathbb{E}_{x_0}\left[\int_{D\times D}\exp\left(|\log|x-y||^{1/3}\right)\hat{\mu}(dx)\hat{\mu}(dy)\right] \\
&\leq \limsup_{\varepsilon\to 0}\mathbb{E}_{x_0}\left[\int_{D\times D}\exp\left(|\log|x-y||^{1/3}\right)\hat{\mu}_\varepsilon(dx)\hat{\mu}_\varepsilon(dy)\right]<\infty.
\end{aligned}$$

For the proof of (3.18), resp. (3.20), we proceed in the exact same way as before. The only difference is that, instead of (3.61), we need to bound from above

$$|\log\varepsilon|\varepsilon^2\mathsf{E}^0_{r_{x,\eta}}\left[e^{2X_t}\mathbf{1}_{\{\forall s\leq t, -X_s+2s+\beta'>0\}}\right],$$

resp.

$$\frac{1}{2-\gamma}\sqrt{|\log\varepsilon|}\varepsilon^{\gamma^2/2}\mathsf{E}^0_{r_{x,\eta}}\left[e^{\gamma X_t}\mathbf{1}_{\{\forall s\leq t, -X_s+2s+\beta'>0\}}\right].$$

This is done in (3.35), resp. (3.37), and we conclude the proof of (3.18), resp. (3.20), along the same lines as above.

□

3.5.2 Cauchy sequence in L^2: proof of Proposition 3.17

This section is devoted to the proof of Proposition 3.17.

Proof of Proposition 3.17. Let A be a Borel set of $\mathbb{R}^2$. Let $\eta = e^{-k_\eta}\in\{e^{-n}, n\geq 1\}$ be small and consider

$$(A\times A)_\eta := \{(x,y)\in A\times A : \forall n\geq 1, D(x,\eta)\cap\partial D(y,e^{-n}) = D(y,\eta)\cap\partial D(x,e^{-n}) = \varnothing\}. \qquad (3.64)$$

If $(x,y) \in (A\times A)_\eta$, the two sequences of circles $(\partial D(x,e^{-n}), n\geq 1)$ and $(\partial D(y,e^{-n}), n \geq 1)$ will not interact between each other inside $D(x,\eta)$ and $D(y,\eta)$. We can write

$$\limsup_{\varepsilon,\varepsilon'\to 0} \mathbb{E}_{x_0}\left[(\hat{\mu}_\varepsilon(A) - \hat{\mu}_{\varepsilon'}(A))^2 \right] \leq 2 \limsup_{\varepsilon\to 0} \int_{(A\times A)\backslash (A\times A)_\eta} \mathbb{E}_{x_0}\left[\hat{\mu}_\varepsilon(dx)\hat{\mu}_\varepsilon(dy) \right] + 2\limsup_{\varepsilon,\varepsilon'\to 0} \int_{(A\times A)_\eta} \mathbb{E}_{x_0}\left[\hat{\mu}_\varepsilon(x) \left(\hat{\mu}_\varepsilon(y) - \hat{\mu}_{\varepsilon'}(y)\right)\right] dxdy.$$

Thanks to (3.19) and because the Lebesgue measure of $(A\times A)\backslash(A\times A)_\eta$ goes to zero as $\eta\to 0$, we know that the first right hand side term goes to zero as $\eta \to 0$. We are going to show that for a fixed η the second right hand side term vanishes. (3.19) provides the upper bound required to apply dominating convergence theorem and we are left to show the pointwise convergence

$$\limsup_{\varepsilon,\varepsilon'\to 0} \mathbb{E}_{x_0}\left[\hat{\mu}_\varepsilon(x) \left(\hat{\mu}_\varepsilon(y) - \hat{\mu}_{\varepsilon'}(y)\right)\right] = 0 \tag{3.65}$$

for a fixed $(x,y)\in (A\times A)_\eta$. Let $\eta' = e^{-k_{\eta'}} \in \{e^{-n}, n\geq 0\}$ be much smaller than η. Let N_y (resp. N_y') be the number of excursions from $\partial D(y,\eta/e)$ to $\partial D(y,\eta)$ before hitting ∂D (resp. before hitting $\partial D(y,R)$). For $i = 1\dots N_y'$ and $\delta \leq \eta/e$, we will denote $L^i_{y,\delta}$ the local time of $\partial D(y,\delta)$ accumulated during the i-th such excursion. Denote by I (resp. I') the subset of $\{1,\dots,N_y\}$ (resp. $\{1,\dots,N_y'\}$) corresponding to the excursions that visited $\partial D(y,\eta')$. First of all, one can show that there exists $N\geq 1$ depending on η such that

$$\limsup_{\varepsilon,\varepsilon'\to 0} \mathbb{E}_{x_0}\left[\hat{\mu}_\varepsilon(x)\hat{\mu}_{\varepsilon'}(y) \mathbf{1}_{\left\{N_y'>N\right\}}\right] \leq \eta.$$

This is a direct consequence of the bound (3.63). Let $\mathcal{F}_{x,y}$ be the sigma-algebra generated by $(L_{x,e^{-n}}(\tau), L_{x,e^{-n}}(\tau_{x,R}), n\geq 0)$, $(L_{y,e^{-n}}(\tau), L_{y,e^{-n}}(\tau_{y,R}), n = 0\dots k_\eta - 1)$, N_y, N_y', I, I', $(L^i_{y,e^{-n}}, i\notin I', n = k_\eta \dots k_{\eta'})$ as well as the starting and exiting point of the excursions from $\partial D(y,\eta/e)$ to $\partial D(y,\eta)$ before hitting $\partial D(y,R)$. Denote $(e/\eta)(r_{y,\eta/e})^2$ (resp. $(e/\eta)(r'_{y,\eta/e})^2$) the local time $L_{y,\eta/e}(\tau)$ (resp. $L_{y,\eta/e}(\tau_{y,R}) - L_{y,\eta/e}(\tau)$), $t = \log(\eta/(e\varepsilon))$, $\beta' = \beta - 2\log(\eta/e)$, $t_0 = \log(e/\eta)$, $t_1 = \log(e\eta'/\eta)$. With a reasoning similar as what we did in the proof of Proposition 3.16, Lemma 3.20, (3.6) and Lemma 3.29 imply that $\mathbb{E}_{x_0}\left[\hat{\mu}_\varepsilon(x)\hat{\mu}_\varepsilon(y)\mathbf{1}_{\left\{N_y'\leq N\right\}}\right]$ is equal to

$$(1\pm p(\eta'/\eta))^N \mathbb{E}_{x_0}\Bigg[\sqrt{|\log\varepsilon|}\varepsilon^2 \left(-\sqrt{\frac{1}{\varepsilon}L_{x,\varepsilon}(\tau_{x,R})} + 2|\log\varepsilon| + \beta \right) e^{2\sqrt{\frac{1}{\varepsilon}L_{x,\varepsilon}(\tau)}} \mathbf{1}_{G_\varepsilon(x)\cap G'_\varepsilon(x)}$$

$$\times \mathbf{1}_{G_{\eta/e}(y)\cap G'_{\eta/e}(y)} \mathbf{1}_{\left\{N_y'\leq N\right\}} \mathsf{IE}^0_{r_{y,\alpha/e}} \otimes \mathsf{IE}^0_{r'_{y,\alpha/e}} \Bigg[\sqrt{|\log\varepsilon|}\varepsilon^2 \left(-\sqrt{X_t^2 + (X_t')^2} + 2t + \beta' \right) e^{2X_t}$$

$$\times f_t(X_s, X_s', s\leq t) \Bigg| \mathcal{F}_{x,y} \Bigg]\Bigg]$$

where

$$f_t(X_s, X'_s, s \le t) := \mathbf{1}_{\left\{\forall s \le t_1 - t_0, \sqrt{X_s^2 + (X'_s)^2 + \sum_{i \notin I'} \frac{e}{\eta} e^s L^i_{y,\eta e^{-s-1}}} \le 2s + \beta' - \frac{\sqrt{s+t_0}}{M \log(2+t_0+s)^2}\right\}}$$
$$\times \mathbf{1}_{\left\{\forall s \in [t_1 - t_0, t], \sqrt{X_s^2 + (X'_s)^2} \le 2s + \beta' - \frac{\sqrt{s+t_0}}{M \log(2+t_0+s)^2}\right\}}.$$

Now, by (3.32), we have

$$\begin{aligned}
&\mathbb{E}^0_{r_{y,\alpha/e}} \otimes \mathbb{E}^0_{r'_{y,\alpha/e}} \left[\sqrt{|\log \varepsilon|} \varepsilon^2 \left(-\sqrt{X_t^2 + (X'_t)^2} + 2t + \beta' \right) e^{2X_t} f_t(X_s, X'_s, s \le t) \mathbf{1}_{\{X_t > 0\}} \middle| \mathcal{F}_{x,y} \right] \\
&= \frac{\sqrt{r_{y,\alpha/e}}}{\sqrt{2}} e^{2 r_{y,\alpha/e}} \frac{\sqrt{|\log \varepsilon|}}{\sqrt{|\log(e\varepsilon/\eta)|}} \left(\frac{\eta}{e} \right)^2 (\beta' - r_{y,\eta/e}) \mathbb{E}^3_{\beta' - r_{y,\alpha/e}} \otimes \mathbb{E}^0_{r'_{y,\alpha/e}} \left[\left(1 - \frac{X_t - \beta'}{2t} \right)^{-1/2} \right. \\
&\times \frac{-\sqrt{(2t - X_t + \beta')^2 + (X'_t)^2} + 2t + \beta'}{X_t} \exp\left(-\frac{3}{8} \int_0^t \frac{ds}{(2s - X_s + \beta')^2} \right) \\
&\left. \times f_t(2s - X_s + \beta', X'_s, s \le t) \middle| \mathcal{F}_{x,y} \right]
\end{aligned}$$

which converges as $\varepsilon \to 0$ (and hence $t \to \infty$) towards

$$\begin{aligned}
&\frac{\sqrt{r_{y,\alpha/e}}}{\sqrt{2}} e^{2 r_{y,\alpha/e}} \left(\frac{\eta}{e} \right)^2 (\beta' - r_{y,\eta/e}) \mathbb{E}^3_{\beta' - r_{y,\alpha/e}} \otimes \mathbb{E}^0_{r'_{y,\alpha/e}} \left[\exp\left(-\frac{3}{8} \int_0^\infty \frac{ds}{(2s - X_s + \beta')^2} \right) \right. \\
&\left. \times f_\infty(2s - X_s + \beta', X'_s, s \ge 0) \middle| \mathcal{F}_{x,y} \right].
\end{aligned}$$

This shows that

$$\limsup_{\varepsilon, \varepsilon' \to 0} (1 + p(\eta'/\eta))^{-N} \mathbb{E}_{x_0} \left[\hat{\hat{\mu}}_\varepsilon(x) \hat{\hat{\mu}}_\varepsilon(y) \mathbf{1}_{\{N'_y \le N\}} \right] - (1 - p(\eta'/\eta))^{-N} \mathbb{E}_{x_0} \left[\hat{\hat{\mu}}_\varepsilon(x) \hat{\hat{\mu}}_{\varepsilon'}(y) \mathbf{1}_{\{N'_y \le N\}} \right]$$

is at most zero. The only quantity depending on η' in the above expression is $p(\eta'/\eta)$ which goes to zero as $\eta' \to 0$. By letting $\eta' \to 0$, we thus obtain

$$\limsup_{\varepsilon, \varepsilon' \to 0} \mathbb{E}_{x_0} \left[\hat{\hat{\mu}}_\varepsilon(x) \hat{\hat{\mu}}_\varepsilon(y) \mathbf{1}_{\{N'_y \le N\}} \right] - \mathbb{E}_{x_0} \left[\hat{\hat{\mu}}_\varepsilon(x) \hat{\hat{\mu}}_{\varepsilon'}(y) \mathbf{1}_{\{N'_y \le N\}} \right] \le 0.$$

This concludes the proof of the fact that $(\hat{\hat{\mu}}_\varepsilon(A), \varepsilon > 0)$ is Cauchy in L^2.

We move on to the proof of the convergence of $(\hat{\hat{m}}_\varepsilon(A), \varepsilon > 0)$ together with the identification of the limit with $\sqrt{2/\pi}\hat{\hat{\mu}}(A)$. Since we already know that $(\hat{\hat{\mu}}_\varepsilon(A), \varepsilon > 0)$ converges in L^2 towards $\hat{\hat{\mu}}(A)$, it is enough to show that

$$\limsup_{\varepsilon \to 0} \mathbb{E}_{x_0} \left[|\hat{\hat{m}}_\varepsilon(A) - \sqrt{2/\pi} \hat{\hat{\mu}}_\varepsilon(A)|^2 \right] = 0.$$

In particular, we don't need to consider "mixed moments" with $\varepsilon' \neq \varepsilon$. As before, we bound

$$\begin{aligned}
&\limsup_{\varepsilon\to 0} \mathbb{E}_{x_0}\left[|\hat{m}_\varepsilon(A) - \sqrt{2/\pi}\hat{\mu}_\varepsilon(A)|^2\right] \\
&\leq \limsup_{\varepsilon\to 0} \int_{(A\times A)\setminus(A\times A)_\eta} \mathbb{E}_{x_0}\left[\hat{m}_\varepsilon(dx)\hat{m}_\varepsilon(dy)\right] + \frac{2}{\pi}\mathbb{E}_{x_0}\left[\hat{\mu}_\varepsilon(dx)\hat{\mu}_\varepsilon(dy)\right] \\
&+ \limsup_{\varepsilon\to 0} \int_{(A\times A)_\eta} \mathbb{E}_{x_0}\left[\hat{m}_\varepsilon(dx)\left(\hat{m}_\varepsilon(dy) - \sqrt{\frac{2}{\pi}}\hat{\mu}_\varepsilon(dy)\right)\right] \\
&+ \sqrt{\frac{2}{\pi}} \limsup_{\varepsilon\to 0} \int_{(A\times A)_\eta} \mathbb{E}_{x_0}\left[\hat{\mu}_\varepsilon(dx)\left(\sqrt{\frac{2}{\pi}}\hat{\mu}_\varepsilon(dy) - \hat{m}_\varepsilon(dy)\right)\right].
\end{aligned}$$

As before, we only need to care about the two last right hand side terms and thanks to (3.18) and (3.19), we only need to show the two following pointwise convergences:

$$\lim_{\varepsilon\to 0} \mathbb{E}_{x_0}\left[\hat{m}_\varepsilon(dx)\left(\hat{m}_\varepsilon(dy) - \sqrt{\frac{2}{\pi}}\hat{\mu}_\varepsilon(dy)\right)\right] = \lim_{\varepsilon\to 0} \mathbb{E}_{x_0}\left[\hat{\mu}_\varepsilon(dx)\left(\sqrt{\frac{2}{\pi}}\hat{\mu}_\varepsilon(dy) - \hat{m}_\varepsilon(dy)\right)\right] = 0 \quad (3.66)$$

where $(x, y) \in (A \times A)_\eta$ is fixed. In both cases, we employ the same technique as before by decomposing the Brownian trajectory according to what happens close to the point y and (3.66) follows from the fact that

$$\mathbb{E}^0_{r_{y,\alpha/e}} \otimes \mathbb{E}^0_{r'_{y,\alpha/e}}\left[|\log\varepsilon|\varepsilon^2 e^{2X_t} f_t(X_s, X'_s, s \leq t)\Big|\mathcal{F}_{x,y}\right]$$

converges to the same limit as

$$\sqrt{\frac{2}{\pi}}\mathbb{E}^0_{r_{y,\alpha/e}} \otimes \mathbb{E}^0_{r'_{y,\alpha/e}}\left[\sqrt{|\log\varepsilon|}\varepsilon^2\left(-\sqrt{X_t^2 + (X'_t)^2} + 2t + \beta'\right) e^{2X_t} f_t(X_s, X'_s, s \leq t)\Big|\mathcal{F}_{x,y}\right].$$

Let us justify this last claim. After using (3.32), we see that we only need to show that

$$\begin{aligned}
&\lim_{t\to\infty} \mathbb{E}^3_{\beta' - r_{y,\alpha/e}} \otimes \mathbb{E}^0_{r'_{y,\alpha/e}}\Bigg[\frac{\sqrt{t}}{X_t}\left(1 - \frac{X_t - \beta'}{2t}\right)_+^{-1/2} \qquad (3.67)\\
&\times \exp\left(-\frac{3}{8}\int_0^t \frac{ds}{(2s - X_s + \beta')^2}\right) f_t(2s - X_s + \beta', X'_s, s \leq t)\Big|\mathcal{F}_{x,y}\Bigg] \\
&= \sqrt{\frac{2}{\pi}} \lim_{t\to\infty} \mathbb{E}^3_{\beta' - r_{y,\alpha/e}} \otimes \mathbb{E}^0_{r'_{y,\alpha/e}}\Bigg[\frac{-\sqrt{(2t - X_t + \beta')^2 + (X'_t)^2} + 2t + \beta'}{X_t}\left(1 - \frac{X_t - \beta'}{2t}\right)_+^{-1/2} \\
&\times \exp\left(-\frac{3}{8}\int_0^t \frac{ds}{(2s - X_s + \beta')^2}\right) f_t(2s - X_s + \beta', X'_s, s \leq t)\Big|\mathcal{F}_{x,y}\Bigg].
\end{aligned}$$

Take $t_2 > t_1 - t_0$ large. We can bound

$$\begin{aligned}
&|f_{t_2}(2s - X_s + \beta', X'_s, s \leq t_2) - f_t(2s - X_s + \beta', X'_s, s \leq t)| \\
&\leq \mathbf{1}_{\left\{\exists s \geq t_2, X_s < \frac{\sqrt{s+t_0}}{M\log(2+t_0+s)^2} \text{ or } X_s \geq 2s+\beta'\right\}} + \mathbf{1}_{\{\exists s \geq t_2, X'_s > 0\}}.
\end{aligned}$$

The difference between the expectation on the left hand side of (3.67) and

$$\mathbb{E}^3_{\beta'-r_{y,\alpha/e}} \otimes \mathbb{E}^0_{r'_{y,\alpha/e}} \left[\frac{\sqrt{t}}{X_t} \left(1 - \frac{X_t - \beta'}{2t} \right)_+^{-1/2} \right.$$
$$\left. \times \exp\left(-\frac{3}{8} \int_0^t \frac{ds}{(2s - X_s + \beta')^2} \right) f_{t_2}(2s - X_s + \beta', X'_s, s \le t_2) \middle| \mathcal{F}_{x,y} \right]$$

is thus at most

$$\mathbb{E}^3_{\beta'-r_{y,\alpha/e}} \otimes \mathbb{E}^0_{r'_{y,\alpha/e}} \left[\frac{\sqrt{t}}{X_t} \left(1 - \frac{X_t - \beta'}{2t} \right)_+^{-1/2} \right.$$
$$\left. \times \left\{ \mathbf{1}_{\left\{ \exists s \ge t_2, X_s < \frac{\sqrt{s+t_0}}{M \log(2+t_0+s)^2} \text{ or } X_s \ge 2s + \beta' \right\}} + \mathbf{1}_{\{\exists s \ge t_2, X'_s > 0\}} \right\} \right].$$

Let $q_1 \in (1,3), q_2 \in (1,2)$ and $q_3 > 1$ be such that $1/q_1 + 1/q_2 + 1/q_3 = 1$. By Hölder's inequality, we can bound the above expression by

$$\mathbb{E}^3_{\beta'-r_{y,\alpha/e}} \left[\frac{t^{q_1/2}}{X_t^{q_1}} \right]^{1/q_1} \mathbb{E}^3_{\beta'-r_{y,\alpha/e}} \left[\left(1 - \frac{X_t - \beta'}{2t} \right)_+^{-q_2/2} \right]^{1/q_2} \left\{ \mathbb{P}^0_{r'_{y,\alpha/e}} (\exists s \ge t_2, X'_s > 0) \right.$$
$$\left. + \mathbb{P}^3_{\beta'-r_{y,\alpha/e}} \left(\exists s \ge t_2, X_s < \frac{\sqrt{s+t_0}}{M \log(2+t_0+s)^2} \text{ or } X_s \ge 2s + \beta' \right) \right\}^{1/q_3}.$$

The first two expectations are bounded by a universal constant by Lemma 3.23 Points 3 and 4. The last term containing the two probabilities goes to zero as $t_2 \to \infty$. Similarly, we can replace

$$\int_0^t \frac{ds}{(2s - X_s + \beta')^2} \quad \text{by} \quad \int_0^{t_2} \frac{ds}{(2s - X_s + \beta')^2}$$

and

$$\left(1 - \frac{X_t - \beta'}{2t} \right)_+^{-1/2} \quad \text{by} \quad 1.$$

We have shown that the left hand side term of (3.67) is equal to $o_{t_2 \to \infty}(1)$ plus

$$\mathbb{E}^3_{\beta'-r_{y,\alpha/e}} \otimes \mathbb{E}^0_{r'_{y,\alpha/e}} \left[\frac{\sqrt{t}}{X_t} \exp\left(-\frac{3}{8} \int_0^{t_2} \frac{ds}{(2s - X_s + \beta')^2} \right) f_{t_2}(2s - X_s + \beta', X'_s, s \le t_2) \middle| \mathcal{F}_{x,y} \right].$$

By conditioning up to t_2 and then by using Lemma 3.23 point 2, we see that the above expectation converges as $t \to \infty$ to

$$\sqrt{\frac{2}{\pi}} \mathbb{E}^3_{\beta'-r_{y,\alpha/e}} \otimes \mathbb{E}^0_{r'_{y,\alpha/e}} \left[\exp\left(-\frac{3}{8} \int_0^{t_2} \frac{ds}{(2s - X_s + \beta')^2} \right) f_{t_2}(2s - X_s + \beta', X'_s, s \le t_2) \middle| \mathcal{F}_{x,y} \right].$$

With a similar reasoning as above, one can show that the expectation on the right hand side of (3.67) converges as $t \to \infty$ to $o_{t_2 \to \infty}(1)$ plus

$$\mathsf{IE}^3_{\beta' - r_{y,\alpha/e}} \otimes \mathsf{IE}^0_{r'_{y,\alpha/e}} \left[\exp\left(-\frac{3}{8} \int_0^{t_2} \frac{ds}{(2s - X_s + \beta')^2} \right) f_{t_2}(2s - X_s + \beta', X'_s, s \le t_2) \middle| \mathcal{F}_{x,y} \right].$$

We have thus shown the left and right hand sides of (3.67) differ by at most some $o_{t_2 \to \infty}(1)$. Since they do not depend on t_2, we obtain the claim (3.67) by letting $t_2 \to \infty$. This concludes the fact that $(\hat{\hat{m}}_\varepsilon(A), \varepsilon > 0)$ converges in L^2 towards $\sqrt{\frac{2}{\pi}} \hat{\hat{\mu}}(A)$.

The fact that for all $\gamma \in (1,2)$, $(\hat{\hat{m}}^\gamma_\varepsilon(A), \varepsilon < \varepsilon_\gamma)$ is a Cauchy sequence in L^2 follows along lines that are very similar to the proof of the fact that $(\hat{\hat{\mu}}_\varepsilon(A), \varepsilon > 0)$ is a Cauchy sequence in L^2. For this reason we omit the details and we now turn to the proof of the convergence of $((2-\gamma)^{-1} \hat{\hat{m}}^\gamma(A), \gamma \in (1,2))$ towards $2\hat{\hat{\mu}}(A)$. Here, we do not restrict ourselves to the sequence $(\gamma_n, n \ge 1)$ as stated in Proposition 3.17 to ease notations. We hope the reader will forgive us for this lack of rigour. By Fatou's lemma,

$$\limsup_{\gamma \to 2} \mathbb{E}_{x_0} \left[\left| 2\hat{\hat{\mu}}(A) - \frac{1}{2-\gamma} \hat{\hat{m}}^\gamma(A) \right|^2 \right] \le \limsup_{\gamma \to 2} \limsup_{\varepsilon \to 0} \mathbb{E}_{x_0} \left[\left| 2\hat{\hat{\mu}}_\varepsilon(A) - \frac{1}{2-\gamma} \hat{\hat{m}}^\gamma_\varepsilon(A) \right|^2 \right]$$

and we aim to show that the above right hand side term vanishes. As before and thanks to (3.19) and (3.20), we only need to show the following two pointwise convergences

$$\limsup_{\gamma \to 2} \limsup_{\varepsilon \to 0} \frac{1}{2-\gamma} \mathbb{E}_{x_0} \left[\hat{\hat{m}}^\gamma_\varepsilon(dx) \left(\frac{1}{2-\gamma} \hat{\hat{m}}^\gamma_\varepsilon(dy) - 2\hat{\hat{\mu}}_\varepsilon(dy) \right) \right] = 0$$

and

$$\limsup_{\gamma \to 2} \limsup_{\varepsilon \to 0} \mathbb{E}_{x_0} \left[\hat{\hat{\mu}}_\varepsilon(dx) \left(\frac{1}{2-\gamma} \hat{\hat{m}}^\gamma_\varepsilon(dy) - 2\hat{\hat{\mu}}_\varepsilon(dy) \right) \right] = 0$$

where $(x,y) \in (A \times A)_\eta$ is fixed. In both cases, this follows from the fact that

$$\frac{1}{2-\gamma} \mathsf{IE}^0_{r_{y,\alpha/e}} \otimes \mathsf{IE}^0_{r'_{y,\alpha/e}} \left[\sqrt{|\log \varepsilon|} \varepsilon^{\gamma^2/2} e^{\gamma X_t} f_t(X_s, X'_s, s \le t) \middle| \mathcal{F}_{x,y} \right] \tag{3.68}$$

converges as $\varepsilon \to 0$ and then $\gamma \to 2$ to the same limit as

$$2\mathsf{IE}^0_{r_{y,\alpha/e}} \otimes \mathsf{IE}^0_{r'_{y,\alpha/e}} \left[\sqrt{|\log \varepsilon|} \varepsilon^2 \left(-\sqrt{X_t^2 + (X'_t)^2 + 2t + \beta'} \right) e^{2X_t} f_t(X_s, X'_s, s \le t) \middle| \mathcal{F}_{x,y} \right]. \tag{3.69}$$

Let us justify this claim. By (3.30), (3.68) is equal to

$$\frac{1}{2-\gamma} \sqrt{r_{y,\alpha/e}} e^{\gamma r_{y,\alpha/e}} \frac{\sqrt{|\log \varepsilon|}}{\sqrt{|\log(e\varepsilon/\eta)|}} \left(\frac{\eta}{e} \right)^{\gamma^2/2} \mathsf{IE}_{r_{y,\alpha/e}} \otimes \mathsf{IE}^0_{r'_{y,\alpha/e}} \left[\left(\frac{t}{X_t + \gamma t} \right)^{1/2} \right.$$
$$\left. \times \exp\left(-\frac{3}{8} \int_0^t \frac{ds}{(X_s + \gamma s)^2} \right) f_t(X_s + \gamma s, X'_s, s \le t) \middle| \mathcal{F}_{x,y} \right].$$

As before, let $t_2 > t_1 - t_0$ be large. One can show in a similar manner as what we did above that

$$\frac{1}{2-\gamma}\mathbb{E}_{r_{y,\alpha/e}} \otimes \mathbb{E}^0_{r'_{y,\alpha/e}} \left[\left(\frac{t}{X_t + \gamma t} \right)^{1/2} \exp\left(-\frac{3}{8}\int_0^t \frac{ds}{(X_s+\gamma s)^2} \right) f_t(X_s + \gamma s, X'_s, s \leq t) \right]$$
$$= o_{t_2\to\infty}(1) + \frac{1}{2-\gamma}\mathbb{E}_{r_{y,\alpha/e}} \otimes \mathbb{E}^0_{r'_{y,\alpha/e}} \left[\left(\frac{t_2}{X_{t_2} + \gamma t_2} \right)^{1/2} \exp\left(-\frac{3}{8}\int_0^{t_2} \frac{ds}{(X_s+\gamma s)^2} \right) \right.$$
$$\left. \times f_{t_2}(X_s + \gamma s, X'_s, s \leq t_2) \mathbf{1}_{\{\forall s \in [t_2,t], X_s < (2-\gamma)s + \beta'\}} \middle| \mathcal{F}_{x,y} \right].$$

Since (see [Res92, Proposition 6.8.1] for instance)

$$\lim_{\gamma\to 2}\lim_{t\to\infty} \frac{1}{2-\gamma} \mathbb{P}_{X_{t_2}}\left(\forall s \leq t, X_s < (2-\gamma)s + \beta' + (2-\gamma)t_2 \right)$$
$$= \lim_{\gamma\to 2} \frac{1}{2-\gamma}\left(1 - e^{-2(2-\gamma)(\beta' - X_{t_2})} \right) = 2(\beta' - X_{t_2}),$$

this shows that the liminf and limsup of (3.68) as $\varepsilon \to 0$ and then $\gamma \to 2$ are equal to $o_{t_2\to\infty}(1)$ plus

$$2\sqrt{r_{y,\alpha/e}} e^{2r_{y,\alpha/e}} \left(\frac{\eta}{e} \right)^2 \mathbb{E}_{r_{y,\alpha/e}} \otimes \mathbb{E}^0_{r'_{y,\alpha/e}} \left[\left(\frac{t_2}{X_{t_2} + \gamma t_2} \right)^{1/2} \right.$$
$$\left. \times \exp\left(-\frac{3}{8}\int_0^{t_2} \frac{ds}{(X_s+2s)^2} \right) (\beta' - X_{t_2}) f_{t_2}(X_s + 2s, X'_s, s \leq t_2) \middle| \mathcal{F}_{x,y} \right].$$

By using (3.30) in the other direction, we see that the above term converges as $t_2 \to \infty$ towards

$$2(\eta/e)^2 \lim_{t_2\to\infty} \mathbb{E}^0_{r_{y,\alpha/e}} \otimes \mathbb{E}^0_{r'_{y,\alpha/e}} \left[\sqrt{t_2} e^{-t_2} \left(-X_{t_2} + 2t_2 + \beta' \right) e^{2X_{t_2}} f_{t_2}(X_s, X'_s, s \leq t_2) \middle| \mathcal{F}_{x,y} \right]$$
$$= 2 \lim_{\varepsilon\to 0} \mathbb{E}^0_{r_{y,\alpha/e}} \otimes \mathbb{E}^0_{r'_{y,\alpha/e}} \left[\sqrt{|\log\varepsilon|}\varepsilon^2 \left(-\sqrt{X_t^2 + (X'_t)^2} + 2t + \beta' \right) e^{2X_t} f_t(X_s, X'_s, s \leq t) \middle| \mathcal{F}_{x,y} \right]$$

recalling that $t = \log(\frac{\eta}{e\varepsilon})$ and since X' will be trapped by zero. We have shown that (3.68) converges as $\varepsilon \to 0$ and then $\gamma \to 2$ to the same limit as (3.69) as wanted. This concludes the proof of the fact that $(2-\gamma)^{-1}\hat{m}^\gamma_\varepsilon(A)$ converges in L^2 as $\varepsilon \to 0$ and then $\gamma \to 2$ towards $2\hat{\mu}(A)$. □

Appendix 3.A Process of Bessel bridges: proof of Lemma 3.13

We prove Lemma 3.13 for completeness. It is a direct consequence of the following:

Lemma 3.30. *For all $\delta \in \{e^{-n}, n \geq 0\}$, let $x \in D \mapsto f_{x,\delta} \in [0,\infty)$ be continuous functions. By enlarging the probability space we are working on if necessary, we can construct a random field $(h_{x,\delta}, x \in D, \delta \in (0,1])$ that is independent of $(B_t, t \leq \tau)$ and such that*

- *for all $x \in D$, and $n \geq 0$, $(h_{x,e^{-t}}, t \in [n, n+1])$ has the law of a zero-dimensional Bessel bridge from $f_{x,e^{-n}}$ to $f_{x,e^{-n-1}}$;*

- *for all* $\delta_0 \in (0,1]$ *and* $x, y \in D$, $(h_{x,\delta}, \delta \leq \delta_0)$ *and* $(h_{y,\delta}, \delta \leq \delta_0)$ *are independent as soon as* $|x-y| \geq 2\delta_0$*;*
- *For all* $n \geq 0$, $(h_{x,e^{-t}}, x \in D, t \in [n, n+1])$ *and* $(h_{x,e^{-t}}, x \in D, t \notin [n, n+1])$ *are independent;*
- *for all* $n \geq 0$ *and* $z \in e^{-n-10}\mathbb{Z}^2 \cap D$, $(h_{x,\delta}, x \in D, \lfloor e^{n+10}x \rfloor = e^{n+10}z, e^{-n-1} \leq \delta \leq e^{-n})$ *is continuous.*

Proof of Lemma 3.30. We start by explaining how to construct a continuous process $(b_t^{u,v}, u, v \geq 0, 0 \leq t \leq 1)$ such that for all $u, v \geq 0$, $(b_t^{u,v}, 0 \leq t \leq 1)$ has the law of a zero-dimensional Bessel bridge from u to v. Let $(b_t^{1\to 0, d=0}, 0 \leq t \leq 1)$, $(b_t^{0\to 1, d=0}, 0 \leq t \leq 1)$ and $(b_t^{0\to 0, d=4n}, 0 \leq t \leq 1)$, $n \geq 1$, be independent Bessel bridges with starting and ending points and dimensions written in superscript. Since 0 is a trap for zero-dimensional Bessel process, $(b_t^{0\to 1, d=0}, 0 \leq t \leq 1)$ is defined as the time reversal of a zero-dimensional Bessel bridge from 1 to 0. For $w \geq 0$, let $(\alpha_{w,n}, n \geq 1)$ be a sequence of random variables such that for all $n \geq 1$,

$$\mathbb{P}\left(\alpha_{w,n} = 1, \forall k \neq n, \alpha_{w,k} = 0\right) = \frac{1}{n!}(w/2)^{2n-1}\Gamma(n) I_1(w)$$

and

$$\mathbb{P}\left(\forall k \geq 1, \alpha_{w,k} = 0\right) = 1 - \sum_{n\geq 1} \mathbb{P}\left(\alpha_{w,n} = 1, \forall k \neq n, \alpha_{w,k} = 0\right).$$

Here I_1 is a modified Bessel function of the first kind and Γ is the Gamma function. By using a single uniform random variable on $[0,1]$, it is easy to build all the variables $\alpha_{w,n}, w \geq 0, n \geq 1$ on the same probability space such that they are independent from the Bessel bridges above and such that for all $n \geq 1$, $w \mapsto \alpha_{w,n}$ is continuous. We now define for all $u, v \geq 0$, and $t \in [0,1]$,

$$b_t^{u,v} = u b_t^{1\to 0, d=0} + v b_t^{0\to 1, d=0} + \sum_{n\geq 1} \alpha_{\sqrt{uv}, n} b_t^{0\to 0, d=4n}.$$

By construction, $(b_t^{u,v}, u, v \geq 0, 0 \leq t \leq 1)$ is a continuous process. Moreover, by [PY82, Theorem (5.8)], for all $u, v \geq 0$, $b^{u,v}$ has the law of a zero-dimensional Bessel bridge from u to v over the time interval $[0,1]$ as desired.

We now explain how to construct the process $(h_{x,\delta}, x \in D, \delta \in (0,1])$. For $n \geq 0$ and $x \in D$, define $x_n := e^{-n-10}\lfloor e^{n+10}x \rfloor \in e^{-n-10}\mathbb{Z}^2$. For all $n \geq 0$ and $z \in e^{-n-10}\mathbb{Z}^2 \cap D$, consider independent continuous processes $(h_{x,\delta}^{n,z}, x \in D, x_n = z, e^{-n-1} \leq \delta \leq e^{-n})$ such that for all $x \in D$ with $x_n = z$, $(h_{x,e^{-t}}^{n,z}, n \leq t \leq n+1)$ has the law of a zero-dimensional Bessel bridge from $f_{x,e^{-n}}$ to $f_{x,e^{-n-1}}$. This countable collection of independent continuous processes can be constructed thanks to the first step above. We now define for all $x \in D$ and $\delta \in (0,1]$, $h_{x,\delta} = h_{x,\delta}^{n,x_n}$ where $n \geq 0$ is such that $e^{-n-1} < \delta \leq e^{-n}$. By construction, the process h satisfies the desired properties. □

Appendix 3.B Semi-continuity of subcritical measures: proof of Proposition 3.3

In this section we explain how we obtain Proposition 3.3. We will only sketch the proof since it follows from [Jeg20a] as well as from arguments having similar flavour as what we already did in this paper.

Proof. We will first truncate the measure to make it bounded in L^2. We will then show that the truncated version is continuous in γ by Kolmogorov's continuity theorem and by L^2 computations. The statement on the non-truncated measures will then follow.

Let $0 < \gamma_- < \gamma_+ < 2$. We are going to study the regularity of $\gamma \in [\gamma_-, \gamma_+] \mapsto m^\gamma$. Recall Notation 3.9 and the definition of the process $(h_{x,\delta}, x \in D, \delta \in (0,1])$. Fix $\bar\gamma \in (\gamma_+, 2)$ very close to γ_+. For $\beta > 0$ large, define for all $\varepsilon = e^{-k}$ and $x \in D$ at distance at least ε from x_0, the good event

$$G_\varepsilon(x) := \{\forall s \in [k_x, k], h_{x,e^{-s}} \leq \bar\gamma s + \beta\}$$

and the modified measures

$$\bar m_\varepsilon^\gamma(dx, \beta) = \mathbf{1}_{G_\varepsilon(x)} m_\varepsilon^\gamma(dx).$$

Since $\bar\gamma > \gamma_+$, one can show that this modification does affect the measures in the L^1 sense:

$$\lim_{\beta\to\infty} \sup_{\gamma\in[\gamma_-,\gamma_+]} \limsup_{\varepsilon\to 0} \mathbb{E}_{x_0}\left[m_\varepsilon^\gamma(D) - \bar m_\varepsilon^\gamma(D,\beta)\right] = 0. \tag{3.70}$$

Moreover, if $\bar\gamma$ is close enough to γ_+, the modified measures are bounded in L^2 (consequence of [Jeg20a, Proposition 4.2]) and we can show with a reasoning similar to what we did in Section 3.5.2 (this does not follow completely from [Jeg20a] since the good events that we define here are slightly different from the ones considered in [Jeg20a]) that for all Borel set A and all $\gamma \in [\gamma_-, \gamma_+]$, $(\bar m_\varepsilon^\gamma(A,\beta), \varepsilon > 0)$ is a Cauchy sequence in L^2. We will denote $\bar m^\gamma(A,\beta)$ the limiting random variable. We can further show that for all Borel set A and for all $\gamma_1, \gamma_2 \in [\gamma_-, \gamma_+]$,

$$\limsup_{\varepsilon\to 0} \mathbb{E}_{x_0}\left[\left(\bar m_\varepsilon^{\gamma_1}(A,\beta) - \bar m_\varepsilon^{\gamma_2}(A,\beta)\right)^2\right] \leq C(\gamma_1-\gamma_2)^2 \tag{3.71}$$

for some $C > 0$ possibly depending on $\beta, \gamma_-, \gamma_+, \bar\gamma$. This follows on the one hand from a reasoning similar to what we have already done to transfer computations from local times to zero-dimensional Bessel process, and on the other hand from the following estimate which is a consequence of (3.30): for all $K > 0$, there exists $C > 0$ why may depend on $K, \beta, \gamma_-, \gamma_+, \bar\gamma$ such that

$$\limsup_{t\to\infty} \sup_{\gamma_1,\gamma_2\in[\gamma_-,\gamma_+]} \sup_{r\in[0,K]} \sqrt{t}\left|\mathsf{E}_r^0\left[\left(e^{-\frac{\gamma_1^2}{2}t}e^{\gamma_1 X_t} - e^{-\frac{\gamma_2^2}{2}t}e^{\gamma_2 X_t}\right)\mathbf{1}_{\{\forall s\leq t, X_s\leq\bar\gamma s+\beta\}}\right]\right| \leq C(\gamma_1-\gamma_2).$$

Let $\mathcal{P} := \{[a,b)\times[c,d) : a,b,c,d \in \mathbb{Q}\}$. $\mathcal{P}$ is a countable pi-system generating the Borel sigma-algebra on $\mathbb{R}^2$. From (3.71) and Kolmogorov's continuity theorem, we deduce that we can build the variables $\bar m^\gamma(A,\beta)$ simultaneously for all $\gamma \in [\gamma_-, \gamma_+]$, $\beta \in \mathbb{N}$ and $A \in \mathcal{P}$ in such a way that for all $\beta \in \mathbb{N}$ and $A \in \mathcal{P}$, $\gamma \in [\gamma_-, \gamma_+] \mapsto \bar m^\gamma(A,\beta)$ is continuous. Let $\bar m^\gamma(A,\infty)$ be the nondecreasing limit of

$(\bar{m}^\gamma(A,\beta), \beta \geq 1)$. A nondecreasing sequence of continuous functions being lower-semicontinous, we have shown that we can build on the same probability space the variables $\bar{m}^\gamma(A,\infty)$, $\gamma \in [\gamma_-, \gamma_+]$, $A \in \mathcal{P}$ such that for all $A \in \mathcal{P}$, $\gamma \in [\gamma_-, \gamma_+] \mapsto \bar{m}^\gamma(A,\infty)$ is lower-semicontinuous. For all $\gamma \in [\gamma_-, \gamma_+]$, $\bar{m}^\gamma$ defines a Borel measure. By (3.70), for all $\gamma \in [\gamma_-, \gamma_+]$, $A \in \mathcal{P}$, $m^\gamma(A) = \bar{m}^\gamma(A,\infty)$ $\mathbb{P}_{x_0}$-a.s. Concluding the proof of Proposition 3.3 is now routine. $\square$

Chapter 4

Multiplicative chaos of the Brownian loop soup

We construct a measure on the thick points of a Brownian loop soup in a bounded domain D of the plane with given intensity $\theta > 0$, which is formally obtained by exponentiating the square root of its occupation field. The measure is constructed via a regularisation procedure, in which loops are killed at a fix rate, allowing us to make use of the Brownian multiplicative chaos measures previously considered in [BBK94, AHS20, Jeg20a]. At the critical intensity $\theta = 1/2$, it is shown that this measure coincides with the hyperbolic cosine of the Gaussian free field, which is closely related to Liouville measure. This allows us to draw several conclusions which elucidate connections between Brownian multiplicative chaos, Gaussian free field and Liouville measure. For instance, it is shown that Liouville-typical points are of infinite loop multiplicity, with the relative contribution of each loop to the overall thickness of the point being described by the Poisson–Dirichlet distribution with parameter $\theta = 1/2$. Conversely, the Brownian chaos associated to each loop describes its microscopic contribution to Liouville measure. Along the way, our proof reveals a surprising exact integrability of the multiplicative chaos associated to a killed Brownian loop soup. We also obtain some estimates on the loop soup which may be of independent interest.

4.1 Introduction and main results

The two-dimensional Gaussian free field (GFF) and its associated Gaussian multiplicative chaos (sometimes called Liouville measure) have been in recent years at the heart of some extraordinary developments, in particular in connection with the study of Liouville quantum gravity. Formally, the multiplicative chaos associated to a field h in a domain $D \subset \mathbb{R}^2$ is a measure of the form

$$\mu_\gamma(dz) = \lim_{\varepsilon \to 0} \varepsilon^{\gamma^2/2} e^{\gamma h_\varepsilon(z)} dz \tag{4.1}$$

where $\gamma \in \mathbb{R}$ is a parameter, h is typically a logarithmically correlated field, and h_ε denotes some regularisation of h at scale ε. The convergence of this procedure (as the regularisation scale ε converges to 0) is by no means obvious; in the case where h is in addition assumed to be Gaussian, this is precisely the purpose of Gaussian multiplicative chaos theory, initially introduced by Kahane [Kah85] in the 1980s to model turbulence (following ideas of Kolmogorov and Mandelbrot) and further considerably developed in the last decade [RV10, DS11, RV11, Sha16, Ber17]. Gaussian multiplicative chaos is a powerful tool to study properties of the underlying field h, particulary in connection with its extreme values. By now, Gaussian multiplicative chaos is a fundamental object in its own right which describes scaling limits arising naturally in many different contexts, including random matrices [FK14, Web15, NSW18, LOS18, BWW18], the Riemann zeta function [SW20], and stochastic volatility models in finance [BDM01] (see also [DRV12]); see the surveys [RV14], [Pow20b] and the book in preparation [BP21] for more context and references.

More recently, it has been shown that an analogous theory can be developed in the case where h describes (at least formally) the square root of the local time (i.e., occupation field) of a Brownian trajectory; see [BBK94, AHS20, Jeg20a, Jeg21, Jeg19]. The construction of the associated multiplicative chaos, a measure which we will denote in the following by $\mathcal{M}^{\wp}$ and which is now termed **Brownian multiplicative chaos** (following the terminology of [Jeg20a]), is one of the first examples (together with [Jun18] which studies random Fourier series with i.i.d. coefficients) of a multiplicative chaos in which the field h is not Gaussian or approximately Gaussian. It is, however, logarithmically correlated as will be clear from the discussion below. More generally, as shown in [Jeg19], given a finite number of independent Brownian trajectories $\wp_1, \dots, \wp_n$, it is possible to define a multiplicative chaos associated to the square root of the *combined* occupation field of $\wp_1, \dots, \wp_n$; the corresponding measure (let us denote it by $\mathcal{M}^{\wp_1,\dots,\wp_n}$ in this introduction) can be thought of as a uniform measure on points that are thick for the combined local times of all paths. A nontrivial fact proved in [Jeg19] is that, sampling from this measure yields a point of multiplicity k (i.e., is visited by exactly k paths) with positive probability for each $1 \leq k \leq n$. More precisely, one can make sense of a measure $\mathcal{M}^{\wp_1 \cap \dots \cap \wp_n}$ which is the restriction of $\mathcal{M}^{\wp_1,\dots,\wp_n}$ to points on the intersection of *all* trajectories; those two types of measures are related by the a.s. identity

$$\mathcal{M}^{\wp_1,\dots,\wp_n} = \sum_{k=1}^{n} \sum_{\{\tilde{\wp}_1,\dots,\tilde{\wp}_k\}} \mathcal{M}^{\tilde{\wp}_1 \cap \dots \cap \tilde{\wp}_k} \tag{4.2}$$

where the second sum runs over all the possible choices of collections $\{\tilde{\wp}_1, \dots, \tilde{\wp}_k\} \subset \{\wp_1, \dots, \wp_n\}$ of pairwise distinct trajectories. This identity corresponds to choosing the trajectories which actually contribute to the overall thickness at a given point x. (However we caution the reader that the identity above is not entirely trivial because the measures $\mathcal{M}^{\tilde{\wp}_1 \cap \dots \cap \tilde{\wp}_k}$ do not require the remaining paths in $\{\wp_1, \dots, \wp_n\} \setminus \{\tilde{\wp}_1, \dots, \tilde{\wp}_k\}$ to avoid this point).

Another, very different approach to the Gaussian free field is provided by the **Brownian loop soup**, first introduced by Lawler and Werner [LW04]. This consists in a Poisson point processes $\mathcal{L}_D^\theta$ of Brownian loops remaining in a domain D, where the intensity measure is of the form $\theta \mu_D^{\text{loop}}$. Here μ_D^{loop} is a certain infinite measure on unrooted loops (see (4.22) for a definition), and the intensity $\theta > 0$ describes roughly speaking the local density of loops. The Brownian loop soup is a fundamental object closely connected to other conformally invariant random processes such as SLE, the conformal loop ensemble CLE, and the Gaussian free field. In particular, the Gaussian free field and the Brownian loop soup with critical intensity parameter $\theta = 1/2$ can be coupled in such a way that they are related via Le Jan's isomorphism ([LJ10, LJ11]), i.e., the occupation field of the loop soup (suitably recentered) is given by half of the square of the Gaussian free field (also suitably recentered). See Section 4.2 for more references on Brownian loop soup and in particular Theorem 4.18 for Le Jan's isomorphism.

The main purpose of this paper is to show how these two *a priori* orthogonal points of view on the Gaussian free field are in fact deeply interwoven. To do so we first extend the construction of [AHS20, Jeg20a] to a finite number of loops, or in fact even to an infinite number of loops but with finite "density", such as the loops of a Brownian loop soup of fixed intensity $\theta > 0$ that are killed, if each loop is killed independently at constant rate $K > 0$. This yields a measure $\mathcal{M}_a^K$ which, informally speaking, can be thought of as the uniform measure on the thick points of the occupation field of this "killed" loop soup. Viewing this killing as an ultraviolet regularisation of the loop soup which converges to the entire loop soup as $K \to \infty$, we show that, after suitable normalisation, the measures $\mathcal{M}_a^K$ converge to a limit $\mathcal{M}_a$ which may be thought of as the **multiplicative chaos associated to the loop soup** of intensity $\theta > 0$ and is the main object of interest in this article.

We then specify this construction to the critical intensity $\theta = 1/2$, and show that this measure coincides with the hyperbolic cosine of the GFF, which is closely related to Liouville measure (essentially, it is an unsigned version of it). This identification may be considered the second main contribution of this paper. Together, these two results allow us to elucidate multiple connections between Gaussian free field, Brownian loop soup and Liouville measure. For instance, we are able to describe precisely the structure of Brownian loops in the vicinity of a Liouville typical point. Conversely, this result allows us to view the Brownian multiplicative chaos of [BBK94, AHS20, Jeg20a] as describing the microscopic contribution of each loop to Liouville measure (or, more precisely, the hyperbolic cosine of the GFF).

4.1.1 Construction of Brownian loop soup multiplicative chaos

Let $a \in (0,2)$ and $\theta > 0$ be respectively a thickness parameter and an intensity parameter. Let $D \subset \mathbb{C}$ be an open bounded simply connected domain and let $\mathcal{L}_D^\theta$ be a Brownian loop soup in D with intensity $\theta \mu_D^{\mathrm{loop}}$. As mentioned above, the first aim of this article is to build the "uniform measure" $\mathcal{M}_a$ on a-thick points of $\mathcal{L}_D^\theta$. We need to start by recalling that for any Brownian-like trajectory $\wp$, there exists a random Borel measure $\mathcal{M}_a^\wp$ supported on a-thick points of $\wp$ [BBK94, AHS20, Jeg20a]. This measure is now known as Brownian multiplicative chaos and can be constructed, for instance, by exponentiating the square root of the local times of $\wp$. Recall also (see Section 4.2.3 for precise definitions) that for any finite number of independent Brownian-like trajectories $\wp_1, \dots, \wp_n$, there exists a measure $\mathcal{M}_a^{\wp_1 \cap \dots \cap \wp_n}$ supported on a-thick points that have been generated by the interaction of the n trajectories [Jeg19].

To build the "uniform measure" on a-thick points of the loop soup, we start by thinning the set of loops that we consider by killing each loop independently of each other at some rate $K > 0$, i.e. each given loop $\wp \in \mathcal{L}_D^\theta$ is killed with probability $1 - e^{-KT(\wp)}$ where $T(\wp)$ denotes the duration of the loop $\wp$. We denote by $\mathcal{L}_D^\theta(K)$ the set of loops that have been killed (note that this differs from the perhaps more standard massive loop soup). Obviously, $\mathcal{L}_D^\theta(K) \to \mathcal{L}_D^\theta$ as $K \to \infty$ in the sense that $\mathcal{L}_D^\theta(K)$ is an increasing collection in $K > 0$ and $\bigcup_{K>0} \mathcal{L}_D^\theta(K) = \mathcal{L}_D^\theta$. Consider

$$\mathcal{M}_a^K := \sum_{n \geq 1} \frac{1}{n!} \sum_{\substack{\wp_1, \dots, \wp_n \in \mathcal{L}_D^\theta(K) \\ \forall i \neq j, \wp_i \neq \wp_j}} \mathcal{M}_a^{\wp_1 \cap \dots \cap \wp_n} \tag{4.3}$$

the measure on a-thick points that have been entirely created by loops in $\mathcal{L}_D^\theta(K)$. This definition is justified by (4.2). Note that the factor $\frac{1}{n!}$ ensures that we count each subset $\{\wp_1, \dots, \wp_n\}$ of loops only once. It is not *a priori* obvious that the left hand side of (4.3) is a finite measure; roughly speaking this comes from the fact that the collection of loops $\mathcal{L}_D^\theta(K)$ has "finite density" for each $K < \infty$ (the number of loops in $\mathcal{L}_D^\theta(K)$ of diameter roughly 2^{-j} close to a point x does not depend on j, which translates into a finite expected total occupation time for $\mathcal{L}_D^\theta(K)$; it is therefore not surprising that the corresponding thick point measure $\mathcal{M}_a^K$ is finite, see e.g. (4.46) for a computation of the expectation which implies a.s. finiteness).

The first result is the construction of the measure $\mathcal{M}_a$, the multiplicative chaos defined by the Brownian loop soup, and which is the main object of this paper.

Theorem 4.1. *Let $\theta > 0$ and $a \in (0,2)$. Then as $K \to \infty$, the convergence*

$$(\log K)^{-\theta} \mathcal{M}_a^K \to \mathcal{M}_a$$

takes place in probability for the topology of weak convergence, where the right hand side is defined by this convergence. Moreover, the limit $\mathcal{M}_a$ satisfies the following properties.

1. *$\mathcal{M}_a$ is non-degenerate: for all open set $A \subset D$, $\mathcal{M}_a(A) \in (0,\infty)$ a.s. Furthermore, denoting by $\mathrm{CR}(z,D)$ the conformal radius of D seen from a point $z \in D$, we have*

$$\mathbb{E}\left[\mathcal{M}_a(\mathrm{d}z)\right] = \frac{1}{2^\theta a^{1-\theta}\Gamma(\theta)} \mathrm{CR}(z,D)^a \mathrm{d}z. \tag{4.4}$$

2. *Measurability: $\mathcal{M}_a$ is independent of the labels underlying the definition of the killed loops and is therefore measurable with respect to the loop soup. More precisely, $\mathcal{M}_a$ is $\sigma(< \mathcal{L}_D^\theta >)$-measurable (see (4.30)).*

3. *Conformal covariance: if $\psi : D \to \widetilde{D}$ is a conformal map between two bounded simply connected domains, then*

$$\left(\mathcal{M}_{a,D} \circ \psi^{-1}\right)(d\tilde{z}) \overset{(d)}{=} \left|(\psi^{-1})'(\tilde{z})\right|^{2+a} \mathcal{M}_{a,\widetilde{D}}(d\tilde{z}).$$

4. *The carrying dimension of $\mathcal{M}_a$[1] is almost surely equal to $2-a$.*

Remark 4.2. We will define in (4.11) below another, simpler approximation to $\mathcal{M}_a$ (essentially just a uniform measure on the thick points of a discrete loop rather, instead of $\mathcal{M}_a^K$). The corresponding convergence result is stated in Theorem 4.12.

Remark 4.3. We also show that for all Borel sets $A, B \subset \mathbb{C}$, $\lim_{K\to\infty} (\log K)^{-2\theta} \mathbb{E}\left[\mathcal{M}_a^K(A)\mathcal{M}_a^K(B)\right]$ is given by

$$\frac{1}{4^\theta a^{1-\theta}\Gamma(\theta)} \int_A \mathrm{d}z \int_B \mathrm{d}z' \,\mathrm{CR}(z,D)^a \,\mathrm{CR}(z',D)^a (2\pi G_D(z,z'))^{1-\theta} I_{\theta-1}\left(4\pi a G_D(z,z')\right), \tag{4.5}$$

[1] Recall that the carrying dimension of a measure μ is given by the infimum of $d > 0$ such that there exists a Borel set A with Hausdorff dimension d and such that $\mu(A) > 0$.

where $I_{\theta-1}$ is a modified Bessel function of the first kind whose definition is recalled in (4.223) and G_D is Green's function in D (4.13). See Corollary 4.42. In particular, for all open set $A \subset D$, $\lim_{K\to\infty} (\log K)^{-2\theta} \mathbb{E}\left[\mathcal{M}_a^K(A)^2\right] < \infty$ if, and only if, $a < 1$. It should be possible to show that one can exchange the expectation and the limit (in the L^2-phase $\{a \in (0,1)\}$, this exchange is straightforward), and this would show that $\mathbb{E}\left[\mathcal{M}_a(A)\mathcal{M}_a(B)\right]$ is given by (4.5). Because of the length of the paper, we preferred to not include a proof of this statement.

Remark 4.4. In Theorem 4.55, we give a stronger form of conformal covariance which concerns not only the measure $\mathcal{M}_a$ but the couple $(\mathcal{L}_D^\theta, \mathcal{M}_a)$.

4.1.2 Multiplicative chaos and hyperbolic cosine of Gaussian free field

We now turn to the connections between the multiplicative chaos measure $\mathcal{M}_a$ associated to the Brownian loop soup and Liouville measure. This will require choosing the intensity of the loop soup to be the critical value $\theta = 1/2$. This value is already known to be special for two distinct (but related) reasons. On the one hand, this is the value such that the (renormalised) occupation field of the loop soup corresponds to the (Wick) square of the Gaussian free field (i.e., Le Jan's isomorphism holds, see Theorem 4.18 in the discrete and Remark 4.19 for the continuum case of interest here). On the other hand, this is also the critical value for the percolation of connected components of the loop soup clusters, as follows from the celebrated work of Sheffield and Werner [SW12]. We show here that in addition, still at $\theta = 1/2$, the associated multiplicative chaos corresponds to the hyperbolic cosine of the Gaussian free field. Formally, this is the measure of the form

$$2\cosh(\gamma h) dz = (e^{\gamma h} + e^{-\gamma h}) dz, \tag{4.6}$$

where $h = \sqrt{2\pi}\varphi$ is a Gaussian free field, and where a and γ are related by the correspondence:

$$\gamma = \sqrt{2a}; \quad a = \frac{\gamma^2}{2}.$$

In other words, the hyperbolic cosine of h is defined in (4.6) as the sum (up to an appropriate multiplicative factor specified below) of the Liouville measures (4.1) with parameters γ and $-\gamma$ respectively (as constructed e.g. in [DS11], [Ber17]). Note that formally, our multiplicative chaos measure $\mathcal{M}_a$ is the exponential of the square root of the (renormalised) occupation field $:\ell:$ of the loop soup $\mathcal{L}_D^{\theta=1/2}$, so it is natural to expect in view of Le Jan's isomorphism, that $\mathcal{M}_a(dz) = e^{\gamma|h|}dz$, which on first inspection does not immediately coincide with the hyperbolic cosine of h. However, since h is not a continuous function, only points where h is either *very negative* or *very positive* contribute to $e^{\gamma|h|}dz$, and it follows that for such points we may indeed write $e^{\gamma|h|} = e^{\gamma h} + e^{-\gamma h}$. The theorem below makes this connection precise.

Let $h = \sqrt{2\pi}\varphi$ (where as before φ is the Gaussian free field in D with zero-boundary conditions whose covariance function is given by $\mathbb{E}\left[\varphi(z)\varphi(w)\right] = G_D(z,w)$). Thus with these notations, $\mathbb{E}(h(z)h(w)) = 2\pi G_D(z,w) \sim -\log|w-z|$ as $w - z \to 0$, which is consistent with the choice of normalisation in Liouville quantum gravity literature (see e.g. [WP21] and [BP21] for an introduction to the Gaussian free field and to Liouville quantum gravity).

Theorem 4.5. *Let $\theta = 1/2$, $a \in (0,2)$ and $\gamma = \sqrt{2a}$. Then $\mathcal{M}_a$ has the same law as*

$$\frac{1}{\sqrt{2\pi a}} \cosh(\gamma h) = \frac{1}{2\sqrt{2\pi a}} \left(e^{\gamma h} + e^{-\gamma h} \right),$$

where $e^{\pm\gamma h}$ is the Liouville measure with parameter $\pm\gamma$ associated with h. More precisely, there is a coupling $(\varphi, \mathcal{L}_D^{1/2}, \mathcal{M}_a)$ between a Gaussian free field φ, a Brownian loop soup with critical intensity $\theta = 1/2$, and a measure $\mathcal{M}_a$ in which the three components are pairwise related as follows:

- *$\mathcal{M}_a$ is the multiplicative chaos measure associated to $\mathcal{L}_D^{1/2}$ as in Theorem 4.1;*
- *$\mathcal{M}_a$ is the hyperbolic cosine of $h = \sqrt{2\pi}\varphi$, i.e., $\mathcal{M}_a = \frac{1}{\sqrt{2\pi a}} \cosh(\gamma h)$;*
- *φ and $\mathcal{L}_D^{1/2}$ satisfy Le Jan's isomorphism, in which the (renormalised) occupation field $:\ell(\mathcal{L}_D^{1/2}):$ of the loop soup $\mathcal{L}_D^{1/2}$ is equal to the (Wick) square of the Gaussian free field φ. That is, $\frac{1}{2}:\varphi^2: = :\ell(\mathcal{L}_D^{1/2}):$ (see Remark 4.19).*

Theorem 4.5 gives a new perspective on Liouville measures by embedding them, or more precisely the hyperbolic cosine of the GFF, in a two-dimensional family of measures indexed by $\theta > 0$ and $\gamma \in (0,2)$.

Remark 4.6. One informal consequence of Theorem 4.5 is that it allows us to describe the contribution of each loop to Liouville measure (or more precisely to the hyperbolic cosine of the GFF): namely, each loop contributes a macroscopic amount (as we will see in Theorem 4.8), given by its Brownian multiplicative chaos, as defined in [Jeg20a] and [AHS20] (see Section 4.A for the extension to Brownian loops).

Remark 4.7. We caution the reader that the relation between the GFF φ and the loop soup $\mathcal{L}_D^{\theta=1/2}$ as stated here (namely, Le Jan's isomorphism) is not sufficient to determine uniquely the joint law of $(\varphi, \mathcal{L}_D^{\theta=1/2})$.

4.1.3 Brownian loops at a typical thick point

Theorem 4.5 raises a number of questions concerning the relations between Brownian loop soup and multiplicative chaos (i.e., hyperbolic cosine of the Gaussian free field or ultimately Liouville measure). Chief among those are questions of the following nature: sample a point z according to the multiplicative chaos measure $\mathcal{M}_a$. What does the loop soup look like in the neighbourhood of such points? In other words (for the value $\theta = 1/2$) what does the Brownian loop soup look like in the vicinity of a Liouville-typical point? Obviously we know that the point z is almost surely γ-thick from the point of view of Liouville measure (see e.g. Theorem 2.4 in [BP21]) so we expect the point z to also have an atypically high local time, and so is also "thick" for the loop soup (this will be formulated precisely below in Theorem 4.11). How do loops combine to create such a thick local time? Does the thickness come from a single loop which visits z very often, or from an infinite number of loops that touch z, with each loop having a typical occupation field (so z is not "thick" with respect to any single loop)? As we see, the answer turns out to be an intermediate scenario. More precisely, we show below that Liouville-typical points are of infinite loop multiplicity, with the relative contribution of each loop to the overall thickness of the point being described by the Poisson—Dirichlet distribution with parameter $\theta = 1/2$ (see e.g. [ABT03] for a definition and some properties of Poisson–Dirichlet distribution).

In fact, the theorem below will hold without restriction over $\theta > 0$, and the parameter of the corresponding Poisson–Dirichlet distribution will precisely be the intensity θ of the loop soup. The behaviour above is encapsulated by the following theorem, which gives a precise description of the so-called "rooted measure". To formulate the result, we will need to decompose the loops touching a point z into excursions (analogous to Itô excursions in one dimension). Let us say that a function of $\mathcal{L}_D^\theta$ is **admissible** if it is invariant under reordering these excursions (see Definition 4.15 for a more precise definition; see also Section 4.2.1 for details concerning the topology on the set of collections of loops).

Let $\{a_1, a_2, \dots\}$ be a random partition of $[0, a]$ distributed according to a Poisson–Dirichlet distribution with parameter θ. Conditionally on this partition, let $\Xi_{a_i}^z$, $i \geq 1$, be independent loops with the following distribution: for all $i \geq 1$, $\Xi_{a_i}^z$ is the concatenation of the loops in a Poisson point process with intensity $2\pi a_i \mu_D^{z,z}$. Here, $\mu_D^{z,z}$ is an infinite measure on loops that go through z (see (4.15)).

Theorem 4.8. *Let $\theta > 0$ and $a \in (0, 2)$. For any nonnegative measurable admissible function F,*

$$\mathbb{E}\left[\int_D F(z, \mathcal{L}_D^\theta) \mathcal{M}_a(dz)\right] = \frac{1}{2^\theta a^{1-\theta} \Gamma(\theta)} \int_D \mathbb{E}\left[F(z, \mathcal{L}_D^\theta \cup \{\Xi_{a_i}^z, i \geq 1\})\right] \mathrm{CR}(z, D)^a \mathrm{d}z \tag{4.7}$$

where the two collections of loops $\mathcal{L}_D^\theta$ and $\{\Xi_{a_i}^z, i \geq 1\}$ appearing in the right hand side term are independent.

Moreover, the joint law of the couple $(\mathcal{L}_D^\theta, \mathcal{M}_a)$ is characterised by

- *$\mathcal{L}_D^\theta$ has the law of a Brownian loop soup in D with intensity θ;*
- *$\mathcal{M}_a$ is measurable w.r.t. the equivalence class $< \mathcal{L}_D^\theta >$ (see (4.30));*
- *(4.7) is satisfied for any nonnegative measurable admissible function F.*

Remark 4.9. Recall that, by Girsanov's theorem, shifting the probability measure by the hyperbolic cosine of the GFF amounts to adding a logarithmic singularity with strength γ to the GFF. More precisely, and using the notations of Theorem 4.5, one has for any bounded measurable function F,

$$\mathbb{E}\left[\int_D F(z, h) \cosh(\gamma h(z)) \mathrm{d}z\right] = \int_D \mathrm{CR}(z, D)^{\gamma^2/2} \mathbb{E}\left[F(z, h + 2\pi\gamma\sigma G_D(z, \cdot))\right] \mathrm{d}z,$$

where σ is a spin independent of h taking values $+1$ or -1 with equal probability $1/2$. Theorem 4.8 above can be seen as explaining the way the Brownian loop soup creates this logarithmic singularity at z. Since here it is easy to check that $\cosh(\gamma h(z)) dz$ is measurable with respect to h, the above identity in fact characterises the joint law of $(h, \cosh(\gamma h))$ (see [Sha16] or (3.30) in [BP21]).

The above result, in conjunction with Theorem 4.5, immediately implies (in the case $\theta = 1/2$) some notable consequences in connection with Le Jan's isomorphism. We state below a simple instance of such a statement. The isomorphism below is closely related to (and in fact could also be deduced from) the isomorphism in [ALS20, Proposition 3.9] where the occupation field of a Poisson point process of boundary-to-boundary excursions is added.

Corollary 4.10. *Let $z \in D$ and let Ξ_a^z be a loop as in Theorem 4.8 independent of the Brownian loop soup $\mathcal{L}_D^{1/2}$ with critical intensity $\theta = 1/2$. Let $:\ell(\mathcal{L}_D^{1/2}):$ denote the (renormalised) occupation field of $\mathcal{L}_D^{1/2}$, and let $\ell(\Xi_a^z)$ denote the occupation measure of Ξ_a^z (which is well defined as a distribution in D, without any centering). Then*

$$:\ell(\mathcal{L}_D^{1/2}): + \ell(\Xi_a^z) \quad \overset{(d)}{=} \quad \frac{1}{2} :\varphi^2: + \gamma\sqrt{2\pi} G_D(z,\cdot)\varphi + \frac{\gamma^2}{2} 2\pi G_D(z,\cdot)^2$$

where, as before, $\gamma = \sqrt{2a}$. In particular, the expectation of $\ell(\Xi_a^z)$ is given by $a 2\pi G_D(z,\cdot)^2$.

4.1.4 Dimension of the set of thick points

The study of the multifractal behaviour of thick points of logarithmically correlated fields has attracted a lot of attention in the past two decades. In particular, the Hausdorff dimension of the set of thick points was established both in the case of planar Brownian motion [DPRZ01] and in the case of the 2D Gaussian free field [HMP10]. Related results were also obtained in the discrete; see [DPRZ01, Ros05, BR07, Jeg20b] for the random walk and [Dav06] for the discrete GFF. Many more articles studied related questions concerning other log-correlated fields; see [Shi15, Arg17] for more references.

We now define precisely a notion of thick points for the loop soup described informally earlier, and state some results concerning these points. We show that with this definition, $\mathcal{M}_a$ is almost surely supported on "a-thick points" of the loop soup. We also compute its Hausdorff dimension (a statement which does not involve the multiplicative chaos). Our definition of thick points is in terms of crossings of annuli. For $z \in D$, $r > 0$ and $\wp \in \mathcal{L}_D^\theta$ a loop, we denote by $N_{z,r}^{\wp}$ the number of upcrossings from $\partial D(z,r)$ to $\partial D(z,er)$ in $\wp$ (since $\wp$ is a loop, this is also equal to the number of downcrossings). Denote also $N_{z,r}^{\mathcal{L}_D^\theta} := \sum_{\wp \in \mathcal{L}_D^\theta} N_{z,r}^{\wp}$.

Theorem 4.11. *Let $\theta > 0$ and $a \in (0,2)$. $\mathcal{M}_a$ is almost surely supported by the set*

$$\mathcal{T}(a) := \left\{ z \in D : \lim_{n\to\infty} \frac{1}{n^2} N_{z,e^{-n}}^{\mathcal{L}_D^\theta} = a \right\}, \tag{4.8}$$

that is, $\mathcal{M}_a(D \setminus \mathcal{T}(a)) = 0$ a.s. Moreover, the Hausdorff dimension of $\mathcal{T}(a)$ equals $2-a$ a.s.

We mention that it would have been possible to quantify the thickness of a point z via the normalised occupation measure of small discs, or circles, centred at z. This would have been closer to the notion of thick points in [DPRZ01] and [Jeg20a]. To keep the paper of a reasonable size, we do not attempt to prove a result for these notions of thick points. Note that, before the current work, it was not even a priori immediately clear that points of infinite loop multiplicity exist with probability one.

In the next section, we establish the scaling limit of the set of thick points of random walk loop soup. In particular, we will obtain in Corollary 4.13 the convergence of the number of discrete thick points when appropriately normalised; as we will see this identifies a nontrivial subpolynomial term which goes beyond the calculation of the exponent $2-a$ corresponding to the above dimension; interestingly this subpolynomial term depends on the intensity θ itself.

4.1.5 Random walk loop soup approximation

As mentioned before, Theorem 4.5 is natural from the point of Le Jan's isomorphism in the continuum. However this relation is far too weak to obtain a proof of this theorem (for instance, it is not even clear at this point whether the hyperbolic cosine is a measurable function of the Wick square of the GFF). Instead, we rely on a discrete approach where the relations hold pointwise, and with no renormalisations, so that this type of difficulties does not arise. This approach also provides a very natural approximation of the multiplicative chaos measure $\mathcal{M}_a$ from a discrete random walk loop soup ([LTF07]): namely, $\mathcal{M}_a$ is the limit of the uniform measure on thick points of the discrete loop soup. Let us now detail this result.

Without loss of generality, assume that the domain D contains the origin. For all $N \geq 1$, we consider a discrete approximation $D_N \subset D \cap \frac{1}{N}\mathbb{Z}^2$ of D by a portion of the square lattice with mesh size $1/N$. Specifically,

$$D_N := \left\{ z \in D \cap \frac{1}{N}\mathbb{Z}^2 : \begin{array}{c} \text{there exists a path in } \frac{1}{N}\mathbb{Z}^2 \text{ from } z \text{ to the origin} \\ \text{whose distance to the boundary of } D \text{ is at least } \frac{1}{N} \end{array} \right\}. \tag{4.9}$$

Let $\mathcal{L}_{D_N}^\theta$ be a random walk loop soup with intensity θ. See Section 4.2.2 for a precise definition. For any vertex $z \in D_N$ and any discrete path $(\wp(t))_{0 \leq t \leq T(\wp)}$ parametrised by continuous time, we denote by $\ell_z(\wp)$ the local time of $\wp$ at z, i.e.

$$\ell_z(\wp) := \int_0^{T(\wp)} \mathbf{1}_{\{\wp(t)=z\}} \mathrm{d}t.$$

With our normalisation,

$$\mathbb{E}\Big[\sum_{\wp \in \mathcal{L}_{D_N}^\theta} \ell_z(\wp) \Big] \sim \frac{\theta}{2\pi} \log N \quad \text{as } N \to \infty.$$

We define the set of a-thick points by

$$\mathcal{T}_N(a) := \Big\{ z \in D_N : \sum_{\wp \in \mathcal{L}_{D_N}^\theta} \ell_z(\wp) \geq \frac{1}{2\pi} a (\log N)^2 \Big\}. \tag{4.10}$$

We encode this set in the following point measure: for all Borel set $A \subset \mathbb{C}$, define

$$\mathcal{M}_a^N(A) := \frac{(\log N)^{1-\theta}}{N^{2-a}} \sum_{z \in \mathcal{T}_N(a)} \mathbf{1}_{\{z \in A\}}. \tag{4.11}$$

In the next result and in the rest of the paper, we will denote

$$c_0 := 2\sqrt{2} e^{\gamma_{\mathrm{EM}}}, \tag{4.12}$$

where γ_{EM} is the Euler–Mascheroni constant (4.216). The constant c_0 arises from the asymptotic behaviour of the discrete Green function on the diagonal; see Lemma 4.93.

Theorem 4.12. *Let $\theta > 0$ and $a \in (0,2)$. The couple $(\mathcal{L}_{D_N}^\theta, \mathcal{M}_a^N)$ converges in distribution towards*

$(\mathcal{L}_D^\theta, 2^\theta c_0^a \mathcal{M}_a)$, relatively to the topology induced by $d_{\mathfrak{L}}$ (4.29) for $\mathcal{L}_{D_N}^\theta$ and the weak topology on $\mathbb{C}$ for $\mathcal{M}_a^N$.

In particular,

Corollary 4.13. *The convergence*

$$\frac{(\log N)^{1-\theta}}{N^{2-a}} \# \mathcal{T}_N(a) \to 2^\theta c_0^a \mathcal{M}_a(D)$$

holds in distribution.

Theorem 4.12 can be seen as an interpolation and extrapolation of the scaling limit results of [Jeg19] and [BL19] concerning, respectively, thick points of finitely many random walk trajectories (informally, $\theta \to 0^+$) and thick points of the discrete GFF ($\theta = 1/2$).

The proof of Theorem 4.12 ends up taking a large part of this article (essentially, all of Part Two). At a high level, the difficulties stem from the fact that (unlike in the continuum) it is very difficult to compare directly two random walk loop soups with different lattice mesh sizes, thereby ruling out the possibility to apply an L^1 convergence argument as in Gaussian multiplicative chaos [Ber17]. Instead, we rely on results of [Jeg19] in which analogous difficulties were resolved in the case of a finite number of random walk trajectories, together with a new discrete description (see Proposition 4.61) of the rooted discrete measure (i.e., a discrete loop soup version of the Girsanov transform) which must be proved by hand. These computations reveal a surprising amount of integrability, which we think is interesting in its own right. Another technical ingredient which we obtain along the way is a strengthening of a KMT-type coupling between the discrete loop soup and the continuum loop soup proved by Lawler and Trujillo-Ferreras [LTF07]. This coupling allows us to show that discrete and continuous loops of *all* mesoscopic scales are close to one another (in contrast with [LTF07], where the comparison holds for sufficiently large mesoscopic scales), provided we are only interested in loops that are localised close to a given point $z \in D$. This coupling is useful to obtain rough estimates on the discrete loop soup such as large deviations for the number of crossings of annuli of a given scale. See Lemma 4.86 for details.

4.1.6 Martingale and exact solvability

Before starting the proofs it is useful to highlight a few nontrivial aspects of the proofs. A crucial idea is the identification of a certain measure-valued martingale $m_a^K(dz)$ with respect to the filtration $\mathcal{F}_K$ generated by $\mathcal{L}_D^\theta(K)$. The definition of this martingale is in itself highly nontrivial and is described in Proposition 4.24. As follows *a posteriori* from our analysis, this martingale corresponds to the conditional expectation of $\mathcal{M}_a$ given $\mathcal{F}_K$. Although it is *a priori* far from clear that this conditional expectation should take the given form, it is nevertheless possible to guess a rough form for this conditional expectation. Indeed, consider the decomposition of the entire loop soup $\mathcal{L}_D^\theta$ into the killed part $(\mathcal{L}_D^\theta(K))$ and its complement. These two parts are independent. Furthermore, by the isomorphism theorem (see Theorem 4.18), the occupation field of the complement is given by one half of the square of a **massive Gaussian free field**. This suggests that $\mathcal{M}_a$ can be described by the sum of two terms.

The first term comes just from the hyperbolic cosine of this massive free field (since it is possible that a point is thick without being visited at all by $\mathcal{L}_D^\theta(K)$). The second term on the other hand describes the possible interactions between these two parts: it measures the contribution of points whose thickness comes in part from the massive free field and in another part from the killed loop soup. This interaction term is thus described by an integral in which the integrand describes the respective thickness of each part; however the precise law of this mixture cannot be easily inferred from combinatorial arguments and was instead obtained by trial and error. We stress however that the appearance of the massive free field (and its hyperbolic cosine) is what makes the ultraviolet regularisation by killing particularly attractive from our point of view.

While these arguments are useful to guess the general rough form of the martingale, they cannot be used to give a proof of the martingale property: rather, the martingale property is the engine that drives the proof and the above explanation may only be seen as a justification after the facts. The proof of the martingale property relies instead on a central observation (stated in Proposition 4.21 and proved in Section 4.5), which allows us to compute *exactly* the expectation of the approximate measure $\mathcal{M}_a^K$ with finite $K < \infty$. This expectation is computed in terms of the hypergeometric function ${}_1F_1$ and the conformal radius of a point. This computation is the result of the triple differentiation of a certain infinite series whose nth term involves an n-dimensional integral, see Lemma 4.33. The fact that such a computation is at all possible is another stroke of luck which suggests the choice of ultraviolet regularisation (by killing as opposed to say, by diameter) is particularly well suited to this problem. The exact solvability which seems to underly this calculation is in fact a constant feature of the paper; as shown in Part Two, analogous remarkable identities hold even at the discrete level. The existence of such exact formulae for the ultraviolet regularisation of the Brownian loop soup by killing seems to not have been noticed before; we hope it may prove useful in other contexts as well.

We end this introduction by pointing out that the results of this paper open the door to a generalisation, in particular to non-half integer values of θ, of constructions from the Euclidean Quantum Field Theory that relate the Wick powers of the GFF, the Gaussian multiplicative chaos and the intersection and self-intersection local times of Brownian paths (see e.g., [Sym65, Sym66, Sym69, Var69, Dyn84b, Dyn84c, Sim74, Wol78b, Wol78a, LG85, LJ11]). We plan to develop this in future works.

Organisation of the paper In the next section, we will give some background on loop soups and measures on paths both in the continuum and in the discrete. We will also recall the definitions of Brownian multiplicative chaos measures. The rest of the paper is then be divided into two main parts dealing with the continuum and the discrete settings respectively. Each of these parts starts with a preliminary section (Sections 4.3 and 4.10 respectively) outlining the proofs of the main theorems at a high level. The structure of each part is then described more thoroughly in these preliminary sections.

Acknowledgements We thank Marcin Lis for raising with two of us (AJ and NB) questions which ultimately triggered our interest in this problem. Part of this work was carried out when AJ visited EA and TL at NYU-ECNU Institute of Mathematical Sciences at NYU Shanghai. The hospitality of that department is gratefully acknowledged.

EA and TL acknowledge the support of the French National Research Agency (ANR) within the project MALIN (ANR-16-CE93-0003). AJ is recipient of a DOC Fellowship of the Austrian Academy of Sciences at the Faculty of Mathematics of the University of Vienna. AJ's research is partly supported by the EPSRC grant EP/L016516/1 for the University of Cambridge Centre for Doctoral Training, the Cambridge Centre for Analysis. NB's work was supported by: University of Vienna start-up grant, and FWF grant P33083 on "Scaling limits in random conformal geometry".

4.2 Background

4.2.1 Measures on Brownian paths and Brownian loop soup

We start first by recalling some basic properties of the Brownian loop soup, mostly to introduce our notations and choice of normalisations.

By Brownian motion we will denote the 2D Brownian motion with infinitesimal generator Δ rather than the standard Brownian motion, which has generator $\frac{1}{2}\Delta$. Let D be an open domain which we may assume to be bounded without loss of generality. Let $p_D(t,z,z)$ denote the transition probability of Brownian motion killed upon leaving the domain D. If

$$p_{\mathbb{C}}(t,z,w) = \frac{1}{4\pi t} \exp\Big(- \frac{|w-z|^2}{4\pi t} \Big)$$

denotes the transition probabilities of this Brownian motion in the full plane, and if $\pi_D(t,z,w)$ denotes the probability that a Brownian bridge of duration t remains in the domain D throughout, then

$$p_D(t,z,w) = p_{\mathbb{C}}(t,z,w)\pi_D(t,z,w).$$

Let $G_D(z,w)$ denote **Green function** of $-\Delta$ on D with Dirichlet 0 boundary conditions; that is,

$$G_D(z,w) = \int_0^\infty p_D(t,z,w)\mathrm{d}t. \tag{4.13}$$

In our normalisation,

$$G_D(z,w) \sim -\frac{1}{2\pi} \log(|w-z|) \tag{4.14}$$

as $|w-z| \to 0$.

Next we recall the definitions of natural measures on Brownian paths and loops. For details, we refer to [Law05, Chapter 5] and [LW04]. Given $z,w \in D$ and $t>0$, let $\mathbb{P}^{z,w}_{D,t}$ denote the probability measure on Brownian bridges from z to w of duration t, conditioned on staying in D. Let $\mu^{z,w}_D$ denote the following measure on continuous path from z to w in D:

$$\mu^{z,w}_D(d\wp) = \int_0^{+\infty} \mathbb{P}^{z,w}_{D,t}(d\wp) p_D(t,z,w)\mathrm{d}t. \tag{4.15}$$

The total mass of $\mu^{z,w}_D$ is $G_D(z,w)$. In particular, it is infinite if $z=w$. The image of $\mu^{z,w}_D$ by time reversal is $\mu_D^{w,z}$. Given a subdomain $D' \subset D$ and $z, w \in D'$,

$$\mu_{D'}^{z,w}(d\wp) = \mathbf{1}_{\{\wp \text{ stays in } D'\}} \mu_D^{z,w}(d\wp). \tag{4.16}$$

Further, if $z \in D$ and $x \in \partial D$, and ∂D is smooth near x, we will denote

$$\mu_D^{z,x}(d\wp) = \lim_{\varepsilon \to 0} \varepsilon^{-1} \mu_D^{z,x+\varepsilon \overrightarrow{n}_x}(d\wp), \tag{4.17}$$

where $\overrightarrow{n}_x$ is the normal unit vector at x pointing inwards. In this way, $\mu_D^{z,x}$ is a measure on interior-to-boundary Brownian excursions from z to x. Its total mass is given by

$$H_D(z,x) = \lim_{\varepsilon \to 0} \varepsilon^{-1} G_D(z, x + \varepsilon \overrightarrow{n}_x). \tag{4.18}$$

This $H_D(z,x)$ is the **Poisson kernel**, the density of the harmonic measure from z. The probability measure $\mu_D^{z,x}/H_D(z,x)$ is the law of the Brownian motion starting from z up to the first hitting time of ∂D, conditioned on hitting ∂D in x. Now, if $x, y \in \partial D$ and ∂D is smooth near x and near y, we similarly define

$$\mu_D^{x,y}(d\wp) = \lim_{\varepsilon \to 0} \varepsilon^{-2} \mu_D^{x+\varepsilon \overrightarrow{n}_x, y+\varepsilon \overrightarrow{n}_y}(d\wp). \tag{4.19}$$

In this way, $\mu_D^{x,y}$ is a measure on boundary-to-boundary Brownian excursions from x to y. Its total mass is given by

$$H_D(x,y) = \lim_{\varepsilon \to 0} \varepsilon^{-2} G_D(x + \varepsilon \overrightarrow{n}_x, y + \varepsilon \overrightarrow{n}_y). \tag{4.20}$$

Here, $H_D(x,y)$ is the **boundary Poisson kernel**. Note that $H_D(x,x) = +\infty$.

Notation 4.14. *For any $z \in D$ and $w \in D$, respectively $w \in \partial D$, we will denote by $\wp_D^{z,w}$ a Brownian trajectory distributed according to*

$$\mu_D^{z,w}/G_D(z,w), \quad \textit{respectively} \quad \mu_D^{z,w}/H_D(z,w). \tag{4.21}$$

If $z \in \partial D$ and $w \in D$, we will denote by $\wp_D^{z,w}$ a trajectory which is the time reversal of a path distributed according to $\mu_D^{w,z}/H_D(w,z)$.

The natural measure on Brownian loops in D is

$$\mu_D^{\mathrm{loop}}(d\wp) = \int_D \int_0^{+\infty} \mathbb{P}_{D,t}^{z,z}(d\wp) p_D(t,z,z) \frac{\mathrm{d}t}{t} \mathrm{d}z. \tag{4.22}$$

The measure μ_D^{loop} has an infinite total mass because of the ultraviolet divergence. The measure on loops is invariant under time reversal. Given a subdomain $D' \subset D$,

$$\mu_{D'}^{\mathrm{loop}}(d\wp) = \mathbf{1}_{\{\wp \text{ stays in } D'\}} \mu_D^{\mathrm{loop}}(d\wp). \tag{4.23}$$

The measure μ_D^{loop} can be rewritten as

$$\mu_D^{\mathrm{loop}}(d\wp) = \frac{1}{T(\wp)} \int_D \mu_D^{z,z}(d\wp) \mathrm{d}z, \tag{4.24}$$

where $T(\wp)$ denotes the total duration of a generic path $\wp$.

We will also need in what follows the massive version of the measure on Brownian loops. Let $K > 0$ be a constant. Let $G_{D,K}(z,w)$ denote the **massive Green function** associated to $-\Delta + K$, with Dirichlet 0 boundary conditions. We have that

$$G_{D,K}(z,w) = \int_0^\infty e^{-Kt} p_D(t,z,w) \mathrm{d}t. \tag{4.25}$$

In Quantum Field Theory, K corresponds to the square of a particle mass. In terms of Brownian motion, K is just a killing rate. The massive measure on Brownian loops in D is

$$\mu_{D,K}^{\mathrm{loop}}(d\wp) = e^{-KT(\wp)} \mu_D^{\mathrm{loop}}(d\wp). \tag{4.26}$$

Note that the massive measure on Brownian loops was introduced in early works on Euclidean QFT by Symanzik [Sym65, Sym66, Sym69].

The loops under the measures μ_D^{loop} (4.22) and $\mu_{D,K}^{\mathrm{loop}}$ (4.26) are **rooted**, that is to say the loops $\wp$ have a well defined starting time and end time. However, one usually considers **unrooted loops** [LW04, Law05], that is to say one identifies the loops under circular shifts of the parametrisation. Two rooted loops $\wp$ and $\tilde{\wp}$ correspond to the same unrooted loop if $T(\wp) = T(\tilde{\wp})$, and there is $s \in [0, T(\wp)]$ such that $\tilde{\wp}(t) = \wp(t+s)$ for $t \in [0, T(\wp) - s]$, and $\tilde{\wp}(t) = \wp(t + s - T(\wp))$ for $t \in [T(\wp) - s, T(\wp)]$. We will denote by $\mu_D^{\mathrm{loop}*}$, respectively $\mu_{D,K}^{\mathrm{loop}*}$, the measures on unrooted loops induced by μ_D^{loop}, respectively $\mu_{D,K}^{\mathrm{loop}}$.

By considering unrooted loops, one gains a covariance under conformal maps for $\mu_D^{\mathrm{loop}*}$. Let D and $\widetilde{D}$ be two conformally equivalent open domains and $\psi : D \to \widetilde{D}$ a conformal map. Let $\mathcal{T}_\psi$ be the following transformation of paths induced by ψ. Given $\wp$ a path in D, one applies to $\wp$ the map ψ and performs a change of time $\mathrm{d}s = |\psi'(\wp(t))|^2 \mathrm{d}t$. Then $\mu_{\widetilde{D}}^{\mathrm{loop}*}$ is the image measure of $\mu_D^{\mathrm{loop}*}$ under $\mathcal{T}_\psi$; see [LW04, Proposition 6] and [Law05, Proposition 5.27]. Note that in general, $\mu_{\widetilde{D}}^{\mathrm{loop}}$ is not the image of μ_D^{loop} under $\mathcal{T}_\psi$.

Given $\theta > 0$, a **Brownian loop soup** $\mathcal{L}_D^\theta$, as introduced in [LW04], is a Poisson point process of intensity $\theta \mu_D^{\mathrm{loop}}$. We see it as a random infinite countable collection of Brownian loops in D. We will consider both rooted and unrooted loops, depending on the context, and use the same notation $\mathcal{L}_D^\theta$ in both cases. On simply connected domains, the Brownian loop soups were used in the construction of Conformal Loop Ensembles CLE_κ [SW12]. At the particular value of the intensity parameter $\theta = 1/2$, the loop soup $\mathcal{L}_D^{1/2}$ is related to the continuum Gaussian free field (GFF) and to the CLE_4 [LJ10, LJ11, SW12, QW19, ALS20]. These relations are part of the random walk/Brownian motion representations of the GFF, also known as **isomorphism theorems** [Sym65, Sym66, Sym69, BFS82, Dyn84a, Dyn84b, MR06, Szn12].

Now let us define the loops in $\mathcal{L}_D^\theta$ killed by a killing rate K. Let $U_\wp, \wp \in \mathcal{L}_D^\theta$, be a collection of i.i.d. uniform random variables on $[0,1]$. Given $K > 0$, set

$$\mathcal{L}_D^\theta(K) := \left\{ \wp \in \mathcal{L}_D^\theta : U_\wp < 1 - e^{-KT(\wp)} \right\}. \tag{4.27}$$

The subset $\mathcal{L}_D^\theta(K)$ of $\mathcal{L}_D^\theta$ consists of loops killed by K. The complementary $\mathcal{L}_D^\theta \setminus \mathcal{L}_D^\theta(K)$ is a Poisson point process of intensity $\theta\mu_{D,K}^{\mathrm{loop}}$. In other words it is a massive Brownian loop soup. The construction through the uniform r.v.s $U_\wp$-s allows to couple $\mathcal{L}_D^\theta$ and the $\mathcal{L}_D^\theta(K)$ for all possible K on the same probability space. Moreover, this coupling is monotone: if $K' \leq K$, then $\mathcal{L}_D^\theta(K') \subset \mathcal{L}_D^\theta(K)$ a.s.

It is easy to see that a.s., for every $K > 0$, $\mathcal{L}_D^\theta(K)$ is infinite. However,

$$\mathbb{E}[|\{\wp \in \mathcal{L}_D^\theta(K) | T(\wp) > \varepsilon\}|] \asymp \log(\varepsilon^{-1}), \qquad \mathbb{E}[|\{\wp \in \mathcal{L}_D^\theta(K) | \operatorname{diam}(\wp) > \varepsilon\}|] \asymp \log(\varepsilon^{-1}),$$

whereas for the whole loop soup $\mathcal{L}_D^\theta$,

$$\mathbb{E}[|\{\wp \in \mathcal{L}_D^\theta | T(\wp) > \varepsilon\}|] \asymp \varepsilon^{-1}, \qquad \mathbb{E}[|\{\wp \in \mathcal{L}_D^\theta | \operatorname{diam}(\wp) > \varepsilon\}|] \asymp \varepsilon^{-2}.$$

For the sequel we will need to formalize a topology on collections of unrooted loops. First, let us defined a distance on the continuous paths in $\mathbb{C}$ of finite duration. Given $(\wp_1(t))_{0 \leq t \leq T(\wp_1)}$ and $(\wp_2(t))_{0 \leq t \leq T(\wp_2)}$ such paths, let be the distance

$$d_{\mathrm{paths}}(\wp_1, \wp_2) := |\log(T(\wp_2)/T(\wp_1))| + \max_{0 \leq s \leq 1} |\wp_2(sT(\wp_2)) - \wp_1(sT(\wp_1))|. \tag{4.28}$$

If $\wp_1$ and $\wp_2$ are two rooted loops, i.e. $\wp_1(T(\wp_1)) = \wp_1(0)$ and $\wp_2(T(\wp_2)) = \wp_2(0)$, and if $[\wp_1]$ and $[\wp_2]$ are the corresponding unrooted loops, i.e. the equivalence classes under circular shifts of parametrisation, then let be the distance

$$d_{\mathrm{unrooted}}([\wp_1], [\wp_2]) := \min_{\tilde{\wp} \in [\wp_1]} d_{\mathrm{paths}}(\tilde{\wp}, \wp_2) = \min_{\tilde{\wp} \in [\wp_2]} d_{\mathrm{paths}}(\wp_1, \tilde{\wp}).$$

Now let us consider finite collections of unrooted loops. Here and in the sequel by collection we mean a multiset. The elements of a multiset are unordered, but may come each with a finite multiplicity. A collection can also be empty. Given $\mathcal{L}_1$ and $\mathcal{L}_2$ two such finite collections of unrooted loops on $\mathbb{C}$, we set the distance

$$d_{\mathrm{fin.col.}}(\mathcal{L}_1, \mathcal{L}_2) := \min_{\sigma \in \mathrm{Bij}(\mathcal{L}_1, \mathcal{L}_2)} \sum_{\wp \in \mathcal{L}_1} d_{\mathrm{unrooted}}(\wp, \sigma(\wp))$$

if $\mathcal{L}_1$ and $\mathcal{L}_2$ have same cardinal with multiplicities taken into account, and $d_{\mathrm{fin.col.}}(\mathcal{L}_1, \mathcal{L}_2) = +\infty$ otherwise. In particular, the distance of the empty collection to any non-empty collection is $+\infty$.

Given $z \in \mathbb{C}$ and $r > 0$, let $D(z, r)$ denote the open disc with center z and radius r. Given $\mathcal{L}$ a collection of unrooted loops, not necessarily finite, and $r > 0$, denote

$$\mathcal{L}_{|r} := \{\wp \in \mathcal{L} : \wp \text{ stays in } \overline{D(0,r)}, \operatorname{diam}(\wp) \geq r^{-1}\}.$$

Let $\mathfrak{L}$ be the following space:

$$\mathfrak{L} := \{\mathcal{L} \text{ collection of unrooted loops on } \mathbb{C} : \forall r > 0, \mathcal{L}_{|r} \text{ is finite}\}.$$

The empty collection also belongs to $\mathfrak{L}$. All the collections belonging to $\mathfrak{L}$ are countable. We endow $\mathfrak{L}$

with the following distance:

$$d_{\mathfrak{L}}(\mathcal{L}_1, \mathcal{L}_2) := \int_1^{+\infty} e^{-r} (d_{\text{fin.col.}}((\mathcal{L}_1)_{|r}, (\mathcal{L}_2)_{|r}) \wedge 1) \mathrm{d}r. \tag{4.29}$$

A sequence $(\mathcal{L}_k)_{k \geq 0}$ converges to $\mathcal{L}$ for $d_{\mathfrak{L}}$ if and only if there is a positive increasing sequence $(r_j)_{j \geq 0}$, with $\lim_{j \to +\infty} r_j = +\infty$, such that for every $j \geq 0$,

$$\lim_{k \to +\infty} d_{\text{fin.col.}}((\mathcal{L}_k)_{|r_j}, \mathcal{L}_{|r_j}) = 0.$$

It is easy to see that the induced metric space $(\mathfrak{L}, d_{\mathfrak{L}})$ is complete. Moreover, the finite collections are dense in $\mathfrak{L}$. Further, the finite collections can be approximated by a countable subset of finite collections. Consider for instance the trigonometric series. Thus, the metric space $(\mathfrak{L}, d_{\mathfrak{L}})$ is separable. So, $(\mathfrak{L}, d_{\mathfrak{L}})$ is a Polish space. We will often see the Brownian loop soups $\mathcal{L}_D^\theta$ and $\mathcal{L}_D^\theta(K)$ as r.v.s with values in $\mathfrak{L}$.

4.2.1.0.1 Equivalence relation on $(\mathfrak{L}, d_{\mathfrak{L}})$ and admissible functions We now formalise the notion of functions $F : \mathfrak{L} \to \mathbb{R}$ that are invariant by exchanging the order of the excursions in the loops at a given point $z \in \mathbb{C}$. We will call such functions *z-admissible* functions.

Let $\wp : t \in [0, T(\wp)] \mapsto \wp_t \in \mathbb{C}$ be a continuous path in $\mathbb{C}$ with finite duration and such that $\wp_0 = \wp_{T(\wp)}$. Let $z \in \mathbb{C}$ be a point visited by $\wp$. To $\wp$ and z we can uniquely associate an at most countable collection of excursions $\{e_i^{\wp,z}, i \in I\}$, where by an excursion e we mean a continuous path $(e_t, 0 \leq t \leq \zeta)$ such that $e_0 = e_\zeta = z$ and $e_t \neq z$ for all $t \in (0, \zeta)$, and such that the reunion of all $e_i^{\wp,z}$ coincides with the loop $\wp$. In fact, these excursions inherit from $\wp$ a chronological order but we will not need this.

For a fixed $z \in \mathbb{C}$, we define an equivalence relation $\sim_z$ on unrooted loops by saying that two loops $\wp$ and $\wp'$ are equivalent if, and only if,

- either z is not visited by $\wp$, nor $\wp'$, and in that case the unrooted loops $[\wp]$ and $[\wp']$ agree;
- or z is visited by both $\wp$ and $\wp'$ and the collections of *unordered* excursions $\{e_i^{\wp,z}, i \in I\}$ and $\{e_i^{\wp',z}, i \in I'\}$ coincide.

We will denote $< \wp >_z$ the equivalence class of a loop $\wp$ under the relation $\sim_z$. If $\mathcal{C} \in \mathfrak{L}$ is a collection of loops, we will denote $< \mathcal{C} >_z := \{< \wp >_z, \wp \in \mathcal{C}\}$.

We can now give a precise definition of admissible functions.

Definition 4.15. *Let $z \in \mathbb{C}$. We will say that a function $F : \mathfrak{L} \to \mathbb{R}$ is z-admissible if $F(\cdot)$ is invariant under the relation $\sim_z$, i.e. if for all $\mathcal{C}, \mathcal{C}' \in \mathfrak{L}$, $F(\mathcal{C}) = F(\mathcal{C}')$ as soon as $< \mathcal{C} >_z = < \mathcal{C}' >_z$.*

Functions $F : D \times \mathfrak{L} \to \mathbb{R}$ (resp. $F : D \times D \times \mathfrak{L} \to \mathbb{R}$) are called admissible if for all $z \in D$, $F(z, \cdot)$ is z-admissible (resp. if for all $z, z' \in D$, $F(z, z', \cdot)$ is z-admissible and z'-admissible).

Examples of admissible functions include total time duration, number of crossings of an annulus, etc.

Finally, we introduce the σ-algebra

$$\sigma(\left\langle \mathcal{L}_D^\theta \right\rangle) := \sigma\left(F(\mathcal{L}_D^\theta), F : \mathfrak{L} \to \mathbb{R} \text{ bounded measurable s.t. } \forall z \in \mathbb{C}, F \text{ is } z\text{-admissible} \right). \tag{4.30}$$

It is the σ-algebra generated by the equivalence class of $\mathcal{L}_D^\theta$ where two loops $\wp$ and $\wp'$ are identified if and only if $\wp \sim_z \wp'$ for all $z \in \mathbb{C}$. Note that this σ-algebra is included in $\sigma(\left\langle \mathcal{L}_D^\theta \right\rangle_z)$ for any $z \in D$.

4.2.2 Measures on discrete paths and random walk loop soup

Here we will recall some properties of the continuous-time discrete-space random walk loop soups.

Let $N \geq 1$ be an integer. We will denote $\mathbb{Z}_N := \frac{1}{N}\mathbb{Z}$, and work on the rescaled square lattice $\mathbb{Z}_N^2$. Let Δ_N be the discrete Laplacian on $\mathbb{Z}_N^2$:

$$(\Delta_N f)(z) := N^2 \sum_{\substack{w \in \mathbb{Z}_N^2 \\ |w-z| = \frac{1}{N}}} (f(w) - f(z)), \quad z \in \mathbb{Z}_N^2.$$

Note that with our normalisation, Δ_N converges as $N \to +\infty$ to the continuum Laplacian Δ on $\mathbb{C}$. Let $(X_t^{(N)})_{t \geq 0}$ be the Markov jump process on $\mathbb{Z}_N^2$ with infinitesimal generator Δ_N. In other words, this is the continuous-time simple symmetric random walk, with exponential holding times with mean $\frac{1}{4N^2}$. As $N \to +\infty$, $(X_t^{(N)})_{t \geq 0}$ converges in law to the Brownian motion on $\mathbb{C}$ with infinitesimal generator Δ.

Let D_N be a non-empty subset of $\mathbb{Z}_N^2$. Note that in the sequel we will typically consider sequences $(D_N)_{N \geq 1}$ converging to continuum domains $D \subset \mathbb{C}$ as in (4.9). Let $\tau_{\mathbb{Z}_N^2 \setminus D_N}$ denote the first hitting time of $\mathbb{Z}_N^2 \setminus D_N$ by $X_t^{(N)}$. Denote

$$p_{D_N}(t, z, w) := N^2 \mathbb{P}^z(X_t^{(N)} = w, \ \tau_{\mathbb{Z}_N^2 \setminus D_N} > t), \quad z, w \in D_N.$$

Note that $p_{D_N}(t, z, w) = p_{D_N}(t, w, z)$. Denote

$$G_{D_N}(z, w) = \int_0^{+\infty} p_{D_N}(t, z, w) \mathrm{d}t.$$

If z or w is in $\mathbb{Z}_N^2 \setminus D_N$, we set $G_{D_N}(z, w) = 0$. Defined this way, G_{D_N} is the discrete Green function. It satisfies

$$-\Delta_{N,w} G_{D_N}(z, w) = N^2 \mathbf{1}_{\{z=w\}}, \quad z, w \in D_N,$$

where the notation $\Delta_{N,w}$ indicates that the discrete Laplacian Δ_N is taken with respect to the variable w.

Let $\mathbb{P}_{D_N,t}^{z,w}$ denote the law of $(X_s^{(N)})_{0 \leq s \leq t}$, with $X_0^{(N)} = z$, conditionally on $X_t^{(N)} = w$ and $\tau_{\mathbb{Z}_N^2 \setminus D_N} > t$. Next we recall the discrete analogues of measures (4.15) and (4.22). For details, we refer to [LJ10, LJ11]. The measure $\mu_{D_N}^{z,w}$ will be a measure on nearest-neighbour paths from z to w in D_N, parametrised by continuous time, and of final total duration:

$$\mu_{D_N}^{z,w}(d\wp) = \int_0^{+\infty} \mathbb{P}_{D_N,t}^{z,w}(d\wp) p_{D_N}(t, z, w) \mathrm{d}t. \tag{4.31}$$

The total mass of $\mu_{D_N}^{z,w}$ is G_{D_N}. The image of $\mu_{D_N}^{z,w}$ by time reversal is $\mu_{D_N}^{w,z}$.

In the case when $\mathbb{Z}_N^2 \setminus D_N$ is also non-empty, let ∂D_N denote the subset of $\mathbb{Z}_N^2 \setminus D_N$ made of vertices at graph distance 1 from D_N, i.e. at Euclidean distance $\frac{1}{N}$. Given $z \in D_N$ and $x \in \partial D_N$, denote

$$\mu_{D_N}^{z,x} = N \sum_{\substack{w \in D_N \\ |w-x| = \frac{1}{N}}} \mu_{D_N}^{z,w}. \tag{4.32}$$

Let $H_{D_N}(z,x)$ denote the total mass of the measure $\mu_{D_N}^{z,x}$. We have that

$$H_{D_N}(z,x) = N \sum_{\substack{w \in D_N \\ |w-x| = \frac{1}{N}}} G_{D_N}(z,w). \tag{4.33}$$

Usually, we will add to trajectories under $\mu_{D_N}^{z,x}$ an additional instantaneous jump to x at the end, without local time spent at x. In this way, the probability measure $\mu_{D_N}^{z,x}/H_{D_N}(z,x)$ is actually the distribution of $(X_t^{(N)})_{0 \le t \le \tau_{\mathbb{Z}_N^2 \setminus D_N}}$ given that $X_0^{(N)} = z$ and conditionally on $X_{\tau_{\mathbb{Z}_N^2 \setminus D_N}}^{(N)} = x$. Moreover,

$$H_{D_N}(z,x) = N \mathbb{P}^z (X_{\tau_{\mathbb{Z}_N^2 \setminus D_N}}^{(N)} = x).$$

So we see $H_{D_N}(z,x)$ as the discrete Poisson kernel.

The measure $\mu_{D_N}^{\mathrm{loop}}$ will be a measure on rooted nearest-neighbour loops in D_N, parametrised by continuous time, and of final total duration:

$$\mu_{D_N}^{\mathrm{loop}}(d\wp) = \frac{1}{N^2} \sum_{z \in D_N} \int_0^{+\infty} \mathbb{P}_{D_N,t}^{z,z}(d\wp) p_{D_N}(t,z,z) \frac{\mathrm{d}t}{t}. \tag{4.34}$$

The measure $\mu_{D_N}^{\mathrm{loop}}$ is invariant by time reversal. Note that the total mass of $\mu_{D_N}^{\mathrm{loop}}$ is always infinite because of the ultraviolet divergence. The measure puts an infinite mass on trivial "loops" that stay in one vertex, without performing jumps. To the contrary, $\mu_{D_N}^{\mathrm{loop}}$ puts a finite mass on loops that visit at least two vertices and stay inside a finite box. More precisely, given $z_1, z_2, \dots, z_{2n} \in D_N$, with $|z_i - z_{i-1}| = \frac{1}{N}$ and $|z_{2n} - z_1| = \frac{1}{N}$, the weight given to the set of rooted loops starting from z_1, then successively visiting $z_2, \dots, z_{2n}$, and then returning to z_1 is $(2n)^{-1} 4^{-2n}$. Moreover, conditionally on this discrete skeleton, the holding times are i.i.d. exponential r.v.s with mean $\frac{1}{4N^2}$. Given a subset $D_N' \subset D_N$,

$$\mu_{D_N'}^{z,w} = \mathbf{1}_{\{\wp \text{ stays in } D_N'\}} \mu_{D_N}^{z,w}, \quad z, w \in D_N', \qquad \mu_{D_N'}^{\mathrm{loop}} = \mathbf{1}_{\{\wp \text{ stays in } D_N'\}} \mu_{D_N}^{\mathrm{loop}}.$$

The measure on continuous time discrete space loops (4.34) first appeared in [LJ10, LJ11]. Related measures on discrete time loops appeared in [BFS82, LTF07, LL10].

We will also need a measure $\check{\mu}_{D_N}^{z,w}$ related but different from $\mu_{D_N}^{z,w}$. Given $z, w \in D_N$, denote

$$\check{\mu}_{D_N}^{z,w} = \sum_{\substack{z',w' \in D_N \setminus \{z,w\} \\ |z'-z| = \frac{1}{N} \\ |w'-w| = \frac{1}{N}}} \mu_{D_N \setminus \{z,w\}}^{z',w'}. \tag{4.35}$$

This is a measure on continuous time nearest-neighbour paths from a neighbour of z to a neighbour of w, and staying in $D_N \setminus \{z, w\}$. Actually, to a path under $\check{\mu}_{D_N}^{z,w}$ we will add an initial jump from z to the corresponding neighbour z', and a final jump to w from the corresponding neighbour w'. In this way we get a path from z to w, but with zero holding time in z and w.

We will also consider the massive case. Let $K > 0$ be a constant. Denote $G_{D_N,K}(z,w)$ the massive Green function

$$G_{D_N,K}(z,w) = \int_0^{+\infty} e^{-Kt} p_{D_N}(t,z,w) \mathrm{d}t. \tag{4.36}$$

The massive version of the measure on loops (4.34) is

$$\mu_{D_N,K}^{\text{loop}}(d\wp) = e^{-KT(\wp)} \mu_{D_N}^{\text{loop}}(d\wp),$$

where $T(\wp)$ is the total duration of a loop.

Again, given $\theta > 0$, we will consider Poisson point processes of intensity $\theta \mu_{D_N}^{\text{loop}}$, denoted $\mathcal{L}_{D_N}^\theta$. We will consider both rooted and unrooted loops. These are random countable collections of loops in D_N, known as continuous time **random walk loop soups**. Note that, if D_N is finite, then $\mathcal{L}_{D_N}^\theta$ contains a.s. only finitely many non-trivial loops that visit at least two vertices. However, a.s., for every $z \in D_N$, $\mathcal{L}_{D_N}^\theta$ contains infinitely many trivial "loops" that only stay in z.

Now, consider a constant $K > 0$. Let $U_\wp, \wp \in \mathcal{L}_{D_N}^\theta$, be a collection of i.i.d. uniform random variables on $[0,1]$. Define

$$\mathcal{L}_{D_N}^\theta(K) := \left\{ \wp \in \mathcal{L}_{D_N}^\theta : U_\wp < 1 - e^{-KT(\wp)} \right\}.$$

The subset $\mathcal{L}_{D_N}^\theta(K)$ corresponds to loops killed by the killing rate K. The complementary $\mathcal{L}_{D_N}^\theta \setminus \mathcal{L}_{D_N}^\theta(K)$ is a Poisson point process with intensity measure $\theta \mu_{D_N,K}^{\text{loop}}$. Unlike in the continuum case, $\mathcal{L}_{D_N}^\theta(K)$ is a.s. finite if D_N is finite. This is because

$$\int_0^\varepsilon (1 - e^{-Kt}) \frac{\mathrm{d}t}{t} < +\infty.$$

For a vertex $z \in \mathbb{Z}_N^2$ and a path on $\mathbb{Z}_N^2$ parametrised by continuous time $(\wp(t))_{0 \leq t \leq T(\wp)}$, we denote by $\ell_z(\wp)$ the local time accumulated by $\wp$ at z, i.e.

$$\ell_z(\wp) := \int_0^{T(\wp)} \mathbf{1}_{\{\wp(t) = z\}} \mathrm{d}t.$$

Given $\mathcal{L}$ a collection of path on $\mathbb{Z}_N^2$, we denote

$$\ell_z(\mathcal{L}) := \sum_{\wp \in \mathcal{L}} \ell_z(\wp). \tag{4.37}$$

First we state some Markovian decomposition properties for the measures $\mu_{D_N}^{z,w}$ and $\check{\mu}_{D_N}^{z,w}$. These are elementary, so we do not provide proofs.

Lemma 4.16. *Let $D_N \subset \mathbb{Z}_N^2$ such that both D_N and $\mathbb{Z}_N^2 \setminus D_N$ are non-empty.*

1. *Given $z \in D_N$, under the probability measure $G_{D_N}(z,z)^{-1}\mu_{D_N}^{z,z}(d\wp)$, the local time $\ell_z(\wp)$ is an exponential r.v. with mean $G_{D_N}(z,z)$. Conditionally on $\ell_z(\wp)$, the behaviour of $\wp$ outside z is given by a Poisson point process of excursions from z to z with intensity measure $\ell_z(\wp)\check{\mu}_{D_N}^{z,z}$.*

2. *Let $z, w, z' \in D_N$ such that z' is at a graph distance at least 2 from both z and w, i.e. $|z'-z| > \frac{1}{N}$ and $|z'-w| > \frac{1}{N}$. Then for any bounded measurable function F,*

$$\int \mathbf{1}_{\{\wp \text{ visits } z'\}} F(\wp) \check{\mu}_{D_N}^{z,w}(d\wp) = \int \check{\mu}_{D_N}^{z,z'}(d\wp_1) \int \mu_{D_N \setminus \{z,w\}}^{z',z'}(d\wp) \int \check{\mu}_{D_N}^{z',w}(d\wp_2) F(\wp_1 \wedge \wp \wedge \wp_2),$$

where $\wedge$ denotes the concatenation of paths.

3. *Let $z, w \in D_N$ such that z and w are at a graph distance at least 2, i.e. $|w-z| > \frac{1}{N}$. Then for any bounded measurable function F,*

$$\int F(\wp) \mu_{D_N}^{z,w}(d\wp) = \int \mu_{D_N}^{z,z}(d\wp_1) \int \check{\mu}_{D_N}^{z,w}(d\wp) \int \mu_{D_N \setminus \{z\}}^{w,w}(d\wp_2) F(\wp_1 \wedge \wp \wedge \wp_2).$$

Next we describe the law of the local times of loops in a random walk loop soup. For details, we refer to [LJ10, LJ11].

Proposition 4.17 (Le Jan [LJ10, LJ11])**.** *Let $D_N \subset \mathbb{Z}_N^2$ such that both D_N and $\mathbb{Z}_N^2 \setminus D_N$ are non-empty. Fix $\theta > 0$ and consider the random walk loop soup $\mathcal{L}_{D_N}^\theta$. Given $z \in D_N$, the collection of random times $(\ell_z(\wp))_{\wp \in \mathcal{L}_{D_N}^\theta, \wp \text{ visits } z}$ is a Poisson point process of $(0, +\infty)$ with intensity measure*

$$\mathbf{1}_{\{t>0\}} \theta e^{-t/G_{D_N}(z,z)} \frac{\mathrm{d}t}{t}, \tag{4.38}$$

that is to say these are the jumps of a Gamma subordinator. In particular, $\ell_z(\mathcal{L}_{D_N}^\theta)$ follows a Gamma(θ) distribution with density

$$\mathbf{1}_{\{t>0\}} \frac{1}{\Gamma(\theta) G_{D_N}(z,z)^\theta} t^{\theta-1} e^{-t/G_{D_N}(z,z)}.$$

Conditionally on the family of local times $(\ell_z(\wp))_{\wp \in \mathcal{L}_{D_N}^\theta, \wp \text{ visits } z}$, the loops $\wp$ visiting z are obtained, up to rerooting, by taking independent Poisson point processes of excursions from z to z with respective intensities $\ell_z(\wp)\check{\mu}_{D_N}^{z,z}$. The collections of loops not visiting z is independent from the loops visiting z, and distributed as $\mathcal{L}_{D_N \setminus \{z\}}^\theta$.

Furthermore, given $K > 0$ and $z \in D_N$, the collection of random times $(\ell_z(\wp))_{\wp \in \mathcal{L}_{D_N}^\theta \setminus \mathcal{L}_{D_N}^\theta(K), \wp \text{ visits } z}$

is a Poisson point process of $(0, +\infty)$ *with intensity measure*

$$\mathbf{1}_{\{t>0\}} \theta e^{-t/G_{D_N,K}(z,z)} \frac{\mathrm{d}t}{t}. \tag{4.39}$$

For the particular value of the intensity parameter $\theta = 1/2$, the random walk loop soup $\mathcal{L}_{D_N}^{1/2}$ is related to the discrete Gaussian free field (GFF) through the **Le Jan's isomorphism theorem** [LJ10, LJ11]. Let φ_N denote the discrete (massless) GFF on D_N with condition 0 on $\mathbb{Z}_N^2 \setminus D_N$. It is a random centred Gaussian field with covariance kernel given by the Green function G_{D_N}. Given a constant $K > 0$, there is also the massive discrete GFF $\varphi_{N,K}$, with covariance kernel $G_{D_N,K}$.

Theorem 4.18 (Le Jan [LJ10, LJ11])**.** *Let $D_N \subset \mathbb{Z}_N^2$ such that both D_N and $\mathbb{Z}_N^2 \setminus D_N$ are non-empty. Consider the random walk loop soup $\mathcal{L}_{D_N}^{1/2}$. Then, the occupation field $(\ell_z(\mathcal{L}_{D_N}^{1/2}))_{z \in D_N}$ is distributed as $\frac{1}{2}\varphi_N^2$. Further, given a constant $K > 0$, the occupation field $(\ell_z(\mathcal{L}_{D_N}^{1/2} \setminus \mathcal{L}_{D_N}^{1/2}(K)))_{z \in D_N}$ is distributed as $\frac{1}{2}\varphi_{N,K}^2$.*

Remark 4.19. Note that in dimension 2, Le Jan's isomorphism has a renormalised version in continuum space involving the Wick's square of the continuum GFF [LJ10, LJ11].

4.2.3 Brownian multiplicative chaos

This section recalls some facts about Brownian multiplicative chaos measures. These measures were introduced in [BBK94, AHS20, Jeg20a] in the case of one given Brownian trajectory and can be formally defined as the exponential of the square root of the local time of the trajectory (see [Jeg20a, Theorems 1.1 and 1.2] for a construction that uses an exponential approximation). In the current article, we will need to consider "multipoint" versions of these measures for finitely many independent trajectories. This generalisation has been studied in [Jeg19] and was key in order to characterise the law of Brownian multiplicative chaos. The current article focuses on the subcritical regime, but let us mention that Brownian chaos measures have also been constructed at criticality, i.e. when $a = 2$ (equivalently, $\gamma = 2$); see [Jeg21].

For all $i \geq 1$, let $D_i \subset \mathbb{C}$ be a bounded simply connected domain and let $z_i \in D_i$ be a starting point. Let us consider independent random processes $\wp_i = (\wp_i(t))_{0 \leq t \leq \tau_i}, i \geq 1$, in the plane such that for each $i \geq 1$, the law of $\wp_i$ is locally mutually absolutely continuous with respect to the law of Brownian motion starting at z_i and killed upon exiting for the first time D_i. In order to recall a rigorous definition of the Brownian chaos measures that we will consider in this article, we first introduce local times of circles: for all $i \geq 1$, $z \in D_i$ and $\varepsilon > 0$ be such that $D(z, \varepsilon) \subset D_i$, let

$$L_{z,\varepsilon}^i := \lim_{r \to 0^+} \frac{1}{2r} \int_0^{\tau_i} \mathbf{1}_{\{\varepsilon - r \leq |\wp_i(t) - z| \leq \varepsilon + r\}} \mathrm{d}t.$$

As shown in [Jeg20a, Proposition 1.1], these local times are well-defined simultaneously for all z and ε. Recall that, in the current article, we consider Brownian motion with infinitesimal generator Δ instead of the standard Brownian motion considered in [Jeg20a, Jeg19] which has generator $\frac{1}{2}\Delta$. Because of this difference of normalisation, the local times defined above are 2 times smaller than the local times used in [Jeg20a, Jeg19].

This article will consider the following measures:

- $\mathcal{M}_a^{\wp_1 \cap \dots \cap \wp_n}$, $a \in (0,2)$: measure on a-thick points coming from the interaction of the n trajectories. Each trajectory is required to visit the thick point, but the way the thickness is distributed among the n trajectories is not specified. This measure is defined as the limit in probability, relatively to the topology of weak convergence, of

$$\mathcal{M}_a^{\wp_1 \cap \dots \cap \wp_n}(A) := \lim_{\varepsilon \to 0} |\log \varepsilon| \varepsilon^{-a} \int_A \mathbf{1}_{\left\{ \frac{1}{\varepsilon} \sum_{i=1}^n L^i_{x,\varepsilon} \geq a |\log \varepsilon|^2 \right\}} \mathbf{1}_{\left\{ \forall i = 1 \dots n, L^i_{x,\varepsilon} > 0 \right\}} \mathrm{d}x, \quad A \subset \mathbb{C} \text{ Borel}.$$

  See [Jeg19, Proposition 1.1].

- $\bigcap_{i=1}^n \mathcal{M}_{a_i}^{\wp_i}$, $\sum a_i < 2$: measure supported on the intersection of the support of each measure, the i-th trajectory is required to contribute exactly a_i to the overall thickness. It is defined by:

$$\bigcap_{i=1}^n \mathcal{M}_{a_i}^{\wp_i}(A) := \lim_{\varepsilon \to 0} |\log \varepsilon|^n \varepsilon^{-\sum a_i} \int_A \prod_{i=1}^n \mathbf{1}_{\left\{ \frac{1}{\varepsilon} L^i_{x,\varepsilon} \geq a_i |\log \varepsilon|^2 \right\}} \mathrm{d}x, \quad A \subset \mathbb{C} \text{ Borel},$$

  where the convergence holds in probability relatively to the topology of weak convergence. See [Jeg19, Section 1.4].

These two types of measures are closely related. Indeed, on the one hand, $\bigcap_{i=1}^n \mathcal{M}_{a_i}^{\wp_i}$ is the Brownian chaos measure $\mathcal{M}_{a_n}^{\wp_n}$ with reference measure $\bigcap_{i=1}^{n-1} \mathcal{M}_{a_i}^{\wp_i}$, i.e. $\bigcap_{i=1}^n \mathcal{M}_{a_i}^{\wp_i}$ is also equal to

$$\lim_{\varepsilon \to 0} |\log \varepsilon| \varepsilon^{-a_n} \mathbf{1}_{\left\{ \frac{1}{\varepsilon} L^n_{x,\varepsilon} \geq a_n |\log \varepsilon|^2 \right\}} \bigcap_{i=1}^{n-1} \mathcal{M}_{a_i}^{\wp_i}(dx). \tag{4.40}$$

See [Jeg19, Proposition 1.2 (ii)]. On the other hand, the following disintegration formula holds [Jeg19, Proposition 1.3]:

$$\mathcal{M}_a^{\wp_1 \cap \dots \cap \wp_n} = \int_{\mathsf{a} \in E(a,n)} \mathrm{d}\mathsf{a} \bigcap_{i=1}^n \mathcal{M}_{a_i}^{\wp_i} \tag{4.41}$$

showing that the thickness is uniformly distributed among $\wp_1, \dots, \wp_n$. In this formula and in the remaining of the article, we denote by $E(a,n)$ the $(n-1)$-dimensional simplex: for all $n \geq 1, a > 0$,

$$E(a,n) := \{ \mathsf{a} = (a_1, \dots, a_n) \in (0,a]^n : a_1 + \dots + a_n = a \}. \tag{4.42}$$

This disintegration formula allows us to naturally extend these definitions to "mixed" cases. For instance, for $a + a' < 2$, we define

$$\mathcal{M}_a^{\wp_1 \cap \dots \cap \wp_n} \cap \mathcal{M}_{a'}^{\wp_{n+1} \cap \dots \cap \wp_{n+m}} = \int_{\mathsf{a} \in E(a,n)} \mathrm{d}\mathsf{a} \int_{\mathsf{a}' \in E(a',m)} \mathrm{d}\mathsf{a}' \bigcap_{i=1}^{n+m} \mathcal{M}_{a_i}^{\wp_i}.$$

We finally explain a Girsanov-transform-type result associated to these measures, i.e. the way the law of the paths $\wp_i$ changes after shifting the probability measure by $\mathcal{M}_{a_i}^{\wp_i}(dz)$. For this purpose, we need to specify the laws of the trajectories $\wp_i$, $i \geq 1$. For all $i \geq 1$, let D_i be a bounded simply connected domain, let $x_i \in D$ and let $z_i \in \partial D_i$ be a point where the boundary D_i is locally analytic.

The independent trajectories $\wp_i, i \geq 1$ are then assumed to be Brownian paths from x_i to z_i in D_i, i.e. $\wp_i \sim \mu_{D_i}^{x_i,z_i}/H_{D_i}(x_i,z_i)$ (4.17). Let $n \geq 1$ and $a_i > 0, i = 1 \dots n$, be thickness parameters such that $\sum a_i < 2$.

We will see that this shift amounts to adding infinitely many excursions from z to z that are sampled according to a Poisson point process. Such excursions will play a prominent role in this paper and we define them now.

Notation 4.20. *We will denote by Ξ_a^z (or by $\Xi_a^{z,D}$ when we want to emphasise the dependence in the domain D) the random loop rooted at z, obtained by concatenating a Poisson point process of Brownian excursions from z to z of intensity $2\pi a \mu_D^{z,z}$ (4.15). Such a Poisson point process appears in the description of a Brownian trajectory seen from a typical a-thick point [BBK94, AHS20, Jeg20a]. We will denote by $\wedge$ the concatenation of paths.*

Recall also Notation 4.14. [Jeg19, Proposition 1.4] states that for all bounded measurable function F,

$$\mathbb{E}\left[\int_{\mathbb{C}} F(z,\wp_1,\dots,\wp_n) \bigcap_{i=1}^n \mathcal{M}_{a_i}^{\wp_i}(dz)\right] = (2\pi)^n \int_{\cap D_i} \Big(\prod_{i=1}^n \frac{H_{D_i}(z,z_i)}{H_{D_i}(x_i,z_i)} G_{D_i}(x_i,z)\, \mathrm{CR}(z,D_i)^{a_i} \Big) \times \mathbb{E}\left[F(z, \{\wp_{D_i}^{x_i,z} \wedge \Xi_{a_i}^{z,D_i} \wedge \wp_{D_i}^{z,z_i}\}_{i=1\dots n}) \right] \mathrm{d}z \tag{4.43}$$

where all the paths above are independent. The factor $(2\pi)^n$ is due to the different normalisations of the Green function in [Jeg19] and in the current paper. In words, after the shift, the path $\wp_i$ is distributed as the concatenation of three independent paths: a trajectory $\wp_{D_i}^{x_i,z}$ from x_i to z in D_i; a loop $\Xi_{a_i}^{z,D_i}$ rooted at z going infinitely many times through z; and a path $\wp_{D_i}^{z,z_i}$ from z to z_i. Such a description was already present in the paper [BBK94] in the context of one trajectory.

These results concern Brownian multiplicative chaos associated to independent Brownian trajectories from internal points to boundary points in fixed domains, but they can be extended to the loops in the Brownian loop soup. This will be made clear in Section 4.4.

Part One: Continuum

4.3 High level description of Proof of Theorem 4.1

In this section, we give a high-level description of the proof of Theorem 4.1. We start with the first moment computations for $\mathcal{M}_a^K$. As mentioned in the introduction the first moment is surprisingly explicit, which suggests that there is a certain amount of exact solvability or integrability in this approximation of the loop soup. Indeed we will see that the first moment is expressed in terms of Kummer's confluent hypergeometric function ${}_1F_1(\theta,1,\cdot)$ whose definition is recalled in (4.226) in Appendix 4.C. Recall also that $\mathrm{CR}(z,D)$ denotes the conformal radius of D seen from a point $z \in D$.

Proposition 4.21. *Define for all $u \geq 0$,*

$$\mathsf{F}(u) := \theta \int_0^u e^{-t} {}_1F_1(\theta,1,t)\mathrm{d}t \tag{4.44}$$

and for all $z \in D$,

$$C_K(z) := 2\pi(G_D - G_{D,K})(z,z) = 2\pi \int_0^\infty p_D(t,z,z)(1-e^{-Kt})\mathrm{d}t. \tag{4.45}$$

Then

$$\mathbb{E}[\mathcal{M}_a^K(dz)] = \frac{1}{a}\mathsf{F}\left(C_K(z)a\right)\mathrm{CR}(z,D)^a dz. \tag{4.46}$$

The function $C_K(z)$ plays a prominent role in the following; except for the factor of 2π in front, $C_K(z)$ corresponds to the Green function of loops that are killed and thus one may think of C_K (which also depends on D, even though D does not appear in the notation) as the covariance of the Gaussian field encoding the occupation measure of killed loops. Note that $C_K(z) < \infty$, so that this field is in fact defined pointwise.

Remark 4.22. In Lemma 4.32, we will obtain a more precise version of Proposition 4.21: we will get analogous (but more complicated) expressions when the underlying probability measure has been tilted by $\mathcal{M}_a^K(\mathrm{d}z)$, thereby showing a version of Theorem 4.8 valid even when $K < \infty$. This will then play a crucial role in second moment computations.

Proposition 4.21 allows us to compute asymptotics of the first moment in a relatively straightforward manner.

Lemma 4.23. *We have the following asymptotics:*

1. *There exists $C > 0$ such that for all $u > 0$,*

$$\mathsf{F}(u) \le C \begin{cases} u, & \text{if} \quad u \le 1, \\ u^\theta, & \text{if} \quad u \ge 1, \end{cases} \tag{4.47}$$

Moreover,

$$\lim_{u\to\infty} u^{-\theta}\mathsf{F}(u) = \frac{1}{\Gamma(\theta)}. \tag{4.48}$$

2.

$$\lim_{K\to\infty} \frac{C_K(z)}{\log K} = \frac{1}{2}. \tag{4.49}$$

We note that this justifies the normalisation $(\log K)^{-\theta}$ chosen in the statement of Theorem 4.1. Heuristically, (4.49) can be derived by noting that loops in (4.45) have a duration of order $1/K$ and hence a typical diameter of order $1/\sqrt{K}$, so that C_K corresponds roughly to the Green function G_D evaluated at points x, y separated by $\varepsilon = 1/\sqrt{K}$. Plugging this in (4.14) yields (4.49).

A crucial consequence of this explicit first moment is a positive martingale which plays a key role in our analysis. Recall that by (4.27), the collections $\mathcal{L}_D^\theta(K)$ are coupled on the same probability space for different values of K, and the set of K-killed loops increases with K. We will denote by $\mathcal{F}_K$ the σ-algebra generated by the K-killed loops.

Proposition 4.24. *Define a Borel measure $m_a^K(dz)$ as follows:*

$$m_a^K(dz) := \frac{1}{a^{1-\theta}}\mathrm{CR}(z,D)^a e^{-aC_K(z)}dz + \int_0^a \mathrm{d}\rho \frac{1}{(a-\rho)^{1-\theta}}\mathrm{CR}(z,D)^{a-\rho}e^{-(a-\rho)C_K(z)}\mathcal{M}_\rho^K(dz). \tag{4.50}$$

Then $(m_a^K(dz), K > 0)$ *is a* $(\mathcal{F}_K, K > 0)$*-martingale (that is,* $m_a^K(A)$ *is a martingale in that filtration, for any Borel set* $A \subset D$*).*

We mention that the measure m_a^K is well-defined since we show that the process $a \in (0,2) \mapsto \mathcal{M}_a^K$ is measurable relatively to the topology of weak convergence; see Definition 4.29 and the discussion below.

The proof of Proposition 4.24 will be given in Section 4.5.3 (see also Section 4.8 for an alternative proof). Intuitively (and as follows *a posteriori* from our results and Lévy's martingale convergence theorem), the measure on the left hand side corresponds to the conditional expectation of $\mathcal{M}_a$ given $\mathcal{F}_K$. To understand what the identity (4.50) expresses, or alternatively to motivate the definition of $m_a^K(dz)$, consider for simplicity of this discussion the special case $\theta = 1/2$ where we may use isomorphism theorems for clarity (Theorem 4.18). This conditional expectation should consist of two parts. The first part of the conditional expectation is given by thick points created only by the massive GFF with mass $\sqrt{K}$ (this is the first term in the right hand side). The second part is given by points whose thickness comes from a combination of the massive GFF *and* killed loops. The respective contribution to the overall thickness a of the point is arbitrary in the interval $[0,a]$, resulting in an integral. The variable $\rho \in [0,a]$ of integration corresponds to points which have a thickness of order ρ in the soup of killed loops, and a thickness $a-\rho$ in the massive GFF. This identity is therefore an analogue of Proposition 1.3 in [Jeg19] (see also (4.41)). The presence of the factor $1/(a-\rho)^{1-\theta}$ in front is not straightforward. A posteriori, it may be viewed as describing the "law" of this mixture of thicknesses. See Remark 4.36 for more discussion on this point.

We now assume the conclusion of Proposition 4.24 and see how the proof proceeds. Since $m_a^K(A) \geq 0$ for all Borel set A, we deduce that $(m_a^K, K > 0)$ converges almost surely for the topology of weak convergence towards a Borel measure m_a (see e.g. Section 6 of [Ber17]). We will show that except for a normalisation factor, this is the same as $\mathcal{M}_a$ in the statement of Theorem 4.1. To do this, the main step will be to show that when $K \to \infty$, the integral in the right hand side of (4.50) concentrates around the value $\rho = a$, so that m_a^K is in fact very close to $\mathcal{M}_a^K$ (up to a certain multiplicative constant). This is the content of the following proposition:

Proposition 4.25. *For all Borel set* $A \subset \mathbb{C}$*,*

$$\lim_{K \to \infty} \mathbb{E}\left[\left| m_a^K(A) - \frac{2^\theta \Gamma(\theta)}{(\log K)^\theta} \mathcal{M}_a^K(A) \right| \right] = 0. \tag{4.51}$$

The convergence of $((\log K)^{-\theta} \mathcal{M}_a^K, K > 0)$ follows directly from Propositions 4.24 and 4.25.

We now explain how Proposition 4.25 is obtained. The core of the proof, that we encapsulate in the following result, consists in controlling the oscillations of $\mathcal{M}_a^K$ with respect to the thickness parameter a.

Proposition 4.26. *Let* $a \in (0,2)$ *and* $A \Subset D$*. Then,*

$$\limsup_{\rho \to a} \limsup_{K \to \infty} \sup_f \frac{1}{\|f\|_\infty (\log K)^\theta} \mathbb{E}\left[\left| \int_D f(z) \mathcal{M}_a^K(dz) - \int_D f(z) \mathcal{M}_\rho^K(dz) \right| \right] = 0,$$

where the supremum runs over all bounded, non-zero, non-negative measurable function $f : D \to [0, \infty)$ with compact support included in A.

The proof of Proposition 4.26 will be given in Section 4.7. We now explain how to prove Proposition 4.25 assuming Proposition 4.26.

Proof of Proposition 4.25, assuming Proposition 4.26. Let $A \subset \mathbb{C}$ be a Borel set and for $\delta > 0$, define $A_\delta = A \cap \{z \in D : \mathrm{d}(z, D^c) > \delta\}$. Proposition 4.21 shows that

$$\lim_{\delta \to 0} \limsup_{K \to \infty} \mathbb{E}\left[\left| m_a^K(A) - m_a^K(A_\delta) \right| \right] = \lim_{\delta \to 0} \limsup_{K \to \infty} \frac{1}{(\log K)^\theta} \mathbb{E}\left[\left| \mathcal{M}_a^K(A) - \mathcal{M}_a^K(A_\delta) \right| \right] = 0.$$

Therefore, it is sufficient to show that for all $\delta > 0$,

$$\lim_{K \to \infty} \mathbb{E}\left[\left| m_a^K(A_\delta) - \frac{2^\theta \Gamma(\theta)}{(\log K)^\theta} \mathcal{M}_a^K(A_\delta) \right| \right] = 0.$$

In other words, we can assume that A is compactly included in D. It is then easy to see that one has the crude lower bound:

$$\inf_{z \in A} C_K(z) \geq c \log K. \tag{4.52}$$

(Indeed, if $z \in A_\delta$, then $C_K(z)$ is at least equal to the function $C_K(z)$ associated with a ball of radius δ around z, a quantity which in fact does not depend on z and whose asymptotics is given by Lemma 4.23). Let $\eta > 0$ be small. Proposition 4.21 implies that

$$\begin{aligned} &\mathbb{E}\left[\left| m_K^a(A) - \int_A \int_{a-\eta}^a \frac{\mathrm{d}\rho}{(a-\rho)^{1-\theta}} \mathrm{CR}(z, D)^{a-\rho} e^{-(a-\rho)C_K(z)} \mathcal{M}_\rho^K(dz) \right| \right] \\ &= \frac{1}{a^{1-\theta}} \int_A \mathrm{CR}(z, D)^a e^{-aC_K(z)} \mathrm{d}z + \int_A \int_0^{a-\eta} \frac{\mathrm{d}\rho}{(a-\rho)^{1-\theta}} \mathrm{CR}(z, D)^{a-\rho} e^{-(a-\rho)C_K(z)} \mathbb{E}\left[\mathcal{M}_\rho^K(dz) \right] \\ &= o(1) + \int_A \mathrm{d}z \, \mathrm{CR}(z, D)^a \int_0^{a-\eta} \frac{\mathrm{d}\rho}{\rho(a-\rho)^{1-\theta}} \mathsf{F}\left(C_K(z)\rho \right) e^{-(a-\rho)C_K(z)} \end{aligned} \tag{4.53}$$

as the first integral clearly converges to 0 when $K \to \infty$ using (4.52). Using (4.47) we can bound the second integral by

$$\begin{aligned} &C \int_A \mathrm{d}z \int_0^{a-\eta} \frac{\mathrm{d}\rho}{\rho(a-\rho)^{1-\theta}} \max\left(C_K(z)\rho, C_K(z)^\theta \rho^\theta \right) e^{-(a-\rho)C_K(z)} \\ &\leq C \sup_{z \in A} \{ e^{-\eta C_K(z)} \max(C_K(z), C_K(z)^\theta) \} |A| \int_0^a \frac{\mathrm{d}\rho}{\rho(a-\rho)^{1-\theta}} \max(\rho, \rho^\theta). \end{aligned}$$

The integral is in any case finite since $\theta > 0$ and does not depend on K. Since $C_K(z) \to \infty$, we deduce that the right hand side above tends to zero. Overall, we see that (4.53) tends to 0 as $K \to \infty$.

Hence

$$
\begin{aligned}
&\mathbb{E}\left[\left|m_a^K(A) - \frac{2^\theta \Gamma(\theta)}{(\log K)^\theta} \mathcal{M}_a^K(A)\right|\right] \\
&\leq o(1) + \mathbb{E}\left[\left|\int_A \int_{a-\eta}^a \frac{\mathrm{d}\rho}{(a-\rho)^{1-\theta}} \, \mathrm{CR}(z,D)^{a-\rho} e^{-(a-\rho)C_K(z)} \mathcal{M}_\rho^K(dz) - \frac{2^\theta \Gamma(\theta)}{(\log K)^\theta} \mathcal{M}_a^K(A)\right|\right] \\
&\leq o(1) + \int_{a-\eta}^a \frac{\mathrm{d}\rho}{(a-\rho)^{1-\theta}} \mathbb{E}\left[\left|\int_A \mathrm{CR}(z,D)^{a-\rho} e^{-(a-\rho)C_K(z)} (\mathcal{M}_\rho^K(dz) - \mathcal{M}_a^K(dz))\right|\right] \\
&+ \int_A \mathbb{E}\left[\mathcal{M}_a^K(dz)\right] \left|\int_{a-\eta}^a \frac{\mathrm{d}\rho}{(a-\rho)^{1-\theta}} \, \mathrm{CR}(z,D)^{a-\rho} e^{-(a-\rho)C_K(z)} - \frac{2^\theta \Gamma(\theta)}{(\log K)^\theta}\right|.
\end{aligned}
$$

To control the third term of the above sum, we recall that $\mathbb{E}\left[\mathcal{M}_a^K(dz)\right] \asymp (\log K)^\theta$ (by Proposition 4.21 and Lemma 4.23), and we make a change of variable $\mathrm{CR}(z,D)^{a-\rho} e^{-(a-\rho)C_K(z)} = e^{-t}$. So the third term is bounded by

$$
\begin{aligned}
&C \left|(\log K)^\theta \int_{a-\eta}^a \frac{\mathrm{d}\rho}{(a-\rho)^{1-\theta}} \, \mathrm{CR}(z,D)^{a-\rho} e^{-(a-\rho)C_K(z)} - 2^\theta \Gamma(\theta)\right| \\
&= \left|\left(\frac{\log K}{C_K(z) - \log \mathrm{CR}(z,D)}\right)^\theta \int_0^{\eta(C_K(z) - \log \mathrm{CR}(z,D))} \frac{\mathrm{d}t}{t^{1-\theta}} e^{-t} - 2^\theta \Gamma(\theta)\right|
\end{aligned}
$$

which goes to zero as $K \to \infty$, uniformly in $z \in A$ (see (4.49) and (4.220)). Therefore, the third term of the sum vanishes. To bound the second term we use Proposition 4.26 where the function f is taken to be $f(z) = \mathrm{CR}(z,D)^{a-\rho} e^{-(a-\rho)C_K(z)} \mathbf{1}_{\{z \in A\}}$ (this depends on K, but since the estimate in Proposition 4.26 is uniform, this is not a problem). We obtain that it is bounded by:

$$
o_\eta(1)(\log K)^\theta \int_{a-\eta}^a \frac{\mathrm{d}\rho}{(a-\rho)^{1-\theta}} e^{-c(a-\rho)\log K} \leq C o_\eta(1)
$$

where the term $o_\eta(1)$ can be made arbitrarily small by choosing η sufficiently close to zero, uniformly in K. To conclude, we have proven that

$$
\limsup_{K \to \infty} \mathbb{E}\left[\left|m_a^K(A) - \frac{2^\theta \Gamma(\theta)}{(\log K)^\theta} \mathcal{M}_a^K(A)\right|\right] \leq C o_\eta(1).
$$

Since the above left hand side term does not depend on η, by letting $\eta \to 0$, we deduce that it vanishes. This finishes the proof. □

The rest of Part One is organised as follows:

- Section 4.4: Brownian chaos measures were defined for Brownian trajectories killed upon exiting for the first time a given domain. This section explains how to transfer the definition to loops. This specific choice of definition is important for some proofs in subsequent sections.
- Section 4.5: We study the first moment of $\mathcal{M}_a^K$ and provide a Girsanov-type transform associated to $\mathcal{M}_a^K$ (Lemma 4.32). In particular, this gives an explicit expression for the first moment of $\mathcal{M}_a^K$.

The formula obtained is expressed as a complicated sum of convoluted integrals, but we show in Lemma 4.33 that it reduces to a very simple form as stated in Proposition 4.21 above. Finally, this first moment study culminates in Section 4.5.3 with a proof of the fact that $(m_a^K, K > 0)$ is a martingale.

- Section 4.6: We initiate the study of the second moment of $\mathcal{M}_a^K$ and give in Lemma 4.40 an exact expression for the second moment of the (two-point) rooted measure. The exact formula we obtain is arguably lengthy and the goal of Lemma 4.41 is to analyse its asymptotic behaviour. This section concludes the proof of Proposition 4.26 in the L^2-phase $\{a \in (0, 1)\}$.
- Section 4.7: This section aims to go beyond the L^2-phase to cover the whole subcritical regime $\{a \in (0, 2)\}$. To this end, we introduce a truncation requiring the number of crossings of dyadic annuli to remain below a certain curve. Adding this truncation does change the measure with high probability (Lemma 4.43) and turns the truncated measure bounded in L^2 (Lemma 4.44). The truncated measure is then shown to vary smoothly with respect to the thickness parameter (Lemma 4.45).
- Section 4.8: A proof of Theorem 4.8 is given. As a consequence of our approach, a new proof of Proposition 4.24 is given.
- Section 4.9: A proof that the limiting measure $\mathcal{M}_a$ is independent of the labels underlying the definition of the killing is given (Theorem 4.1, Point 2). We then show that the characterisation of the law of the couple $(\mathcal{L}_D^\theta, \mathcal{M}_a)$ given in Theorem 4.8 implies the conformal covariance of this couple (Theorem 4.55). Finally, the conformal covariance of the measure is shown to implies its almost sure positivity (Theorem 4.1, Point 1).
- Appendix 4.A: This section handles some technicalities concerning measurability of Brownian chaos measures w.r.t. starting points, ending points, domains and thickness levels.

4.4 Multiplicative chaos for finitely many loops

Brownian multiplicative chaos measures have been defined for Brownian trajectories confined to a given domain (for instance, killed upon exiting for the first time the domain). The purpose of this section is to explain that we can also define these measures for the loops coming from the Brownian loop soup. This is not a difficult task, but some proofs (not the results) in the subsequent sections depend on the precise definition that we will take.

The rough strategy is to cut the loops into two pieces for which we can define a Brownian chaos. We decided to do this by rooting the loops at the point with minimal imaginary part. We will restrict ourselves to loops with height larger than a given threshold ε and we first want to describe the law of this collection of loops. We start by introducing a few notations.

Notation 4.27. *For any $\wp \in \mathcal{L}_D^\theta$, we denote by*

$$\mathrm{mi}(\wp) := \inf\{\mathrm{Im}(\wp(t)) : t \in [0, T(\wp)]\}, \quad \mathrm{Mi}(\wp) := \sup\{\mathrm{Im}(\wp(t)) : t \in [0, T(\wp)]\}, \tag{4.54}$$

and

$$h(\wp) := \mathrm{Mi}(\wp) - \mathrm{mi}(\wp) \tag{4.55}$$

the height, or vertical displacement, of $\wp$. *We also write*

$$\mathrm{mi}(D) := \inf\{\mathrm{Im}(z) : z \in D\} \quad \textit{and} \quad \mathrm{Mi}(D) := \sup\{\mathrm{Im}(z) : z \in D\},$$

and for any real numbers $y < y'$,

$$\mathbb{H}_y := \{z \in \mathbb{C} : \mathrm{Im}(z) > y\} \quad \textit{and} \quad S_{y,y'} := \{z \in \mathbb{C} : y < \mathrm{Im}(z) < y'\}. \tag{4.56}$$

Consider now the collection of loops with height larger than some given $\varepsilon > 0$:

$$\mathcal{L}^\theta_{D,\varepsilon} := \{\wp \in \mathcal{L}^\theta_D : h(\wp) > \varepsilon\}. \tag{4.57}$$

In Lemma 4.28 below, we describe the law of $\mathcal{L}^\theta_{D,\varepsilon}$. To do that, for each $\wp \in \mathcal{L}^\theta_{D,\varepsilon}$, we will root $\wp$ at the unique point $z_\perp$ where the imaginary part of $\wp$ is at its minimum. We will then stop the loop when its height becomes for the first time larger than ε:

$$\tau_\varepsilon(\wp) := \inf\{t \in [0, T(\wp)] : \mathrm{Im}(\wp(t)) \geq \mathrm{mi}(\wp) + \varepsilon\}.$$

The loop will therefore be decomposed into two parts:

$$\wp_{\varepsilon,1} := (\wp(t))_{0 \leq t \leq \tau_\varepsilon} \quad \text{and} \quad \wp_{\varepsilon,2} := (\wp(t))_{\tau_\varepsilon \leq t \leq T(\wp)}. \tag{4.58}$$

By construction, $\wp_{\varepsilon,1}$ is an excursion from $z_\perp$ to $z_\varepsilon := \wp(\tau_\varepsilon)$ in the domain $D \cap S_{\mathrm{mi}(\wp),\mathrm{mi}(\wp)+\varepsilon}$ and $\wp_{\varepsilon,2}$ is an excursion from the internal point z_ε to the boundary point $z_\perp$ in the domain $D \cap \mathbb{H}_{\mathrm{mi}(\wp)}$. See Figure 4.1.

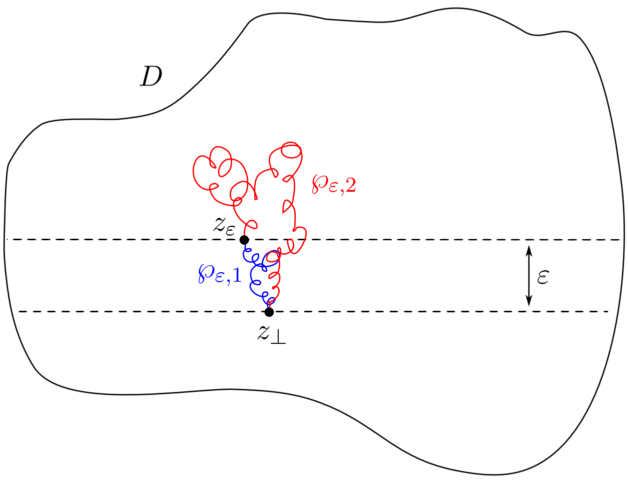


Figure 4.1: Rooting a loop at the point with minimal imaginary part.

We can now describe the law of $\mathcal{L}^\theta_{D,\varepsilon}$.

Lemma 4.28. *$\#\mathcal{L}^\theta_{D,\varepsilon}$ is a Poisson random variable with mean given by $\theta\mu_D^{\mathrm{loop}}(h(\wp)>\varepsilon)$, with*

$$\mu_D^{\mathrm{loop}}(h(\wp)>\varepsilon) = \int_{\mathrm{mi}(D)}^{\mathrm{Mi}(D)-\varepsilon} \mathrm{d}m \int_{D\cap(\mathbb{R}+im)} \mathrm{d}z_1 \int_{D\cap(\mathbb{R}+i(m+\varepsilon))} \mathrm{d}z_2 H_{D\cap S_{m,m+\varepsilon}}(z_1,z_2) H_{D\cap\mathbb{H}_m}(z_2,z_1),$$

where $H_{D\cap S_{m,m+\varepsilon}}(z_1,z_2)$ is a boundary Poisson kernel (4.20) in $D\cap S_{m,m+\varepsilon}(z_1,z_2)$ and $H_{D\cap\mathbb{H}_m}(z_2,z_1)$ is a Poisson kernel (4.18) in $D\cap\mathbb{H}_m$. Conditioned on $\{\#\mathcal{L}^\theta_{D,\varepsilon}=n\}$, $\mathcal{L}^\theta_{D,\varepsilon}$ is composed of n i.i.d. loops with common law given by

$$\mathbf{1}_{\{h(\cdot)>\varepsilon\}}\mu_D^{\mathrm{loop}}(\cdot)/\mu_D^{\mathrm{loop}}(\{\wp : h(\wp)>\varepsilon\}). \tag{4.59}$$

Moreover, if $\wp$ is distributed according to the law (4.59) above, then the law of $(z_\perp, z_\varepsilon, \wp_{1,\varepsilon}, \wp_{2,\varepsilon})$ is described as follows:

1. *Conditioned on $(z_\perp, z_\varepsilon)$ and denoting $m=\mathrm{Im}(z_\perp)$, $\wp_{1,\varepsilon}$ and $\wp_{2,\varepsilon}$ are two independent Brownian trajectories distributed according to*

$$\mu^{z_\perp,z_\varepsilon}_{D\cap S_{m,m+\varepsilon}}/H_{D\cap S_{m,m+\varepsilon}}(z_\perp,z_\varepsilon) \quad \textit{and} \quad \mu^{z_\varepsilon,z_\perp}_{D\cap H_m}/H_{D\cap H_m}(z_\varepsilon,z_\perp)$$

   respectively.

2. *The joint law of $(z_\perp, z_\varepsilon)$ is given by: for all bounded measurable function $F:\mathbb{C}^4\to\mathbb{R}$,*

$$\mathbb{E}\left[F(z_\perp,z_\varepsilon)\right] = \frac{1}{Z}\int_{\mathrm{mi}(D)}^{\mathrm{Mi}(D)-\varepsilon} \mathrm{d}m \int_{D\cap(\mathbb{R}+im)} \mathrm{d}z_1 \int_{D\cap(\mathbb{R}+i(m+\varepsilon))} \mathrm{d}z_2 \tag{4.60}$$
$$H_{D\cap S_{m,m+\varepsilon}}(z_1,z_2)H_{D\cap\mathbb{H}_m}(z_2,z_1)F(z_1,z_2)$$

Proof. Since D is bounded, we may assume without loss of generality that D is contained in the upper half-plane $\mathbb{H}=\mathbb{H}_0$. Next, we consider the measure on loops on $\mathbb{H}$, $\mu_{\mathbb{H}}^{\mathrm{loop}}$, and root the loops at their lowest imaginary part. According to [LW04, Proposition 7], $\mu_{\mathbb{H}}^{\mathrm{loop}}$ then disintegrates as

$$\int_0^{+\infty} \mathrm{d}m \int_{\mathbb{R}+im} \mathrm{d}z_1 \; \mu^{z_1,z_1}_{\mathbb{H}_m},$$

where $\mu^{z_1,z_1}_{\mathbb{H}_m}$ is given by (4.19). Further, a path γ under a measure $\mu^{z_1,z_1}_{\mathbb{H}_m}$ with $h(\wp)>\varepsilon$ can be decomposed as

$$\int \mathbf{1}_{\{h(\wp)>\varepsilon\}} F(\wp)\mu^{z_1,z_1}_{\mathbb{H}_m}(d\wp) = \int_{\mathbb{R}+i(m+\varepsilon)} \mathrm{d}z_2 \iint F(\wp_1\wedge\wp_2)\mu^{z_1,z_2}_{S_{m,m+\varepsilon}}(d\wp_1)\mu^{z_2,z_1}_{\mathbb{H}_m}(d\wp_2).$$

This is similar to decompositions appearing in [Law05, Section 5.2]. So one gets the lemma in the case of the upper half-plane $\mathbb{H}$. The case of a domain $D\subset\mathbb{H}$ can be obtained by using the restriction property (4.23). Indeed, given $z_1\in D\cap(\mathbb{R}+im)$ and $z_2\in D\cap(\mathbb{R}+i(m+\varepsilon))$, we have that

$$\mu^{z_1,z_2}_{D\cap S_{m,m+\varepsilon}}(d\wp_1) = \mathbf{1}_{\{\wp_1 \text{ stays in } D\}}\mu^{z_1,z_2}_{S_{m,m+\varepsilon}}(d\wp_1), \qquad \mu^{z_2,z_1}_{D\cap\mathbb{H}_m}(d\wp_2) = \mathbf{1}_{\{\wp_2 \text{ stays in } D\}}\mu^{z_2,z_1}_{\mathbb{H}_m}(d\wp_2). \quad \square$$

From this lemma, it becomes clear that we can define Brownian chaos associated to the loops as soon as we are able to define it for independent Brownian trajectories with random domains, starting points and ending points. We explain this carefully in Appendix 4.A; see especially Lemma 4.56. We can now give a precise definition of Brownian multiplicative chaos associated to the loops in $\mathcal{L}_D^\theta(K)$. We start by fixing $\varepsilon > 0$. For any $\wp \in \mathcal{L}_{D,\varepsilon}^\theta$, we denote by $\wp_{2,\varepsilon}$ the second part of the trajectory defined in (4.58). Thanks to Lemmas 4.28 and 4.56, we can define

$$\mathcal{M}_a^{K,\varepsilon} := \sum_{n \geq 1} \frac{1}{n!} \sum_{\substack{\wp^{(1)}, \dots, \wp^{(n)} \in \mathcal{L}_{D,\varepsilon}^\theta \cap \mathcal{L}_D^\theta(K) \\ \forall i \neq j, \wp^{(i)} \neq \wp^{(j)}}} \mathcal{M}_a^{\wp_{2,\varepsilon}^{(1)} \cap \dots \cap \wp_{2,\varepsilon}^{(n)}}. \tag{4.61}$$

Definition 4.29. *$\mathcal{M}_a^K$ is defined as being the nondecreasing limit of $\mathcal{M}_a^{K,\varepsilon}$ as $\varepsilon \to 0$.*

This procedure not only defines $\mathcal{M}_a^K$ for a fixed a, but defines it as a measurable process, viewed as a function of $a \in (0,2)$, relatively to the topology of weak convergence. Indeed, Lemma 4.56 gives not only the measurability of the measures w.r.t. the starting points, ending points and domains, but also w.r.t. the thickness level a. This justifies for instance that the martingale m_a^K, defined in Proposition 4.24, is well-defined.

4.5 First moment computations and rooted measure

The goal of this section will be to give a proof of Proposition 4.21 and Proposition 4.24. We will also state and prove in Lemma 4.32 a generalisation of Proposition 4.21, which describes the law of the loop soup after reweighting by our measure $\mathcal{M}_a^K(dz)$ (4.3).

4.5.1 Preliminaries

We will consider a finite number of Brownian-like trajectories $\wp_1, \dots, \wp_n$ and consider their distribution seen from a typical thick point z generated by the interaction of the n trajectories.

Recall Definition 4.15 where admissible functions are defined. We also recall that Ξ_a^z denotes the loop rooted at z obtained by gluing a Poisson point process of Brownian excursions from z to z with intensity measure $2\pi a \mu_D^{z,z}$ (4.15). The goal of this section is to prove:

Lemma 4.30. *For any $n \geq 1$ and any nonnegative measurable function F which is admissible,*

$$\int \mu_D^{\mathrm{loop}}(d\wp_1) \dots \mu_D^{\mathrm{loop}}(d\wp_n) F(z, \wp_1, \dots, \wp_n) \mathcal{M}_a^{\wp_1 \cap \dots \cap \wp_n}(dz)$$
$$= \mathrm{CR}(z,D)^a \int_{\mathsf{a} \in E(a,n)} \frac{\mathrm{d}\mathsf{a}}{a_1 \dots a_n} \mathbb{E}\left[F(z, \Xi_{a_1}^z, \dots, \Xi_{a_n}^z)\right] \mathrm{d}z, \tag{4.62}$$

where $(\Xi_{a_i}^z)_{1 \leq i \leq n}$ are independent.

In particular, note that when $n = 1$ the expected mass of the Brownian chaos generated by a single loop coming from the Brownian loop soup is finite; however this becomes infinite as soon as $n \geq 2$.

Before starting the proof of this lemma, we point out that the emergence of the process Ξ_a^z can be guessed (at least in the case $\theta = 1/2$) thanks to isomorphisms theorems (from [ALS20, Proposition 3.9], but see also Corollary 4.10) in which the Gaussian free field has nonzero boundary conditions.

We also comment on the method of proof. A natural approach to this lemma would be to exploit the identity (4.24) which relates the loop measure μ_D^{loop} in terms of excursion measures $\mu_D^{z,z}$, and then to approximate these excursion measures $\mu_D^{z,z}$ by the more well-behaved $\mu_D^{z,w}$, then letting $w \to z$. Indeed, Girsanov-type transforms of chaos measures associated to trajectories sampled according to $\mu_D^{z,w}/G_D(z,w)$ have been obtained in [AHS20], and would lead (formally) relatively quickly and painlessly to formulae such as (4.62).

Unfortunately this appealing approach suffers from a subtle but serious technical drawback, which is that this does not tie in well with our chosen definition for $\mathcal{M}_a^{\wp_1 \cap \dots \cap \wp_n}$ in Section 4.4. The issue is that it is not obvious that the chaos measures associated to excursions to soups of excursions sampled from $\mu^D(z,w)$ converge to the chaos measure $\mathcal{M}_a^{\wp_1 \cap \dots \cap \wp_n}$ defined in Section 4.4. Even if such a convergence could be proved (so that one might take this as the definition of $\mathcal{M}_a^{\wp_1 \cap \dots \cap \wp_n}$) it would not be clear that the limit would be measurable with respect to the collection of loops $\wp_1, \dots, \wp_n$. Unfortunately this measurability is a crucial feature, and so a different route must be taken. The approach we use in Section 4.4 does not suffer from this problem: indeed, although the idea is here again to reduce the loops to excursions, these excursions are measurably defined from $\wp_1, \dots, \wp_n$.

The proof of Lemma 4.30 below may therefore at first sight look a little unnatural and somewhat mysterious: the idea is to start from the answer (i.e., from the right-hand side of (4.62)), write down the explicit law of the decomposition of each loop in Ξ_a^z into excursions according to their point with lowest imaginary part (this is the content of Lemma 4.31), and check that this agrees after simplifications with the left hand side of (4.62).

Lemma 4.31. *Let $z \in D$, $a > 0$ and F be a nonnegative measurable function which is admissible. Then, $\mathbb{E}\left[F(\Xi_a^{z,D})\right]$ is equal to*

$$2\pi a \int_{\mathrm{mi}(D)}^{\mathrm{Im}(z)} \mathrm{d}m \int_{(\mathbb{R}+im)\cap D} \mathrm{d}z_\perp \frac{\mathrm{CR}(z, D \cap \mathbb{H}_m)^a}{\mathrm{CR}(z,D)^a} H_{D\cap\mathbb{H}_m}(z,z_\perp)^2 \mathbb{E}\left[F(\wp_{D\cap\mathbb{H}_m}^{z,z_\perp} \wedge \wp_{D\cap\mathbb{H}_m}^{z_\perp,z} \wedge \Xi_a^{z,D\cap\mathbb{H}_m})\right]$$

where the loops $\wp_{D\cap\mathbb{H}_m}^{z,z_\perp}$, $\wp_{D\cap\mathbb{H}_m}^{z_\perp,z}$ and $\Xi_a^{z,D\cap\mathbb{H}_m}$ are independent and distributed as in Notations 4.20 and 4.14.

In words, this lemma states that the point $z_\perp$ of $\Xi_a^{z,D}$ with minimal imaginary part has a density with respect to Lebesgue measure given by the above expression. Moreover, the law of $\Xi_a^{z,D}$ conditionally on $z_\perp \in \mathbb{R} + im$ is given by the concatenation of two independents paths: the original path Ξ_a^z in the smaller domain $D \cap \mathbb{H}_m$ and a loop $\wp_{D\cap\mathbb{H}_m}^{z,z_\perp} \wedge \wp_{D\cap\mathbb{H}_m}^{z_\perp,z}$ in $D \cap \overline{\mathbb{H}}_m$ joining z and $z_\perp$. We point out that it is not immediately obvious that the right hand side defines a probability law (i.e., is equal to 1 when $F \equiv 1$) but this can be seen directly using variational considerations on the conformal radius of z in $D \cap \mathbb{H}_m$ as m varies.

Proof of Lemma 4.31. By density-type arguments, we can assume that F is continuous (recall that the topology on the space of continuous paths is the one associated to the distance d_{paths} (4.28)).

We first observe that it is enough to prove Lemma 4.31 in the case of the upper half plane $\mathbb{H}$. Indeed, let us assume that the result holds in that case and let D be a bounded simply connected domain. By translating D if necessary, we can assume that D is contained in $\mathbb{H}$. It is an easy computation to show the result for D from the result for $\mathbb{H}$ as soon as we know the following two restriction properties:

$$\mathbb{E}\left[F(\Xi_a^{z,\mathbb{H}})\mathbf{1}_{\{\Xi_a^{z,\mathbb{H}}\subset D\}}\right] = \frac{\mathrm{CR}(z,D)^a}{\mathrm{CR}(z,\mathbb{H})^a}\mathbb{E}\left[F(\Xi_a^{z,D})\right] \tag{4.63}$$

and for any $m > \mathrm{mi}(D)$ and $z_\perp \in (\mathbb{R}+im)\cap D$,

$$\mathbb{E}\left[F(\wp_{\mathbb{H}_m}^{z,z_\perp}\wedge\wp_{\mathbb{H}_m}^{z_\perp,z})\mathbf{1}_{\{\wp_{\mathbb{H}_m}^{z,z_\perp}\wedge\wp_{\mathbb{H}_m}^{z_\perp,z}\subset D\}}\right] = \frac{H_{D\cap\mathbb{H}_m}(z,z_\perp)^2}{H_{\mathbb{H}_m}(z,z_\perp)^2}\mathbb{E}\left[F(\wp_{D\cap\mathbb{H}_m}^{z,z_\perp}\wedge\wp_{D\cap\mathbb{H}_m}^{z_\perp,z})\right]. \tag{4.64}$$

(4.64) is a mere reformulation of the restriction property (4.16) on measures. To conclude the transfer of the result to general domains, let us prove (4.63). It turns out that it is also a consequence of (4.16). Indeed, by continuity of F,

$$\mathbb{E}\left[F(\Xi_a^{z,\mathbb{H}})\mathbf{1}_{\{\Xi_a^{z,\mathbb{H}}\subset D\}}\right] = \lim_{w\to\infty} e^{-2\pi a G_\mathbb{H}(z,w)}\sum_{n\geq 0}\frac{(2\pi G_\mathbb{H}(z,w))^n}{n!}\mathbb{E}\left[F(\wp_{\mathbb{H},1}^{z,w}\wedge\dots\wedge\wp_{\mathbb{H},n}^{z,w})\mathbf{1}_{\{\forall i=1\dots n,\wp_{\mathbb{H},i}^{z,w}\subset D\}}\right]$$

where $\wp_{\mathbb{H},i}^{z,w}$, $i=1\dots n$, are i.i.d. and distributed according to (4.21). By the restriction property (4.16), we further have

$$\begin{aligned}\mathbb{E}\left[F(\Xi_a^{z,\mathbb{H}})\mathbf{1}_{\{\Xi_a^{z,\mathbb{H}}\subset D\}}\right] &= \lim_{w\to\infty} e^{-2\pi a G_\mathbb{H}(z,w)}\sum_{n\geq 0}\frac{(2\pi G_D(z,w))^n}{n!}\mathbb{E}\left[F(\wp_{D,1}^{z,w}\wedge\dots\wedge\wp_{D,n}^{z,w})\right]\\ &= \left(\lim_{w\to\infty} e^{-2\pi a(G_\mathbb{H}(z,w)-G_D(z,w))}\right)\mathbb{E}\left[F(\Xi_a^{z,D})\right] = \frac{\mathrm{CR}(z,D)^a}{\mathrm{CR}(z,\mathbb{H})^a}\mathbb{E}\left[F(\Xi_a^{z,D})\right]\end{aligned}$$

This shows (4.63).

The rest of the proof is dedicated to showing Lemma 4.31 in the case of the upper half plane $\mathbb{H}$. By continuity of F, we have

$$\mathbb{E}\left[F(\Xi_a^{z,\mathbb{H}})\right] = \lim_{w\to z} e^{-2\pi a G_\mathbb{H}(z,w)}\sum_{n\geq 1}\frac{(2\pi a G_\mathbb{H}(z,w))^n}{n!}\mathbb{E}\left[F(\wp_{\mathbb{H},1}^{z,w}\wedge\dots\wedge\wp_{\mathbb{H},n}^{z,w})\right]. \tag{4.65}$$

By symmetry,

$$\mathbb{E}\left[F(\wp_{\mathbb{H},1}^{z,w}\wedge\dots\wedge\wp_{\mathbb{H},n}^{z,w})\right] = \mathbb{E}\left[F(\wp_{\mathbb{H},1}^{z,w}\wedge\dots\wedge\wp_{\mathbb{H},n}^{z,w})|\forall i=1\dots n-1, \mathrm{mi}(\wp_{\mathbb{H},n}^{z,w}) < \mathrm{mi}(\wp_{\mathbb{H},i}^{z,w})\right]. \tag{4.66}$$

To make the n trajectories independent, we will condition further on $\min_{i=1\dots n}\mathrm{mi}(\wp_{\mathbb{H},i}^{z,w})$. Let us first compute its distribution. For all $m\in(0,\mathrm{Im}(z))$, we have

$$\mathbb{P}\left(\min_{i=1\dots n}\mathrm{mi}(\wp_{\mathbb{H},i}^{z,w}) > m\right) = \mathbb{P}\left(\wp_\mathbb{H}^{z,w}\subset\mathbb{H}_m\right)^n = G_{\mathbb{H}_m}(z,w)^n G_\mathbb{H}(z,w)^{-n}.$$

The Green function in the upper half plane is explicit and is equal to

$$G_\mathbb{H}(z,w) = \frac{1}{2\pi}\log\frac{|z-\bar{w}|}{|z-w|},\quad G_{\mathbb{H}_m}(z,w) = \frac{1}{2\pi}\log\frac{|z-\bar{w}-2im|}{|z-w|}.$$

By differentiating w.r.t. m, we deduce that the density of $\min_{i=1\dots n} \mathrm{mi}(\wp_{\mathbb{H},i}^{z,w})$ is given by

$$\frac{n}{\pi} \frac{G_{\mathbb{H}_m}(z,w)^{n-1}}{G_{\mathbb{H}}(z,w)^n} \frac{\mathrm{Im}(z-\bar{w})-2m}{|z-\bar{w}-2im|^2} \mathrm{d}m.$$

We now want to expand (4.66). Conditioned on $\mathrm{mi}(\wp_{\mathbb{H},n}^{z,w}) = \min_{i=1\dots n} \mathrm{mi}(\wp_{\mathbb{H},i}^{z,w}) = m$, the n trajectories are independent with the following distributions: the first $n-1$ trajectories are trajectories from z to w in $\mathbb{H}_m$ with law $\mu_{\mathbb{H}_m}^{z,w}/G_{\mathbb{H}_m}(z,w)$ and the last trajectory $\wp_{\min}$ which reaches the lowest level is distributed as follows:

$$\mathbb{E}\left[f(\wp_{\min})\right] = \frac{1}{Z_m(z,w)} \int_{\mathbb{R}+im} \mathrm{d}z_\perp H_{\mathbb{H}_m}(z,z_\perp) H_{\mathbb{H}_m}(w,z_\perp) \mathbb{E}\left[f(\wp_{\mathbb{H}_m}^{z,z_\perp} \wedge \wp_{\mathbb{H}_m}^{z_\perp,w})\right].$$

In the above equation, $Z_m(z,w)$ is the normalising constant

$$Z_m(z,w) = \int_{\mathbb{R}+im} H_{\mathbb{H}_m}(z,z_\perp) H_{\mathbb{H}_m}(w,z_\perp) \mathrm{d}z_\perp.$$

Overall, this shows that

$$\mathbb{E}\left[F(\wp_{\mathbb{H},1}^{z,w} \wedge \dots \wedge \wp_{\mathbb{H},n}^{z,w})\right] = \frac{n}{\pi} \frac{G_{\mathbb{H}_m}(z,w)^{n-1}}{G_{\mathbb{H}}(z,w)^n} \int_0^{\mathrm{Im}(z)} \mathrm{d}m \frac{\mathrm{Im}(z-\bar{w})-2m}{|z-\bar{w}-2im|^2} \frac{1}{Z_m(z,w)}$$
$$\times \int_{\mathbb{R}+im} \mathrm{d}z_\perp H_{\mathbb{H}_m}(z,z_\perp) H_{\mathbb{H}_m}(z,w) \mathbb{E}\left[F(\wp_{\mathbb{H}_m,1}^{z,w} \wedge \dots \wedge \wp_{\mathbb{H}_m,n-1}^{z,w} \wedge \wp_{\mathbb{H}_m}^{z,z_\perp} \wedge \wp_{\mathbb{H}_m}^{z_\perp,w})\right].$$

Plugging this back in (4.65), we have

$$\mathbb{E}\left[F(\Xi_a^{z,\mathbb{H}})\right] = 2a \lim_{w\to z} e^{-2\pi a G_{\mathbb{H}}(z,w)} \sum_{n\geq 1} \frac{(2\pi a G_{\mathbb{H}_m}(z,w))^{n-1}}{(n-1)!} \int_0^{\mathrm{Im}(z)} \mathrm{d}m \frac{\mathrm{Im}(z-\bar{w})-2m}{|z-\bar{w}-2im|^2} \frac{1}{Z_m(z,w)}$$
$$\times \int_{\mathbb{R}+im} \mathrm{d}z_\perp H_{\mathbb{H}_m}(z,z_\perp) H_{\mathbb{H}_m}(z,w) \mathbb{E}\left[F(\wp_{\mathbb{H}_m,1}^{z,w} \wedge \dots \wedge \wp_{\mathbb{H}_m,n-1}^{z,w} \wedge \wp_{\mathbb{H}_m}^{z,z_\perp} \wedge \wp_{\mathbb{H}_m}^{z_\perp,w})\right]$$
$$= 2a \lim_{w\to z} e^{-2\pi a (G_{\mathbb{H}}(z,w) - G_{\mathbb{H}_m}(z,w))} \int_0^{\mathrm{Im}(z)} \mathrm{d}m \frac{\mathrm{Im}(z-\bar{w})-2m}{|z-\bar{w}-2im|^2} \frac{1}{Z_m(z,w)}$$
$$\times \int_{\mathbb{R}+im} \mathrm{d}z_\perp H_{\mathbb{H}_m}(z,z_\perp) H_{\mathbb{H}_m}(z,w) \mathbb{E}\left[F(\Xi_a^{(z,w),\mathbb{H}_m} \wedge \wp_{\mathbb{H}_m}^{z,z_\perp} \wedge \wp_{\mathbb{H}_m}^{z_\perp,w})\right]$$

where in the last line we wrote $\Xi_a^{(z,w),\mathbb{H}_m}$ for a trajectory which consists in the concatenation (at z say) of all the excursion in a Poisson point process with intensity $2\pi a \mu_{\mathbb{H}_m}^{z,w}$. At this stage, it is not a loop, but it converges to $\Xi_a^{z,\mathbb{H}_m}$ as $w \to z$. We are now ready to take the limit $w \to z$. Firstly,

$$e^{-2\pi a (G_{\mathbb{H}}(z,w) - G_{\mathbb{H}_m}(z,w))} \to \mathrm{CR}(z,\mathbb{H}_m)^a / \mathrm{CR}(z,\mathbb{H})^a.$$

Secondly, since the Poisson kernel is explicit in the upper half plane

$$H_{\mathbb{H}_m}(z,z_\perp) = \frac{1}{\pi} \frac{\mathrm{Im}(z)-m}{|z-z_\perp|^2},$$

we can compute

$$\lim_{w\to z} Z_m(z,w) = \frac{1}{\pi^2}\int_{\mathbb{R}} \frac{(\mathrm{Im}(z)-m)^2}{(x^2+(\mathrm{Im}(z)-m)^2)^2}\mathrm{d}x = \frac{1}{\pi^2}\frac{1}{\mathrm{Im}(z)-m}\int_{\mathbb{R}}\frac{1}{(x^2+1)^2}\mathrm{d}x = \frac{1}{2\pi}\frac{1}{\mathrm{Im}(z)-m}.$$

Therefore, as $w\to z$, we have

$$\frac{\mathrm{Im}(z-\bar{w})-2m}{|z-\bar{w}-2im|^2}\frac{1}{Z_m(z,w)} \to \pi$$

By dominated convergence theorem, we obtain that

$$\mathbb{E}\left[F(\Xi_a^{z,\mathbb{H}})\right] = 2\pi a\frac{\mathrm{CR}(z,\mathbb{H}_m)^a}{\mathrm{CR}(z,\mathbb{H})^a}\int_0^{\mathrm{Im}(z)}\mathrm{d}m\int_{\mathbb{R}+im}\mathrm{d}z_\perp H_{\mathbb{H}_m}(z,z_\perp)^2\mathbb{E}\left[F(\Xi_a^{z,\mathbb{H}_m}\wedge\wp_{\mathbb{H}_m}^{z,z_\perp}\wedge\wp_{\mathbb{H}_m}^{z_\perp,z})\right]$$

which concludes the proof. □

Proof of Lemma 4.30. By density-type arguments, we can assume that F is continuous. By definition, we can rewrite the left hand side of (4.62) as

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\int \mu_D^{\mathrm{loop}}(d\wp^1)\dots\mu_D^{\mathrm{loop}}(d\wp^n)\mathbf{1}_{\{\forall i=1\dots\varepsilon,h(\wp^i)>\varepsilon\}}F(z,\wp^1,\dots,\wp^n)\mathcal{M}_a^{\wp^1_{\varepsilon,2}\cap\dots\cap\wp^n_{\varepsilon,2}}(dz)\\
&=\lim_{\varepsilon\to 0}\mu_D^{\mathrm{loop}}(h(\wp)>\varepsilon)^n\mathbb{E}\left[F(z,\wp^1_{\varepsilon,2},\dots,\wp^n_{\varepsilon,2})\mathcal{M}_a^{\wp^1_{\varepsilon,2}\cap\dots\cap\wp^n_{\varepsilon,2}}(dz)\right]
\end{aligned}$$

where in the second line, $\wp^i_{\varepsilon,2}$, $i=1\dots n$, are i.i.d. trajectories with law (4.59) described in Lemma 4.28. Note also that in the second line we used the continuity of F and the fact that the first portion of the trajectory $\wp_{\varepsilon,1}$ vanishes as $\varepsilon\to 0$. We are going to expand this expression with the help of Lemma 4.28. The term $\mu_D^{\mathrm{loop}}(h(\wp)>\varepsilon)$ and the partition function Z in (4.60) will cancel out and we obtain that the left hand side of (4.62) is equal to (we write below with some abuse of notation a product of integrals instead of multiple integrals)

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\prod_{i=1}^n\int_{\mathrm{mi}(D)}^{\mathrm{Mi}(D)-\varepsilon}\mathrm{d}m^i\int_{(\mathbb{R}+im^i)\cap D}\mathrm{d}z_\perp^i\int_{(\mathbb{R}+i(m^i+\varepsilon))\cap D}\mathrm{d}z_\varepsilon^i H_{S_{m^i,m^i+\varepsilon}}(z_\perp^i,z_\varepsilon^i)H_{D\cap\mathbb{H}_{m^i}}(z_\varepsilon^i,z_\perp^i) \qquad (4.67)\\
&\times\mathbb{E}\left[F(z,(\wp_{D\cap\mathbb{H}_{m^i}}^{z_\varepsilon^i,z_\perp^i})_{i=1\dots n})\mathcal{M}_a^{\cap\wp_{D\cap\mathbb{H}_{m^i}}^{z_\varepsilon^i,z_\perp^i}}(dz)\right].
\end{aligned}$$

The trajectories $\wp_{D\cap\mathbb{H}_{m^i}}^{z_\varepsilon^i,z_\perp^i}$ are independent Brownian trajectories with law as in (4.21). By (4.43), the last expectation above is equal to

$$\begin{aligned}
&(2\pi)^n\int_{\mathsf{a}\in E(a,n)}\mathrm{d}\mathsf{a}\prod_{i=1}^n\mathrm{CR}(z,D\cap\mathbb{H}_{m^i})^{a_i}G_{D\cap\mathbb{H}_{m^i}}(z_\varepsilon^i,z)\frac{H_{D\cap\mathbb{H}_{m^i}}(z,z_\perp^i)}{H_{D\cap\mathbb{H}_{m^i}}(z_\varepsilon^i,z_\perp^i)} \qquad (4.68)\\
&\times\mathbb{E}\left[F(z,(\wp_{D\cap\mathbb{H}_{m^i}}^{z_\varepsilon^i,z}\wedge\Xi_{a_i}^{z,D\cap\mathbb{H}_{m^i}}\wedge\wp_{D\cap\mathbb{H}_{m^i}}^{z,z_\perp^i})_{i=1\dots n})\right]dz
\end{aligned}$$

where all the trajectories above are independent. When $\varepsilon\to 0$, $z_\varepsilon^i\to z_\perp^i$ and it is easy to see that $\wp_{D\cap\mathbb{H}_{m^i}}^{z_\varepsilon^i,z}\wedge\wp_{D\cap\mathbb{H}_{m^i}}^{z,z_\perp^i}$ converges in distribution to a loop $\wp_{D\cap\mathbb{H}_{m^i}}^{z_\perp^i,z}\wedge\wp_{D\cap\mathbb{H}_{m^i}}^{z,z_\perp^i}$ that is the concatenation of two independent paths distributed as in Notations 4.14. This loop will play the role of the loop whose imaginary part reaches the minimum among all loops in $\Xi_{a_i}^{z,\mathbb{H}}$ (see Lemma 4.31). Coming back to (4.67) and (4.68), we see that the Poisson kernels $H_{D\cap\mathbb{H}_{m^i}}(z_\varepsilon^i, z_\perp^i)$ appearing in both equations cancel out. Noticing that as soon as $\mathrm{Im}(z) > m^i + \varepsilon$,

$$\int_{(\mathbb{R}+i(m^i+\varepsilon))\cap D} H_{S_{m^i,m^i+\varepsilon}}(z_\perp^i, z_\varepsilon^i) G_{D\cap\mathbb{H}_{m^i}}(z_\varepsilon^i, z) \mathrm{d}z_\varepsilon^i = H_{D\cap\mathbb{H}_{m^i}}(z, z_\perp^i),$$

we overall obtain that the left hand side of (4.62) is equal to

$$(2\pi)^n \int_{\mathsf{a}\in E(a,n)} \mathrm{d}\mathsf{a} \prod_{i=1}^n \int_{\mathrm{mi}(D)}^{\mathrm{Im}(z)} \mathrm{d}m^i \int_{(\mathbb{R}+im^i)\cap D} \mathrm{d}z_\perp^i \, \mathrm{CR}(z, D\cap\mathbb{H}_{m^i})^{a_i} H_{D\cap\mathbb{H}_{m^i}}(z, z_\perp^i)^2$$
$$\times \mathbb{E}\left[F(z, (\wp_{D\cap\mathbb{H}_{m^i}}^{z_\perp^i, z} \wedge \wp_{D\cap\mathbb{H}_{m^i}}^{z, z_\perp^i} \wedge \Xi_a^{z, D\cap\mathbb{H}_{m^i}})_{i=1\ldots n}) \right].$$

Lemma 4.31 identifies this last expression with the right hand side of (4.62). This concludes the proof. □

4.5.2 First moment (Girsanov transform)

We now start the proof of Proposition 4.21 as well as describing the way the loop soup changes when one shifts the probability measure by $\mathcal{M}_a^K(dz)$. The following result is the analogue of Theorem 4.8 at the approximation level. It is a quick consequence of Lemma 4.30.

Lemma 4.32. *For any bounded measurable admissible function F,*

$$\mathbb{E}\left[F(z, \mathcal{L}_D^\theta) \mathcal{M}_a^K(dz) \right] =$$
$$\mathrm{CR}(z, D)^a \sum_{n\geq 1} \frac{\theta^n}{n!} \int_{\mathsf{a}\in E(a,n)} \frac{\mathrm{d}\mathsf{a}}{a_1 \ldots a_n} \mathbb{E}\left[\prod_{i=1}^n \left(1 - e^{-KT(\Xi_{a_i}^z)}\right) F(z, \mathcal{L}_D^\theta \cup \{\Xi_{a_i}^z, i = 1\ldots n\}) \right] dz,$$

where $(\Xi_{a_i}^z)_{1\leq i\leq n}$ are independent and independent of $\mathcal{L}_D^\theta$.

Proof of Lemma 4.32. By definition of $\mathcal{M}_a^K$ in (4.3) and monotone convergence, we want to compute

$$\mathbb{E}\left[\sum_{\substack{\wp_1,\ldots,\wp_n\in\mathcal{L}_D^\theta \\ \forall i\neq j, \wp_i\neq\wp_j}} \prod_{i=1}^n \left(1 - e^{-KT(\wp_i)}\right) F(z, \mathcal{L}_D^\theta) \mathcal{M}_a^{\wp_1\cap\cdots\cap\wp_n}(dz) \right]. \tag{4.69}$$

By Palm's formula applied to the Poisson point process $\mathcal{L}_D^\theta$, we can rewrite (4.69) as

$$\theta^n \int \mu_D^{\mathrm{loop}}(d\wp_1) \ldots \mu_D^{\mathrm{loop}}(d\wp_n) \prod_{i=1}^n \left(1 - e^{-KT(\wp_i)}\right) \mathbb{E}_{\mathcal{L}_D^\theta}\left[F(z, \mathcal{L}_D^\theta \cup \{\wp_1, \ldots, \wp_n\}) \right] \mathcal{M}_a^{\wp_1\cap\cdots\cap\wp_n}(dz).$$

By Lemma 4.30, this is equal to

$$\mathrm{CR}(z, D)^a \theta^n \int_{\mathsf{a}\in E(a,n)} \frac{\mathrm{d}\mathsf{a}}{a_1 \ldots a_n} \mathbb{E}\left[\prod_{i=1}^n \left(1 - e^{-KT(\Xi_{a_i}^z)}\right) F(z, \mathcal{L}_D^\theta \cup \{\Xi_{a_i}^z, i = 1\ldots n\}) \right] dz.$$

This concludes the proof of Lemma 4.32. □

We will get Proposition 4.21 simply by taking a function F depending only on z in Lemma 4.32. Before this, we first state a lemma which shows that (somewhat miraculously, in our opinion) the integrals appearing in Lemma 4.32 can be computed explicitly in terms of hypergeometric functions; this is where the function F comes from in our results.

Lemma 4.33. *The function* F *defined in* (4.44) *can be expressed as follows: for all* $u \geq 0$*,*

$$\mathsf{F}(u) = \sum_{n\geq 1} \frac{1}{n!}\theta^n \int_{\mathsf{a}\in E(1,n)} \frac{\mathrm{d}a_1 \dots \mathrm{d}a_{n-1}}{a_1 \dots a_n} \prod_{i=1}^n \left(1 - e^{-ua_i}\right). \tag{4.70}$$

Proof of Lemma 4.33. For all $u \geq 0$, let $\hat{\mathsf{F}}(u)$ denote the right hand side of (4.70). We will show that $\hat{\mathsf{F}}(u) = \mathsf{F}(u)$. Note that we have

$$\frac{\mathrm{d}}{\mathrm{d}u} \prod_{i=1}^n (1 - e^{-ua_i}) = \sum_{j=1}^n e^{-ua_j} \prod_{i\neq j}(1 - e^{-ua_i})$$

and by symmetry we deduce that

$$\begin{aligned}
\hat{\mathsf{F}}'(u) &= \theta e^{-u} + \sum_{n\geq 2} \frac{\theta^n}{(n-1)!} \int_{a_i>0, a_1+\dots+a_{n-1}<1} \frac{\mathrm{d}a_1 \dots \mathrm{d}a_{n-1}}{a_1 \dots a_{n-1}} e^{-u(1-(a_1+\dots+a_{n-1}))} \prod_{i=1}^{n-1}(1 - e^{-ua_i}) \\
&= \theta e^{-u} \left(1 + \sum_{n\geq 2} \frac{\theta^{n-1}}{(n-1)!} \int_{a_i>0, a_1+\dots+a_{n-1}<1} \frac{\mathrm{d}a_1 \dots \mathrm{d}a_{n-1}}{a_1 \dots a_{n-1}} \prod_{i=1}^{n-1} (e^{ua_i} - 1) \right).
\end{aligned} \tag{4.71}$$

Differentiating further,

$$\begin{aligned}
\frac{1}{\theta}\frac{\mathrm{d}}{\mathrm{d}u}(e^u \hat{\mathsf{F}}'(u)) &= \theta \sum_{n\geq 2} \frac{\theta^{n-2}}{(n-2)!} \int_{a_i>0, a_1+\dots+a_{n-2}<1} \frac{\mathrm{d}a_1 \dots \mathrm{d}a_{n-2}}{a_1 \dots a_{n-2}} \prod_{i=1}^{n-2} (e^{ua_i} - 1) \\
&\qquad \times \int_0^{1-(a_1+\dots+a_{n-2})} e^{ua_{n-1}} \mathrm{d}a_{n-1} \\
&= \frac{\theta e^u}{u} \sum_{n\geq 2} \frac{\theta^{n-2}}{(n-2)!} \int_{a_i>0, a_1+\dots+a_{n-2}<1} \frac{\mathrm{d}a_1 \dots \mathrm{d}a_{n-2}}{a_1 \dots a_{n-2}} \prod_{i=1}^{n-2} (1 - e^{-ua_i}) \\
&\quad - \frac{\theta}{u} \sum_{n\geq 2} \frac{\theta^{n-2}}{(n-2)!} \int_{a_i>0, a_1+\dots+a_{n-2}<1} \frac{\mathrm{d}a_1 \dots \mathrm{d}a_{n-2}}{a_1 \dots a_{n-2}} \prod_{i=1}^{n-2} (e^{ua_i} - 1).
\end{aligned}$$

By (4.71), we see that the second term in the right hand side is equal to $-e^u \hat{\mathsf{F}}'(u)/u$. We now define the function $\mathsf{G}(u)$ to be the first term in the right hand side, multiplied by ue^{-u}/θ. Thus we have

$$\frac{1}{\theta} e^u (\hat{\mathsf{F}}'(u) + \hat{\mathsf{F}}''(u)) = \frac{\theta e^u}{u} \mathsf{G}(u) - \frac{e^u}{u} \hat{\mathsf{F}}'(u). \tag{4.72}$$

We further have

$$\begin{aligned}
\mathsf{G}'(u) &= \theta \sum_{n\geq 3} \frac{\theta^{n-3}}{(n-3)!} \int_{a_i>0, a_1+\dots+a_{n-3}<1} \frac{\mathrm{d}a_1 \dots \mathrm{d}a_{n-3}}{a_1 \dots a_{n-3}} \prod_{i=1}^{n-3} (1-e^{-ua_i}) \\
&\qquad \times \int_0^{1-(a_1+\dots+a_{n-3})} e^{-ua_{n-2}} \mathrm{d}a_{n-2} \\
&= \frac{\theta}{u} \sum_{n\geq 3} \frac{\theta^{n-3}}{(n-3)!} \int_{a_i>0, a_1+\dots+a_{n-3}<1} \frac{\mathrm{d}a_1 \dots \mathrm{d}a_{n-3}}{a_1 \dots a_{n-3}} \prod_{i=1}^{n-3} (1-e^{-ua_i}) \\
&- \frac{\theta e^{-u}}{u} \sum_{n\geq 3} \frac{\theta^{n-3}}{(n-3)!} \int_{a_i>0, a_1+\dots+a_{n-3}<1} \frac{\mathrm{d}a_1 \dots \mathrm{d}a_{n-3}}{a_1 \dots a_{n-3}} \prod_{i=1}^{n-3} (e^{ua_i}-1) \\
&= \frac{\theta}{u}\mathsf{G}(u) - \frac{1}{u}\hat{\mathsf{F}}'(u)
\end{aligned}$$

by definition of G and (4.71). Reformulating,

$$\frac{\mathrm{d}}{\mathrm{d}u}\left(\frac{\mathsf{G}(u)}{u^\theta}\right) = \frac{\mathsf{G}'(u) - \theta u^{-1}\mathsf{G}(u)}{u^\theta} = -\frac{\hat{\mathsf{F}}'(u)}{u^{\theta+1}}.$$

Thanks to (4.72), we deduce that

$$\frac{1}{\theta}\frac{\mathrm{d}}{\mathrm{d}u}\left(u^{1-\theta}(\hat{\mathsf{F}}'(u) + \hat{\mathsf{F}}''(u))\right) = -\frac{1}{u^\theta}\hat{\mathsf{F}}''(u)$$

and

$$(1-\theta)\hat{\mathsf{F}}'(u) + \hat{\mathsf{F}}''(u) + u(\hat{\mathsf{F}}''(u) + \hat{\mathsf{F}}'''(u)) = 0.$$

By looking at the solutions of this equation (see [AS84, Section 13.1]), we deduce that there exist $c_1, c_2 \in \mathbb{R}$ such that

$$\hat{\mathsf{F}}'(u) = c_1 e^{-u} U(\theta, 1, u) + c_2 e^{-u} {}_1F_1(\theta, 1, u)$$

where $U(\theta, 1, u) = \frac{1}{\Gamma(\theta)} \int_0^\infty e^{-ut} t^{\theta-1}(1+t)^{-\theta} \mathrm{d}t$ is Tricomi's confluent hypergeometric function and ${}_1F_1(\theta, 1, u) = \sum_{n=0}^\infty \frac{\theta(\theta+1)\dots(\theta+n-1)}{n!^2} u^n$ is Kummer's confluent hypergeometric function. With (4.71), we see that $\hat{\mathsf{F}}'(u) \to \theta$ as $u \to 0$. Hence $c_1 = 0$ and $c_2 = \theta$. We have proven that

$$\hat{\mathsf{F}}'(u) = \theta e^{-u} {}_1F_1(\theta, 1, u)$$

and, therefore, $\hat{\mathsf{F}} = \mathsf{F}$. This concludes the proof of Lemma 4.33. □

We can now conclude with a proof of Proposition 4.21.

Proof of Proposition 4.21. By Lemma 4.32 applied to the function $F = F(z)$ depending only on z, and by doing the change of variable $b_i = a_i/a$, we have

$$\mathbb{E}[\mathcal{M}_a^K(dz)] = \frac{1}{a}\sum_{n\geq 1} \frac{\theta^n}{n!} \int_{\mathsf{b}\in E(1,n)} \frac{\mathrm{d}\mathsf{b}}{b_1 \dots b_n} \prod_{i=1}^n \mathbb{E}\left[1 - e^{-KT(\Xi^z_{a\cdot b_i})}\right] \mathrm{CR}(z, D)^a \mathrm{d}z.$$

By Palm's formula and by recalling the definition (4.45) of $C_K(z)$, we have

$$\mathbb{E}\left[1 - e^{-KT(\Xi^z_{a\cdot b_i})}\right] = 1 - \exp\left(2\pi a \cdot b_i \int_0^\infty p_D(t,z,z)(e^{-Kt} - 1)dt\right) = 1 - \exp\left(-C_K(z) a \cdot b_i\right). \quad (4.73)$$

With Lemma 4.33, we conclude that

$$\mathbb{E}[\mathcal{M}_a^K(dz)] = \frac{1}{a}\mathsf{F}(C_K(z)a)\,\mathrm{CR}(z,D)^a \mathrm{d}z.$$

This concludes the proof. □

4.5.3 The crucial martingale

We now turn to the proof of Proposition 4.24. We will see that it is the consequence of the following two lemmas. We will first state these two lemmas, then show how they imply Proposition 4.24, and then prove the two lemmas.

The first lemma shows that the function F, defined in (4.44) and appearing in the first moment of $\mathcal{M}_a^K$, solves some integral equation. As we will see, this equation is precisely what is required in order to show that the expectation of the martingale is constant.

Lemma 4.34. *For all $a \geq 0$ and $v \geq 0$,*

$$\int_0^a \frac{\mathrm{d}\rho}{\rho(a-\rho)^{1-\theta}} e^{\rho v}\mathsf{F}(\rho v) + \frac{1}{a^{1-\theta}} = \frac{e^{av}}{a^{1-\theta}}. \quad (4.74)$$

Let $K' < K$. The second lemma expresses the measure $\mathcal{M}_a^K$ in terms of $\mathcal{M}_\rho^{K'}$, $\rho \in (0,a)$. Denote by $\mathcal{M}_a^{K,K'}$ the measure on a-thick points of loops in $\mathcal{L}_D^\theta(K)\backslash\mathcal{L}_D^\theta(K')$, i.e.

$$\mathcal{M}_a^{K,K'} := \sum_{n\geq 1} \frac{1}{n!} \sum_{\substack{\wp_1,\dots,\wp_n \in \mathcal{L}_D^\theta(K)\backslash\mathcal{L}_D^\theta(K') \\ \forall i\neq j, \wp_i \neq \wp_j}} \mathcal{M}_a^{\wp_1\cap\dots\cap\wp_n}.$$

For any $\rho \in (0,a)$, denote also $\mathcal{M}_{a-\rho}^{K,K'} \cap \mathcal{M}_\rho^{K'}$ the measure on thick points, where the total thickness a comes from a combination of loops in $\mathcal{L}_D^\theta(K) \setminus \mathcal{L}_D^\theta(K')$ (with thickness $a-\rho$) and loops in $\mathcal{L}_D^\theta(K')$ (with thickness ρ). More precisely,

$$\mathcal{M}_{a-\rho}^{K,K'} \cap \mathcal{M}_\rho^{K'} := \sum_{n,m\geq 1} \frac{1}{n!m!} \sum_{\substack{\wp_1,\dots,\wp_n \in \mathcal{L}_D^\theta(K)\backslash\mathcal{L}_D^\theta(K') \\ \forall i\neq j, \wp_i \neq \wp_j}} \sum_{\substack{\wp'_1,\dots,\wp'_m \in \mathcal{L}_D^\theta(K') \\ \forall i\neq j, \wp'_i \neq \wp'_j}} \mathcal{M}_{a-\rho}^{\wp_1\cap\dots\cap\wp_n} \cap \mathcal{M}_\rho^{\wp'_1\cap\dots\cap\wp'_m},$$

where $\mathcal{M}_{a-\rho}^{\wp_1\cap\dots\cap\wp_n} \cap \mathcal{M}_\rho^{\wp'_1\cap\dots\cap\wp'_m}$ is defined in Section 4.2.3. We recall that $\mathcal{M}_{a-\rho}^{K,K'} \cap \mathcal{M}_\rho^{K'}$ may be viewed as the Brownian chaos generated by $\mathcal{M}_{a-\rho}^{K,K'}$ with respect to an intensity measure σ, which is itself an (independent) Brownian chaos generated by $\mathcal{M}_\rho^{K'}$; see (4.40).

We claim:

Lemma 4.35. *Let $K' < K$. We can decompose*

$$\mathcal{M}_a^K = \mathcal{M}_a^{K,K'} + \mathcal{M}_a^{K'} + \int_0^a \mathrm{d}\rho \, \mathcal{M}_{a-\rho}^{K,K'} \cap \mathcal{M}_\rho^{K'}. \tag{4.75}$$

Remark 4.36. By taking $K \to \infty$, and writing K instead of K', it should be possible to deduce from Lemma 4.35 and from our results, a posteriori, that we have an identity of the type:

$$\nu_a^K + \int_0^a \nu_{a-\rho}^K \cap \mathcal{M}_\rho^K = \nu_a. \tag{4.76}$$

Here, the measure ν_a^K, is (informally) the uniform measure on thick points of the non-killed loop soup, and $\nu_{a-\rho}^K \cap \mathcal{M}_\rho^K$ would be a uniform measure on thick points created by both measures; both would need to be defined carefully. One should further expect that ν_a^K coincides with the exponential measure on such points except for a factor of the form $1/a^{1-\theta}$ (this can heuristically be understood in the case $\theta = 1/2$ as coming from the tail of the Gaussian distribution).

Accepting the above, we see that (4.76) is consistent with the martingale in Proposition 4.24. The identity (4.76) is in fact what motivated us to define the martingale in Proposition 4.24.

Let us see how Proposition 4.24 follows from Lemmas 4.34 and 4.35.

Proof of Proposition 4.24. Let $K' < K$. We first note that

$$\mathbb{E}\left[\mathcal{M}_a^{K,K'}(dz)\right] = \frac{1}{a}\mathsf{F}\left(aC_K(z) - aC_{K'}(z)\right) e^{-aC_{K'}(z)} \, \mathrm{CR}(z,D)^a \mathrm{d}z. \tag{4.77}$$

Indeed, the only difference with the expectation of $\mathcal{M}_a^K$ is that loops are required to survive the K'-killing, so that $1 - e^{-KT(\wp)}$ is replaced by $e^{-K'T(\wp)} - e^{-KT(\wp)}$ and we find that

$$\mathbb{E}\left[\mathcal{M}_a^{K,K'}(dz)\right] = \sum_{n\geq 1} \frac{\theta^n}{n!} \int_{\mathsf{a} \in E(a,n)} \mathrm{d}\mathsf{a} \prod_{i=1}^n \frac{e^{-a_i C_{K'}(z)} - e^{-a_i C_K(z)}}{a_i} \mathrm{CR}(z,D)^a \mathrm{d}z.$$

(4.77) then follows by factorising by $\prod_i e^{-a_i C_{K'}(z)} = e^{-aC_{K'}(z)}$ and by Lemma 4.33.

By (4.75) and properties of the intersection measure (in particular (1.6) in [Jeg19]), we have

$$\begin{aligned}
\mathbb{E}\left[\mathcal{M}_\rho^K(dz)|\mathcal{F}_{K'}\right] &= \mathcal{M}_\rho^{K'} + \mathbb{E}\left[\mathcal{M}_\rho^{K,K'}\right] + \int_0^\rho \mathrm{d}\beta \, \mathbb{E}\left[\mathcal{M}_{\rho-\beta}^{K,K'}(z)\right] \mathcal{M}_\beta^{K'}(dz) \\
&= \mathcal{M}_\rho^{K'} + \frac{1}{\rho}\mathsf{F}\left(\rho C_K(z) - \rho C_{K'}(z)\right) e^{-\rho C_{K'}(z)} \, \mathrm{CR}(z,D)^\rho \mathrm{d}z \\
&+ \int_0^\rho \mathrm{d}\beta \frac{1}{\rho - \beta} \mathsf{F}\left((\rho-\beta)(C_K(z) - C_{K'}(z))\right) e^{-(\rho-\beta)C_{K'}(z)} \, \mathrm{CR}(z,D)^{\rho-\beta} \mathcal{M}_\beta^{K'}(dz).
\end{aligned}$$

Hence the conditional expectation $\mathbb{E}\left[m_a^K|\mathcal{F}_{K'}\right]$ is equal to

$$
\begin{aligned}
&a^{\theta-1}\,\mathrm{CR}(z,D)^a e^{-aC_K(z)}dz + \int_0^a \mathrm{d}\rho \frac{1}{(a-\rho)^{1-\theta}}\,\mathrm{CR}(z,D)^{a-\rho}e^{-(a-\rho)C_K(z)}\mathcal{M}_\rho^{K'}(dz) \qquad (4.78)\\
&+\int_0^a \frac{\mathrm{d}\rho}{\rho(a-\rho)^{1-\theta}}\,\mathrm{CR}(z,D)^a e^{-(a-\rho)C_K(z)-\rho C_{K'}(z)}\mathsf{F}\left(\rho C_K(z)-\rho C_{K'}(z)\right)dz\\
&+\int_0^a \mathrm{d}\rho \int_0^\rho \mathrm{d}\beta \frac{1}{(\rho-\beta)(a-\rho)^{1-\theta}}\,\mathrm{CR}(z,D)^{a-\beta}e^{-(a-\rho)C_K(z)-(\rho-\beta)C_{K'}(z)}\\
&\qquad\qquad \times \mathsf{F}\left((\rho-\beta)(C_K(z)-C_{K'}(z))\right)\mathcal{M}_\beta^{K'}(dz).
\end{aligned}
$$

By Lemma 4.34 (with $v=C_K(z)-C_{K'}(z)$), the sum of the first and third terms in (4.78) is equal to

$$
\frac{1}{a^{1-\theta}}\,\mathrm{CR}(z,D)^a e^{-aC_{K'}(z)}dz.
$$

On the other hand, by exchanging the two integrals, the fourth term is equal to

$$
\begin{aligned}
&\int_0^a \mathrm{d}\beta\,\mathrm{CR}(z,D)^{a-\beta}e^{-(a-\beta)C_K(z)}\mathcal{M}_\beta^{K'}(dz)\\
&\quad\times \int_\beta^a \frac{\mathrm{d}\rho}{(\rho-\beta)(a-\rho)^{1-\theta}}e^{(\rho-\beta)(C_K(z)-C_{K'}(z))}\mathsf{F}\left((\rho-\beta)(C_K(z)-C_{K'}(z))\right).
\end{aligned}
$$

By Lemma 4.34, the integral with respect to ρ is equal to

$$
\frac{e^{(a-\beta)(C_K(z)-C_{K'}(z))}-1}{(a-\beta)^{1-\theta}}
$$

implying that the fourth term of (4.78) is equal to

$$
\begin{aligned}
&\int_0^a \frac{\mathrm{d}\beta}{(a-\beta)^{1-\theta}}\,\mathrm{CR}(z,D)^{a-\beta}e^{-(a-\beta)C_{K'}(z)}\mathcal{M}_\beta^{K'}(dz)\\
&-\int_0^a \frac{\mathrm{d}\beta}{(a-\beta)^{1-\theta}}\,\mathrm{CR}(z,D)^{a-\beta}e^{-(a-\beta)C_K(z)}\mathcal{M}_\beta^{K'}(dz).
\end{aligned}
$$

This second integral cancels with the second term of (4.78). Overall, this shows that

$$
\begin{aligned}
&\mathbb{E}\left[m_a^K(dz)|\mathcal{F}_{K'}\right]\\
&=\frac{1}{a^{1-\theta}}\,\mathrm{CR}(z,D)^a e^{-aC_{K'}(z)}dz + \int_0^a \frac{\mathrm{d}\beta}{(a-\beta)^{1-\theta}}\,\mathrm{CR}(z,D)^{a-\beta}e^{-(a-\beta)C_{K'}(z)}\mathcal{M}_\beta^{K'}(dz)\\
&=m_a^{K'}(dz).
\end{aligned}
$$

This concludes the proof of Proposition 4.24. □

The rest of the section is devoted to the proofs of Lemmas 4.34 and 4.35. We start with Lemma 4.34.

Proof of Lemma 4.34. By doing the change of variable $\rho = a\beta$, it is enough to show that

$$\int_0^1 \mathrm{d}\beta \frac{1}{\beta(1-\beta)^{1-\theta}} e^{av\beta} \mathsf{F}(av\beta) = e^{av} - 1.$$

Recall that for all $u \geq 0$,

$$\mathsf{F}(u) = \theta \int_0^u \mathrm{d}t \; e^{-t} \sum_{n=0}^{\infty} \frac{\theta^{(n)}}{n!^2} t^n$$

where we have let

$$\theta^{(0)} := 1 \qquad \text{and} \qquad \theta^{(n)} := \theta(\theta+1)\dots(\theta+n-1), \quad n \geq 1. \tag{4.79}$$

By exchanging the integral and the sum, we find that for all $u \geq 0$,

$$\mathsf{F}(u) = \theta \sum_{n=0}^{\infty} \frac{\theta^{(n)}}{n!} \left(1 - e^{-u} \sum_{k=0}^{n} \frac{u^k}{k!} \right) = \theta e^{-u} \sum_{n=0}^{\infty} \frac{\theta^{(n)}}{n!} \sum_{k=n+1}^{\infty} \frac{u^k}{k!}.$$

Hence

$$\int_0^1 \mathrm{d}\beta \frac{1}{\beta(1-\beta)^{1-\theta}} e^{av\beta} \mathsf{F}(av\beta) = \theta \sum_{n=0}^{\infty} \frac{\theta^{(n)}}{n!} \sum_{k=n+1}^{\infty} \frac{(av)^k}{k!} \int_0^1 \mathrm{d}\beta \frac{\beta^{k-1}}{(1-\beta)^{1-\theta}}.$$

Now, by (4.222), for all $k \geq 1$,

$$\int_0^1 \mathrm{d}\beta \frac{\beta^{k-1}}{(1-\beta)^{1-\theta}} = \frac{(k-1)!\Gamma(\theta)}{\Gamma(k+\theta)} = \frac{(k-1)!}{\theta^{(k)}},$$

which implies that

$$\begin{aligned} \int_0^1 \mathrm{d}\beta \frac{1}{\beta(1-\beta)^{1-\theta}} e^{av\beta} \mathsf{F}(av\beta) &= \theta \sum_{n=0}^{\infty} \frac{\theta^{(n)}}{n!} \sum_{k=n+1}^{\infty} \frac{1}{k.\theta^{(k)}} (av)^k \\ &= \theta \sum_{k=1}^{\infty} \frac{1}{k.\theta^{(k)}} (av)^k \sum_{n=0}^{k-1} \frac{\theta^{(n)}}{n!}. \end{aligned}$$

Furthermore, we can easily show by induction that

$$\sum_{n=0}^{k-1} \frac{\theta^{(n)}}{n!} = \frac{1}{\theta} \frac{\theta^{(k)}}{(k-1)!}.$$

We can thus conclude that

$$\int_0^1 \mathrm{d}\beta \frac{1}{\beta(1-\beta)^{1-\theta}} e^{av\beta} \mathsf{F}(av\beta) = \sum_{k=1}^{\infty} \frac{1}{k!} (av)^k = e^{av} - 1$$

as desired. □

We now turn to the proof of Lemma 4.35.

Proof of Lemma 4.35. We have

$$\begin{aligned}\mathcal{M}_a^K &= \sum_{n\geq 1} \frac{1}{n!} \sum_{\wp_1 \neq \ldots \neq \wp_n \in \mathcal{L}_D^\theta(K)} \mathcal{M}_a^{\wp_1 \cap \cdots \cap \wp_n} \\ &= \sum_{n\geq 1} \frac{1}{n!} \sum_{k=0}^n \binom{n}{k} \sum_{\substack{\wp_1 \neq \ldots \neq \wp_k \in \mathcal{L}_D^\theta(K') \\ \wp_{k+1} \neq \ldots \neq \wp_n \in \mathcal{L}_D^\theta(K) \setminus \mathcal{L}_D^\theta(K')}} \mathcal{M}_a^{\wp_1 \cap \cdots \cap \wp_n}\end{aligned}$$

The terms $k=0$ and $k=n$ give rise to $\mathcal{M}_a^{K,K'}$ and $\mathcal{M}_a^{K'}$ respectively. By Proposition 1.3 in [Jeg19] (applied to Brownian loops instead of Brownian motions, although as explained in Section 4.A this is justified), we can disintegrate

$$\mathcal{M}_a^{\wp_1 \cap \cdots \cap \wp_n} = \int_0^a \mathrm{d}\rho \, \mathcal{M}_\rho^{\wp_1 \cap \cdots \cap \wp_k} \cap \mathcal{M}_{a-\rho}^{\wp_{k+1} \cap \cdots \cap \wp_n}.$$

Therefore, letting m be $n-k$,

$$\begin{aligned}&\mathcal{M}_a^K = \mathcal{M}_a^{K,K'} + \mathcal{M}_a^{K'} \\ &+ \int_0^a \mathrm{d}\rho \Big(\sum_{k\geq 1} \tfrac{1}{k!} \sum_{\wp_1 \neq \ldots \neq \wp_k \in \mathcal{L}_D^\theta(K')} \mathcal{M}_\rho^{\wp_1 \cap \cdots \cap \wp_k} \Big) \cap \Big(\sum_{m\geq 1} \tfrac{1}{m!} \sum_{\wp_1 \neq \ldots \neq \wp_m \in \mathcal{L}_D^\theta(K) \setminus \mathcal{L}_D^\theta(K')} \mathcal{M}_{a-\rho}^{\wp_1 \cap \cdots \cap \wp_m} \Big) \\ &= \mathcal{M}_a^{K,K'} + \mathcal{M}_a^{K'} + \int_0^a \mathrm{d}\rho \, \mathcal{M}_\rho^{K'} \cap \mathcal{M}_{a-\rho}^{K,K'}.\end{aligned}$$

This concludes the proof of Lemma 4.35. □

4.6 Second moment computations and multi-point rooted measure

The goal of this section is to initiate the study of the second moment.

4.6.1 Preliminaries

We start off by giving the analogue of Lemma 4.30 in the second moment case. A new process of excursions will come into play, which we describe now. We first introduce the following special function:

$$\mathsf{B}(u) := \sum_{k\geq 1} \frac{u^k}{k!(k-1)!}, \quad u \geq 0. \tag{4.80}$$

B can be expressed in terms of the modified Bessel function of the first kind I_1 (see (4.223)), but it is more convenient to give a name to the function B instead of I_1, since it comes up in many places below.

Let $z, z' \in D$ be two distinct points and let $a, a' > 0$. We consider the cloud (meaning the point process) of excursions $\Xi_{a,a'}^{z,z'}$ such that for all $k \geq 1$,

$$\mathbb{P}\left(\#\Xi_{a,a'}^{z,z'} = 2k \right) = \frac{1}{\mathsf{B}\Big((2\pi)^2 a a' G_D(z,z')^2\Big)} \frac{(2\pi\sqrt{aa'} G_D(z,z'))^{2k}}{k!(k-1)!}, \tag{4.81}$$

and conditionally on $\{\#\Xi_{a,a'}^{z,z'} = 2k\}$, $\Xi_{a,a'}^{z,z'}$ is composed of $2k$ independent and identically distributed excursions from z to z', with common law $\mu_D^{z,z'}/G_D(z,z')$ (4.15). Note that $\Xi_{a,a'}^{z,z'}$ is **not** a Poisson point process of excursions, since (4.81) is not the Poisson distribution. However, one can see that it becomes asymptotically Poisson (conditioned to be even), in the limit when $z \to z'$. This fact will not be needed in what follows but is useful to guide the intuition. The parity condition implicit in (4.81) is crucial, since it allows us to combine these excursions into loops that visit both z and z'.

Recall the notion of admissible functions intoduced in Definition 4.15 and also Notation 4.20 where the loops Ξ_a^z are defined.

Lemma 4.37. *Let $z, z' \in D$. Let $0 < a, a' < 2$. Let $n, m \geq 1$, $l \in \{0, \dots, n \wedge m\}$ and $F = F(z, z', \wp_1, \dots, \wp_n, \wp'_{l+1}, \dots, \wp'_m)$ be a bounded measurable admissible function of two points and $n+m-l$ loops. We have*

$$\begin{aligned}
&\int \mu_D^{\mathrm{loop}}(d\wp_1)\dots\mu_D^{\mathrm{loop}}(d\wp_n)\mu_D^{\mathrm{loop}}(d\wp'_{l+1})\dots\mu_D^{\mathrm{loop}}(d\wp'_m) \\
&\quad \mathcal{M}_a^{\wp_1\cap\dots\wp_n}(dz)\mathcal{M}_{a'}^{\wp_1\cdots\cap\wp_l\cap\wp'_{l+1}\cap\dots\wp'_m}(dz')F(z,z',\wp_1,\dots,\wp_n,\wp'_{l+1},\dots,\wp'_m) \\
&= \mathrm{CR}(z,D)^a\,\mathrm{CR}(z',D)^{a'}\int_{\substack{\mathsf{a}\in E(a,n)\\ \mathsf{a}'\in E(a',m)}} \frac{\mathrm{d}\mathsf{a}}{a_1\dots a_n}\frac{\mathrm{d}\mathsf{a}'}{a'_1\dots a'_m}\prod_{i=1}^{l}\mathsf{B}\left((2\pi)^2 a_i a'_i G_D(z,z')^2\right) \\
&\quad \times \mathbb{E}\left[F\Big(z,z',\{\Xi_{a_i,a'_i}^{z,z'}\wedge\Xi_{a_i}^z\wedge\Xi_{a'_i}^{z'}\}_{i=1}^l;\{\Xi_{a_i}^z\}_{i=l+1}^n;\{\Xi_{a'_i}^{z'}\}_{i=l+1}^m\Big)\right]\mathrm{d}z\mathrm{d}z'
\end{aligned} \tag{4.82}$$

where all the random variables appearing above are independent and $\wedge$ denotes concatenation in some order (the precise order does not matter by admissibility).

Before we start with the proof of this lemma, we make a few comments on its meaning. Note that in the left hand side, we can think of z and z' respectively as having been sampled from Brownian chaos measures associated with loops which can overlap: namely, $\wp_1, \dots, \wp_l$ are common to both collections. The right hand side expresses the law that results from this conditioning (or more precisely reweighting): we get not only the Poisson point processes of excursions $\Xi_{a_i}^z$ and $\Xi_{a'_i}^{z'}$ which already appeared in Lemma 4.30, but also an independent *non-Poissonian* collection of excursions joining z and z' with law given by (4.81).

We encapsulate the heart of the proof Lemma 4.37 in Lemma 4.38 below. For $a, a' \in (0,2)$, $z \in D$, let $\mathcal{M}_{a'}^{\Xi_a^z}$ denote the measure on a'-thick points generated by the loop Ξ_a^z (recall Notation 4.20). More precisely,

$$\mathcal{M}_{a'}^{\Xi_a^z} := \sum_{k\geq 1}\frac{1}{k!}\sum_{\substack{\wp_1,\dots,\wp_k\\ \text{excursions of } \Xi_a^z\\ \forall i\neq j, \wp_i\neq\wp_j}} \mathcal{M}_{a'}^{\wp_1\cap\dots\cap\wp_k}. \tag{4.83}$$

Lemma 4.38. *Let $z \in D$, $a, a' \in (0,2)$. For any nonnegative measurable admissible function F,*

$$\mathbb{E}\left[\mathcal{M}_{a'}^{\Xi_a^z}(dz')F(z',\Xi_a^z)\right] = \frac{1}{a'}\mathrm{CR}(z',D)^{a'}\mathsf{B}((2\pi)^2 a a' G_D(z,z')^2)\mathbb{E}\left[F(z',\Xi_a^z\wedge\Xi_{a'}^{z'}\wedge\Xi_{a,a'}^{z,z'})\right]\mathrm{d}z'. \tag{4.84}$$

We now explain how Lemma 4.37 is obtained from this result. We will then prove Lemma 4.38.

Proof of Lemma 4.37. By Lemma 4.30, we can rewrite the left hand side of (4.82) as

$$\mathrm{CR}(z,D)^a \mathrm{d}z \int_{\mathsf{a}\in E(a,n)} \frac{\mathrm{d}\mathsf{a}}{a_1 \dots a_n} \int \mu_D^{\mathrm{loop}}(d\wp'_{l+1}) \dots \mu_D^{\mathrm{loop}}(d\wp'_m) \times \mathbb{E}\Big[\mathcal{M}_{a'}^{\Xi^z_{a_1}\cap\dots\cap\Xi^z_{a_l}\cap\wp'_{l+1}\cap\dots\cap\wp_m}(dz') F(z,z',\{\Xi^z_{a_i}\}_{i=1\dots n};\{\wp'_i\}_{i=l+1\dots m}) \Big].$$

Concluding the proof is then routine: we use the disintegration formula (4.41) to specify the thickness of each trajectory, Lemma 4.30 and Lemma 4.38. We omit the details. □

The rest of this section is dedicated to the proof of Lemma 4.38. As in the first moment computations made in Section 4.5.1, we will need to have an understanding of the processes of loops involved in Lemma 4.38 seen from their point with minimal imaginary part. Lemma 4.31 already achieves such a description for Ξ^z_a. We now completes the picture by doing it for loops $\wp_D^{z,z'} \wedge \wp_D^{z',z}$ appearing in the definition of $\Xi^{z,z'}_{a,a'}$.

Lemma 4.39. *Let $z, z' \in D$ be distinct points. For all nonnegative measurable function F,*

$$\mathbb{E}\left[F(\wp_D^{z,z'} \wedge \wp_D^{z',z})\right] = \frac{1}{G_D(z,z')^2} \int_{\mathrm{mi}(D)}^{m_z} \mathrm{d}m G_{D\cap\mathbb{H}_m}(z,z') \int_{(\mathbb{R}+im)\cap D} \mathrm{d}z_\perp H_{D\cap\mathbb{H}_m}(z',z_\perp) H_{D\cap\mathbb{H}_m}(z,z_\perp) \times \mathbb{E}\left[F(\wp^{z,z'}_{D\cap\mathbb{H}_m} \wedge \wp^{z',z_\perp}_{D\cap\mathbb{H}_m} \wedge \wp^{z_\perp,z}_{D\cap\mathbb{H}_m}) + F(\wp^{z,z_\perp}_{D\cap\mathbb{H}_m} \wedge \wp^{z_\perp,z'}_{D\cap\mathbb{H}_m} \wedge \wp^{z',z}_{D\cap\mathbb{H}_m}) \right]$$

We mention that the first (resp. second) term in the above expectation corresponds to the case where the minimum of the loop $\wp_D^{z,z'} \wedge \wp_D^{z',z}$ is achieved by the second piece $\wp_D^{z',z}$ (resp. first piece $\wp_D^{z,z'}$).

Proof. The proof is similar to the proof of Lemma 4.31. We first notice that, by restriction arguments, it is enough to show the result for the upper half plane. We then show it exploiting explicit expressions for the Green function and the Poisson kernel. We do not provide more details. □

We finally prove Lemma 4.38.

Proof of Lemma 4.38. By density-type arguments, we can assume that F is continuous. Let $m_z := \mathrm{Im}(z)$ and let $\Xi^z_{a,\varepsilon} := \{\wp \text{ excursion in } \Xi^z_a : \mathrm{mi}(\wp) < m_z - \varepsilon\}$ be the set of excursions in Ξ^z_a which go below $\mathbb{H}_{m_z-\varepsilon}$ (recall Notation 4.27). In this proof, we will, with some abuse of notations, denote by $\Xi^z_{a,\varepsilon}$ both the set of excursions and the loop obtained as the concatenation of all these excursions. $\#\Xi^z_{a,\varepsilon}$ is a Poisson variable with mean $2\pi a \mu_D^{z,z}(\mathrm{mi}(\wp) < m_z - \varepsilon)$ and conditioned on $\#\Xi^z_{a,\varepsilon} = n$, $\Xi^z_{a,\varepsilon}$ is composed of n i.i.d. excursions with common distribution $\wp_\varepsilon$ that we describe now. We root $\wp_\varepsilon$ at the point $z_\perp$ with minimal imaginary part and, recalling Notation 4.14, we have for any bounded measurable function F,

$$\mathbb{E}\left[F(\wp_\varepsilon)\right] = \frac{1}{Z_\varepsilon} \int_{\mathrm{mi}(D)}^{m_z-\varepsilon} \mathrm{d}m \int_{(\mathbb{R}+im)\cap D} \mathrm{d}z_\perp H_{D\cap\mathbb{H}_m}(z,z_\perp)^2 \mathbb{E}\left[F(\wp^{z,z_\perp}_{D\cap\mathbb{H}_m} \wedge \wp^{z_\perp,z}_{D\cap\mathbb{H}_m}) \right], \tag{4.85}$$

where Z_ε is the normalising constant

$$Z_\varepsilon = \int_{\mathrm{mi}(D)}^{m_z-\varepsilon} \mathrm{d}m \int_{(\mathbb{R}+im)\cap D} \mathrm{d}z_\perp H_{D\cap\mathbb{H}_m}(z,z_\perp)^2.$$

Note that $Z_\varepsilon = \mu_D^{z,z}(\mathrm{mi}(\wp) < m_z - \varepsilon)$. We can now start the computation of the left hand side of (4.84). By continuity of F, it is equal to

$$\lim_{\varepsilon\to 0} e^{-2\pi a Z_\varepsilon} \sum_{n=0}^{\infty} \frac{(2\pi a Z_\varepsilon)^n}{n!} \mathbb{E}\left[\mathcal{M}_{a'}^{\Xi^z_{a,\varepsilon}}(dz')F(z',\Xi^z_{a,\varepsilon}) | \#\Xi^z_{a,\varepsilon} = n\right] \tag{4.86}$$
$$= \lim_{\varepsilon\to 0} e^{-2\pi a Z_\varepsilon} \sum_{n=0}^{\infty} \frac{(2\pi a Z_\varepsilon)^n}{n!} \sum_{k=1}^{n} \binom{n}{k} \mathbb{E}\left[\mathcal{M}_{a'}^{\wp_\varepsilon^1\cap\dots\cap\wp_\varepsilon^k}(dz')F(z',\wp_\varepsilon^1\wedge\dots\wedge\wp_\varepsilon^n)\right]$$

where $\wp_\varepsilon^i, i = 1\dots n$, are i.i.d. trajectories distributed according to (4.85). The binomial coefficient corresponds to the number of ways to choose k trajectories that actually visit z' among the collection of n trajectories. We now use the disintegration formula (4.41) to specify the contribution of each of the k trajectories. To ease notations, in the following computations we denote by $D_i = D \cap \mathbb{H}_{m^i}$ and we write with some abuse of notation a product of integrals instead of multiple integrals. Also, by independence, the shift by the above intersection measure will not have any impact on $\wp_\varepsilon^{k+1},\dots,\wp_\varepsilon^n$. We will therefore remove these trajectories from the computations and add them back when it will be necessary. We have

$$\mathbb{E}\left[\mathcal{M}_{a'}^{\wp_\varepsilon^1\cap\dots\cap\wp_\varepsilon^k}(dz')F(z',\wp_\varepsilon^1\wedge\dots\wedge\wp_\varepsilon^k)\right] = \frac{1}{Z_\varepsilon^k}\int_{\mathsf{a}'\in E(a',k)} \mathrm{d}\mathsf{a}' \left(\prod_{i=1}^k \int_{\mathrm{mi}(D)}^{m_z-\varepsilon} \mathrm{d}m^i \int_{(\mathbb{R}+im^i)\cap D} \mathrm{d}z_\perp^i\right) \tag{4.87}$$
$$\times \left(\prod_{i=1}^k H_{D_i}(z,z_\perp^i)^2\right)\mathbb{E}\left[\bigcap_{i=1}^k \mathcal{M}_{a_i'}^{\wp_{D_i}^{z,z_\perp^i}\wedge\wp_{D_i}^{z_\perp^i,z}}(dz')F\left(z',\bigwedge_{i=1}^k(\wp_{D_i}^{z,z_\perp^i}\wedge\wp_{D_i}^{z_\perp^i,z})\right)\right].$$

We decompose for all $i = 1\dots k$,

$$\mathcal{M}_{a_i'}^{\wp_{D_i}^{z,z_\perp^i}\wedge\wp_{D_i}^{z_\perp^i,z}} = \mathcal{M}_{a_i'}^{\wp_{D_i}^{z,z_\perp^i}} + \mathcal{M}_{a_i'}^{\wp_{D_i}^{z_\perp^i,z}} + \mathcal{M}_{a_i'}^{\wp_{D_i}^{z,z_\perp^i}\cap\wp_{D_i}^{z_\perp^i,z}}$$

and we then expand

$$\mathbb{E}\left[\bigcap_{i=1}^k \mathcal{M}_{a_i'}^{\wp_{D_i}^{z,z_\perp^i}\wedge\wp_{D_i}^{z_\perp^i,z}}(dz')F\left(z',\bigwedge_{i=1}^k(\wp_{D_i}^{z,z_\perp^i}\wedge\wp_{D_i}^{z_\perp^i,z})\right)\right] \tag{4.88}$$
$$= \sum_{I_1,I_2,I_3} \mathbb{E}\left[\bigcap_{i\in I_1}\mathcal{M}_{a_i'}^{\wp_{D_i}^{z,z_\perp^i}} \cap \bigcap_{i\in I_2}\mathcal{M}_{a_i'}^{\wp_{D_i}^{z_\perp^i,z}} \cap \bigcap_{i\in I_3}\mathcal{M}_{a_i'}^{\wp_{D_i}^{z,z_\perp^i}\cap\wp_{D_i}^{z_\perp^i,z}}(dz')F\left(z',\bigwedge_{i=1}^k(\wp_{D_i}^{z,z_\perp^i}\wedge\wp_{D_i}^{z_\perp^i,z})\right)\right]$$

where the sum runs over all partition of $\{1,\dots,k\}$ in three subsets I_1, I_2 and I_3. Recall that the disintegration formula (4.41) yields

$$\mathcal{M}_{a_i'}^{\wp_{D_i}^{z,z_\perp^i}\cap\wp_{D_i}^{z_\perp^i,z}} = \int_0^{a_i'} \mathrm{d}\alpha_i'\, \mathcal{M}_{a_i'-\alpha_i'}^{\wp_{D_i}^{z,z_\perp^i}} \cap \mathcal{M}_{\alpha_i'}^{\wp_{D_i}^{z_\perp^i,z}}. \tag{4.89}$$

Now, by (4.43), the expectation in the left hand side of (4.88) is equal to

$$\sum_{I_1,I_2,I_3} \Big(\prod_{i \in I_1 \cup I_2} \frac{H_{D_i}(z',z_\perp^i)}{H_{D_i}(z,z_\perp^i)} 2\pi G_{D_i}(z,z') \operatorname{CR}(z',D_i)^{a_i'} \Big) \Big(\prod_{i \in I_3} a_i' \frac{H_{D_i}(z',z_\perp^i)^2}{H_{D_i}(z,z_\perp^i)^2} (2\pi)^2 G_{D_i}(z,z')^2 \operatorname{CR}(z',D_i)^{a_i'} \Big) \tag{4.90}$$
$$\times \mathbb{E}\Big[F\Big(z', \bigwedge_{i \in I_1} (\wp_{D_i}^{z,z'} \wedge \wp_{D_i}^{z',z_\perp^i} \wedge \wp_{D_i}^{z_\perp^i,z} \wedge \Xi_{a_i'}^{z',D_i}) \wedge \bigwedge_{i \in I_2} (\wp_{D_i}^{z,z_\perp^i} \wedge \wp_{D_i}^{z_\perp^i,z'} \wedge \wp_{D_i}^{z',z} \wedge \Xi_{a_i'}^{z',D_i})$$
$$\wedge \bigwedge_{i \in I_3} (\wp_{D_i}^{z,z'} \wedge \wp_{D_i}^{z',z} \wedge \Xi_{a_i}^{z',D_i} \wedge \wp_{D_i}^{z',z_\perp^i} \wedge \wp_{D_i}^{z_\perp^i,z'})\Big)\Big].$$

Note that the a_i' in the product over $i \in I_3$ comes from the integration of α_i' in (4.89). When we will plug this back in (4.87), we will have to multiply everything with the product of Poisson kernel $H_{D_i}(z,z_\perp^i)^2$. This latter product times the two products in parenthesis in (4.90) can be rewritten as

$$\Big(\prod_{i \in I_1 \cup I_2} H_{D_i}(z',z_\perp^i) H_{D_i}(z,z_\perp^i) 2\pi G_{D_i}(z,z') \operatorname{CR}(z',D_i)^{a_i'} \Big)$$
$$\times \Big(\prod_{i \in I_3} a_i' H_{D_i}(z',z_\perp^i)^2 (2\pi)^2 G_{D_i}(z,z')^2 \operatorname{CR}(z',D_i)^{a_i'} \Big)$$
$$= (2\pi)^k \operatorname{CR}(z',D)^{a'} G_D(z,z')^{2k} \Big(\prod_{i \in I_1 \cup I_2} \Big\{ H_{D_i}(z',z_\perp^i) H_{D_i}(z,z_\perp^i) \frac{G_{D_i}(z,z')}{G_D(z,z')^2} \Big\} \frac{\operatorname{CR}(z',D_i)^{a_i'}}{\operatorname{CR}(z',D)^{a_i'}} \Big)$$
$$\times \Big(\prod_{i \in I_3} \Big\{ 2\pi a_i' H_{D_i}(z',z_\perp^i)^2 \frac{\operatorname{CR}(z',D_i)^{a_i'}}{\operatorname{CR}(z',D)^{a_i'}} \Big\} \frac{G_{D_i}(z,z')^2}{G_D(z,z')^2} \Big).$$

We recognise in the brackets above the density of the point $z_\perp^i$ with minimal imaginary part in the loops $\wp_D^{z,z',i} \wedge \wp_D^{z',z,i}$ ($i \in I_1 \cup I_2$) and $\Xi_{a_i'}^{z,D}$ ($i \in I_3$); see Lemmas 4.39 and 4.31. The term $\operatorname{CR}(z',D_i)^{a_i'} / \operatorname{CR}(z',D)^{a_i'}$ (resp. $G_{D_i}(z,z')^2 / G_D(z,z')^2$) corresponds to the probability for $\Xi_{a_i'}^{z',D}$ (resp. $\wp_D^{z,z',i} \wedge \wp_D^{z',z,i}$) to stay in D_i. To make this more precise, we introduce the following events: for all $i = 1 \dots k$, let $E_1^i(\varepsilon)$, resp. $E_2^i(\varepsilon)$ and $E_3^i(\varepsilon)$, be the event that the minimal height among the trajectories $\wp_D^{z,z',i}$, $\wp_D^{z',z,i}$ and $\Xi_{a_i'}^{z',D}$ is smaller than $m_z - \varepsilon = \operatorname{Im}(z) - \varepsilon$ and is reached by the first, resp. second and third. Lemmas 4.39 and 4.31 imply that for all $\beta \in \{1,2\}$, for all $i \in I_\beta$,

$$\int_{\operatorname{mi}(D)}^{m_z - \varepsilon} \mathrm{d}m^i \int_{(\mathbb{R}+im^i)\cap D} \mathrm{d}z_\perp^i \Big\{ H_{D_i}(z',z_\perp^i) H_{D_i}(z,z_\perp^i) \frac{G_{D_i}(z,z')}{G_D(z,z')^2} \Big\} \frac{\operatorname{CR}(z',D_i)^{a_i'}}{\operatorname{CR}(z',D)^{a_i'}}$$
$$\times \mathbb{E}\left[F(z', \wp_{D_i}^{z,z'} \wedge \wp_{D_i}^{z',z_\perp^i} \wedge \wp_{D_i}^{z_\perp^i,z} \wedge \Xi_{a_i'}^{z',D_i}) \right]$$
$$= \mathbb{E}\left[F(z', \wp_D^{z,z',i} \wedge \wp_D^{z',z,i} \wedge \Xi_{a_i'}^{z',D}) \mathbf{1}_{E_\beta^i(\varepsilon)} \right]$$

and that for all $i \in I_3$,

$$\int_{\mathrm{mi}(D)}^{m_z-\varepsilon} \mathrm{d}m^i \int_{(\mathbb{R}+im^i)\cap D} \mathrm{d}z_\perp^i \Big\{ 2\pi a_i' H_{D_i}(z', z_\perp^i)^2 \frac{\mathrm{CR}(z', D_i)^{a_i'}}{\mathrm{CR}(z', D)^{a_i'}} \Big\} \frac{G_{D_i}(z,z')^2}{G_D(z,z')^2}$$
$$\times \mathbb{E}\left[F(z', \wp_{D_i}^{z,z'} \wedge \wp_{D_i}^{z',z} \wedge \Xi_{a_i}^{z',D_i} \wedge \wp_{D_i}^{z',z_\perp^i} \wedge \wp_{D_i}^{z_\perp^i, z'}) \right]$$
$$= \mathbb{E}\left[F(z', \wp_D^{z,z',i} \wedge \wp_D^{z',z,i} \wedge \Xi_{a_i'}^{z',D}) \mathbf{1}_{E_3^i(\varepsilon)} \right].$$

Overall, and going back to (4.87), we have obtained that

$$\mathbb{E}\left[\mathcal{M}_{a'}^{\wp_\varepsilon^1 \cap \dots \cap \wp_\varepsilon^k}(dz') F(z', \wp_\varepsilon^1 \wedge \dots \wedge \wp_\varepsilon^k) \right] = \frac{1}{Z_\varepsilon^k} (2\pi)^k G_D(z,z')^{2k} \int_{\mathsf{a}' \in E(a',k)} \mathrm{d}\mathsf{a}'$$
$$\times \sum_{I_1,I_2,I_3} \mathbb{E}\left[F\Big(z', \bigwedge_{i=1}^k (\wp_D^{z,z',i} \wedge \wp_D^{z',z,i} \wedge \Xi_{a_i'}^{z',D})\Big) \prod_{i \in I_1} \mathbf{1}_{E_1^i(\varepsilon)} \prod_{i \in I_2} \mathbf{1}_{E_2^i(\varepsilon)} \prod_{i \in I_3} \mathbf{1}_{E_3^i(\varepsilon)} \right].$$

Plugging this back in (4.86) and remembering that we have to add the trajectories $\wp_\varepsilon^{k+1}, \dots, \wp_\varepsilon^n$, we see that the left hand side of (4.84) is equal to

$$\lim_{\varepsilon \to 0} e^{-2\pi a Z_\varepsilon} \sum_{n=0}^\infty \sum_{k=1}^n \frac{(2\pi a Z_\varepsilon)^{n-k}}{(n-k)!k!} (2\pi \sqrt{a} G_D(z,z'))^{2k} \int_{\mathsf{a}' \in E(a',k)} \mathrm{d}\mathsf{a}'$$
$$\times \sum_{I_1,I_2,I_3} \mathbb{E}\left[F\Big(z', \bigwedge_{i=1}^k (\wp_D^{z,z',i} \wedge \wp_D^{z',z,i} \wedge \Xi_{a_i'}^{z',D}) \wedge \wp_\varepsilon^{k+1} \wedge \dots \wedge \wp_\varepsilon^n\Big) \prod_{\beta \in \{1,2,3\}} \prod_{i \in I_\beta} \mathbf{1}_{E_\beta^i(\varepsilon)} \right]$$
$$= \sum_{k=1}^\infty \frac{(2\pi \sqrt{a} G_D(z,z'))^{2k}}{k!} \int_{\mathsf{a}' \in E(a',k)} \mathrm{d}\mathsf{a}'$$
$$\times \sum_{I_1,I_2,I_3} \mathbb{E}\left[F\Big(z', \bigwedge_{i=1}^k (\wp_D^{z,z',i} \wedge \wp_D^{z',z,i} \wedge \Xi_{a_i'}^{z',D}) \wedge \Xi_a^{z,D}\Big) \prod_{\beta \in \{1,2,3\}} \prod_{i \in I_\beta} \mathbf{1}_{E_\beta^i(\varepsilon=0)} \right].$$

Since,

$$\sum_{I_1,I_2,I_3} \prod_{i \in I_\beta} \mathbf{1}_{E_\beta^i(\varepsilon=0)} = 1,$$

we can use additivity of Poisson point processes and then the fact that the Lebesgue measure of the simplex $E(a',k)$ is equal to $(a')^{k-1}/(k-1)!$ to obtain that the left hand side of (4.84) is equal to

$$\sum_{k=1}^\infty \frac{(2\pi \sqrt{a} G_D(z,z'))^{2k}}{k!} \mathbb{E}\left[F\Big(z', \bigwedge_{i=1}^k (\wp_D^{z,z',i} \wedge \wp_D^{z',z,i}) \wedge \Xi_{a'}^{z',D} \wedge \Xi_a^{z,D}\Big) \right] \int_{\mathsf{a}' \in E(a',k)} \mathrm{d}\mathsf{a}'$$
$$= \frac{1}{a'} \sum_{k=1}^\infty \frac{(2\pi \sqrt{a a'} G_D(z,z'))^{2k}}{k!(k-1)!} \mathbb{E}\left[F\Big(z', \bigwedge_{i=1}^k (\wp_D^{z,z',i} \wedge \wp_D^{z',z,i}) \wedge \Xi_{a'}^{z',D} \wedge \Xi_a^{z,D}\Big) \right].$$

This is the right hand side of (4.84) which concludes the proof. □

4.6.2 Second moment

Combining Lemma 4.37 with a Palm formula type of argument, we obtain the following expression for the second moment of functionals of our measure.

Lemma 4.40. *For any bounded measurable admissible function $F = F(z, z', \mathcal{L})$ of a pair of points z, z' and a collection loops $\mathcal{L}$, we have:*

$$\mathbb{E}[F(z, z', \mathcal{L}_D^\theta)\mathcal{M}_a^K(dz)\mathcal{M}_{a'}^K(dz')] = \mathrm{CR}(z, D)^a \, \mathrm{CR}(z', D)^{a'}$$
$$\times \sum_{\substack{n,m\geq 1\\ 0\leq l\leq n\wedge m}} \frac{\theta^{n+m-l}}{(n-l)!(m-l)!l!} \int_{\substack{\mathsf{a}\in E(a,n)\\ \mathsf{a}'\in E(a',m)}} \frac{\mathsf{da}}{a_1 \dots a_n} \frac{\mathsf{da}'}{a_1' \dots a_m'} \prod_{i=1}^l \mathsf{B}\left((2\pi)^2 a_i a_i' G_D(z,z')^2\right)$$
$$\times \mathbb{E}\Bigg[\prod_{i=1}^l \left(1 - e^{-KT(\Xi^{z,z'}_{a_i,a_i'} \wedge \Xi^z_{a_i} \wedge \Xi^{z'}_{a_i'})}\right) \prod_{i=l+1}^n \left(1 - e^{-KT(\Xi^z_{a_i})}\right) \prod_{i=l+1}^m \left(1 - e^{-KT(\Xi^{z'}_{a_i'})}\right)$$
$$F\left(z, z', \mathcal{L}_D^\theta \cup \{\Xi^{z,z'}_{a_i,a_i'} \wedge \Xi^z_{a_i} \wedge \Xi^{z'}_{a_i'}\}_{i=1}^l \cup \{\Xi^z_{a_i}\}_{i=l+1}^n \cup \{\Xi^{z'}_{a_i'}\}_{i=l+1}^m\right)\Bigg] \mathrm{d}z\mathrm{d}z'$$

where all the above processes are independent.

Proof. In what follows, to shorten notations, we will write with some abuse of notation "$\wp_1 \neq \dots \neq \wp_n \in \mathcal{L}_D^\theta(K)$" instead of "$\wp_1, \dots, \wp_n \in \mathcal{L}_D^\theta(K)$ and for all $i \neq j$, $\wp_i \neq \wp_j$". By definition of $\mathcal{M}_a^K$, $\mathbb{E}F(z, z', \mathcal{L}_D^\theta)\mathcal{M}_a^K(dz)\mathcal{M}_{a'}^K(dz')$ is equal to

$$\sum_{n,m\geq 1} \frac{1}{n!m!} \mathbb{E} \sum_{\substack{\wp_1\neq\dots\neq\wp_n\in\mathcal{L}_D^\theta(K)\\ \wp_1'\neq\dots\neq\wp_m'\in\mathcal{L}_D^\theta(K)}} F(z,z',\mathcal{L}_D^\theta)\mathcal{M}_a^{\wp_1\cap\dots\wp_n}(dz)\mathcal{M}_{a'}^{\wp_1'\cap\dots\wp_m'}(dz')$$
$$= \sum_{n,m\geq 1} \frac{1}{n!m!} \sum_{l=0}^{n\wedge m} l! \binom{n}{l}\binom{m}{l}$$
$$\times \mathbb{E} \sum_{\substack{\wp_1\neq\dots\neq\wp_n\neq\wp_{l+1}'\neq\dots\neq\wp_m'\in\mathcal{L}_D^\theta(K)\\ \forall i=1\dots l, \wp_i'=\wp_i}} F(z,z',\mathcal{L}_D^\theta)\mathcal{M}_a^{\wp_1\cap\dots\wp_n}(dz)\mathcal{M}_{a'}^{\wp_1'\cap\dots\wp_m'}(dz').$$

l represents the number of loops that are in both sets of loops. $\binom{n}{l}$ (resp. $\binom{m}{l}$) is the number of ways to choose a subset of l loops in the set of n loops (resp. m loops). $l!$ is then the number of ways to map one subset to the other. Fix now, $n, m \geq 1$ and $l \in \{0, \dots, n\wedge m\}$. By Palm's formula, the expectation above is equal to θ^{n+m-l} times

$$\int \mu_D^{\mathrm{loop}}(d\wp_1)\dots\mu^{\mathrm{loop}}(\wp_n)\mu_D^{\mathrm{loop}}(d\wp_{l+1}')\dots\mu^{\mathrm{loop}}(\wp_m') \prod_{i=1}^n \left(1 - e^{-KT(\wp_i)}\right) \prod_{i=l+1}^m \left(1 - e^{-KT(\wp_i')}\right)$$
$$\mathbb{E}\left[F(z,z',\mathcal{L}_D^\theta \cup \{\wp_i\}_{i=1}^n \cup \{\wp_i'\}_{i=l+1}^m\right] \mathcal{M}_a^{\wp_1\cap\dots\wp_n}(dz)\mathcal{M}_{a'}^{\wp_1\dots\cap\wp_l\cap\wp_{l+1}'\cap\dots\wp_m'}(dz')$$

where the expectation is only with respect to $\mathcal{L}_D^\theta$. Lemma 4.37 concludes the proof. $\square$

In particular, Lemma 4.40 gives an explicit formula for the second moment. Indeed, we have already seen that

$$\mathbb{E}\left[e^{-KT(\Xi_a^z)}\right] = e^{-aC_K(z)}.$$

Moreover,

$$\begin{aligned}\mathbb{E}\left[e^{-KT(\Xi_{a,a'}^{z,z'})}\right] &= \frac{1}{\mathsf{B}((2\pi)^2 aa' G_D(z,z')^2)} \sum_{k=1}^{\infty} \frac{(2\pi\sqrt{aa'}G_D(z,z'))^{2k}}{k!(k-1)!} \mathbb{E}\left[e^{-KT(z\to z')}\right]^{2k} \\ &= \frac{\mathsf{B}\left((2\pi)^2 aa' G_D(z,z')^2 \mathbb{E}\left[e^{-KT(z\to z')}\right]^2\right)}{\mathsf{B}((2\pi)^2 aa' G_D(z,z')^2)}\end{aligned}$$

where in the above $T(z \to z')$ is the running time of an excursion from z to z' distributed according to $\mu_D^{z,z'}/G_D(z,z')$. We further notice that

$$G_D(z,z')\mathbb{E}\left[e^{-KT(z\to z')}\right] = G_{D,K}(z,z').$$

Overall, this shows that $\mathbb{E}[\mathcal{M}_a^K(dz)\mathcal{M}_{a'}^K(dz')]$ is equal to

$$\begin{aligned}&\mathrm{CR}(z,D)^a \,\mathrm{CR}(z',D)^{a'} \sum_{\substack{n,m\geq 1 \\ 0\leq l\leq n\wedge m}} \frac{\theta^{n+m-l}}{(n-l)!(m-l)!l!} \int_{\substack{\mathsf{a}\in E(a,n) \\ \mathsf{a}'\in E(a',m)}} \frac{d\mathsf{a}}{a_1\dots a_n} \frac{d\mathsf{a}'}{a_1'\dots a_m'} \\ &\times \prod_{i=1}^{l} \left(\mathsf{B}\left((2\pi)^2 a_i a_i' G_D(z,z')^2\right) - e^{-a_i C_K(z) - a_i' C_K(z')} \mathsf{B}\left((2\pi)^2 a_i a_i' G_D(z,z')^2\right)\right) \\ &\times \prod_{i=l+1}^{n}\left(1-e^{-a_i C_K(z)}\right) \prod_{i=l+1}^{m}\left(1-e^{-a_i' C_K(z')}\right) dz dz'. \end{aligned} \tag{4.91}$$

The purpose of the next section is to study the asymptotic properties of this expression. This will basically conclude the proof of Theorem 4.1 in the L^2-phase, but this will also be useful in order to go beyond this phase to cover the whole L^1-phase.

4.6.3 Simplifying the second moment

Let $a, a' > 0$. Recall the definition (4.80) of B and define for all $u, u', v \geq 0$,

$$\begin{aligned}\mathsf{H}_{a,a'}(u,u',v) := &\sum_{\substack{n,m\geq 1 \\ 0\leq l\leq n\wedge m}} \frac{\theta^{n+m-l}}{(n-l)!(m-l)!l!} \\ &\int_{\substack{\mathsf{a}\in E(a,n) \\ \mathsf{a}'\in E(a',m)}} d\mathsf{a}d\mathsf{a}' \prod_{i=1}^{l} \frac{\mathsf{B}(va_i a_i')}{a_i a_i'} \prod_{i=l+1}^{n} \frac{1-e^{-ua_i}}{a_i} \prod_{i=l+1}^{m} \frac{1-e^{-u'a_i'}}{a_i'}.\end{aligned} \tag{4.92}$$

v will be taken to be a multiple of $G_D(z,z')^2$, whereas u and u' will coincide with $C_K(z)$ and $C_K(z')$ respectively.

To get an upper bound on the second moment, we will start from the expression (4.91), and bound the second line in that expression with a quantity that does not depend on K. We do so simply by ignoring the second term in the product, which leads to the expression for H in (4.92). Intuitively, this amounts to ignoring the requirements that the loops that visit both z and z' are killed. Indeed, since z and z' are typically macroscopically far away, such loops will be killed with high probability and so ignoring the requirement gives us a good upper bound.

Lemma 4.41. *Let $a, a' > 0$. There exists $C > 0$ such that for all $u, u' \geq 1, v > 0$,*

$$\mathsf{H}_{a,a'}(u, u', v) \leq C(uu')^\theta v^{1/4-\theta/2} e^{2\sqrt{vaa'}}. \tag{4.93}$$

Moreover, for all $v > 0$,

$$\lim_{u,u'\to\infty} \frac{\mathsf{H}_{a,a'}(u, u', v)}{(uu')^\theta} = \frac{1}{\Gamma(\theta)} \left(\frac{aa'}{v}\right)^{\frac{\theta-1}{2}} I_{\theta-1}\left(2\sqrt{vaa'}\right). \tag{4.94}$$

In particular, when $\theta = 1/2$, for all $v > 0$,

$$\lim_{u,u'\to\infty} \frac{\mathsf{H}_{a,a'}(u, u', v)}{\sqrt{uu'}} = \frac{1}{\pi\sqrt{aa'}} \cosh(2\sqrt{vaa'}). \tag{4.95}$$

Proof of Lemma 4.41. We start off by doing the change of variable $(n, m, l) \leftarrow (n - l, m - l, l)$ and obtain using Lemma 4.33 that $\mathsf{H}_{a,a'}(u, u', v)$ is equal to

$$\begin{aligned} &\sum_{\substack{n,m\geq 1 \text{ and } l\geq 0 \\ \text{or } n=m=0 \text{ and } l\geq 1}} \frac{\theta^{n+m+l}}{n!m!l!} \int_{\substack{\mathsf{a}\in E(a,n+l) \\ \mathsf{a}'\in E(a',m+l)}} \mathrm{d}\mathsf{a}\mathrm{d}\mathsf{a}' \prod_{i=1}^l \frac{\mathsf{B}(va_ia_i')}{a_ia_i'} \prod_{i=l+1}^{n+l} \frac{1-e^{-ua_i}}{a_i} \prod_{i=l+1}^{m+l} \frac{1-e^{-u'a_i'}}{a_i'} \\ &= \sum_{l\geq 0} \frac{\theta^l}{l!} \int_{\substack{\mathsf{a}\in E(a,l+1) \\ \mathsf{a}'\in E(a',l+1)}} \mathrm{d}\mathsf{a}\mathrm{d}\mathsf{a}' \frac{\mathsf{F}(ua_{l+1})}{a_{l+1}} \frac{\mathsf{F}(u'a_{l+1}')}{a_{l+1}'} \prod_{i=1}^l \frac{\mathsf{B}(va_ia_i')}{a_ia_i'} + \sum_{l\geq 1} \frac{\theta^l}{l!} \int_{\substack{\mathsf{a}\in E(a,l) \\ \mathsf{a}'\in E(a',l)}} \mathrm{d}\mathsf{a}\mathrm{d}\mathsf{a}' \prod_{i=1}^l \frac{\mathsf{B}(va_ia_i')}{a_ia_i'}. \end{aligned} \tag{4.96}$$

Let us explain briefly where this comes from. The first term is the "off-diagonal" term corresponding to $n, m \geq 1$ and $l \geq 0$, while the second term is the "on-diagonal" term corresponding to $n = m = 0$ and $l \geq 1$. Furthermore, to get the expression of the first term, we reason as follows. The term $\mathrm{d}\mathsf{a}$ in the first line concerns $n + l$ variables, $a_1, \ldots, a_{n+l}$, whose sum is fixed equal to a. We freeze $a_1, \ldots, a_l$ and first integrate over $a_{l+1}, \ldots a_{l+n}$. We may call a_{l+1} the sum of these n variables; thus $a_1 + \ldots + a_{l+1} = a$. Summing over n and applying Lemma 4.33 we recognise the expression for $\mathsf{F}(ua_{l+1})/a_{l+1}$. The same can be done separately for $\mathrm{d}\mathsf{a}'$, leading us to the claimed expression.

Now, let $l \geq 1$, $\alpha, \alpha' > 0$ and let us note that by definition of B,

$$\int_{\substack{\mathsf{a}\in E(\alpha,l) \\ \mathsf{a}'\in E(\alpha',l)}} \mathrm{d}\mathsf{a}\mathrm{d}\mathsf{a}' \prod_{i=1}^l \frac{\mathsf{B}(va_ia_i')}{a_ia_i'} = \int_{\substack{\mathsf{a}\in E(\alpha,l) \\ \mathsf{a}'\in E(\alpha',l)}} \mathrm{d}\mathsf{a}\mathrm{d}\mathsf{a}' \sum_{k_1,\ldots,k_l\geq 1} v^{k_1+\cdots+k_l} \prod_{i=1}^l \frac{(a_ia_i')^{k_i-1}}{k_i!(k_i-1)!}.$$

Using the fact that for all $\beta > 0$, $k, k' \geq 1$,

$$\int_0^\beta \frac{x^{k-1}}{(k-1)!} \frac{(\beta-x)^{k'-1}}{(k'-1)!} \mathrm{d}x = \frac{\beta^{k+k'-1}}{(k+k'-1)!},$$

we find by induction that

$$\int_{\mathsf{a} \in E(\alpha,l)} d\mathsf{a} \prod_{i=1}^{l} \frac{a_i^{k_i-1}}{(k_i-1)!} = \frac{\alpha^{k_1+\dots+k_l-1}}{(k_1+\dots+k_l-1)!}.$$

Hence

$$\begin{aligned}\int_{\substack{\mathsf{a} \in E(\alpha,l)\\ \mathsf{a}' \in E(\alpha',l)}} d\mathsf{a} d\mathsf{a}' \prod_{i=1}^{l} \frac{\mathsf{B}(v a_i a_i')}{a_i a_i'} &= \sum_{k_1,\dots,k_l \geq 1} v^{k_1+\dots+k_l} \frac{(\alpha\alpha')^{k_1+\dots+k_l-1}}{(k_1+\dots+k_l-1)!^2} \prod_{i=1}^{l} \frac{1}{k_i} \\ &= \sum_{k \geq l} v^k \frac{(\alpha\alpha')^{k-1}}{(k-1)!^2} \sum_{\substack{k_1,\dots,k_l \geq 1\\ k_1+\dots+k_l=k}} \prod_{i=1}^{l} \frac{1}{k_i}\end{aligned}$$

and

$$\sum_{l \geq 1} \frac{\theta^l}{l!} \int_{\substack{\mathsf{a} \in E(\alpha,l)\\ \mathsf{a}' \in E(\alpha',l)}} d\mathsf{a} d\mathsf{a}' \prod_{i=1}^{l} \frac{\mathsf{B}(v a_i a_i')}{a_i a_i'} = \sum_{k \geq 1} v^k \frac{(\alpha\alpha')^{k-1}}{(k-1)!^2} \sum_{l=1}^{k} \frac{\theta^l}{l!} \sum_{\substack{k_1,\dots,k_l \geq 1\\ k_1+\dots+k_l=k}} \prod_{i=1}^{l} \frac{1}{k_i}.$$

Looking at the series expansion of $(1-x)^{-\theta}$ near 0 and recalling the definition (4.79) of $\theta^{(k)}$, we see that for all $k \geq 1$,

$$\sum_{l=1}^{k} \frac{\theta^l}{l!} \sum_{\substack{k_1,\dots,k_l \geq 1\\ k_1+\dots+k_l=k}} \prod_{i=1}^{l} \frac{1}{k_i} = \frac{\theta^{(k)}}{k!}.$$

We deduce that

$$\sum_{l \geq 1} \frac{\theta^l}{l!} \int_{\substack{\mathsf{a} \in E(\alpha,l)\\ \mathsf{a}' \in E(\alpha',l)}} d\mathsf{a} d\mathsf{a}' \prod_{i=1}^{l} \frac{\mathsf{B}(v a_i a_i')}{a_i a_i'} = \sum_{k \geq 1} v^k \frac{(\alpha\alpha')^{k-1}}{(k-1)!^2} \frac{\theta^{(k)}}{k!}. \tag{4.97}$$

Taking $\alpha = a$, $\alpha' = a'$, this gives an expression for the second term of (4.96). As for the first term in (4.96), it can be computed in a similar manner: namely, we get

$$\frac{\mathsf{F}(ua)}{a} \frac{\mathsf{F}(u'a')}{a'} + \sum_{k \geq 1} \frac{v^k}{(k-1)!^2} \frac{\theta^{(k)}}{k!} \int_0^a d\alpha \int_0^{a'} d\alpha' (\alpha\alpha')^{k-1} \frac{\mathsf{F}(u(a-\alpha))}{a-\alpha} \frac{\mathsf{F}(u'(a'-\alpha'))}{a'-\alpha'}. \tag{4.98}$$

with $\alpha = a - a_{l+1} = a_1 + \dots a_l$ and, respectively, $\alpha' = a' - a'_{l+1} = a'_1 + \dots + a'_l$.

We then use (4.47) and (4.222) to bound

$$\int_0^a \alpha^{k-1} \frac{\mathsf{F}(u(a-\alpha))}{a-\alpha} d\alpha \leq C u^\theta \int_0^a \frac{\alpha^{k-1}}{(a-\alpha)^{1-\theta}} d\alpha = C u^\theta a^{k-1+\theta} \frac{(k-1)!}{\theta^{(k)}}.$$

We finally find that the first term of (4.96) is at most

$$C(uu')^\theta (aa')^{\theta-1} + C(uu')^\theta (aa')^{\theta-1} \sum_{k \geq 1} \frac{(vaa')^k}{k!\theta^{(k)}} = C(uu')^\theta (aa'/v)^{\theta/2-1/2} \Gamma(\theta) I_{\theta-1}(2\sqrt{vaa'}).$$

The second term of (4.96) can be bounded by $\cosh(2\sqrt{vaa'})$. This concludes the proof of (4.93). (4.94) follows as well by using the asymptotic $\mathsf{F}(w) \sim w^\theta/\Gamma(\theta)$ as $w \to \infty$ and by applying dominated convergence theorem in (4.98). (4.95) follows from (4.224) and (4.221). $\square$

As a consequence, we obtain the following estimates on the second moment of $\mathcal{M}_a^K$.

Corollary 4.42. *There exists $C > 0$ such that for all $K \geq 1$, $z, z' \in D$, $a, a' \in (0,2)$,*

$$\mathbb{E}\left[\mathcal{M}_a^K(dz)\mathcal{M}_{a'}^K(dz')\right] \leq C(aa')^{\theta/2-3/4}(\log K)^{2\theta} G_D(z,z')^{1/2-\theta} \exp\left(4\pi\sqrt{aa'}G_D(z,z')\right).$$

Moreover,

$$\lim_{K\to\infty} \frac{\mathbb{E}\left[\mathcal{M}_a^K(dz)\mathcal{M}_{a'}^K(dz')\right]}{(\log K)^{2\theta}} = \frac{1}{4^\theta\Gamma(\theta)}\left(\frac{\sqrt{aa'}}{2\pi G_D(z,z')}\right)^{\theta-1} I_{\theta-1}\left(4\pi\sqrt{aa'}G_D(z,z')\right).$$

In particular, when $\theta = 1/2$,

$$\lim_{K\to\infty} \frac{\mathbb{E}\left[\mathcal{M}_a^K(dz)\mathcal{M}_{a'}^K(dz')\right]}{\log K} = \frac{1}{2\pi\sqrt{aa'}} \cosh\left(4\pi\sqrt{aa'}G_D(z,z')\right).$$

4.7 Going beyond the L^2-phase

The goal of this section is to prove Proposition 4.26. We now describe the proof at a high level. When the thickness parameter a is smaller than 1, we can directly apply Cauchy–Schwarz inequality and control the second moment. (This could be done directly using Corollary 4.42). However when $a \in [1,2)$, the second moment blows up. The broad strategy is by now well understood and consists in introducing "good events", similar to [Ber17]. In our context, this good event at a given point $z \in D$ will take the following form: we will require that the total number of crossings of each dyadic annulus centred at z is upper bounded at each scale by some given scale-dependent quantity (see (4.99)). On the one hand, adding these events does not change the measure with high probability (Lemma 4.43). On the other hand, the measure restricted to the good events has a finite second moment which varies smoothly with respect to the thickness parameter (Lemma 4.45).

In the entire section, we will fix a set $A \Subset D$ compactly included in D. We will always restrict our attention to points lying in A and the estimates that we obtain may depend on A. We will only provide the proof of Proposition 4.26 in the case $\rho < a$ (which is in fact all that we use). The case $\rho > a$ would be similar as we have assumed $a > 0$.

We start by defining the "**good events**" that we will work with. For any countable collection $\mathcal{C}$ of Brownian-like loops and for any $r > 0$ and $z \in D$, we define $N_{z,r}^{\mathcal{C}}$ to be the number of crossings from $\partial D(z,r)$ to $\partial D(z,er)$ in $\mathcal{C}$ (upward crossings, we do not count the way back). That is, $N_{z,r}^{\mathcal{C}} = \sum_{\wp\in\mathcal{C}} N_{z,r}^{\wp}$, and $N_{z,r}^{\wp}$ is the number of upcrossings of the interval $[r, er]$ by the function $|\wp(\cdot) - z|$. Note that this is an admissible functional of $\mathcal{C}$ and z.

Recall that the parameter $a \in (0,2)$ is the thickness parameter which is fixed throughout this paper. We now choose $a < b < 2$ sufficiently close to a (in a way which will be specified later). Let $r_0 \in (0,1)$

be small. For a given $z \in D$, we consider the good event

$$\mathcal{G}_K(z) := \left\{ \forall r \in \{e^{-n}, n \geq 1\} \cap (0, r_0) : N_{z,r}^{\mathcal{L}_D^\theta(K)} \leq b(\log r)^2 \right\}. \tag{4.99}$$

As will be clear from what follows from Lemma 4.44, for typical (in the sense of $\mathcal{M}_\rho^K$) points, we expect the number of crossings to be roughly $a(\log r)^2$, since the aspect ratio of the annulus is e. Given these good events, we also define the modified version of $\mathcal{M}_\rho^K$, $\rho \in (0, a]$ as follows:

$$\tilde{\mathcal{M}}_\rho^K(dz) := \mathbf{1}_{\mathcal{G}_K(z)} \mathcal{M}_\rho^K(dz). \tag{4.100}$$

Note that we use the same parameter b in the definition of the good event for *all* $\rho \leq a$ above.

Proposition 4.26 will follow quickly from the following intermediate results.

Lemma 4.43. *There exists $w_1 : (0,1) \to (0,\infty)$ such that $w_1(r_0) \to 0$ as $r_0 \to 0$ and such that for all bounded measurable function $f : D \to \mathbb{R}$ with compact support included in A, for all $\rho \in [a/2, a]$ and $K \geq 1$,*

$$\mathbb{E}\left[\left| \int_D f(z) \mathcal{M}_\rho^K(dz) - \int_D f(z) \tilde{\mathcal{M}}_\rho^K(dz) \right| \right] \leq w_1(r_0) \left\| f \right\|_\infty (\log K)^\theta.$$

To analyse the behaviour of $\tilde{\mathcal{M}}_\rho^K$, a key role will be played by the following estimate.

Lemma 4.44. *Let $\eta \in [0, 2-a)$. If b is close enough to a, then*

$$\sup_{\rho \in [a/2,a]} \sup_{K \geq 1} \frac{1}{(\log K)^{2\theta}} \int_{A \times A} \frac{1}{|z - z'|^\eta} \mathbb{E}\left[\tilde{\mathcal{M}}_\rho^K(dz) \tilde{\mathcal{M}}_\rho^K(dz') \right] < \infty.$$

Together with Frostman's lemma, this essentially shows that any set S which supports $\mathcal{M}_a$ (or, more precisely, $\tilde{\mathcal{M}}_a$ but this has no impact by Lemma 4.43) has dimension at least $2-a$. We will also use this estimate (with $\eta = 0$) to show the following control, which is the main required estimate for Proposition 4.26.

Lemma 4.45. *Let $r_0 \in (0,1)$ be fixed. If b is close enough to a, then*

$$\limsup_{\rho \to a^-} \limsup_{K \geq 1} \sup_f \left\| f \right\|_\infty^{-2} (\log K)^{-2\theta} \mathbb{E}\left[\left(\int_D f(z) \tilde{\mathcal{M}}_\rho^K(dz) - \int_D f(z) \tilde{\mathcal{M}}_a^K(dz) \right)^2 \right] = 0,$$

where the supremum is over all bounded, non-zero, non-negative measurable function $f : D \to [0,\infty)$ with compact support included in A.

Let us first briefly check that Lemmas 4.43 and 4.45 allow us to conclude the proof of Proposition 4.26.

Proof of Proposition 4.26. Let $f : D \to \mathbb{R}$ be a bounded measurable function with compact support included in A and let $K \geq 1$, $\rho \in [a/2, a]$. By Lemma 4.43,

$$\begin{aligned} &\mathbb{E}\left[\left| \int_D f(z) \mathcal{M}_\rho^K(dz) - \int_D f(z) \mathcal{M}_a^K(dz) \right| \right] \\ &\leq 2 w_1(r_0) \left\| f \right\|_\infty (\log K)^\theta + \mathbb{E}\left[\left| \int_D f(z) \tilde{\mathcal{M}}_\rho^K(dz) - \int_D f(z) \tilde{\mathcal{M}}_a^K(dz) \right| \right]. \end{aligned}$$

Lemma 4.45 and Cauchy–Schwarz allow us to control the second right hand side term, so that

$$\limsup_{\rho \to a} \limsup_{K \to \infty} \sup_f \frac{1}{\|f\|_\infty (\log K)^\theta} \mathbb{E}\left[\left| \int_D f(z) \mathcal{M}_\rho^K(dz) - \int_D f(z) \mathcal{M}_a^K(dz) \right| \right] \leq 2 w_1(r_0).$$

Since the left hand side term is independent of r_0, by letting $r_0 \to 0$, we deduce that it vanishes. This concludes the proof. □

The rest of this section is devoted to the proof of the three intermediate lemmas.

4.7.1 Number of crossings in the processes of excursions

We start by studying the number of crossings in the processes of excursions that appear in the second moment computations. Recall that these processes are defined in Notation 4.20 and in (4.81). Let $z, z' \in D$ and $r > 0$ be such that $|z - z'| > er$. We are going to study $N_{z,r}^{\mathcal{C}}$ for $\mathcal{C} = \Xi_a^z, \Xi_a^{z'}$ or $\Xi_{a,a'}^{z,z'}$. We start off with the first two variables. We can decompose

$$N_{z,r}^{\Xi_a^z} = \sum_{i=1}^{P} G_i \quad \text{and} \quad N_{z,r}^{\Xi_a^{z'}} = \sum_{i=1}^{P'} G_i'$$

where P (resp. P') is the Poisson random variable corresponding to the number of excursions in Ξ_a^z (resp. $\Xi_a^{z'}$) that touch $\partial D(z, er)$ (resp. $\partial D(z, r)$) and G_i (resp. G_i'), $i \geq 1$, are i.i.d. random variables, independent of P (resp. P'), and distributed according to the number of crossings from $\partial D(z, r)$ to $\partial D(z, er)$ in a path distributed according to $\mu_D^{z,z}(\cdot | \tau_{z,er} < \infty)$ (resp. $\mu_D^{z',z'}(\cdot | \tau_{z,r} < \infty)$)

If the domain D were a disc centred at z, then, by rotational invariance and Markov property, the G_i's would be geometric random variables. In general, this is only asymptotically true as $r \to 0$. We recall that we fix a set $A \Subset D$ compactly included in D during the whole Section 4.7.

Lemma 4.46. *1. P and P' are Poisson random variables with means given by*

$$\mathbb{E}[P] = a \log \frac{\mathrm{CR}(z, D)}{er} \quad \textit{and} \quad \mathbb{E}[P'] = a \log \mathrm{CR}(z', D) - a \xi_{D \setminus D(z,r)}(z', z')$$

where $w \mapsto \xi_{D \setminus D(z,r)}(z', w)$ is the harmonic extension of $w \in \partial D \cup \partial D(z, r) \mapsto \log |z' - w|$ in the domain $D \setminus D(z, r)$. In particular, for all $z, z' \in A, r > 0$ such that $e^2 \leq |z - z'|/r \leq e^3$,

$$\mathbb{E}[P] = a \log \frac{1}{r} + O(1) \quad \textit{and} \quad \mathbb{E}[P'] = a \log \frac{1}{r} + O(1). \tag{4.101}$$

2. Let $z, z' \in A, r > 0$ be such that $e^2 \leq |z - z'|/r \leq e^3$. The random variable G_i is stochastically dominated by G_+ and stochastically dominates G_- where $G_\pm$ are geometric random variables with success probabilities

$$p_\pm = \frac{1 + o(1)}{|\log r|}.$$

There exist $C_+, C_- \in \mathbb{R}$, $u_+(r), u_-(r) \in \mathbb{R}$ that go to zero as $r \to 0$ such that for all $k \geq 1$,

$$\left(1 - \frac{1+u_-(r)}{|\log r|}\right)^{k-1} \left(1 + \frac{C_-}{\log r}\right) \leq \mathbb{P}\left(G_i' \geq k\right) \leq \left(1 - \frac{1+u_+(r)}{|\log r|}\right)^{k-1} \left(1 + \frac{C_+}{\log r}\right). \quad (4.102)$$

(The quantities $C_+, C_-, u_+(r), u_-(r)$ and the implicit constants in $O(1)$ and $o(1)$ may depend on A.)

Proof. 1. We will rely on the following (probably well known) fact about Green function in a domain U (which however may be a non simply connected domain) with Dirichlet boundary conditions on ∂U: we claim that

$$G_U(z,w) = -\frac{1}{2\pi} \log|z-w| + \xi_U(z,w), \quad (4.103)$$

where $\xi_U(z,\cdot)$ is the harmonic extension of $\frac{1}{2\pi}\log|z-\cdot|$ from ∂U to U. Furthermore, when U is simply connected and $z = w$ then $\xi_U(z,z) = \frac{1}{2\pi}\log \mathrm{CR}(z,U)$ (see, e.g., (1.4) in [BP21]). To see this, observe that the difference between the two functions on the left and on the right hand sides of (4.103) is harmonic in w (except possibly at $w = z$) and is at most $o(\log|z-w|)$ when $w \to z$ (for instance, one may use domain monotonicity to see this). This difference also has zero boundary condition on ∂U. An application of the optional stopping theorem therefore shows that this difference is identically zero on U.

We obtain the mean of P by considering all trajectories that start from z and leave $D(z,er)$; equivalently we can subtract from all trajectories those that stay in $D(z,er)$ and get the desired asymptotics from Dirichlet Green function asymptotics:

$$\begin{aligned} 2\pi a \mu_D^z(\tau_{z,er} < \infty) &= 2\pi a \lim_{w\to z} \mu_D^{z,w}(\tau_{z,er} < \infty) \\ &= 2\pi a \lim_{w \to z} G_D(z,w) - G_{D(z,er)}(z,w) = a \log \frac{\mathrm{CR}(z,D)}{er}. \end{aligned}$$

The mean of P' can be computed similarly using (4.103). (4.101) then follows.

2. Consider a Brownian motion starting from a point on $\partial D(z,er)$, conditioned to hit z before exiting D. This is a Markov process (it can be described through a certain h-transform, where $h(x) = G_D(x,z)$). By the strong Markov property and elementary properties of h-transforms, we can stochastically dominate G_i by a geometric random variable whose success probability is given by

$$p_+ := 1 - \min_{x \in \partial D(z,r)} \frac{\mu_D^{x,z}(\tau_{z,er} < \infty)}{G_D(x,z)} = 1 - \min_{x\in\partial D(z,r)} \frac{1}{G_D(x,z)} \mathbb{E}_x\left[G_D(X_{\tau_{z,er}}, z)\mathbf{1}_{\{\tau_{z,er}<\infty\}}\right].$$

Here $\mathbb{E}_x$ denotes the expectation with respect to a Brownian motion starting from x. Hence

$$p_+ = 1 - \frac{\log\frac{\mathrm{CR}(z,D)}{er} + o(1)}{\log\frac{\mathrm{CR}(z,D)}{r} + o(1)} = \frac{1}{\log\frac{\mathrm{CR}(z,D)}{r}} + o\left(\frac{1}{\log r}\right)$$

where the $o(1)$ terms are uniform over z restricted to A. The lower bound is similar with minima replaced by maxima.

We now turn to the case of G_i'. For all $k \geq 1$, using again elementary properties of the h-transform,

$$\begin{aligned} \mathbb{P}\left(G_i' \geq k\right) &\geq \min_{x \in \partial D(z,er)} \mathbb{P}_x\left(\tau_{z,r} < \tau_{\partial D}\right)^{k-1} \min_{y \in \partial D(z,r)} \frac{G_D(y,z')}{G_D(x,z')} \\ &= \left(1 - \frac{1+o(1)}{|\log r|}\right)^{k-1} \left(1 + \frac{O(1)}{\log r}\right), \end{aligned}$$

as desired. □

We now state three corollaries of Lemma 4.46. The first corollary will be used in the proof of Lemma 4.43 whereas the third one will be used in the proof of Lemma 4.44. The second one will be useful in order to show that $\mathcal{M}_a$ is supported by $\mathcal{T}(a)$ almost surely (Theorem 4.11). We will only prove the first corollary, since it is the most difficult one to prove and the proofs of the other two only require small adaptations.

Note that, in Corollary 4.47, we will need to take into account the killing associated to the mass. On the other hand, in Corollaries 4.49 and 4.48, this will not be necessary thanks to FKG-inequality for Poisson point processes (see [Jan84, Lemma 2.1]).

Corollary 4.47. *Let $u \in (0,1/2)$. There exists $C(u) > 0$ such that for all $z \in A$, $r \in (0,1)$ and $\rho > 0$,*

$$\mathbb{E}\left[\left(1 - e^{-KT(\Xi_\rho^z)}\right) e^{\frac{u}{|\log r|} N_{z,r}^{\Xi_\rho^z}}\right] \leq \left(1 - e^{-\rho(3/2C_K(z) + C(u)|\log r|)}\right) \exp\left(\rho \frac{u}{1-u}(1+o(1))|\log r|\right) \tag{4.104}$$

where $o(1) \to 0$ as $r \to 0$ and may depend on u and A.

By FKG-inequality for Poisson point processes, the expectation on the left hand side of (4.104) is at least the product of the expectation of each of the two terms which behaves like (as we will see in the proof below)

$$\left(1 - e^{-\rho C_K(z))}\right) \exp\left(\rho \frac{u}{1-u}(1+o(1))|\log r|\right).$$

The content of Corollary 4.47 is therefore that upper bound matches the lower bound with the only difference that $C_K(z)$ becomes the larger value $3/2C_K(z) + C(u)|\log r|$.

Proof. Since $\frac{u}{|\log r|} N_{z,r}^{\Xi_\rho^z}$ and $KT(\Xi_\rho^z)$ are additive functions of Ξ_ρ^z, Palm formula gives that the left hand side of (4.104) is equal to

$$\begin{aligned} &\exp\left(2\pi\rho \int \mu_D^{z,z}(d\wp)\left(e^{\frac{u}{|\log r|}N_{z,r}^{\wp}} - 1\right)\right) - \exp\left(2\pi\rho \int \mu_D^{z,z}(d\wp)\left(e^{\frac{u}{|\log r|}N_{z,r}^{\wp} - KT(\wp)} - 1\right)\right) \\ &= \mathbb{E}\left[e^{\frac{u}{|\log r|}N_{z,r}^{\Xi_\rho^z}}\right]\left(1 - \exp\left(2\pi\rho \int \mu_D^{z,z}(d\wp) e^{\frac{u}{|\log r|}N_{z,r}^{\wp}}\left(e^{-KT(\wp)} - 1\right)\right)\right). \end{aligned}$$

Our goal now is to bound from above

$$2\pi \int \mu_D^{z,z}(d\wp) e^{\frac{u}{|\log r|}N_{z,r}^{\wp}}\left(1 - e^{-KT(\wp)}\right).$$

We can rewrite it as

$$C_K(z) + 2\pi \int \mu_D^{z,z}(d\wp) \left(e^{\frac{u}{|\log r|} N_{z,r}^{\wp}} - 1 \right) \left(1 - e^{-KT(\wp)} \right)$$

and by bounding for $x > 1$ and $y \in (0,1)$, $(x-1)(1-y) \leq ((x-1)^2 + (1-y)^2)/2 \leq (x^2-1)/2 + (1-y)/2$, we obtain that it is at most

$$C_K(z) + \frac{1}{2} C_K(z) + \frac{1}{2} 2\pi \int \mu_D^{z,z}(d\wp) \left(e^{\frac{2u}{|\log r|} N_{z,r}^{\wp}} - 1 \right).$$

We denote G a random variable whose law is given by $N_{z,r}^{\wp}$ where $\wp$ is a trajectory distributed according to $\mu_D^{z,z}(\cdot | \tau_{z,er}) / \mu_D^{z,z}(\tau_{z,er} < \infty)$. Thanks to Lemma 4.46 point 2, an easy computation with geometric random variables shows that

$$\mathbb{E}\left[e^{\frac{2u}{|\log r|} G} - 1 \right] = \frac{1}{1-2u} - 1 + o(1) = \frac{2u}{1-2u} + o(1).$$

With Lemma 4.46 point 1, this implies that

$$2\pi \int \mu_D^{z,z}(d\wp) \left(e^{\frac{2u}{|\log r|} N_{z,r}^{\wp}} - 1 \right) = 2\pi \mu_D^{z,z}(\tau_{z,er} < \infty) \mathbb{E}\left[e^{\frac{2u}{|\log r|} G} - 1 \right] \leq C(u) |\log r|.$$

The same reasoning shows that

$$\mathbb{E}\left[e^{\frac{u}{|\log r|} N_{z,r}^{\Xi_\rho^z}} \right] = \exp\left(\rho \frac{u}{1-u} (1 + o(1)) |\log r| \right).$$

Wrapping things up, we have proven that the left hand side of (4.104) is at most

$$\left(1 - e^{-\rho(3/2 C_K(z) + C(u)|\log r|)} \right) \exp\left(\rho \frac{u}{1-u} (1 + o(1)) |\log r| \right).$$

This concludes the proof. □

Corollary 4.48. *There exist $\gamma > 0$ and $r_0 > 0$ that may depend on a, b and A such that for all $r \in (0, r_0)$ and $z \in A$,*

$$\mathbb{P}\left(N_{z,r}^{\Xi_a^z} < \left\{ a - \frac{b-a}{2} \right\} (\log r)^2 \right) \leq r^\gamma. \tag{4.105}$$

Corollary 4.49. *Let $a > 0$, $z, z' \in A, r > 0$ be such that $e^2 \leq |z - z'|/r \leq e^3$. Fix a parameter $u > 0$. Then,*

$$\mathbb{E}\left[\exp\left(-\frac{u}{|\log r|} N_{z,r}^{\Xi_a^z} \right) \right] = \exp\left(-a \frac{u}{1+u} (1 + o(1)) |\log r| \right) \tag{4.106}$$

and

$$\mathbb{E}\left[\exp\left(-\frac{u}{|\log r|} N_{z,r}^{\Xi_a^{z'}} \right) \right] = \exp\left(-a \frac{u}{1+u} (1 + o(1)) |\log r| \right) \tag{4.107}$$

where the $o(1)$ terms tend to 0 as $r \to 0$ and may depend on A and u.

We now move on to the study of $N_{z,r}^{\Xi_{a,a'}^{z,z'}}$, again in the setting where $e^2 \leq |z - z'|/r \leq e^3$. It is convenient to first view the trajectories in $\Xi_{a,a'}^{z,z'}$ as excursions from z' to z (rather than a mixture of equal number of excursions going from z to z' and vice-versa). When we time-reverse an excursion, an upcrossing becomes a downcrossing. Since two upcrossings are necessarily separated by a downcrossing, the error in counting the upcrossings when we fix the direction of the excursion as being from z' to z is at most 1. We can decompose

$$N_{z,r}^{\Xi_{a,a'}^{z,z'}} = \sum_{i=1}^{\#\Xi_{a,a'}^{z,z'}} G_i^z(r)$$

where $G_i^z(r)$, $i \geq 1$, are i.i.d. random variables, independent of $\#\Xi_{a,a'}^{z,z'}$, that correspond to the number of crossings from $\partial D(z,r)$ to $\partial D(z,er)$ for a trajectory distributed according to $\mu_D^{z',z}/G_D(z',z)$. In the same vein as in Lemma 4.46, $1 + G_i^z(r)$ dominates and is dominated stochastically a geometric random variable with success probability

$$p = \frac{1+o(1)}{|\log r|}. \tag{4.108}$$

Note in particular that since $r \to 0$, $p \to 0$.

When we then consider the quantity which is really of interest to us, i.e., the number of crossings of the annulus $D(z,er) \setminus D(z,r)$ associated with the *loops* coming from concatenating the pairs of excursions in $\Xi_{a,a'}^{z,z'}$, the resulting error from having considered the upcrossings of the reverse excursions instead of those of the original excursions in (4.108) is therefore negligible.

In particular, we obtain:

Lemma 4.50. *Let $a, a' > 0$, $z, z' \in A, r > 0$ be such that $e^2 \leq |z - z'|/r \leq e^3$. Fix a parameter $u > 0$. Then*

$$\mathbb{E}\left[\exp\left(-\frac{u}{|\log r|} N_{z,r}^{\Xi_{a,a'}^{z,z'}}\right)\right] = \frac{\mathsf{B}\left((2\pi)^2 aa' G_D(z,z')^2 \frac{1+o(1)}{(1+u)^2}\right)}{\mathsf{B}\left((2\pi)^2 aa' G_D(z,z')^2\right)}, \tag{4.109}$$

where the $o(1)$ term tends to 0 as $r \to 0$ (and may depend on A and $u > 0$).

Proof. For all $c > 0$, we have trivially

$$\mathbb{E}\left[c^{\#\Xi_{a,a'}^{z,z'}}\right] = \frac{\mathsf{B}\left((2\pi)^2 aa' G_D(z,z')^2 c^2\right)}{\mathsf{B}\left((2\pi)^2 aa' G_D(z,z')^2\right)},$$

where we have used the definition of $\Xi_{a,a'}^{z,z'}$ in (4.81) and the definition of B just above. Therefore, applying this with $c = \mathbb{E}(e^{-uG/(|\log r|)})$ (where G is the geometric random variable coming from (4.108)) concludes the proof. □

4.7.2 Proof of Lemma 4.43 (typical points are not thick)

Before we begin the proof of Lemma 4.43, we will require an estimate which says that a Lebesgue-typical, fixed point z is not thick for the measure $\mathcal{M}_a^K$. We will need to show this in a somewhat quantitative way, and uniformly in K. For orientation, the number of crossings $N_{z,r}^{\mathcal{L}_D^\theta}$ of the annulus of scale r around z roughly corresponds to the local time regularised at scale r around z accumulated by $\mathcal{L}_D^\theta$, and so is roughly of the order of the square of the GFF. For a typical point, we expect this to be roughly $\log 1/r$. For a Liouville typical point, this would instead be of the order of $(\log 1/r)^2$. The deviation probability below may thus be expected to decay polynomially. Let us finally mention that it will be important for us to nail the right exponent in order to obtain the upper bound on the dimension of the set $\mathcal{T}(a)$ of a-thick points (Theorem 4.11).

Lemma 4.51. *For any $\lambda \in (0,1)$, there exists $r_\lambda > 0$ such that for all $r \in (0, r_\lambda)$, $z \in D$ and $u > 0$,*

$$\mathbb{P}\left(N_{z,r}^{\mathcal{L}_D^\theta} \geq u (\log r)^2 \right) \leq r^{\lambda u}. \tag{4.110}$$

Proof of Lemma 4.51. First of all, $N_{z,r}^{\mathcal{L}_D^\theta}$ is stochastically dominated by $N_{z,r}^{\mathcal{L}_U^\theta}$ where U is the disc centred at z with radius being equal to the diameter of D. Without loss of generality, we can therefore assume that the domain D is the unit disc $\mathbb{D}$ and that z is the origin. In the remaining of the proof, we will write $N_r^\cdot$ instead of $N_{z,r}^\cdot$.

For $0 < r_1 < r_2$, we will denote by $A(r_1, r_2)$ the annulus $r_2\mathbb{D} \setminus r_1\mathbb{D}$. For all $k = 1, \dots, k_{\max} := \lfloor -\log r \rfloor - 1$, consider the set of "loops at scale k"

$$\mathcal{L}_k := \{ \wp \in \mathcal{L}_\mathbb{D}^\theta : \wp \subset e^{k+1} r \mathbb{D}, \wp \text{ crosses } A(e^{k-1/2} r, e^k r) \}.$$

We can decompose

$$N_r^{\mathcal{L}_D^\theta} = \sum_{k=1}^{k_{\max}} \sum_{\wp \in \mathcal{L}_k} N_r^\wp.$$

We now make three observations. Firstly, by thinning property of Poisson point processes, $\mathcal{L}_k$, $k = 1 \dots k_{\max}$, are independent collections of loops. Secondly, conditioned on $\#\mathcal{L}_k$, $\mathcal{L}_k$ is composed of $\#\mathcal{L}_k$ i.i.d. loops with law

$$\frac{\mathbf{1}_{\{\wp \subset e^{k+1} r\mathbb{D}, \wp \text{ crosses } A(e^{k-1/2} r, e^k r)\}} \mu_\mathbb{D}^{\text{loop}}(d\wp)}{\mu_\mathbb{D}^{\text{loop}}(\{\wp \subset e^{k+1} r\mathbb{D}, \wp \text{ crosses } A(e^{k-1/2} r, e^k r)\})}. \tag{4.111}$$

Finally, for each k, $\#\mathcal{L}_k$ is a Poisson random variable whose mean is, by scaling invariance of the Brownian loop measure, given by

$$\mu_{e^{k+1} r\mathbb{D}}^{\text{loop}}(\{\wp \text{ crosses } A(e^{k-1/2} r, e^k r)\}) = \mu_\mathbb{D}^{\text{loop}}(\{\wp \text{ crosses } A(e^{-3/2}, e^{-1})\}).$$

Therefore $\mathbb{E}[\#\mathcal{L}_k]$ is a finite quantity that does not depend on k or r. Let $P_k, k = 1 \dots k_{\max}$, be i.i.d. Poisson random variables with the above mean. We have decomposed

$$N_r^{\mathcal{L}_D^\theta} = \sum_{k=1}^{k_{\max}} \sum_{i=1}^{P_k} N_r^{\wp_i^k}$$

where for all k and i, $\wp_i^k$ are independent and distributed according to (4.111). Let $\lambda \in (0,1)$ be a parameter. We have

$$\mathbb{E}\left[\exp\left(\frac{\lambda}{|\log r|} N_r^{\mathcal{L}_D^\theta} \right) \right] = \prod_{k=1}^{k_{\max}} \exp\left\{ \mathbb{E}[P] \left(\mathbb{E}\left[e^{\frac{\lambda}{|\log r|} N_r^{\wp^k}} \right] - 1 \right) \right\}. \tag{4.112}$$

The rest of the proof is dedicated to showing that for all $k = 1 \dots k_{\max}$,

$$\mathbb{E}\left[e^{\frac{\lambda}{|\log r|} N_r^{\wp^k}} \right] \leq 1 + C_\lambda / |\log r| \tag{4.113}$$

for some constant C_λ depending only on λ. Indeed, this will imply that

$$\mathbb{E}\left[\exp\left(\frac{\lambda}{|\log r|} N_r^{\mathcal{L}_D^\theta} \right) \right] \leq e^{\mathbb{E}[P] C_\lambda}$$

and the proof of Lemma (4.51) will be completed by Markov inequality.

We now turn to the proof of (4.113). Let $k \in \{1, \dots, k_{\max}\}$. We are going to describe the law (4.111) by rooting the loop $\wp^k$ at the unique point z where its modulus is maximal. We will denote $R = |z|$ and w the first hitting point of $e^{k-1/2} r \mathbb{D}$. The law (4.111) can be disintegrated as

$$\frac{1}{Z_k} \int_{e^k r}^{e^{k+1} r} R \, \mathrm{d}R \int_{R\partial\mathbb{D}} \frac{\mathrm{d}z}{2\pi R} \mathbf{1}_{\{\wp \text{ crosses } A(e^{k-1/2} r, e^k r)\}} \mu_{R\mathbb{D}}^{z,z}(d\wp),$$

where the measure $\mu_{R\mathbb{D}}^{z,z}(d\wp)$ is given by (4.19) and Z_k is the normalising constant. This decomposition is somewhat similar to [LW04, Proposition 8]. Further, the measure $\mathbf{1}_{\{\wp \text{ crosses } A(e^{k-1/2} r, e^k r)\}} \mu_{R\mathbb{D}}^{z,z}(d\wp)$ is the image of the measure

$$\int_{e^{k-1/2} r \partial\mathbb{D}} \mathrm{d}w \; \mu_{A(e^{k-1/2} r, R)}^{z,w}(d\wp_1) \mu_{R\mathbb{D}}^{w,z}(d\wp_2)$$

under the concatenation $(\wp_1, \wp_2) \mapsto \wp_1 \wedge \wp_2$. This is similar to decompositions appearing in [Law05, Section 5.2]. Moreover, in this decomposition, $N_r^{\wp_1 \wedge \wp_2} = N_r^{\wp_2}$. It follows that for any bounded measurable function $F : \mathbb{R} \to \mathbb{R}$, we have

$$\mathbb{E}\left[F(N_r^{\wp^k}) \right] = \frac{1}{Z_k} \int_{e^k r}^{e^{k+1} r} R \, \mathrm{d}R \int_{R\partial\mathbb{D}} \frac{\mathrm{d}z}{2\pi R} \int_{e^{k-1/2} r \partial\mathbb{D}} \mathrm{d}w H_{A(e^{k-1/2} r, R)}(z, w) H_{R\mathbb{D}}(w, z) \mathbb{E}_{R\mathbb{D}}^{w,z}[F(N_r^\wp)] \tag{4.114}$$

where $\mathbb{E}_{R\mathbb{D}}^{w,z}$ is the expectation associated to the law $\mu_{R\mathbb{D}}^{w,z}(\cdot)/H_{R\mathbb{D}}(w,z)$ (4.17) and Z_k is the normalising constant

$$Z_k = \int_{e^k r}^{e^{k+1} r} R \, \mathrm{d}R \int_{R\partial\mathbb{D}} \frac{\mathrm{d}z}{2\pi R} \int_{e^{k-1/2} r \partial\mathbb{D}} \mathrm{d}w H_{A(e^{k-1/2} r, R)}(z, w) H_{R\mathbb{D}}(w, z).$$

Let $n \geq 1$ and denote $\mathbb{P}^w$ the law of planar Brownian motion $(B_t)_{t \geq 0}$ starting from w and τ_n the first time that $r\partial\mathbb{D}$ is reached after having already crossed the annulus $A(r, er)$ $n-1$ times in the upward direction. We also denote by $\tau_{R\partial\mathbb{D}}$ the first hitting time of $R\partial\mathbb{D}$. The conditional law $\mathbb{P}_{R\mathbb{D}}^{w,z}$ can be expressed as an h-transform of $\mathbb{P}^w$ as follows:

$$\mathbb{P}_{R\mathbb{D}}^{w,z}(N_r^\wp \geq n) = \mathbb{E}^w \left[\frac{H_{R\mathbb{D}}(B_{\tau_n}, z)}{H_{R\mathbb{D}}(w, z)} \mathbf{1}_{\{\tau_n < \tau_{R\partial\mathbb{D}}\}} \right].$$

Therefore,

$$\mathbb{P}_{R\mathbb{D}}^{w,z}(N_r^\wp \geq n) \leq \frac{\max_{|y|=r} H_{R\mathbb{D}}(y, z)}{H_{R\mathbb{D}}(w, z)} \mathbb{P}^w \left(\tau_n < \tau_{R\partial\mathbb{D}} \right). \tag{4.115}$$

Since $(\log|B_t|)_{t\geq 0}$ is a martingale, for all $0<r_1<r_2<r_3$, for all $x \in r_2\partial\mathbb{D}$, we have

$$\mathbb{P}^x\left(\tau_{r_1\partial\mathbb{D}} < \tau_{r_3\partial\mathbb{D}}\right) = \frac{\log(r_3/r_2)}{\log(r_3/r_1)}$$

and by strong Markov property, we deduce that

$$\mathbb{P}^w\left(\tau_n < \tau_{R\partial\mathbb{D}}\right) = \frac{\log(R/e^{k-1/2}r)}{\log(R/r)}\left(\frac{\log(R/er)}{\log(R/r)}\right)^{n-1}.$$

Moreover, by Harnack inequality, the ratio of Poisson kernels in (4.115) can be bounded by some constant independent of k and r. Recalling that $R \in [e^k r, e^{k+1} r]$, this shows that

$$\mathbb{P}^{w,z}_{R\mathbb{D}}(N_r^{\wp} \geq n) \leq \frac{C}{k}\left(1-\frac{1}{k}\right)^{n-1}.$$

Going back to (4.114), we have proven that when $\wp^k$ is distributed according to (4.111), then for all $n\geq 1$,

$$\mathbb{P}\left(N_r^{\wp^k} \geq n\right) \leq \frac{C}{k}\left(1-\frac{1}{k}\right)^{n-1}.$$

Since $\lambda<1$ and $k \leq |\log r|$, we deduce from the above bound that

$$\mathbb{E}\left[e^{\frac{\lambda}{|\log r|}N_r^{\wp^k}}\right] \leq 1 + \frac{C}{k}\left(-\frac{|\log r|}{\lambda}\log\left(1-\frac{1}{k}\right)-1\right)^{-1} \leq 1 + \frac{C_\lambda}{|\log r|}$$

for some constant C_λ that depends only on λ. This proves (4.113) and concludes the proof of Lemma 4.51. □

We are now ready to prove Lemma 4.43.

Proof of Lemma 4.43. Let $f: D \to \mathbb{R}$ be a bounded measurable function with compact support included in A, $\rho \in [a/2,a]$ and $K \geq 1$. By a union bound, we have

$$\mathbb{E}\left[\left|\int_D f(z)\mathcal{M}_\rho^K(dz) - \int_D f(z)\tilde{\mathcal{M}}_\rho^K(dz)\right|\right] \leq \sum_{\substack{r=e^{-n}\\ n\geq\lceil\log(1/r_0)\rceil}} \int_D |f(z)|\,\mathbb{E}\left[\mathcal{M}_\rho^K(dz)\mathbf{1}_{\left\{N_{z,r}^{\mathcal{L}_D^\theta(K)}>b(\log r)^2\right\}}\right].$$

Let $r=e^{-n}$ for some $n \geq \lceil \log(1/r_0)\rceil$. By Lemma 4.32, we have

$$\mathbb{E}\left[\mathcal{M}_\rho^K(dz)\mathbf{1}_{\left\{N_{z,r}^{\mathcal{L}_D^\theta(K)}>b(\log r)^2\right\}}\right] = \mathrm{CR}(z,D)^a\sum_{n\geq 1}\frac{\theta^n}{n!}\int_{\mathsf{a}\in E(a,n)}\frac{\mathrm{d}\mathsf{a}}{a_1\dots a_n} \tag{4.116}$$
$$\times\,\mathbb{E}\left[\prod_{i=1}^n\left(1-e^{-KT(\Xi^z_{a_i})}\right)\mathbf{1}_{\left\{\sum_{i=1}^n N_{z,r}^{\Xi^z_{a_i}}+N_{z,r}^{\mathcal{L}_D^\theta(K)}>b(\log r)^2\right\}}\right]\mathrm{d}z.$$

Let $u\in(0,1/2)$ be a parameter. By an exponential Markov inequality, we can bound the above indicator function by

$$\mathbf{1}_{\left\{\sum_{i=1}^n N_{z,r}^{\Xi_{a_i}^z} > \left(\rho + \frac{b-a}{2}\right)(\log r)^2\right\}} + \mathbf{1}_{\left\{N_{z,r}^{\mathcal{L}_D^\theta(K)} > \frac{b-a}{2}(\log r)^2\right\}}$$
$$\leq e^{-u\left(\rho+\frac{b-a}{2}\right)|\log r|} \prod_{i=1}^n \exp\left(\frac{u}{|\log r|} N_{z,r}^{\Xi_{a_i}^z}\right) + \mathbf{1}_{\left\{N_{z,r}^{\mathcal{L}_D^\theta(K)} > \frac{b-a}{2}(\log r)^2\right\}}.$$

By Corollary 4.47 and then by using the fact that $\sum a_i = \rho$, the expectation on the right hand side of (4.116) is therefore at most

$$e^{-u\left(\rho+\frac{b-a}{2}\right)|\log r|} \prod_{i=1}^n \left(1 - e^{-a_i(3/2C_K(z)+C(u)|\log r|)}\right) \exp\left(a_i \frac{u}{1-u}(1+o(1))|\log r|\right)$$
$$+ \prod_{i=1}^n \left(1 - e^{-a_i C_K(z)}\right) \mathbb{P}\left(N_{z,r}^{\mathcal{L}_D^\theta(K)} > \frac{b-a}{2}(\log r)^2\right)$$
$$= \exp\left\{-\left(u\left(\rho + \frac{b-a}{2}\right) - \rho \frac{u}{1-u}(1+o(1))\right)|\log r|\right\} \prod_{i=1}^n \left(1 - e^{-a_i(3/2C_K(z)+C(u)|\log r|)}\right) \tag{4.117}$$
$$+ \prod_{i=1}^n \left(1 - e^{-a_i C_K(z)}\right) \mathbb{P}\left(N_{z,r}^{\mathcal{L}_D^\theta(K)} > \frac{b-a}{2}(\log r)^2\right).$$

By choosing u small enough, we can ensure

$$u\left(\rho + \frac{b-a}{2}\right) - \rho \frac{u}{1-u} = \frac{b-a}{2}u - \rho \frac{u^2}{1-u}$$

to be strictly positive. Therefore, if r is small enough, the first exponential in (4.117) can be bounded by r^γ for some $\gamma > 0$ depending on u, a and b (recall that $\rho \in [a/2, a]$). We use Lemma 4.51 to bound the probability in (4.117) by r^γ for some $\gamma > 0$. With Lemma 4.33 we therefore see that

$$\mathbb{E}\left[\mathcal{M}_\rho^K(dz)\mathbf{1}_{\left\{N_{z,r}^{\mathcal{L}_D^\theta(K)} > b(\log r)^2\right\}}\right] \leq \frac{1}{\rho}\mathrm{CR}(z,D)^a r^\gamma \left(\mathsf{F}\left(\rho(3/2C_K(z) + C(u)|\log r|)\right) + \mathsf{F}\left(\rho C_K(z)\right)\right)$$

Using the inequality $\mathsf{F}(u) \leq Cu^\theta$, we conclude that

$$\mathbb{E}\left[\mathcal{M}_\rho^K(dz)\mathbf{1}_{\left\{N_{z,r}^{\mathcal{L}_D^\theta(K)} > b(\log r)^2\right\}}\right] \leq C(\log K)^\theta r^c$$

for some $C, c > 0$ that may depend on a, b and A. Finally,

$$\mathbb{E}\left[\left|\int_D f(z)\mathcal{M}_\rho^K(dz) - \int_D f(z)\tilde{\mathcal{M}}_\rho^K(dz)\right|\right]$$
$$\leq C\left\|f\right\|_\infty (\log K)^\theta \sum_{\substack{r=e^{-n} \\ n \geq \lceil \log(1/r_0) \rceil}} r^c \leq C\left\|f\right\|_\infty (\log K)^\theta (r_0)^c.$$

This concludes the proof. ☐

4.7.3 Proof of Lemma 4.44 (uniform integrability after truncation)

Proof. Let $\rho \in [a/2, a]$ and let $z, z' \in A$. The constants appearing in this proof may depend on a, b, r_0 and A, but will be uniform in z, z' and ρ. We want to bound from above

$$\mathbb{E}\left[\tilde{\mathcal{M}}_\rho^K(dz)\tilde{\mathcal{M}}_\rho^K(dz')\right].$$

If $|z-z'| \geq r_0$, we simply bound this by

$$\mathbb{E}\left[\tilde{\mathcal{M}}_\rho^K(dz)\tilde{\mathcal{M}}_\rho^K(dz')\right] \leq C(\log K)^{2\theta}\mathrm{d}z\mathrm{d}z'$$

by Corollary 4.42, where $C>0$ is some constant depending on r_0. We now assume that $|z-z'|<r_0$ and we let $r \in (0,r_0)\cap\{e^{-n}, n\geq 1\}$ be such that $e^2 \leq |z-z'|/r \leq e^3$. By Lemma 4.40, $\mathbb{E}\left[\tilde{\mathcal{M}}_\rho^K(dz)\tilde{\mathcal{M}}_\rho^K(dz')\right]$ is at most

$$C \sum_{\substack{n,m\geq 1 \\ 0\leq l\leq n\wedge m}} \frac{1}{(n-l)!(m-l)!l!}\theta^{n+m-l} \int_{\substack{\mathsf{a}\in E(\rho,n)\\ \mathsf{a}'\in E(\rho,m)}} \frac{\mathrm{d}\mathsf{a}}{a_1\dots a_n}\frac{\mathrm{d}\mathsf{a}'}{a_1'\dots a_m'} \prod_{i=1}^l \mathsf{B}\left((2\pi)^2 a_i a_i' G_D(z,z')^2\right)$$
$$\times \mathbb{E}\left[F\left((\Xi_{a_i,a_i'}^{z,z'} \wedge \Xi_{a_i}^z \wedge \Xi_{a_i'}^{z'})_{i=1\dots l}, (\Xi_{a_i}^z)_{i=l+1\dots n}, (\Xi_{a_i'}^{z'})_{i=l+1\dots m}\right)\right] \mathrm{d}z\mathrm{d}z'$$

with $F(\wp_1,\dots,\wp_n,\wp_{l+1}',\dots,\wp_m')$ being equal to

$$\prod_{i=1}^n \left(1-e^{-KT(\wp_i)}\right) \prod_{i=l+1}^m \left(1-e^{-KT(\wp_i')}\right) \mathbf{1}_{\left\{\sum_{i=1}^n N_{z,r}^{\wp_i} + \sum_{i=l+1}^m N_{z,r}^{\wp_i'} \leq b(\log r)^2\right\}}.$$

Now, let $u = 2\sqrt{\rho/b}-1$, which is positive if b is close enough to a, and observe that $F(\wp_1,\dots,\wp_n, \wp_{l+1}',\dots,\wp_m')$ is bounded from above by

$$F_u(\wp_1,\dots,\wp_n,\wp_{l+1}',\dots,\wp_m') := e^{bu|\log r|}\exp\left(-\frac{u}{|\log r|}\sum_{i=1}^n N_{z,r}^{\wp_i} - \frac{u}{|\log r|}\sum_{i=l+1}^m N_{z,r}^{\wp_i'}\right)$$
$$\times \prod_{i=l+1}^n \left(1-e^{-KT(\wp_i)}\right)\prod_{i=l+1}^m\left(1-e^{-KT(\wp_i')}\right).$$

Here we both neglect the killing part for $\wp_1\dots\wp_l$ and we bound the indicator function in the spirit of an exponential Markov inequality. We have

$$\mathbb{E}\left[F_u\left((\Xi_{a_i,a_i'}^{z,z'} \wedge \Xi_{a_i}^z \wedge \Xi_{a_i'}^{z'})_{i=1\dots l}, (\Xi_{a_i}^z)_{i=l+1\dots n}, (\Xi_{a_i'}^{z'})_{i=l+1\dots m}\right)\right]$$
$$= e^{bu|\log r|}\prod_{i=1}^l \mathbb{E}\left[\exp\left(-\tfrac{u}{|\log r|}N_{z,r}^{\Xi_{a_i,a_i'}^{z,z'}}\right)\right]\mathbb{E}\left[\exp\left(-\tfrac{u}{|\log r|}N_{z,r}^{\Xi_{a_i}^z}\right)\right]\mathbb{E}\left[\exp\left(-\tfrac{u}{|\log r|}N_{z,r}^{\Xi_{a_i'}^{z'}}\right)\right]$$
$$\prod_{i=l+1}^n \mathbb{E}\left[\left(1-e^{-KT(\Xi_{a_i}^z)}\right)\exp\left(-\tfrac{u}{|\log r|}N_{z,r}^{\Xi_{a_i}^z}\right)\right]\prod_{i=l+1}^m \mathbb{E}\left[\left(1-e^{-KT(\Xi_{a_i'}^{z'})}\right)\exp\left(-\tfrac{u}{|\log r|}N_{z,r}^{\Xi_{a_i}^{z'}}\right)\right].$$

By FKG-inequality for Poisson point processes (see [Jan84, Lemma 2.1]),

$$\mathbb{E}\left[\left(1-e^{-KT(\Xi^z_{a_i})}\right)\exp\left(-\tfrac{u}{|\log r|}N^{\Xi^z_{a_i}}_{z,r}\right)\right] \le \mathbb{E}\left[1-e^{-KT(\Xi^z_{a_i})}\right]\mathbb{E}\left[\exp\left(-\tfrac{u}{|\log r|}N^{\Xi^z_{a_i}}_{z,r}\right)\right]. \tag{4.118}$$

Recall that (see (4.106) and (4.107))

$$\mathbb{E}\left[\exp\left(-\tfrac{u}{|\log r|}N^{\Xi^z_{a_i}}_{z,r}\right)\right] = e^{-\frac{u+o(1)}{1+u}a_i|\log r|} \quad \text{and} \quad \mathbb{E}\left[\exp\left(-\tfrac{u}{|\log r|}N^{\Xi^{z'}_{a'_i}}_{z,r}\right)\right] = e^{-\frac{u+o(1)}{1+u}a'_i|\log r|}$$

and (see (4.109))

$$\mathsf{B}\left((2\pi)^2 a_i a'_i G_D(z,z')^2\right)\mathbb{E}\left[\exp\left(-\tfrac{u}{|\log r|}N^{\Xi^{z,z'}_{a_i,a'_i}}_{z,r}\right)\right] = \mathsf{B}\left((2\pi)^2 a_i a'_i G_D(z,z')^2\frac{1+o(1)}{(1+u)^2}\right).$$

The $o(1)$ above go to zero as $r\to 0$. In what follows, to ease notations, we will not write the $o(1)$. This is of no importance: alternatively, one can increase slightly the value of the thickness parameter ρ and absorb the $o(1)$ in doing so. We continue the computations and find that

$$\begin{aligned}
&\prod_{i=1}^l \mathsf{B}\left((2\pi)^2 a_i a'_i G_D(z,z')^2\right)\mathbb{E}\left[F\left((\Xi^{z,z'}_{a_i,a'_i}\wedge\Xi^z_{a_i}\wedge\Xi^{z'}_{a'_i})_{i=1\dots l},(\Xi^z_{a_i})_{i=l+1\dots n},(\Xi^{z'}_{a'_i})_{i=l+1\dots m}\right)\right]\\
&\le e^{bu|\log r|}\exp\left(-\frac{u}{1+u}\left(\sum_{i=1}^n a_i+\sum_{i=1}^m a'_i\right)|\log r|\right)\prod_{i=l+1}^n\left(1-e^{-a_iC_K(z)}\right)\prod_{i=l+1}^m\left(1-e^{-a'_iC_K(z')}\right)\\
&\times\prod_{i=1}^l\mathsf{B}\left((2\pi)^2a_ia'_iG_D(z,z')^2/(1+u)^2\right).
\end{aligned}$$

Since $\sum_{i=1}^n a_i = \sum_{i=1}^m a'_i = \rho$, we have found that $\mathbb{E}\left[\tilde{\mathcal{M}}^K_\rho(dz)\tilde{\mathcal{M}}^K_\rho(dz')\right]$ is at most

$$\begin{aligned}
&e^{\left(bu-2\rho\frac{u}{1+u}\right)|\log r|}\sum_{\substack{n,m\ge1\\0\le l\le n\wedge m}}\frac{1}{(n-l)!(m-l)!l!}\theta^{n+m-l}\int_{\substack{\mathsf{a}\in E(\rho,n)\\ \mathsf{a}'\in E(\rho,m)}}\mathrm{d}\mathsf{a}\mathrm{d}\mathsf{a}'\prod_{i=l+1}^n\frac{1-e^{-a_iC_K(z)}}{a_i}\\
&\times\prod_{i=l+1}^m\frac{1-e^{-a'_iC_K(z')}}{a'_i}\prod_{i=1}^l\frac{\mathsf{B}\left((2\pi)^2a_ia'_iG_D(z,z')^2/(1+u)^2\right)}{a_ia'_i}\mathrm{d}z\mathrm{d}z'\\
&=e^{\left(bu-2\rho\frac{u}{1+u}\right)|\log r|}\mathsf{H}_{\rho,\rho}(C_K(z),C_K(z'),(2\pi)^2G_D(z,z')^2/(1+u)^2)\mathrm{d}z\mathrm{d}z'
\end{aligned}$$

where the function $\mathsf{H}_{\rho,\rho}$ is defined in (4.92). By (4.93), we can further bound from above the expectation $\mathbb{E}\left[\tilde{\mathcal{M}}^K_\rho(dz)\tilde{\mathcal{M}}^K_\rho(dz')\right]$ by

$$C(\log K)^{2\theta}e^{\left(bu-2\rho\frac{u}{1+u}\right)|\log r|}G_D(z,z')^{1/2-\theta}e^{4\pi\rho G_D(z,z')/(1+u)}\mathrm{d}z\mathrm{d}z'.$$

Recalling that r has been chosen in such a way that $2\pi G_D(z,z') = |\log r| + O(1)$ and that $u = 2\sqrt{\rho/b} - 1$, we conclude that $\mathbb{E}\left[\tilde{\mathcal{M}}_\rho^K(dz)\tilde{\mathcal{M}}_\rho^K(dz')\right]$ is at most

$$C(\log K)^{2\theta}|\log r|^{1/2-\theta}\exp\left(\left(b - 2(\sqrt{b}-\sqrt{\rho})^2\right)|\log r|\right)\mathrm{d}z\mathrm{d}z' \leq C(\log K)^{2\theta}|z-z'|^{-b}\mathrm{d}z\mathrm{d}z'.$$

Since b can be made arbitrary close to a, this concludes the proof. □

4.7.4 Proof of Lemma 4.45 (convergence)

Proof. Assume that b is close enough to a so that Lemma 4.44 holds for some $\eta > 0$. Let $f : D \to [0,\infty)$ be a non-negative bounded measurable function with compact support included in A and let $a' \in [a/2, a]$. We have

$$\begin{aligned}
&\mathbb{E}\left[\left(\int f\mathrm{d}\tilde{\mathcal{M}}_{a'}^K - \int f\mathrm{d}\tilde{\mathcal{M}}_a^K\right)^2\right] \\
&= \mathbb{E}\left[\int f\mathrm{d}\tilde{\mathcal{M}}_{a'}^K\left(\int f\mathrm{d}\tilde{\mathcal{M}}_{a'}^K - \int f\mathrm{d}\tilde{\mathcal{M}}_a^K\right)\right] + \mathbb{E}\left[\int f\mathrm{d}\tilde{\mathcal{M}}_a^K\left(\int f\mathrm{d}\tilde{\mathcal{M}}_a^K - \int f\mathrm{d}\tilde{\mathcal{M}}_{a'}^K\right)\right].
\end{aligned}$$

Let $\eta > 0$ be small. Since f is non-negative, we can bound

$$\begin{aligned}
&\mathbb{E}\left[\int f\mathrm{d}\tilde{\mathcal{M}}_a^K\left(\int f\mathrm{d}\tilde{\mathcal{M}}_a^K - \int f\mathrm{d}\tilde{\mathcal{M}}_{a'}^K\right)\right] \leq \|f\|_\infty^2 \int_{A\times A}\mathbf{1}_{\{|z-z'|\leq\eta\}}\mathbb{E}\left[\tilde{\mathcal{M}}_a^K(dz)\tilde{\mathcal{M}}_a^K(dz')\right] \\
&+ \int_{A\times A} f(z)f(z')\mathbf{1}_{\{|z-z'|>\eta\}}\mathbb{E}\left[\tilde{\mathcal{M}}_a^K(dz)\left(\tilde{\mathcal{M}}_a^K(dz') - \tilde{\mathcal{M}}_{a'}^K(dz')\right)\right].
\end{aligned}$$

Thanks to Lemma 4.44, we know that

$$\lim_{\eta\to 0}\limsup_{K\to\infty}\frac{1}{(\log K)^{2\theta}}\int_{A\times A}\mathbf{1}_{\{|z-z'|\leq\eta\}}\mathbb{E}\left[\tilde{\mathcal{M}}_a^K(dz)\tilde{\mathcal{M}}_a^K(dz')\right] = 0.$$

We now deal with the second term. Let $z, z' \in A$ such that $|z-z'| > \eta$. We start by claiming that there exist $C > 0, r_1 \in (0, r_0)$ that may depend on η and $b-a$ such that

$$(\log K)^{-2\theta}\mathbb{E}\left[\tilde{\mathcal{M}}_a^K(dz)\tilde{\mathcal{M}}_a^K(dz')\right] \leq \eta + (\log K)^{-2\theta}\mathbb{E}\left[\hat{\mathcal{M}}_a^K(dz)\hat{\mathcal{M}}_a^K(dz')\right] \tag{4.119}$$

where $\hat{\mathcal{M}}_a^K(dz)$ is defined similarly as $\tilde{\mathcal{M}}_a^K(dz)$ but with the good event restricting the number of crossings of annulus for $r \in (r_1, r_0)$ instead of $r \in (0, r_0)$. We omit the proof of this claim since it follows along similar lines as the proof of Lemma 4.43. The point is that since z and z' are at distance macroscopic, there will be only a finite number of excursions between z and z' so that (4.119) boils down to Lemma 4.43. The conclusion of these preliminaries is that we have bounded

$$\begin{aligned}
&(\log K)^{-2\theta}\|f\|_\infty^{-2}\mathbb{E}\left[\int f\mathrm{d}\tilde{\mathcal{M}}_a^K\left(\int f\mathrm{d}\tilde{\mathcal{M}}_a^K - \int f\mathrm{d}\tilde{\mathcal{M}}_{a'}^K\right)\right] \\
&\leq (\log K)^{-2\theta}\int_{A\times A}\frac{f(z)f(z')}{\|f\|_\infty^2}\mathbf{1}_{\{|z-z'|>\eta\}}\mathbb{E}\left[\hat{\mathcal{M}}_a^K(dz)\left(\hat{\mathcal{M}}_a^K(dz') - \hat{\mathcal{M}}_{a'}^K(dz')\right)\right] + o_{\eta\to 0}(1)
\end{aligned} \tag{4.120}$$

where $o_{\eta\to 0}(1) \to 0$ as $\eta \to 0$, uniformly in $K \geq 1, a' \in [a/2, a]$ and f.

Now let $z, z' \in A$ such that $|z - z'| > \eta$. By Lemma 4.40, $\mathbb{E}\left[\hat{\mathcal{M}}_a^K(dz)\hat{\mathcal{M}}_{a'}^K(dz')\right]$ is equal to

$$\frac{1}{aa'} \mathrm{CR}(z,D)^a \mathrm{CR}(z',D)^{a'} \sum_{\substack{n,m\geq 1 \\ 0\leq l\leq n\wedge m}} \frac{\theta^{n+m-l}}{(n-l)!(m-l)!l!} \int_{\substack{\mathsf{a}\in E(1,n)\\ \mathsf{a}'\in E(1,m)}} \mathsf{d}\mathsf{a}\mathsf{d}\mathsf{a}'$$
$$\mathbb{E} \prod_{i=1}^{l} \frac{1-e^{-KT(\Xi^{z,z'}_{aa_i,a'a'_i})-KT(\Xi^{z}_{aa_i})-KT(\Xi^{z'}_{a'a'_i})}}{a_i a'_i} \prod_{i=l+1}^{n} \frac{1-e^{-KT(\Xi^z_{aa_i})}}{a_i} \prod_{i=l+1}^{m} \frac{1-e^{-KT(\Xi^{z'}_{a'a'_i})}}{a'_i}$$
$$F\left(\bigwedge_{i=1}^{l} \Xi^{z,z'}_{aa_i,a'a'_i} \wedge \bigwedge_{i=1}^{n} \Xi^z_{aa_i} \wedge \bigwedge_{i=1}^{m} \Xi^{z'}_{a'a'_i} \wedge \mathcal{L}_D^\theta \right) \prod_{i=1}^{l} \mathsf{B}\left((2\pi)^2 aa'a_ia'_iG_D(z,z')^2\right)$$

where

$$F(\mathcal{C}) := \mathbf{1}_{\left\{\forall r\in\{e^{-n},n\geq 1\}\cap(r_1,r_0), N^{\mathcal{C}}_{z,r}\leq b(\log r)^2 \text{ and } N^{\mathcal{C}}_{z',r}\leq b(\log r)^2\right\}}.$$

We develop further this expression according to the number $2k_i$ of excursions in $\Xi^{z,z'}_{aa_i,a'a'_i}, i = 1 \dots l$. In particular, $\Xi^{z,z'}_{k_i}$ will denote the concatenation of $2k_i$ i.i.d. trajectories distributed according to $\mu_D^{z,z'}/G_D(z,z')$. $\mathbb{E}\left[\hat{\mathcal{M}}_a^K(dz)\hat{\mathcal{M}}_{a'}^K(dz')\right]$ is equal to

$$\frac{1}{aa'} \mathrm{CR}(z,D)^a \mathrm{CR}(z',D)^{a'} \sum_{\substack{n,m\geq 1 \\ 0\leq l\leq n\wedge m}} \frac{\theta^{n+m-l}}{(n-l)!(m-l)!l!} \int_{\substack{\mathsf{a}\in E(1,n)\\ \mathsf{a}'\in E(1,m)}} \mathsf{d}\mathsf{a}\mathsf{d}\mathsf{a}' \sum_{k_1,\dots,k_l\geq 1}$$
$$\mathbb{E} \prod_{i=1}^{l} \frac{1-e^{-KT(\Xi^{z,z'}_{k_i})-KT(\Xi^{z}_{aa_i})-KT(\Xi^{z'}_{a'a'_i})}}{a_i a'_i} \prod_{i=l+1}^{n} \frac{1-e^{-KT(\Xi^z_{aa_i})}}{a_i} \prod_{i=l+1}^{m} \frac{1-e^{-KT(\Xi^{z'}_{a'a'_i})}}{a'_i}$$
$$F\left(\bigwedge_{i=1}^{l} \Xi^{z,z'}_{k_i} \wedge \bigwedge_{i=1}^{n} \Xi^z_{aa_i} \wedge \bigwedge_{i=1}^{m} \Xi^{z'}_{a'a'_i} \wedge \mathcal{L}_D^\theta \right) \prod_{i=1}^{l} \frac{(2\pi\sqrt{aa'a_ia'_i}G_D(z,z'))^{2k_i}}{k_i!(k_i-1)!}$$

In what follows, we will naturally couple the PPP of excursions away of z' by decomposing $\Xi^{z'}_{aa_i} = \Xi^{z'}_{a'a_i} \wedge \Xi^{z'}_{(a-a')a_i}$ (recall that $a' \leq a$). We can then decompose

$$\frac{a}{\mathrm{CR}(z,D)^a}\mathbb{E}\left[\hat{\mathcal{M}}_a^K(dz)\left(\frac{a}{\mathrm{CR}(z',D)^a}\hat{\mathcal{M}}_a^K(dz') - \frac{a'}{\mathrm{CR}(z',D)^{a'}}\hat{\mathcal{M}}_{a'}^K(dz')\right)\right] = S_1 + S_2 + S_3$$

where

$$S_1 = \sum_{\substack{n,m\geq 1\\ 0\leq l\leq n\wedge m}} \frac{\theta^{n+m-l}}{(n-l)!(m-l)!l!} \int_{\substack{\mathsf{a}\in E(1,n)\\ \mathsf{a}'\in E(1,m)}} \mathrm{d}\mathsf{a}\mathrm{d}\mathsf{a}'$$
$$\sum_{k_1,\dots,k_l\geq 1} \left\{ \prod_{i=1}^l \frac{(2\pi a\sqrt{a_i a_i'} G_D(z,z'))^{2k_i}}{k_i!(k_i-1)!} - \prod_{i=1}^l \frac{(2\pi\sqrt{aa'a_i a_i'} G_D(z,z'))^{2k_i}}{k_i!(k_i-1)!} \right\}$$
$$\mathbb{E} \prod_{i=1}^l \frac{1-e^{-KT(\Xi_{k_i}^{z,z'})-KT(\Xi_{aa_i}^z)-KT(\Xi_{aa_i'}^{z'})}}{a_i a_i'} \prod_{i=l+1}^n \frac{1-e^{-KT(\Xi_{aa_i}^z)}}{a_i} \prod_{i=l+1}^m \frac{1-e^{-KT(\Xi_{aa_i'}^{z'})}}{a_i'}$$
$$F\left(\bigwedge_{i=1}^l \Xi_{k_i}^{z,z'} \wedge \bigwedge_{i=1}^n \Xi_{aa_i}^z \wedge \bigwedge_{i=1}^m \Xi_{aa_i'}^{z'} \wedge \mathcal{L}_D^\theta \right),$$

$$S_2 = \sum_{\substack{n,m\geq 1\\ 0\leq l\leq n\wedge m}} \frac{\theta^{n+m-l}}{(n-l)!(m-l)!l!} \int_{\substack{\mathsf{a}\in E(1,n)\\ \mathsf{a}'\in E(1,m)}} \mathrm{d}\mathsf{a}\mathrm{d}\mathsf{a}' \prod_{i=1}^l \mathsf{B}\left((2\pi)^2 aa'a_i a_i' G_D(z,z')^2\right)$$
$$\mathbb{E}\Bigg\{ \prod_{i=1}^l \frac{1-e^{-KT(\Xi_{aa_i,a'a_i'}^{z,z'})-KT(\Xi_{aa_i}^z)-KT(\Xi_{aa_i'}^{z'})}}{a_i a_i'} \prod_{i=l+1}^n \frac{1-e^{-KT(\Xi_{aa_i}^z)}}{a_i} \prod_{i=l+1}^m \frac{1-e^{-KT(\Xi_{aa_i'}^{z'})}}{a_i'}$$
$$- \prod_{i=1}^l \frac{1-e^{-KT(\Xi_{aa_i,a'a_i'}^{z,z'})-KT(\Xi_{aa_i}^z)-KT(\Xi_{a'a_i'}^{z'})}}{a_i a_i'} \prod_{i=l+1}^n \frac{1-e^{-KT(\Xi_{aa_i}^z)}}{a_i} \prod_{i=l+1}^m \frac{1-e^{-KT(\Xi_{a'a_i'}^{z'})}}{a_i'} \Bigg\}$$
$$\times F\left(\bigwedge_{i=1}^l \Xi_{aa_i,a'a_i'}^{z,z'} \wedge \bigwedge_{i=1}^n \Xi_{aa_i}^z \wedge \bigwedge_{i=1}^m \Xi_{aa_i'}^{z'} \wedge \mathcal{L}_D^\theta \right)$$

and

$$S_3 = \sum_{\substack{n,m\geq 1\\ 0\leq l\leq n\wedge m}} \frac{\theta^{n+m-l}}{(n-l)!(m-l)!l!} \int_{\substack{\mathsf{a}\in E(1,n)\\ \mathsf{a}'\in E(1,m)}} \mathrm{d}\mathsf{a}\mathrm{d}\mathsf{a}' \prod_{i=1}^l \mathsf{B}\left((2\pi)^2 aa'a_i a_i' G_D(z,z')^2\right)$$
$$\mathbb{E} \prod_{i=1}^l \frac{1-e^{-KT(\Xi_{aa_i,a'a_i'}^{z,z'})-KT(\Xi_{aa_i}^z)-KT(\Xi_{a'a_i'}^{z'})}}{a_i a_i'} \prod_{i=l+1}^n \frac{1-e^{-KT(\Xi_{aa_i}^z)}}{a_i} \prod_{i=l+1}^m \frac{1-e^{-KT(\Xi_{a'a_i'}^{z'})}}{a_i'}$$
$$\times \left\{ F\left(\bigwedge_{i=1}^l \Xi_{aa_i,a'a_i'}^{z,z'} \wedge \bigwedge_{i=1}^n \Xi_{aa_i}^z \wedge \bigwedge_{i=1}^m \Xi_{aa_i'}^{z'} \wedge \mathcal{L}_D^\theta \right) - F\left(\bigwedge_{i=1}^l \Xi_{aa_i,a'a_i'}^{z,z'} \wedge \bigwedge_{i=1}^n \Xi_{aa_i}^z \wedge \bigwedge_{i=1}^m \Xi_{a'a_i'}^{z'} \wedge \mathcal{L}_D^\theta \right) \right\}. \tag{4.121}$$

We now claim that for all $i \in \{1,2,3\}$, uniformly in $z, z' \in A$ with $|z-z'| > \eta$,

$$\limsup_{a'\to a} \limsup_{K\to\infty} \frac{1}{(\log K)^{2\theta}} S_i = 0. \tag{4.122}$$

For S_1 and S_2, this follows by bounding the function F by one and then by noting that we obtained explicit expressions for the limit in K that are continuous with respect to the thickness parameters.

See Corollary 4.42. We now explain how to deal with S_3. We notice that on the event that none of the excursions of $\bigwedge_{i=1}^l \Xi^{z'}_{(a-a')a'_i}$ hits the circle $\partial D(z', r_1)$, the difference of the function F appearing in (4.121) vanishes. Since $0 \le F \le 1$, we can therefore bound this difference by the indicator of the complement of this event. After applying a union bound, we find that

$$
\begin{aligned}
&\mathbb{E} \prod_{i=1}^l \frac{1-e^{-KT(\Xi^{z,z'}_{aa_i,a'a'_i})-KT(\Xi^z_{aa_i})-KT(\Xi^{z'}_{a'a'_i})}}{a_i a'_i} \prod_{i=l+1}^n \frac{1-e^{-KT(\Xi^z_{aa_i})}}{a_i} \prod_{i=l+1}^m \frac{1-e^{-KT(\Xi^{z'}_{a'a'_i})}}{a'_i} \\
&\times \left\{ F\left(\bigwedge_{i=1}^l \Xi^{z,z'}_{aa_i,a'a'_i} \wedge \bigwedge_{i=1}^n \Xi^z_{aa_i} \wedge \bigwedge_{i=1}^m \Xi^{z'}_{aa'_i} \wedge \mathcal{L}_D^\theta \right) - F\left(\bigwedge_{i=1}^l \Xi^{z,z'}_{aa_i,a'a'_i} \wedge \bigwedge_{i=1}^n \Xi^z_{aa_i} \wedge \bigwedge_{i=1}^m \Xi^{z'}_{a'a'_i} \wedge \mathcal{L}_D^\theta \right) \right\} \\
&\le \mathbb{E} \prod_{i=1}^l \frac{1-e^{-KT(\Xi^{z,z'}_{aa_i,a'a'_i})-KT(\Xi^z_{aa_i})-KT(\Xi^{z'}_{a'a'_i})}}{a_i a'_i} \prod_{i=l+1}^n \frac{1-e^{-KT(\Xi^z_{aa_i})}}{a_i} \prod_{i=l+1}^m \frac{1-e^{-KT(\Xi^{z'}_{a'a'_i})}}{a'_i} \\
&\times \sum_{j=1}^m \sum_{\wp \in \Xi^{z'}_{(a-a')a'_j}} \mathbf{1}_{\{\wp \text{ hits } \partial D(z', r_1)\}} \\
&\le \prod_{i=1}^l \frac{1}{a_i a'_i} \prod_{i=l+1}^n \frac{1-\mathbb{E} e^{-KT(\Xi^z_{aa_i})}}{a_i} \prod_{i=l+1}^m \frac{1-\mathbb{E} e^{-KT(\Xi^{z'}_{a'a'_i})}}{a'_i} \sum_{j=1}^m \mathbb{E} \sum_{\wp \in \Xi^{z'}_{(a-a')a'_j}} \mathbf{1}_{\{\wp \text{ hits } \partial D(z', r_1)\}}.
\end{aligned}
$$

Since

$$
\begin{aligned}
\sum_{j=1}^m \mathbb{E} \sum_{\wp \in \Xi^{z'}_{(a-a')a_j}} \mathbf{1}_{\{\wp \text{ hits } \partial D(z', r_1)\}} &= \sum_{j=1}^m 2\pi (a-a') a'_j \mu_D^{z',z'} (\tau_{\partial D(z', r_1)} < \infty) \\
&= 2\pi (a-a') a' \mu_D^{z',z'} (\tau_{\partial D(z', r_1)} < \infty) \le C(a-a')
\end{aligned}
$$

for some constant $C > 0$ which may depend on r_1, we have obtained that

$$
\begin{aligned}
S_3 \le C(a-a') &\sum_{\substack{n,m \ge 1 \\ 0 \le l \le n \wedge m}} \frac{\theta^{n+m-l}}{(n-l)!(m-l)!l!} \int_{\substack{\mathsf{a} \in E(1,n) \\ \mathsf{a}' \in E(1,m)}} d\mathsf{a} d\mathsf{a}' \prod_{i=1}^l \frac{\mathsf{B}\left((2\pi)^2 a a' a_i a'_i G_D(z,z')^2\right)}{a_i a'_i} \\
&\times \prod_{i=l+1}^n \frac{1-e^{-aa_i C_K(z)}}{a_i} \prod_{i=l+1}^m \frac{1-e^{-a'a'_i C_K(z')}}{a'_i}.
\end{aligned}
$$

By Lemma 4.41, this is at most $C(a-a')(\log K)^{2\theta}$ for some constant $C > 0$ that may depend on r_1 and η. This finishes the proof of (4.122) for S_3.

To conclude, we have proven that

$$
\limsup_{a' \to a} \limsup_{K \to \infty} (\log K)^{-2\theta} \mathbb{E}\left[\hat{\mathcal{M}}_a^K(dz) \left(\frac{a}{\mathrm{CR}(z', D)^a} \hat{\mathcal{M}}_a^K(dz') - \frac{a'}{\mathrm{CR}(z', D)^{a'}} \hat{\mathcal{M}}_{a'}^K(dz') \right) \right] = 0,
$$

uniformly over $z, z' \in A$ with $|z - z'| > \eta$. Hence

$$
\limsup_{a' \to a} \limsup_{K \to \infty} (\log K)^{-2\theta} \mathbb{E}\left[\hat{\mathcal{M}}_a^K(dz) \left(\hat{\mathcal{M}}_a^K(dz') - \hat{\mathcal{M}}_{a'}^K(dz') \right) \right] = 0.
$$

Coming back to (4.120), this implies that

$$\limsup_{a'\to a} \limsup_{K\to\infty} (\log K)^{-2\theta} \|f\|_\infty^{-2} \mathbb{E}\left[\int f \mathrm{d}\tilde{\mathcal{M}}_a^K \left(\int f \mathrm{d}\tilde{\mathcal{M}}_a^K - \int f \mathrm{d}\tilde{\mathcal{M}}_{a'}^K \right) \right] \leq o_{\eta\to 0}(1).$$

Since the left hand side term does not depend on η, it has to be non positive. Similarly, the same statement holds true when one exchanges a and a' in the expectation above so that

$$\limsup_{a'\to a} \limsup_{K\to\infty} (\log K)^{-2\theta} \|f\|_\infty^{-2} \mathbb{E}\left[\left(\int f \mathrm{d}\tilde{\mathcal{M}}_{a'}^K - \int f \mathrm{d}\tilde{\mathcal{M}}_a^K \right)^2 \right] \leq 0.$$

This concludes the proof. □

4.7.5 Proof of Theorem 4.11 (thick points)

We conclude Section 4.7 with a proof of Theorem 4.11.

Proof of Theorem 4.11 and Point 4 of Theorem 4.1. We first start by showing that $\mathcal{M}_a$ is supported on $\mathcal{T}(a)$. To this end, let us denote

$$\mathcal{T}(a, r_0, \eta) := \left\{ z \in D : \forall r \in \{e^{-n}, n \geq 1\} \cap (0, r_0) : \left| \frac{1}{n^2} N_{z,r}^{\mathcal{L}_D^\theta} - a \right| \leq \eta \right\}.$$

Let $A \Subset D$ be a Borel set compactly included in D. We argue that the following version of Lemma 4.43 holds:

$$\lim_{r_0\to 0} \limsup_{K\to\infty} \frac{1}{(\log K)^\theta} \mathbb{E}\left[\int_A \mathbf{1}_{\{z \notin \mathcal{T}(a, r_0, \eta)\}} \mathcal{M}_a^K(dz) \right] = 0.$$

The only difference with Lemma 4.43 is that we require the number of crossings of annuli to be, not only not too large, but also not too small. To prove that the number of crossings is not too small, we use the same approach as what we did to prove Lemma 4.43 and we use FKG-inequality and Corollary 4.48 instead of Corollary 4.47. We omit the details. This shows that $\mathbf{1}_A \mathcal{M}_a$ is almost surely supported by

$$\bigcup_{r_0>0} \mathcal{T}(a, r_0, \eta) = \left\{ z \in D : \exists r_0 > 0, \forall r \in \{e^{-n}, n \geq 1\} \cap (0, r_0) : \left| \frac{1}{n^2} N_{z,r}^{\mathcal{L}_D^\theta} - a \right| \leq \eta \right\}.$$

Since this is true for all $\eta > 0$ and for all $A \Subset D$, this concludes the proof that $\mathcal{M}_a(D \setminus \mathcal{T}(a)) = 0$ a.s.

We now turn to the proof of the claims concerning the carrying dimension of $\mathcal{M}_a$ and the Hausdorff dimension of $\mathcal{T}(a)$. We start with the lower bound and we let $\eta \in [0, 2-a)$, $A \Subset D$ and we assume that b is close enough to a so that Lemma 4.45 holds. Let us denote $\tilde{\mathcal{M}}_{a,r_0}$ the limit of $(\log K)^{-\theta} \tilde{\mathcal{M}}_a^K$ (we keep track of the dependence in r_0). By Lemma 4.44 and by Fatou's lemma, the energy

$$e_{r_0}(A) := \int_{A\times A} \frac{1}{|z-z'|^\eta} \tilde{\mathcal{M}}_{a,r_0}(dz) \tilde{\mathcal{M}}_{a,r_0}(dz')$$

has finite expectation and is therefore almost surely finite. Moreover, Lemma 4.43 and Fatou's lemma also show that

$$\lim_{r_0\to 0} \mathbb{E}\left[\mathcal{M}_a(A) - \tilde{\mathcal{M}}_{a,r_0}(A) \right] = 0.$$

The following event has therefore full probability measure

$$E := \bigcap_A \left\{ \liminf_{n\to\infty} \mathcal{M}_a(A) - \tilde{\mathcal{M}}_{a,e^{-n}}(A) = 0 \quad \text{and} \quad \forall r_0 \in \{e^{-n}, n \geq 1\}, e_{r_0}(A) < \infty \right\}$$

where the intersection runs over all set A of the form $\{z \in D, \mathrm{dist}(z, \partial D) > e^{-n}\}, n \geq 1$. Now, let $B \subset D$ be a Borel set such that $\mathcal{M}_a(B) > 0$. There exists some set A of the above form such that $\mathcal{M}_a(B \cap A) > 0$. Moreover, since for all $r_0 > 0$,

$$\mathcal{M}_a(B \cap A) - \tilde{\mathcal{M}}_{a,r_0}(B \cap A) \leq \mathcal{M}_a(A) - \tilde{\mathcal{M}}_{a,r_0}(A),$$

we see that on the event E, we can find $r_0 \in \{e^{-n}, n \geq 1\}$, such that $\tilde{\mathcal{M}}_{a,r_0}(B \cap A) > 0$. But because on the event E, the energy $e_{r_0}(A)$ is finite, Frostman's lemma implies that the Hausdorff dimension of $B \cap A$ is at least η. To wrap things up, we have proven that almost surely, for all Borel set B such that $\mathcal{M}_a(B) > 0$, the Hausdorff dimension of B is at least η. Since η can be made arbitrary close to $2 - a$, this concludes the lower bound on the carrying dimension of $\mathcal{M}_a$. The lower bound on the dimension of $\mathcal{T}(a)$ follows since we have already proven that $\mathcal{M}_a$ is almost surely supported on $\mathcal{T}(a)$.

We now turn to the upper bound. We will show that the Hausdorff dimension of $\mathcal{T}(a)$ is almost surely at most $2 - a$. Since $\mathcal{M}_a(D \setminus \mathcal{T}(a)) = 0$ a.s., this will also provide the upper bound on the carrying dimension of $\mathcal{M}_a$ and it will conclude the proof. Let $\delta > 0$ and denote by $\mathcal{H}_{2-a+\delta}$ the $(2 - a + \delta)$-Hausdorff measure. Let $\eta > 0$ be much smaller than δ. We first notice that

$$\mathcal{T}(a) \subset \bigcup_{N\geq 1} \bigcap_{n \geq N} \left\{ z \in D : \frac{1}{n^2} N^{\mathcal{L}_D^\theta}_{z,e^{-n}} > a - \eta \right\} \subset \bigcap_{N\geq 1} \bigcup_{n \geq N} \left\{ z \in D : \frac{1}{n^2} N^{\mathcal{L}_D^\theta}_{z,e^{-n}} > a - \eta \right\}$$

and we can therefore bound,

$$\mathcal{H}_{2-a+\delta}(\mathcal{T}(a)) \leq \lim_{N\to\infty} \sum_{n\geq N} \mathcal{H}_{2-a+\delta} \left(\left\{ z \in D : \frac{1}{n^2} N^{\mathcal{L}_D^\theta}_{z,e^{-n}} > a - \eta \right\} \right).$$

Now, let $n \geq 1$ be large and denote by $r_n = e^{-n}$. Let $\{z_i, i \in I\} \subset D$ be a maximal $r_n^{1+\eta}$-net of D (in particular, $\#I \asymp r_n^{-2(1+\eta)}$). If $z \in D$ is such that $|z - z_i| < r_n^{1+\eta}$, we notice that the annulus $D(z, er_n) \setminus D(z, r_n)$ contains the annulus $D(z_i, er_n - r_n^{1+\eta}) \setminus D(z_i, r_n + r_n^{1+\eta})$, and therefore, the number of crossings in $\mathcal{L}_D^\theta$ of the former annulus is smaller or equal than the number of crossings of the latter. This shows that we can cover

$$\left\{ z \in D : \frac{1}{n^2} N^{\mathcal{L}_D^\theta}_{z,e^{-n}} > a - \eta \right\} \subset \bigcup_{i \in I} \left\{ z \in D(z_i, e^{-(1+\eta)n}), \frac{1}{n^2} N^{\mathcal{L}_D^\theta}_{z_i,r_n,\eta} > a - \eta \right\}$$

where we have denoted by $N^{\mathcal{L}_D^\theta}_{z_i,r_n,\eta}$ the number of upcrossings of $D(z_i, er_n - r_n^{1+\eta}) \setminus D(z_i, r_n + r_n^{1+\eta})$ in $\mathcal{L}_D^\theta$. Let $\lambda \in (0, 1)$ be close to 1. An immediate adaptation of Lemma 4.51 to annuli with slightly different radii, shows that if n is large enough, then

$$\mathbb{P}\left(\frac{1}{n^2} N^{\mathcal{L}_D^\theta}_{z_i,r_n,\eta} > a - \eta \right) \leq r_n^{\lambda(a-\eta)}.$$

Therefore

$$\begin{aligned}&\mathbb{E}\left[\mathcal{H}_{2-a+\delta}\left(\left\{z \in D : \frac{1}{n^2} N_{z,e^{-n}}^{\mathcal{L}_D^\theta} > a - \eta\right\}\right)\right] \\ &\leq C\mathbb{E}\left[\sum_{i \in I} \left(r_n^{1+\eta}\right)^{2-a+\delta} \mathbf{1}_{\left\{\frac{1}{n^2} N_{z_i,r_n,\eta}^{\mathcal{L}_D^\theta} > a-\eta\right\}}\right] \leq C r_n^{(1+\eta)(-a+\delta)+\lambda(a-\eta)}.\end{aligned}$$

By choosing λ and η close enough to 1 and 0, respectively, we can ensure the above power to be larger than $\delta/2$ (δ is fixed for now). We have proven that

$$\mathbb{E}\left[\mathcal{H}_{2-a+\delta}(\mathcal{T}(a)\right] \leq C \lim_{N \to \infty} \sum_{n \geq N} r_n^{\delta/2} = 0$$

and the Hausdorff dimension of $\mathcal{T}(a)$ is at most $2 - a + \delta$ a.s. This concludes the proof. □

4.8 Poisson–Dirichlet distribution

The aim of this section is two-fold: proving Theorem 4.8 as well as giving a new perspective on the martingale $(m_a^K(dz), K \geq 0)$. Indeed, it is likely that Theorem 4.8 could be also proven as a consequence of the discrete approximation of $\mathcal{M}_a$ (Theorem 4.12) and as a consequence of Proposition 4.17. We decided to take another route which remains in the continuum setting. The advantage of this approach is that it gives an independent proof of the fact that $(m_a^K(dz), K \geq 0)$ is a martingale. This is close in spirit to Lyons' approach [Lyo97] to the Biggins martingale convergence theorem for spatial branching processes originally established by Biggins [Big77].

We first prove (4.7). Recall from Section 4.2.1 that, conditionally on $\mathcal{L}_D^\theta$, $\{U_\wp, \wp \in \mathcal{L}_D^\theta\} =: \mathcal{U}$ denotes a collection of i.i.d. uniform random variables on $[0,1]$. We will prove that for any nonnegative measurable admissible function F,

$$\mathbb{E}\left[\int_D F(z, \mathcal{L}_D^\theta, \mathcal{U}) \mathcal{M}_a(dz)\right] = \frac{1}{2^\theta a^{1-\theta}\Gamma(\theta)} \int_D \mathbb{E}\left[F(z, \mathcal{L}_D^\theta \cup \Xi_{\underline{a}}, \mathcal{U} \cup \mathcal{U}_{\underline{a}})\right] \mathrm{CR}(z,D)^a \mathrm{d}z \qquad (4.123)$$

where in the RHS, the two collections of loops $\mathcal{L}_D^\theta$ and $\Xi_{\underline{a}} := \{\Xi_{a_i}^z, i \geq 1\}$ are independent, and, conditionally on everything else, $\mathcal{U}_{\underline{a}}$ denotes a collection of i.i.d. uniform random variables on $[0,1]$ indexed by $\Xi_{\underline{a}}$. This equation may seem stronger but is actually equivalent to (4.7). We recall that $\mathcal{L}_D^\theta(K) = \left\{\wp \in \mathcal{L}_D^\theta : U_\wp < 1 - e^{-KT(\wp)}\right\}$ denotes the loops killed at rate K and we further introduce $\mathcal{U}(K) := \{U_\wp, \wp \in \mathcal{L}_D^\theta(K)\}$. Conditionally on $\mathcal{L}_D^\theta(K)$, we see that $\mathcal{U}(K)$ is a collection of independent random variables where $U_\wp$ is uniformly distributed in $[0, 1-e^{-KT(\wp)}]$. By the monotone class theorem, it suffices to prove (4.123) for $F(z, \mathcal{L}_D^\theta, \mathcal{U}) = \mathbf{1}_{\{z \in A\}} G(\mathcal{L}_D^\theta(K), \mathcal{U}(K))$ for G an arbitrary nonnegative measurable function, $A \subset D$ a Borel set and $K > 0$.

Recall the definition of $\mathcal{M}_a(A)$ in Theorem 4.1. By Proposition 4.25, $\mathcal{M}_a(A)$ is the (L^1 by Proposition 4.24) limit of $\frac{1}{2^\theta \Gamma(\theta)} m_a^K(A)$ where we recall that

$$m_a^K(dz) := \frac{1}{a^{1-\theta}} \mathrm{CR}(z,D)^a e^{-aC_K(z)} dz + \int_0^a \mathrm{d}\rho \frac{1}{(a-\rho)^{1-\theta}} \mathrm{CR}(z,D)^{a-\rho} e^{-(a-\rho)C_K(z)} \mathcal{M}_\rho^K(dz).$$

We want to compute the LHS of (4.123) for $\mathbf{1}_{\{z\in A\}}G(\mathcal{L}_D^\theta(K),\mathcal{U}(K))$ instead of $F(z,\mathcal{L}_D^\theta,\mathcal{U})$. Since $(m_a^K(A),\, K\geq 0)$ is a uniformly integrable martingale by Proposition 4.24,

$$\int_D \mathbb{E}\left[\mathbf{1}_{\{z\in A\}}G(\mathcal{L}_D^\theta(K),\mathcal{U}(K))\mathcal{M}_a(\mathrm{d}z)\right] = \frac{1}{2^\theta\Gamma(\theta)}\mathbb{E}\left[G(\mathcal{L}_D^\theta(K),\mathcal{U}(K))m_a^K(A)\right]. \tag{4.124}$$

Set $G^{(K)} := G(\mathcal{L}_D^\theta(K),\mathcal{U}(K))$ for concision. From the expression of $m_a^K(\mathrm{d}z)$, we have

$$\mathbb{E}\left[G^{(K)}m_a^K(\mathrm{d}z)\right] = \mathbb{E}_1 + \mathbb{E}_2 \tag{4.125}$$

with

$$\mathbb{E}_1 := \frac{1}{a^{1-\theta}}\,\mathrm{CR}(z,D)^a e^{-aC_K(z)}\mathbb{E}\left[G^{(K)}\right]dz, \tag{4.126}$$

$$\mathbb{E}_2 := \int_0^a \mathrm{d}\rho\frac{1}{(a-\rho)^{1-\theta}}\,\mathrm{CR}(z,D)^{a-\rho}e^{-(a-\rho)C_K(z)}\mathbb{E}\left[G^{(K)}\mathcal{M}_\rho^K(dz)\right]. \tag{4.127}$$

To compute $\mathbb{E}_2$, we first need to compute $\mathbb{E}\left[G^{(K)}\mathcal{M}_\rho^K(dz)\right]$ for $\rho\in[0,a]$. Recall that by Lemma 4.32, for any nonnegative measurable function F,

$$\mathbb{E}\left[F(z,\mathcal{L}_D^\theta)\mathcal{M}_\rho^K(dz)\right] = \\ \mathrm{CR}(z,D)^\rho\sum_{n\geq 1}\frac{\theta^n}{n!}\int_{\underline{\rho}\in E(\rho,n)}\frac{\mathrm{d}\underline{\rho}}{\rho_1\cdots\rho_n}\mathbb{E}\left[\prod_{i=1}^n\left(1-e^{-KT(\Xi_{\rho_i}^z)}\right)F\left(z,\mathcal{L}_D^\theta\cup\Xi_{\underline{\rho}}^z\right)\right]\mathrm{d}z,$$

where $\underline{\rho}=(\rho_1,\dots,\rho_n)$, $\mathrm{d}\underline{\rho}=\mathrm{d}\,\rho_1\dots\mathrm{d}\,\rho_{n-1}$ and $\Xi_{\underline{\rho}}^z := (\Xi_{\rho_i}^z)_{1\leq i\leq n}$ is independent of $\mathcal{L}_D^\theta$. We rewrite the RHS in a slightly different form. First, the term of index n in the sum is equal to

$$\theta^n\int_{\underline{\rho}\in E(\rho,n),\rho_1<\cdots<\rho_n}\frac{\mathrm{d}\underline{\rho}}{\rho_1\cdots\rho_n}\mathbb{E}\left[\prod_{i=1}^n\left(1-e^{-KT(\Xi_{\rho_i}^z)}\right)F(z,\mathcal{L}_D^\theta\cup\Xi_{\underline{\rho}}^z)\right]\mathrm{d}z.$$

Secondly, recall from (4.73) that $E[1-e^{-KT(\Xi_{\rho_i}^z)}]=1-e^{-\rho_i C_K(z)}$. Hence

$$\mathbb{E}\left[\prod_{i=1}^n\left(1-e^{-KT(\Xi_{\rho_i}^z)}\right)F(z,\mathcal{L}_D^\theta\cup\Xi_{\underline{\rho}}^z)\right] = \prod_{i=1}^n\left(1-e^{-\rho_i C_K(z)}\right)\mathbb{E}\left[F(z,\mathcal{L}_D^\theta\cup\widehat{\Xi}_{\underline{\rho}}^z)\right]$$

where $\widehat{\Xi}_{\underline{\rho}}^z := \{\widehat{\Xi}_{\rho_i}^z, i=1\dots n\}$ is a collection of independent loops independent of $\mathcal{L}_D^\theta$, and $\widehat{\Xi}_{\rho_i}^z$ has the distribution of $\Xi_{\rho_i}^z$ biased by $1-e^{-KT(\Xi_{\rho_i}^z)}$. Combining the two, we see that

$$\mathbb{E}\left[F(z,\mathcal{L}_D^\theta)\mathcal{M}_\rho^K(dz)\right] = \\ \mathrm{CR}(z,D)^\rho\sum_{n\geq 1}\theta^n\int_{\underline{\rho}\in E(\rho,n),\rho_1<\ldots<\rho_n}\mathrm{d}\underline{\rho}\prod_{i=1}^n\frac{1-e^{-\rho_i C_K(z)}}{\rho_i}\mathbb{E}\left[F\left(z,\mathcal{L}_D^\theta\cup\widehat{\Xi}_{\underline{\rho}}^z\right)\right]\mathrm{d}z.$$

Again, one can actually take a function $F(z,\mathcal{L}_D^\theta,\mathcal{U})$. From the proof of Lemma 4.32 and the previous lines, one can check that the function F in the RHS will turn into $F(z,\mathcal{L}_D^\theta\cup\widehat{\Xi}_{\underline{\rho}}^z,\mathcal{U}\cup\widehat{\mathcal{U}}_{\underline{\rho}})$ where $\widehat{\mathcal{U}}_{\underline{\rho}}=\{\widehat{U}_\wp,\wp\in\widehat{\Xi}_{\underline{\rho}}^z\}$ is conditionally on everything else a collection of independent random variables, with $\widehat{U}_\wp$ being uniform in $[0, 1-e^{-KT(\wp)}]$. Taking for $F(z, \mathcal{L}_D^\theta, \mathcal{U})$ the function $G^{(K)} = G(\mathcal{L}_D^\theta(K), \mathcal{U}(K))$, it implies that

$$\mathbb{E}\left[G^{(K)}\mathcal{M}_\rho^K(dz)\right] =$$
$$\mathrm{CR}(z,D)^\rho \sum_{n\geq 1} \theta^n \int_{\underline{\rho}\in E(\rho,n),\rho_1<\ldots<\rho_n} \mathrm{d}\underline{\rho} \prod_{i=1}^n \frac{1-e^{-\rho_i C_K(z)}}{\rho_i} \mathbb{E}\left[G(\mathcal{L}_D^\theta(K)\cup\widehat{\Xi}^z_{\underline{\rho}}, \mathcal{U}(K)\cup\widehat{\mathcal{U}}_{\underline{\rho}})\right]\mathrm{d}z.$$

From the expression of $\mathbb{E}_2$ in (4.127), after a change of variables $(\rho_1,\ldots,\rho_{n-1},\rho)\to(\rho_1,\ldots,\rho_n)$, we get

$$\mathbb{E}_2 = \mathrm{CR}(z,D)^a \sum_{n\geq 1}\theta^n$$
$$\int_{\rho_1<\ldots<\rho_n,\rho<a} \frac{e^{-(a-\rho)C_K(z)}}{(a-\rho)^{1-\theta}} \prod_{i=1}^n \mathrm{d}\,\rho_i \,\frac{1-e^{-\rho_i C_K(z)}}{\rho_i} \mathbb{E}\left[G(\mathcal{L}_D^\theta(K)\cup\widehat{\Xi}^z_{\underline{\rho}}, \mathcal{U}(K)\cup\widehat{\mathcal{U}}_{\underline{\rho}})\right]\mathrm{d}z$$

where $\rho := \rho_1+\ldots+\rho_n$ in the integral. We will reinterpret this equality via the following lemma whose proof is deferred to the end of this section.

Lemma 4.52. *Let $\{a_1,a_2,\ldots\}$ be a random partition of $[0,a]$ distributed according to a Poisson-Dirichlet distribution with parameter θ. Let $u>0$. Remove each atom a_i independently with probability e^{-ua_i}. Denote by $\widehat{a}_1<\ldots<\widehat{a}_N$ the remaining atoms (there are only a finite number of them). Then*

$$P(N=0) = e^{-ua}$$

and for any integer $n\geq 1$, and $0<\rho_1<\ldots<\rho_n$ with $\rho := \rho_1+\ldots+\rho_n$,

$$P(N=n, \widehat{a}_1\in \mathrm{d}\,\rho_1,\ldots,\widehat{a}_n\in\mathrm{d}\,\rho_n) = \frac{a^{1-\theta}}{(a-\rho)^{1-\theta}} e^{-u(a-\rho)} \prod_{i=1}^n \frac{\theta}{\rho_i}(1-e^{-u\rho_i})\mathrm{d}\,\rho_i\,. \tag{4.128}$$

Using the lemma with $u = C_K(z)$ and with the notation of the lemma, we get that

$$\mathbb{E}_2 = \frac{1}{a^{1-\theta}}\,\mathrm{CR}(z,D)^a \mathbb{E}\left[G(\mathcal{L}_D^\theta(K)\cup\widehat{\Xi}^z_{\underline{\hat{a}}}, \mathcal{U}(K)\cup\widehat{\mathcal{U}}_{\underline{\hat{a}}})\mathbf{1}_{\{N\geq 1\}}\right]\mathrm{d}z$$

where $\widehat{\Xi}^z_{\underline{\hat{a}}} = \{\widehat{\Xi}^z_{\hat{a}_i}, i=1\ldots N\}$ and $\widehat{\mathcal{U}}_{\underline{\hat{a}}} := \{\widehat{U}_\wp, \wp\in\widehat{\Xi}^z_{\underline{\hat{a}}}\}$ with natural notation. We also have

$$\mathbb{E}_1 = \frac{1}{a^{1-\theta}}\,\mathrm{CR}(z,D)^a \mathbb{E}\left[G(\mathcal{L}_D^\theta(K), \mathcal{U}(K))\mathbf{1}_{\{N=0\}}\right].$$

From (4.125) , we get that

$$\mathbb{E}\left[G^{(K)} m_a^K(\mathrm{d}z)\right] = \frac{1}{a^{1-\theta}}\,\mathrm{CR}(z,D)^a \mathbb{E}\left[G(\mathcal{L}_D^\theta(K)\cup\widehat{\Xi}^z_{\underline{\hat{a}}}, \mathcal{U}(K)\cup\widehat{\mathcal{U}}_{\underline{\hat{a}}})\right]\mathrm{d}z \tag{4.129}$$

with the convention that $\widehat{\Xi}^z_{\underline{\hat{a}}}$ and $\widehat{\mathcal{U}}_{\underline{\hat{a}}}$ are empty when $N=0$. We observe that in the expectation in the RHS, the loop soup $\mathcal{L}_D^\theta(K)\cup\widehat{\Xi}^z_{\underline{\hat{a}}}$ and the random variables $\mathcal{U}(K)\cup\widehat{\mathcal{U}}_{\underline{\hat{a}}}$ are distributed respectively as the collection of loops killed at rate K in the loop soup $\mathcal{L}_D^\theta\cup\Xi_{\underline{a}}$, and the random variables $\mathcal{U}\cup\mathcal{U}_{\underline{a}}$

restricted to the killed loops. Integrating over $z \in A$, and recalling (4.124), it shows that we proved (4.123) for $F(z, \mathcal{L}_D, \mathcal{U}) = \mathbf{1}_{\{z \in A\}} G(\mathcal{L}_D(K), \mathcal{U}(K))$. Note that (4.129) also proves that $(m_a^K(dz), K > 0)$ is a martingale, independently of the first proof given in Section 4.5.3.

With Proposition 4.54 below, it shows that the couple $(\mathcal{L}_D^\theta, \mathcal{M}_a)$ satisfies the three points of Theorem 4.8. The fact that they characterize the law is standard, see [BBK94, AHS20]. Fix $\mathcal{L}_D^\theta$. We need to show that if $\widetilde{\mathcal{M}}_a$ is another Borel measure which is measurable with respect to $< \mathcal{L}_D^\theta >$ and verifies (4.7), then $\widetilde{\mathcal{M}}_a = \mathcal{M}_a$ a.s. We define $\widehat{\mathcal{M}}_a := \widetilde{\mathcal{M}}_a - \mathcal{M}_a$. By (4.7) applied to $\widetilde{\mathcal{M}}_a$ and $\mathcal{M}_a$, the expectation of $\int_D F(z, \mathcal{L}_D^\theta) \widehat{\mathcal{M}}_a$ is zero for any bounded measurable admissible function F. Take $F(z, \mathcal{L}_D^\theta) = \widehat{\mathcal{M}}_a(A) \mathbf{1}_{\{z \in A\}} \mathbf{1}_{\left\{|\widehat{\mathcal{M}}_a(A)| < c\right\}}$ where $c > 0$ and A is a Borel set. We get that $\mathbb{E}\left[\widehat{\mathcal{M}}_a(A)^2 \mathbf{1}_{\left\{|\widehat{\mathcal{M}}_a(A)| < c\right\}}\right] = 0$ is zero, hence that $\mathbb{E}\left[\widehat{\mathcal{M}}_a(A)^2\right] = 0$ by monotone convergence, so that $\widetilde{\mathcal{M}}_a(A) = \mathcal{M}_a(A)$ a.s. It completes the proof of the characterization.

Proof of Lemma 4.52. That $\mathbb{P}(N = 0) = e^{-ua}$ is clear so we only prove (4.128). Let $\{a_1, a_2, \dots\}$ be a random partition of $[0, a]$ distributed according to a Poisson-Dirichlet distribution with parameter θ. The atoms $\{a_1, a_2, \dots\}$ can be constructed via the jumps of a Gamma subordinator. More precisely, consider a Poisson point process $\{p_1, p_2, \dots\}$ on $\mathbb{R}_+$ with intensity $\mathbf{1}_{\{x>0\}} \frac{\theta}{x} e^{-x} \mathrm{d}x$. Let $\Sigma := \sum_{i \geq 1} p_i$ be the sum of the atoms of the PPP. Then, the collection $\{a \frac{p_1}{\Sigma}, a \frac{p_2}{\Sigma}, \dots\}$ is independent of Σ and distributed as $\{a_1, a_2, \dots\}$. One can also say that the atoms $\{p_1, p_2, \dots\}$ conditioned on $\Sigma = a$ are distributed as $\{a_1, a_2, \dots\}$. Using this representation, we remove each atom p_i of the PPP independently with probability $e^{-u p_i}$. The remaining atoms form a PPP of intensity $\mathbf{1}_{\{x>0\}} \frac{\theta}{x} e^{-x} (1 - e^{-ux}) \mathrm{d}x$. Notice that

$$\int_0^\infty \frac{\theta}{x} e^{-x} (1 - e^{-ux}) \mathrm{d}x = \theta \ln(u + 1).$$

In particular, the set of remaining atoms is finite a.s. Let N_p be its cardinality, and when $N_p \geq 1$, let $\hat{p}_1 < \dots < \hat{p}_{N_p}$ these atoms ordered increasingly. For $n \geq 1$, and $0 < \rho_1 < \dots < \rho_n$,

$$P(N_p = n, \hat{p}_1 \in \mathrm{d}\,\rho_1, \dots, \hat{p}_n \in \mathrm{d}\,\rho_n) = (u+1)^{-\theta} \prod_{i=1}^n \frac{\theta}{\rho_i} e^{-\rho_i} (1 - e^{-u \rho_i}) \mathrm{d}\,\rho_i\,.$$

The removed atoms are independent of the remaining atoms and form a PPP of intensity $\frac{\theta}{x} e^{-(u+1)x} \mathrm{d}x$. It is the Lévy measure of a Gamma$(\theta, u+1)$ subordinator. In particular, the sum of all these atoms, which is $\Sigma - \sum_{i=1}^{N_p} \hat{p}_i$, has the Gamma$(\theta, u+1)$ distribution, with density $\frac{(u+1)^\theta}{\Gamma(\theta)} s^{\theta - 1} e^{-(u+1)s} \mathrm{d}s$. It implies that , with $\rho := \sum_{i=1}^n \rho_i$,

$$P(N_p = n, \hat{p}_1 \in \mathrm{d}\,\rho_1, \dots, \hat{p}_n \in \mathrm{d}\,\rho_n, \, \Sigma \in \mathrm{d}a) = \frac{1}{\Gamma(\theta)} (a - \rho)^{\theta - 1} e^{-(u+1)(a-\rho)} \prod_{i=1}^n \frac{\theta}{\rho_i} e^{-\rho_i} (1 - e^{-u \rho_i}) \mathrm{d}\,\rho_i \, \mathrm{d}a.$$

Dividing by the probability that Σ is in $\mathrm{d}a$, which is $\frac{1}{\Gamma(\theta)} a^{\theta-1} e^{-a} \mathrm{d}a$, we proved that

$$P(N_p = n, \hat{p}_1 \in \mathrm{d}\,\rho_1, \dots, \hat{p}_n \in \mathrm{d}\,\rho_n \mid \Sigma = a) = \left(1 - \frac{\rho}{a}\right)^{\theta - 1} e^{-u(a-\rho)} \prod_{i=1}^n \frac{\theta}{\rho_i} (1 - e^{-u \rho_i}) \mathrm{d}\,\rho_i\,.$$

By the discussion at the beginning of the proof, we know that the distribution of $(N_p, \hat{p}_1, \dots, \hat{p}_{N_p})$ conditionally on $\Sigma = a$ is the one of $(N, \hat{a}_1, \dots, \hat{a}_N)$. The lemma follows. □

4.9 Measurability, conformal covariance and positivity

We will start this section by proving that the measure $\mathcal{M}_a$ is measurable w.r.t. the Brownian loop soup, and even w.r.t. the smaller sigma algebra $\sigma(\left\langle \mathcal{L}_D^\theta \right\rangle)$ defined in (4.30). This will prove Point 2 in Theorem 4.1. From this, we will obtain the characterisation of the joint law of $(\mathcal{L}_D^\theta, \mathcal{M}_a)$ as stated in Theorem 4.8. This characterisation will allow us to obtain the conformal covariance of the measure (actually a stronger version of it) which is the content of Point 3 in Theorem 4.1. In the last part of this section, we will use the conformal invariance of the measure to deduce its almost sure positivity, i.e. Theorem 4.1, Point 1.

4.9.1 Measurability

The purpose of this section is to prove Theorem 4.1, Point 2. In Appendix 4.A, we show that, essentially by definition, for all $K > 0$, $\mathcal{M}_a^K$ is measurable w.r.t. $\sigma(\left\langle \mathcal{L}_D^\theta(K) \right\rangle)$; see Lemma 4.57. Hence, this section consists in showing that the limiting measure $\mathcal{M}_a$ does not depend on the labels underlying the definition of killed loops.

Consider the Brownian loop soup $\mathcal{L}_D^\theta$. Since D is bounded, one can order the loops in the decreasing order of their diameter, $(\hat{\wp}_i)_{i \geq 1}$. Let $(U_i)_{i \geq 1}$ be an i.i.d. sequence of uniform r.v.s in $[0, 1]$, independent from $\mathcal{L}_D^\theta$. Given $K > 0$, we consider that $\mathcal{L}_D^\theta(K)$ is constructed according to (4.27), with the r.v. U_i associated to the loop $\hat{\wp}_i$. Let $\mathcal{F}_{\text{loops}}$ be the σ-algebra generated by the Brownian loop soup $\mathcal{L}_D^\theta$, where the loops are considered to be unrooted. It is the Borel σ-algebra for the topology on collections of unrooted loops described in Section 4.2.1. For $m \geq 1$, denote $\mathcal{F}_m$ the σ-algebra generated by $\left\langle \mathcal{L}_D^\theta \right\rangle$ and the r.v.s $(U_i)_{1 \leq i \leq m}$, and $\check{\mathcal{F}}_m$ the σ-algebra generated by $\left\langle \mathcal{L}_D^\theta \right\rangle$ and the r.v.s $(U_i)_{i > m}$. By Lemma 4.57, the random measure $\mathcal{M}_a$ is measurable with respect to $\check{\mathcal{F}}_1$. We want to show that $\mathcal{M}_a$ admits a modification coinciding a.s. with $\mathcal{M}_a$ which is measurable with respect to $\sigma(\left\langle \mathcal{L}_D^\theta \right\rangle)$.

Lemma 4.53. *For every $m \geq 1$, $\mathcal{M}_a$ is measurable with respect to $\check{\mathcal{F}}_m$.*

Proof. For $K > 0$, denote

$$\mathcal{M}_a^{K,m} := \sum_{n \geq 1} \frac{1}{n!} \sum_{\substack{\wp_1, \dots, \wp_n \\ \in \mathcal{L}_D^\theta(K) \cup \{\hat{\wp}_i, i = 1 \dots m-1\} \\ \forall i \neq j, \wp_i \neq \wp_j}} \mathcal{M}_a^{\wp_1 \cap \dots \cap \wp_n}.$$

Introducing this measure is useful since $\mathcal{M}_a^{K,m}$ is independent of the first m labels $U_i, i = 1 \dots m$: the m biggest loops will be always included, without having to check whether $U_i < 1 - e^{-KT(\hat{\wp}_i)}$ or not. By Lemma 4.57, the random measure $\mathcal{M}_a^{K,m}$ is measurable with respect to $\check{\mathcal{F}}_m$. Moreover, a.s. for K large enough, we have for all $i \in \{1, \dots, m\}$, $U_i < 1 - e^{-KT(\hat{\wp}_i)}$. Thus, if K is large enough, $\mathcal{M}_a^{K,m} = \mathcal{M}_a^K$ and $(\log K)^{-\theta} \mathcal{M}_a^{K,m}$ converges in probability as $K \to +\infty$ to $\mathcal{M}_a$. This shows that $\mathcal{M}_a$ is $\check{\mathcal{F}}_m$-measurable. □

Proposition 4.54. *A.s., we have that* $\mathbb{E}[\mathcal{M}_a | \left\langle \mathcal{L}_D^\theta \right\rangle] = \mathcal{M}_a$*. In particular,* $\mathcal{M}_a$ *admits a modification coinciding a.s. with* $\mathcal{M}_a$ *which is measurable with respect to* $\left\langle \mathcal{L}_D^\theta \right\rangle$*.*

Proof. Lemma (4.53) ensures that for every $m \geq 1$, $\mathbb{E}[\mathcal{M}_a | \mathcal{F}_m] = \mathbb{E}[\mathcal{M}_a | \left\langle \mathcal{L}_D^\theta \right\rangle]$ a.s. Further, as $m \to +\infty$, $\mathbb{E}[\mathcal{M}_a | \mathcal{F}_m]$ converges to $\mathcal{M}_a$ a.s. and in L^1. This concludes. □

4.9.2 Conformal covariance

Let $\psi : D \to \tilde{D}$ be a conformal map between two bounded simply connected domains. Recall that in Section 4.2.1, we introduced the transformation $\mathcal{T}_\psi$ on paths defined by

$$\mathcal{T}_\psi : (\wp(t), 0 \leq t \leq T(\wp)) \mapsto \left(\psi(\wp(S_{\psi,\wp}^{-1}(t)), 0 \leq t \leq S_{\psi,\wp}(T(\wp)) \right)$$

where

$$S_{\psi,\wp}(t) = \int_0^t |\psi'(\wp(s))|^2 \mathrm{d}s.$$

For any collection $\mathcal{C}$ of loops in D, we define $\mathcal{T}_\psi \mathcal{C} := \{\mathcal{T}_\psi \wp, \wp \in \mathcal{C}\}$.

Theorem 4.55. $(\mathcal{T}_\psi \mathcal{L}_D^\theta, |(\psi^{-1})'(\tilde{z})|^{-2-a} \mathcal{M}_{a,D} \circ \psi^{-1}(d\tilde{z}))$ *and* $(\mathcal{L}_{\tilde{D}}^\theta, \mathcal{M}_{a,\tilde{D}})$ *have the same joint distribution.*

Proof. We are going to use the characterisation of the joint law of $(\mathcal{L}_{\tilde{D}}^\theta, \mathcal{M}_{a,\tilde{D}})$ given in Theorem 4.8 and we need to check that $(\mathcal{T}_\psi \mathcal{L}_D^\theta, |\psi'(\psi^{-1}(\tilde{z}))|^{2+a} \mathcal{M}_{a,D} \circ \psi^{-1})$ satisfies the three properties therein. By conformal invariance of the unrooted loop measure $\mu_D^{\text{loop}*}$, $\mathcal{T}_\psi \mathcal{L}_D^\theta$ has the same law as $\mathcal{L}_{\tilde{D}}^\theta$. This shows the first property. The second property concerning the measurability is clear since it is stable under conformal transformations. To conclude, we need to check the third property. Let $F : \tilde{D} \times \mathfrak{L}_{\tilde{D}} \to \mathbb{R}$ be a nonnegative measurable admissible function. By definition of the pushforward of $\mathcal{M}_{a,D}$, we have

$$\begin{aligned} &\mathbb{E}\left[\int_{\tilde{D}} F(\tilde{z}, \mathcal{T}_\psi \mathcal{L}_D^\theta) |\psi'(\psi^{-1}(\tilde{z}))|^{2+a} \mathcal{M}_{a,D} \circ \psi^{-1}(d\tilde{z}) \right] \\ &= \mathbb{E}\left[\int_D F(\psi(z), \mathcal{T}_\psi \mathcal{L}_D^\theta) |\psi'(z)|^{2+a} \mathcal{M}_{a,D}(dz) \right]. \end{aligned} \tag{4.130}$$

Since $(z, \mathcal{L}) \in D \times \mathfrak{L}_D \mapsto F(\psi(z), \mathcal{T}_\psi \mathcal{L}) |\psi'(z)|^{2+a} \in \mathbb{R}$ is a nonnegative measurable admissible function, we can apply Theorem 4.8 to obtain that the left hand side of (4.130) is equal to

$$\frac{1}{2^\theta a^{1-\theta} \Gamma(\theta)} \int_D \mathbb{E}\left[F(\psi(z), \mathcal{T}_\psi(\mathcal{L}_D^\theta \cup \{\Xi_{a_i,D}^z, i \geq 1\})) \right] \mathrm{CR}(z, D)^a |\psi'(z)|^{2+a} \mathrm{d}z.$$

Above, we wrote $\Xi_{a_i,D}^z$ instead of $\Xi_{a_i}^z$ to emphasise that the underlying domain is D. By doing the change of variable $\tilde{z} = \psi(z)$, and because $\mathrm{CR}(z, D)|\psi'(z)| = \mathrm{CR}(\tilde{z}, \tilde{D})$, we obtain that the left hand side of (4.130) is equal to

$$\frac{1}{2^\theta a^{1-\theta} \Gamma(\theta)} \int_{\tilde{D}} \mathbb{E}\left[F(\tilde{z}, \mathcal{T}_\psi(\mathcal{L}_D^\theta \cup \{\Xi_{a_i,D}^{\psi^{-1}(\tilde{z})}, i \geq 1\})) \right] \mathrm{CR}(\tilde{z}, \tilde{D})^a \mathrm{d}\tilde{z}.$$

Since the image of the measure $\mu_D^{\psi^{-1}(\tilde{z}), \psi^{-1}(\tilde{z})}$ under $\mathcal{T}_\psi$ is the measure $\mu_{\tilde{D}}^{\tilde{z},\tilde{z}}$ (see [Law05, Proposition

5.5]), and by conformal invariance of $\mathcal{L}_D^\theta$, we can rewrite

$$\mathbb{E}\left[F(\tilde{z}, \mathcal{T}_\psi(\mathcal{L}_D^\theta \cup \{\Xi_{a_i,D}^{\psi^{-1}(\tilde{z})}, i \geq 1\}))\right] = \mathbb{E}\left[F(\tilde{z}, \mathcal{L}_{\tilde{D}}^\theta \cup \{\Xi_{a_i,\tilde{D}}^{\tilde{z}}, i \geq 1\})\right].$$

To wrap things up, we have proven that

$$\begin{aligned} &\mathbb{E}\left[\int_{\tilde{D}} F(\tilde{z}, \mathcal{T}_\psi \mathcal{L}_D^\theta) |\psi'(\psi^{-1}(\tilde{z}))|^{2+a} \mathcal{M}_{a,D} \circ \psi^{-1}(d\tilde{z})\right] \\ &= \frac{1}{2^\theta a^{1-\theta}\Gamma(\theta)} \int_{\tilde{D}} \mathbb{E}\left[F(\tilde{z}, \mathcal{L}_{\tilde{D}}^\theta \cup \{\Xi_{a_i,\tilde{D}}^{\tilde{z}}, i \geq 1\})\right] \mathrm{CR}(\tilde{z}, \tilde{D})^a \mathrm{d}\tilde{z} \end{aligned}$$

which is the third property characterising the joint law of $(\mathcal{L}_{\tilde{D}}^\theta, \mathcal{M}_{a,\tilde{D}})$. This concludes the proof. □

4.9.3 Positivity

We conclude this section with the proof of Theorem 4.1, Point 1.

Proof of Theorem 4.1, Point 1. The claim that, for all open set $A \subset D$, $\mathcal{M}_a(A)$ is finite almost surely, is clear since the total mass of $\mathcal{M}_a$ has finite expectation. We will therefore focus on proving that for all open set $A \subset D$, $\mathcal{M}_a(A) > 0$ almost surely. Let A be such a set and let A_1 and A_2 be two disjoint subsets of A that are scaled copies of A, i.e. we can write $A_i = f_i(A)$ where f_i, $i = 1, 2$, are affine functions. In what follows, we keep track of the domain D where the loop soup lives by writing $\mathcal{M}_{a,D}$ instead of $\mathcal{M}_a$. By only keeping loops that are contained in A, and by restriction property of Brownian loop soup, we see that $\mathcal{M}_{a,D}(A)$ stochastically dominates $\mathcal{M}_{a,A}(A)$. It is therefore sufficient to show that $\mathbb{P}(\mathcal{M}_{a,A}(A) = 0) = 0$. Similarly, by only keeping loops that are contained in $A_1 \cup A_2$, we obtain that

$$\mathbb{P}(\mathcal{M}_{a,A}(A) = 0) \leq \mathbb{P}(\mathcal{M}_{a,A_1 \cup A_2}(A_1 \cup A_2) = 0).$$

Since $\mathcal{M}_{a,A_1\cup A_2}(A_1 \cup A_2)$ is distributed like the independent sum $\mathcal{M}_{a,A_1}(A_1) + \mathcal{M}_{a,A_2}(A_2)$, we can rewrite the probability on the right hand side as the product of $\mathbb{P}(\mathcal{M}_{a,A_i}(A_i) = 0)$, $i = 1, 2$. Now, by conformal covariance and because the A_i's are affine transformations of A, $\mathbb{P}(\mathcal{M}_{a,A_i}(A_i) = 0) = \mathbb{P}(\mathcal{M}_{a,A}(A) = 0)$, $i = 1, 2$. We have therefore shown that

$$\mathbb{P}(\mathcal{M}_{a,A}(A) = 0) \leq \mathbb{P}(\mathcal{M}_{a,A}(A) = 0)^2.$$

Since this probability is strictly smaller than one (the expectation $\mathcal{M}_{a,A}(A)$ is positive), it has to vanish. This concludes the proof. □

Appendix 4.A Measurability of Brownian multiplicative chaos

This section deals with some technicalities concerning the measurability of the Brownian chaos measures w.r.t. the starting points, ending points, domains and thickness levels.

Denote by $\mathscr{M}$ the set of Borel measures on $\mathbb{C}$ equipped with the topology of weak convergence, and by $\mathscr{C}$ the set of continuous trajectories in the plane with finite duration equipped with the topology induced by d_{paths} (4.28). Recall the definition (4.54) of $\mathrm{mi}(D)$ and $\mathrm{Mi}(D)$ and the definition (4.56) of the half plane $\mathbb{H}_m$. Denote by $\mathscr{S}$ the set

$$\mathscr{S} := \{(m, x_0, z) \in (\mathrm{mi}(D), \mathrm{Mi}(D)) \times D \times D : x_0 \in \mathbb{H}_m, \mathrm{Im}(z) = m\}$$

equipped with its Borel σ-algebra. Let $n \geq 1$. We consider a stochastic process

$$(m_i, x_i, z_i)_{i=1\dots n} \in \mathscr{S}^n \mapsto (\wp_{D \cap \mathbb{H}_{m_i}}^{x_i, z_i})_{i=1\dots n} \in \mathscr{C}$$

such that for all $(m_i, x_i, z_i)_{i=1\dots n} \in \mathscr{S}^n$, $\wp_{D \cap \mathbb{H}_{m_i}}^{x_i, z_i}$, $i = 1 \dots n$, are independent Brownian trajectories from x_i to z_i in the domain $D \cap \mathbb{H}_{m_i}$, i.e. distributed according to $\mu_{D \cap \mathbb{H}_{m_i}}^{x_i, z_i} / H_{D \cap \mathbb{H}_{m_i}}(x_i, z_i)$ (4.17). We consider a measurable version of this stochastic process, that is a version such that

$$(\omega, (m_i, x_i, z_i)_{i=1\dots n}) \in \Omega \times \mathscr{S}^n \mapsto (\wp_{D \cap \mathbb{H}_{m_i}}^{x_i, z_i})_{i=1\dots n}(\omega) \in \mathscr{C}$$

is measurable (Ω stands here for the underlying probability space). In the next result, we consider the multiplicative chaos measures associated to the above Brownian paths. The subset $I \subset \{1 \dots n\}$ encodes the trajectories involved and we will need to consider all these measures jointly in I.

Lemma 4.56. *The process*

$$(a, (m_i, x_i, z_i)_{i=1\dots n}) \in (0, 2) \times \mathscr{S}^n \mapsto \left(\mathcal{M}_a^{\cap_{i \in I} \wp_{D \cap \mathbb{H}_{m_i}}^{x_i, z_i}} \right)_{I \subset \{1 \dots n\}} \in \prod_{I \subset \{1 \dots n\}} \mathscr{M}^{\# I} \tag{4.131}$$

is measurable.

Let us comment that the process (4.131) should actually possess a continuous modification, but showing such a regularity is actually far from being simple (see Proposition 1.2 and Remark 1.1 of [Jeg21]) and will not be needed in this article.

Proof. The Brownian chaos measures are defined as the pointwise limit of measures that are clearly measurable w.r.t. the path (see Section 4.2.3 or [Jeg19, Proposition 1.1]). Therefore, the process (4.131) is measurable as a pointwise limit of measurable processes. □

We finish this section by showing that the measure $\mathcal{M}_a^K$ on thick points of the massive loop soup $\mathcal{L}_D(K)$ is measurable w.r.t. $\sigma(\langle \mathcal{L}_D^\theta \rangle)$ (4.30), a σ-algebra smaller than the one generated by $\mathcal{L}_D^\theta$.

Lemma 4.57. *The measure $\mathcal{M}_a^K$ is measurable w.r.t. $\sigma(\langle \mathcal{L}_D^\theta(K) \rangle)$.*

Proof. For all $\varepsilon > 0$, $n \geq 1$ and pairwise distinct loops $\wp^{(1)}, \dots, \wp^{(n)} \in \mathcal{L}_{D,\varepsilon}^\theta \cap \mathcal{L}_D^\theta(K)$, the measure $\mathcal{M}_a^{\wp_{2,\varepsilon}^{(1)} \cap \dots \cap \wp_{2,\varepsilon}^{(n)}}$ is a measurable function of the occupation measures of $\wp_{2,\varepsilon}^{(i)}$, $i = 1 \dots n$. This is a consequence of [Jeg19, Proposition 1.1]. Therefore, for all $\varepsilon > 0$, $\mathcal{M}_a^{K,\varepsilon}$ is measurable w.r.t. the σ-algebra $\mathcal{F}_\varepsilon$ generated by the occupation measure of $\wp_{2,\varepsilon}, \wp \in \mathcal{L}_D^\theta(K)$. We conclude by noticing that $\cap_{\varepsilon > 0} \sigma(\mathcal{F}_\delta, \delta \in (0, \varepsilon))$ is included in the σ-algebra generated by the occupation measure of $\wp, \wp \in \mathcal{L}_D^\theta(K)$. This proves Lemma 4.57 since the occupation measure of a loop $\wp$ is a function of its equivalence class $\langle \wp \rangle$. □

Part Two: Discrete

4.10 Reduction

The purpose of this section is to explain the main lines of the proof of Theorem 4.12. The idea is to use the result of [Jeg19] that shows that the scaling limit of the set of thick points of planar random walk, killed upon exiting for the first time a given domain, is described by Brownian multiplicative chaos. In order to use this result, we will first compare the discrete measures and the continuum measures at the "approximation level", i.e. for loops killed by the mass. We will then show that the discrete measures with and without mass can be compared.

For $K > 0$, define the set of a-thick points of $\mathcal{L}^\theta_{D_N}(K)$ by

$$\mathcal{T}_{N,K}(a) := \left\{ z \in D_N : \ell_z(\mathcal{L}^\theta_{D_N}(K)) \geq \frac{1}{2\pi} a (\log N)^2 \right\} \tag{4.132}$$

(see (4.37) for the definition of $\ell_z(\mathcal{L}^\theta_{D_N}(K))$) and the associated point measure

$$\mathcal{M}_a^{N,K}(A) := \frac{\log N}{N^{2-a}} \sum_{z \in \mathcal{T}_{N,K}(a)} \mathbf{1}_{\{z \in A\}}. \tag{4.133}$$

Importantly, the normalisation of the measure is the same as in the case of a single random walk trajectory (see (1.1) of [Jeg19]). Without the mass cutoff, there are much more loops and the measure has to be tamed a bit more (see (4.11)).

Our first step will be to prove:

Proposition 4.58. *There exists a universal constant $c_0 > 0$, given by (4.12), such that, for any fixed $K > 0$, $(\mathcal{L}^\theta_{D_N}(K), \mathcal{M}_a^{N,K})$ converges in distribution as $N \to \infty$ towards $(\mathcal{L}^\theta_D(K), c_0^a \mathcal{M}_a^K)$. The underlying topologies are the topology induced by the distance $d_{\mathfrak{L}}$ (4.29) and the topology of weak convergence of measures.*

The second step will be to control the effect of the mass on the measure:

Proposition 4.59. *For any Borel set $A \subset \mathbb{C}$,*

$$\limsup_{K \to \infty} \limsup_{N \to \infty} \mathbb{E}\left[\left| \mathcal{M}_a^N(A) - \frac{2^\theta}{(\log K)^\theta} \mathcal{M}_a^{N,K}(A) \right| \right] = 0.$$

Moreover, $d_{\mathfrak{L}}(\mathcal{L}^\theta_{D_N}, \mathcal{L}^\theta_{D_N}(K))$ goes to zero in probability as $N \to \infty$ and then $K \to \infty$.

We can now prove Theorems 4.12 and 4.5.

Proof of Theorem 4.12. This is an immediate consequence of Theorem 4.1 and Propositions 4.58 and 4.59. □

Proof of Theorem 4.5. By Le Jan's isomorphism (Theorem 4.18), we can couple a discrete GFF φ_N in D_N and a random walk loop soup $\mathcal{L}^{1/2}_{D_N}$ with critical intensity in such a way that the occupation field $\ell(\mathcal{L}^{1/2}_{D_N})$ and $\frac{1}{2}\varphi_N^2$ coincide. Let $\mathcal{M}_a^N$ be the measure defined as in (4.11) and $C_0(D)$ be the space of continuous functions on $\bar{D}$ that vanish on ∂D. We view φ_N as a random element of $\mathbb{R}^{C_0(D)}$ by setting for all $f \in C_0(D)$,

$$\varphi_N f := \frac{1}{N^2} \sum_{z \in D_N} \varphi_N(z) f(z).$$

We are going to show that $(\varphi_N, \mathcal{L}_{D_N}^{1/2}, 2^{-\theta} c_0^{-a} \mathcal{M}_a^N)$ converges in distribution (along a subsequence) towards a triplet that satisfies all the relations required by Theorem 4.5. The topologies associated to $\mathcal{L}_{D_N}^{1/2}$ and $\mathcal{M}_a^N$ are the same ones as in Theorem 4.12 and the topology associated to φ_N is the product topology on $\mathbb{R}^{C_0(D)}$. To establish such a result, we only need to argue that

(i) $(\mathcal{L}_{D_N}^{1/2}, 2^{-\theta} c_0^{-a} \mathcal{M}_a^N) \xrightarrow{(d)} (\mathcal{L}_D^{1/2}, \mathcal{M}_a)$ where $\mathcal{M}_a$ is the multiplicative chaos associated to $\mathcal{L}_D^{1/2}$ from Theorem 4.1;

(ii) $(\varphi_N, 2^{-\theta} c_0^{-a} \mathcal{M}_a^N) \xrightarrow{(d)} (\varphi, \frac{1}{\sqrt{2\pi a}} \cosh(\gamma h))$ where $\cosh(\gamma h)$ is the hyperbolic cosine associated to $h = \sqrt{2\pi} \varphi$;

(iii) $(\mathcal{L}_{D_N}^{1/2}, \varphi_N) \xrightarrow{(d)} (\mathcal{L}_D^{1/2}, \varphi)$ along a subsequence $(N_k)_{k \geq 1}$, where the Brownian loop soup $\mathcal{L}_D^{1/2}$ and the GFF φ satisfy Le Jan's identity: $:\ell(\mathcal{L}_D^{1/2}): = \frac{1}{2} :\varphi^2:$.

Indeed, assume these three convergences. The law of $(\varphi_{N_k}, \mathcal{L}_{D_{N_k}}^{1/2}, 2^{-\theta} c_0^{-a} \mathcal{M}_a^{N_k})$ is tight since each of the three components converges. Let $(\varphi_\infty, \mathcal{L}_{D,\infty}^{1/2}, \mathcal{M}_{a,\infty})$ be any subsequential limit. The three pairwise convergences above suffice to identify the law of this triplet: φ_∞ is a GFF in D and $\mathcal{L}_{D,\infty}^{1/2}$ is a critical Brownian loop soup in D related by Le Jan's identity; $\mathcal{M}_{a,\infty}$ is measurable w.r.t. $\mathcal{L}_{D,\infty}^{\theta}$ and is the associated multiplicative chaos; $\mathcal{M}_{a,\infty}$ is measurable w.r.t. φ_∞ and is the associated hyperbolic cosine.

To conclude the proof, we need to explain where (i)-(iii) come from. (i) is the content of Theorem 4.12. (ii) is a quick consequence of [BL19] as we are about to explain. By definition (4.11) of $\mathcal{M}_a^N$ and because the occupation field of $\mathcal{L}_{D_N}^{1/2}$ is equal to $\varphi_N^2/2$ (without any normalisation), $\mathcal{M}_a^N$ is equal to $\eta_\gamma^N + \eta_{-\gamma}^N$ where $\gamma = \sqrt{2a}$ and $\eta_{\pm\gamma}^N$ are the measures defined by

$$\eta_{\pm\gamma}^N(A) := \frac{\sqrt{\log N}}{N^{2-\gamma^2/2}} \sum_{z \in D_N} \mathbf{1}_{\{z \in A\}} \mathbf{1}_{\left\{\pm \varphi_N(z) \geq \frac{\gamma}{\sqrt{2\pi}} \log N\right\}}, \quad A \subset \mathbb{R}^2 \text{ Borel set}.$$

By [BL19, Theorems 2.1 and 2.5], there exists some universal constant $c_* > 0$ such that η_γ^N converges in distribution to $c_* e^{\gamma h}$ where $h = \sqrt{2\pi}\varphi$ is a Gaussian free field in D. This convergence can be easily extended to the joint convergence of $(\varphi_N, \eta_\gamma^N)$ to $(\varphi, c_* e^{\gamma h})$. Indeed, this extension follows from a simple use of Girsanov's theorem and the details can be found in the proof of [BGL20, Lemma 6.9] in a slightly different setting. The two convergences $(\varphi_N, \eta_\gamma^N) \to (\varphi, c_* e^{\gamma h})$ and $(\varphi_N, \eta_{-\gamma}^N) \to (\varphi, c_* e^{-\gamma h})$, plus the fact that the limiting measures are measurable w.r.t. the underlying GFF φ, imply the joint convergence $(\varphi_N, \eta_\gamma^N, \eta_{-\gamma}^N) \to (\varphi, c_* e^{\gamma h}, c_* e^{-\gamma h})$. In particular, $(\varphi_N, \eta_\gamma^N + \eta_{-\gamma}^N) \to (\varphi, 2c_* \cosh(\gamma h))$ as desired in (ii). The value of the constant c_* can be computed looking at the first moment.

Finally, let us prove (iii). This is an immediate consequence of the two joint convergences

$$(\mathcal{L}_{D_N}^{1/2}, \ell(\mathcal{L}_{D_N}^{1/2}) - \mathbb{E}\ell(\mathcal{L}_{D_N}^{1/2})) \xrightarrow{(d)} (\mathcal{L}_D^{1/2}, :\ell(\mathcal{L}_D^{1/2}):)$$

and

$$(\varphi_N, \varphi_N^2 - \mathbb{E}\varphi_N^2) \xrightarrow{(d)} (\varphi, :\varphi^2:)$$

along a subsequence $(N'_k)_{k\geq 1}$ (see the proof of [QW19, Lemma 6]). Indeed, these convergences implies tightness of the quadruple $(\mathcal{L}_{D_N}^{1/2}, \ell(\mathcal{L}_{D_N}^{1/2}) - \mathbb{E}\ell(\mathcal{L}_{D_N}^{1/2}), \varphi_N, 1/2\varphi_N^2 - 1/2\mathbb{E}\varphi_N^2)$ along (N'_k). Let $(N_k)_{k\geq 1}$ be a further subsequence of $(N'_k)_{k\geq 1}$ such that the quadruple above converges towards some

$$(\mathcal{L}_D^{1/2}, :\ell(\mathcal{L}_D^{1/2}):, \varphi, 1/2 :\varphi^2:)$$

along (N_k). To conclude, we only need to make sure that the second and fourth components of the limiting variable agree. Our specific choice of coupling between $\mathcal{L}_{D_N}^{1/2}$ and φ_N ensures that this is always true at the discrete level. Therefore, it is also true in the limit. □

Remark 4.60. Notice that the above argument does not establish convergence since Le Jan's isomorphism (as noted earlier) does not uniquely determine the joint law of the free field and loop soup. Nevertheless, the subsequential limit satisfies the relations stated in Theorem 4.5.

The remaining of Part Two is organised as follows. In Section 4.11, we give exact expressions for the first two moments associated to $\mathcal{M}_a^N$ and $\mathcal{M}_a^{N,K}$, as well as describing the associated conditional laws of the random walk loop soup $\mathcal{L}_{D_N}^\theta$. These exact formulae will be instrumental in the proof of Proposition 4.59 which is achieved in Section 4.12. Finally, Section 4.13 is dedicated to the proof of Proposition 4.58.

4.11 Exact expressions

In this section we will give the expressions of the first and second moments for $\mathcal{M}_a^N$ and $\mathcal{M}_a^{N,K}$, as well as give the corresponding conditional laws of the random walk loop soup $\mathcal{L}_{D_N}^\theta$.

4.11.1 First moment (discrete Girsanov)

Recall the definition (4.12) of the constant c_0 which appears in the asymptotic of the Green function on the diagonal; see (4.219).

In this section D_N will be just a subset of $\mathbb{Z}_N^2$, with both D_N and $\mathbb{Z}_N^2 \setminus D_N$ non-empty. For $z \in D_N$, denote

$$\mathrm{CR}_N(z, D_N) := N c_0^{-1} e^{-(\log N)^2/(2\pi G_{D_N}(z,z))}.$$

As the notation suggests, we will use $\mathrm{CR}_N(z, D_N)$ in a situation where it converges to a conformal radius as $N \to +\infty$; see (4.219). Let $q_N(z)$ be the ratio

$$q_N(z) := \frac{\log N}{2\pi G_{D_N}(z,z)}.$$

If $N \to +\infty$ and the Euclidean distance from z to $\mathbb{Z}_N^2 \setminus D_N$ is bounded away from 0, then $q_N(z) \to 1$; see Lemma 4.93.

Given $z, w \in D_N$, we will denote by $\tilde{\mu}^{z,w}_{D_N}$ the renormalised measure

$$\tilde{\mu}^{z,w}_{D_N} := \Big(\frac{1}{2\pi}\log N\Big)^2 \check{\mu}^{z,w}_{D_N}, \tag{4.134}$$

where $\check{\mu}^{z,w}_{D_N}$ is given by (4.35). Given $z \in D_N$ and $a > 0$, we will denote by $\Xi^z_{N,a}$ the random loop in D_N, obtained by concatenating a Poisson point process of continuous time random walk excursions from z to z of intensity $2\pi a \tilde{\mu}^{z,z}_{D_N}$, and having a local time in z

$$\ell_z(\Xi^z_{N,a}) = \frac{1}{2\pi} a (\log N)^2.$$

As in Section 4.5, we will consider admissible functions F which do not depend on the order of excursions in a loop.

The following proposition is merely a rephrasing of Proposition 4.17 in terms of the random discrete measure $\mathcal{M}^N_a$ given by (4.11). It is to be compared to Theorem 4.8 for the continuum setting. The Poisson-Dirichlet partition that appears below comes from the Gamma subordinator (4.38).

Proposition 4.61. *Fix $z \in D_N$ and $a > 0$. For any bounded measurable admissible function F,*

$$\begin{aligned} &\mathbb{E}\left[F(\mathcal{L}^\theta_{D_N})\mathcal{M}^N_a(\{z\})\right] \\ &= \frac{1}{N^2\Gamma(\theta)} q_N(z)^\theta \log N \int_a^{+\infty} \rho^{\theta-1} \frac{c_0^\rho}{N^{\rho-a}} \mathrm{CR}_N(z, D_N)^\rho \mathbb{E}\left[F(\mathcal{L}^\theta_{D_N\setminus\{z\}} \cup \{\Xi^z_{N,a_i}, i \geq 1\})\right] \mathrm{d}\rho, \end{aligned}$$

where on the right-hand side, $\mathcal{L}^\theta_{D_N\setminus\{z\}}$ and $\{\Xi^z_{N,a_i}, i \geq 1\}$ are independent, the $(a_i)_{i\geq 1}$ is a Poisson-Dirichlet partition $\mathrm{PD}(0,\theta)$ of $[0,\rho]$, and the Ξ^z_{N,a_i} are conditionally independent given $(a_i)_{i\geq 1}$.

Now let us consider the massive case. Fix $K > 0$ a constant. For $z \in D_N$, denote

$$q_{N,K}(z) := \frac{\log N}{2\pi G_{D_N,K}(z,z)}, \qquad C_{N,K}(z) := 2\pi(G_{D_N}(z,z) - G_{D_N,K}(z,z)),$$

$$J_{N,K}(z) := \int_0^{+\infty} (e^{-t/G_{D_N}(z,z)} - e^{-t/G_{D_N,K}(z,z)}) \frac{\mathrm{d}t}{t}.$$

Again, $q_{N,K}(z)$ tends to 1 if $N \to +\infty$ and the Euclidean distance from z to $\mathbb{Z}^2_N \setminus D_N$ is bounded away from 0.

Lemma 4.62. *For every $z \in D_N$ and $a > 0$,*

$$\begin{aligned} \mathbb{E}\left[e^{-KT(\Xi^z_{N,a})}\right] &= e^{-(2\pi)^{-1} a (\log N)^2 (G_{D_N,K}(z,z)^{-1} - G_{D_N}(z,z)^{-1})} \\ &= e^{-q_N(z) q_{N,K}(z) C_{N,K}(z) a}. \end{aligned}$$

Proof. The expectation above is simply given by the ratio between (4.39) and (4.38) for $t = \frac{1}{2\pi} a (\log N)^2$. □

The following proposition is to be compared to Lemma 4.32 and Proposition 4.21.

Proposition 4.63. *For any bounded measurable admissible function F,*

$$\begin{aligned}&\mathbb{E}\left[F(\mathcal{L}_{D_N}^\theta)\mathcal{M}_a^{N,K}(\{z\})\right] \\ &= \frac{\log N}{N^2}e^{-\theta J_{N,K}(z)}\int_a^{+\infty}\mathrm{d}\rho\frac{c_0^\rho}{N^{\rho-a}}\mathrm{CR}_N(z,D_N)^\rho\sum_{n\geq 1}\frac{\theta^n}{n!}\int_{\mathsf{a}\in E(\rho,n)}\frac{\mathrm{d}\mathsf{a}}{a_1\dots a_n} \\ &\times\mathbb{E}\left[\prod_{i=1}^n\left(1-e^{-KT(\Xi_{N,a_i}^z)}\right)F(\mathcal{L}_{D_N\setminus\{z\}}^\theta\cup\widetilde{\mathcal{L}}_{D_N,K,z}^\theta\cup\{\Xi_{N,a_i}^z,i=1\dots n\})\right],\end{aligned}\tag{4.135}$$

where on the right-hand side, the three collections of loops $\mathcal{L}_{D_N\setminus\{z\}}^\theta$, $\widetilde{\mathcal{L}}_{D_N,K,z}^\theta$ and $\{\Xi_{N,a_i}^z, i=1\dots n\}$ are independent, the different Ξ_{N,a_i}^z are independent, and $\widetilde{\mathcal{L}}_{D_N,K,z}^\theta$ is distributed as the loops in $\mathcal{L}_{D_N}^\theta\setminus\mathcal{L}_{D_N}^\theta(K)$ visiting z. In particular,

$$\mathbb{E}\left[\mathcal{M}_a^{N,K}(\{z\})\right]=\frac{\log N}{N^2}e^{-\theta J_{N,K}(z)}\int_a^{+\infty}\frac{c_0^\rho}{N^{\rho-a}}\mathrm{CR}_N(z,D_N)^\rho\mathsf{F}(q_N(z)q_{N,K}(z)C_{N,K}(z)\rho)\frac{\mathrm{d}\rho}{\rho},\tag{4.136}$$

where F is given by (4.44).

Proof. The second identity follows from the first one and Lemma 4.62, by taking $F=1$. See also Lemma 4.33.

Regarding the identity (4.135), observe that the loops in $\mathcal{L}_{D_N}^\theta(K)$ visiting z form a Poisson point process which is a.s. finite, regardless of D_N being finite or not. For instance, the intensity measure for $(\ell_z(\wp))_{\wp\in\mathcal{L}_{D_N}^\theta(K),\wp\text{ visits }z}$ is

$$\mathbf{1}_{\{t>0\}}\theta(e^{-t/G_{D_N}(z,z)}-e^{-t/G_{D_N,K}(z,z)})\frac{\mathrm{d}t}{t},$$

which is the difference between (4.38) and (4.39). Its total mass is finite, equal to $\theta J_{N,K}(z)$. We obtain (4.135) by summing over the values of $\#\{\wp\in\mathcal{L}_{D_N}^\theta(K):\wp\text{ visits }z\}$. We skip the details. □

As a corollary,

Corollary 4.64. *Let D be an open bounded simply connected domain, $(D_N)_N$ be a discrete approximation of D as in (4.9) and $f:D\to[0,\infty)$ be a nonnegative bounded continuous function. Then*

$$\sup_{N\geq 1}\mathbb{E}\left[\left\langle\mathcal{M}_a^N,f\right\rangle\right]<\infty\quad\text{and}\quad\lim_{N\to\infty}\mathbb{E}\left[\left\langle\mathcal{M}_a^N,f\right\rangle\right]=c_0^a\frac{a^{\theta-1}}{\Gamma(\theta)}\int_D f(z)\,\mathrm{CR}(z,D)^a\mathrm{d}z.$$

Moreover,

$$\sup_{N\geq 1}\mathbb{E}\left[\left\langle\mathcal{M}_a^{N,K},f\right\rangle\right]<\infty\quad\text{and}\quad\lim_{N\to\infty}\mathbb{E}\left[\left\langle\mathcal{M}_a^{N,K},f\right\rangle\right]=c_0^a\frac{\mathsf{F}(C_K(z)a)}{a}\int_D f(z)\,\mathrm{CR}(z,D)^a\mathrm{d}z.$$

Proof. With Propositions 4.61 and 4.63 in hand, checking this corollary is a simple computation. □

4.11.2 Second moment (two-point discrete Girsanov)

Here we will deal with the second moments of $\mathcal{M}_a^N$ and $\mathcal{M}_a^{N,K}$.

Given $z \in D_N$, the Green function on $D_N \setminus \{z\}$ can be expressed as follows:

$$G_{D_N \setminus \{z\}}(z', w') = G_{D_N}(z', w') - \frac{G_{D_N}(z', z) G_{D_N}(z, w')}{G_{D_N}(z, z)}. \tag{4.137}$$

Given $z \neq w \in D_N$, denote

$$q_N(z, w) := \frac{(\log N)^2}{4\pi^2 (G_{D_N}(z, z) G_{D_N}(w, w) - G_{D_N}(z, w)^2)}.$$

Let $\widetilde{G}_{D_N}(z, w)$ denote the total mass of the measure $\tilde{\mu}^{z,w}_{D_N}$.

Lemma 4.65. *Let $z, w \in D_N$ such that the graph distance on $\mathbb{Z}^2_N$ between z and w is at least 2, i.e. $|w - z| > \frac{1}{N}$. Then,*

$$\widetilde{G}_{D_N}(z, w) = q_N(z, w) G_{D_N}(z, w).$$

Proof. From (3) in Lemma 4.16 follows that the total mass of $\check{\mu}^{z,w}_{D_N}$ equals

$$\frac{G_{D_N}(z, w)}{G_{D_N}(z, z) G_{D_N \setminus \{z\}}(w, w)} = \frac{G_{D_N}(z, w)}{G_{D_N}(z, z) G_{D_N}(w, w) - G_{D_N}(z, w)^2}.$$

□

Given $z \neq z' \in D_N$, denote

$$\mathrm{CR}_{N,z}(z', D_N) := N c_0^{-1} e^{-(\log N)^2 / (2\pi G_{D_N \setminus \{z\}}(z', z'))}.$$

Given $z \neq z' \in D_N$ and $a' > 0$, let $\Xi^{z'}_{N,z,a'}$ denote the the random loop in $D_N \setminus \{z\}$, obtained by concatenating a Poisson point process of continuous time random walk excursions from z' to z' of intensity $2\pi a' \tilde{\mu}^{z',z'}_{D_N \setminus \{z\}}$, and having a local time in z'

$$\ell_{z'}(\Xi^{z'}_{N,z,a'}) = \frac{1}{2\pi} a' (\log N)^2.$$

By construction, $\Xi^{z'}_{N,z,a'}$ does not visit z. Applying Lemma 4.62 to $D_N \setminus \{z\}$, we have that

$$\mathbb{E}\left[e^{-KT(\Xi^{z'}_{N,z,a'})} \right] = e^{-a' C_{N,K,z}(z')} \tag{4.138}$$

where

$$C_{N,K,z}(z') := \frac{(\log N)^2}{4\pi^2 G_{D_N \setminus \{z\}}(z', z') G_{D_N \setminus \{z\}, K}(z', z')} 2\pi \left(G_{D_N \setminus \{z\}}(z', z') - G_{D_N \setminus \{z\}, K}(z', z') \right) \tag{4.139}$$

where we recall that the massive Green function is defined in (4.36).

Lemma 4.66. *Let $z, z' \in D_N$ such that the graph distance on $\mathbb{Z}^2_N$ between z and z' is at least 2, i.e.*

$|z'-z| > \frac{1}{N}$. *Let $a>0$. Then for any bounded measurable function F,*

$$\int \mathbf{1}_{\{\wp \text{ visits } z'\}} F(\wp) \tilde{\mu}^{z,z}_{D_N}(d\wp) = 2\pi \int_0^{+\infty} \mathrm{d}a' \frac{c_0^{a'}}{N^{a'}} \mathrm{CR}_{N,z}(z',D_N)^{a'} \int \tilde{\mu}^{z,z'}_{D_N}(d\wp_1) \int \tilde{\mu}^{z',z}_{D_N}(d\wp_2) \mathbb{E}\left[F(\wp_1 \wedge \Xi^{z'}_{N,z,a'} \wedge \wp_2) \right],$$

where $\wedge$ denotes the concatenation of paths.

Proof. This is a consequence of Lemma 4.16. By applying (2) in Lemma 4.16 in the case $z=w$, and then (1) in Lemma 4.16 for the measure $\mu^{z',z'}_{D_N\setminus\{z\}}$, we get that

$$\begin{aligned} &\int \mathbf{1}_{\{\wp \text{ visits } z'\}} F(\wp) \tilde{\mu}^{z,z}_{D_N}(d\wp) \\ &= \frac{4\pi^2}{(\log N)^2} \int_0^{+\infty} \mathrm{d}t e^{-t/G_{D_N\setminus\{z\}}(z',z')} \int \tilde{\mu}^{z,z'}_{D_N}(d\wp_1) \int \tilde{\mu}^{z',z}_{D_N}(d\wp_2) \\ &\times \mathbb{E}\left[F(\wp_1 \wedge \Xi^{z'}_{N,z,2\pi t(\log N)^{-2}} \wedge \wp_2) \right]. \end{aligned} \tag{4.140}$$

We conclude by performing the change of variables $a' = 2\pi t (\log N)^{-2}$. ☐

Given $z \neq z' \in D_N$ and $a,a'>0$, let $\Xi^{z,z'}_{N,a,a'}$ denote the random collection of an even number of excursions from z to z' with the following law. For all $k \geq 1$,

$$\mathbb{P}\left(\# \Xi^{z,z'}_{N,a,a'} = 2k \right) = \frac{1}{\mathsf{B}\Big((2\pi)^2 a a' \widetilde{G}_{D_N}(z,z')^2 \Big)} \frac{(2\pi \sqrt{aa'} \widetilde{G}_{D_N}(z,z'))^{2k}}{k!(k-1)!}, \tag{4.141}$$

where B is given by (4.80), and conditionally on $\{\#\Xi^{z,z'}_{N,a,a'} = 2k\}$, $\Xi^{z,z'}_{N,a,a'}$ is composed of $2k$ i.i.d. excursions with common law $\tilde{\mu}^{z,z'}_{D_N} / \widetilde{G}_{D_N}(z,z')$. As in Section 4.6, we will consider admissible function, invariant under reordering of excursions.

Lemma 4.67. *Let $z,z' \in D_N$ such that the graph distance on $\mathbb{Z}^2_N$ between z and z' is at least 2, i.e. $|z'-z| > \frac{1}{N}$. Let $a>0$. Then for any bounded measurable admissible function F,*

$$\begin{aligned} &\mathbb{E}\left[\mathbf{1}_{\{\Xi^z_{N,a} \text{ visits } z'\}} F(\Xi^z_{N,a}) \right] \\ &= e^{-a(2\pi)^3 \widetilde{G}_{D_N}(z,z')^2 G_{D_N\setminus\{z\}}(z',z')/(\log N)^2} \int_0^{+\infty} \frac{\mathrm{d}a'}{a'} \frac{c_0^{a'}}{N^{a'}} \mathrm{CR}_{N,z}(z',D_N)^{a'} \mathsf{B}((2\pi)^2 a a' \widetilde{G}_{D_N}(z,z')^2) \\ &\times \mathbb{E}\left[F(\Xi^{z,z'}_{N,a,a'} \wedge \Xi^z_{N,z',a} \wedge \Xi^{z'}_{N,z,a'}) \right], \end{aligned}$$

where $\Xi^{z,z'}_{N,a,a'}$, $\Xi^z_{N,z',a}$ and $\Xi^{z'}_{N,z,a'}$ are independent.

Proof. In $\Xi^z_{N,a}$, the excursions away from z visiting z' are independent from those not visiting z'. The concatenation of the excursions not visiting z' is distributed as $\Xi^z_{N,z',a}$. The excursions visiting z' form a Poisson point process which is a.s. finite. According to (4.140), the total mass of the corresponding intensity measure is

$$2\pi a \times \frac{4\pi^2}{(\log N)^2} \widetilde{G}_{D_N}(z,z')^2 G_{D_N\setminus\{z\}}(z',z').$$

According to Lemma 4.66, and excursion that goes k times there and back between z and z' can be decomposed into $2k$ excursion between z and z' and k excursions $\Xi^{z'}_{N,z,a'_1},\dots,\Xi^{z'}_{N,z,a'_k}$ from z' to z' not visiting z. The "thicknesses" (i.e. renormalised local times) $a'_1,\dots,a'_k$ are random and i.i.d. The excursions $\Xi^{z'}_{N,z,a'_1},\dots,\Xi^{z'}_{N,z,a'_k}$ are conditionally independent given $(a'_1,\dots,a'_k)$. The concatenation $\Xi^{z'}_{N,z,a'_1}\wedge\cdots\wedge\Xi^{z'}_{N,z,a'_k}$ is distributed as $\Xi^{z'}_{N,z,a'}$ where $a'=a'_1+\cdots+a'_k$. The $2k$ excursions from z to z' are i.i.d., independent from $a'_1,\dots,a'_k$ and $\Xi^{z'}_{N,z,a'_1},\dots,\Xi^{z'}_{N,z,a'_k}$, each one distributed according to $\tilde\mu^{z,z'}_{D_N}/\widetilde{G}_{D_N}(z,z')$. The distribution of $(a'_1,\dots,a'_k)$ on the event that $\Xi^z_{N,a}$ performs k travels from z to z' (and k back) is

$$\begin{aligned}&\mathbf{1}_{\{a'_1,\dots,a'_k>0\}}e^{-a(2\pi)^3\widetilde{G}_{D_N}(z,z')^2G_{D_N\setminus\{z\}}(z',z')/(\log N)^2}\frac{(2\pi)^{2k}a^k\widetilde{G}_{D_N}(z,z')^{2k}}{k!}\\ &\times\prod_{i=1}^k\Big(\frac{c_0^{a'_i}}{N^{a'_i}}\mathrm{CR}_{N,z}(z',D_N)^{a'_i}\mathrm{d}a'_i\Big).\end{aligned}$$

The induced distribution on $a'=a'_1+\cdots+a'_k$ is

$$\mathbf{1}_{\{a'>0\}}e^{-a(2\pi)^3\widetilde{G}_{D_N}(z,z')^2G_{D_N\setminus\{z\}}(z',z')/(\log N)^2}\frac{(2\pi)^{2k}(aa')^k\widetilde{G}_{D_N}(z,z')^{2k}}{k!(k-1)!}\frac{c_0^{a'}}{N^{a'}}\mathrm{CR}_{N,z}(z',D_N)^{a'}\frac{\mathrm{d}a'}{a'}.$$

One recognizes above the k-th term in the expansion of $\mathsf{B}((2\pi)^2aa'\widetilde{G}_{D_N}(z,z')^2)$; see (4.80). This concludes. $\square$

Next we consider the loop measure $\mu^{\mathrm{loop}}_{D_N}$ (4.34) and the decomposition of loops that visit two given vertices z and z'.

Lemma 4.68. *Let $z,z'\in D_N$ such that the graph distance on $\mathbb{Z}^2_N$ between z and z' is at least 2, i.e. $|z'-z|>\frac{1}{N}$. Then for any bounded measurable admissible function F,*

$$\begin{aligned}&\int\mathbf{1}_{\{\wp\text{ visits }z\text{ and }z'\}}F(\gamma)\mu^{\mathrm{loop}}_{D_N}(d\wp)\\ &=\int_0^{+\infty}\frac{\mathrm{d}a}{a}\mathrm{CR}_{N,z'}(z,D_N)^a\int_0^{+\infty}\frac{\mathrm{d}a'}{a'}\mathrm{CR}_{N,z}(z',D_N)^{a'}\frac{c_0^{a+a'}}{N^{a+a'}}\mathsf{B}\big((2\pi)^2aa'\widetilde{G}_{D_N}(z,z')^2\big)\\ &=\mathbb{E}\left[F(\Xi^{z,z'}_{N,a,a'}\wedge\Xi^z_{N,z',a}\wedge\Xi^{z'}_{N,z,a'})\right].\end{aligned}$$

Proof. From Proposition 4.17 it follows that

$$\begin{aligned}&\int\mathbf{1}_{\{\wp\text{ visits }z\text{ and }z'\}}F(\gamma)\mu^{\mathrm{loop}}_{D_N}(d\wp)\\ &\qquad=\int_0^{+\infty}\frac{\mathrm{d}a}{a}e^{-a(\log N)^2/(2\pi G_{D_N}(z,z))}\mathbb{E}\left[\mathbf{1}_{\{\Xi^z_{N,a}\text{ visits }z'\}}F(\Xi^z_{N,a})\right].\end{aligned}$$

By combining with Lemma 4.67, we get that this further equals to

$$\int_0^{+\infty} \frac{\mathrm{d}a}{a} e^{-a(\log N)^2/(2\pi G_{D_N}(z,z))} e^{-a(2\pi)^3 \widetilde{G}_{D_N}(z,z')^2 G_{D_N\setminus\{z\}}(z',z')/(\log N)^2}$$
$$\times \int_0^{+\infty} \frac{\mathrm{d}a'}{a'} \frac{c_0^{a'}}{N^{a'}} \mathrm{CR}_{N,z}(z',D_N)^{a'} \mathsf{B}\big((2\pi)^2 a a' \widetilde{G}_{D_N}(z,z')^2\big)$$
$$\times \mathbb{E}\left[F(\Xi_{N,a,a'}^{z,z'} \wedge \Xi_{N,z',a}^{z} \wedge \Xi_{N,z,a'}^{z'})\right].$$

We have that

$$\begin{aligned}
&\frac{(\log N)^2}{2\pi G_{D_N}(z,z)} + \frac{(2\pi)^3 \widetilde{G}_{D_N}(z,z')^2 G_{D_N\setminus\{z\}}(z',z')}{(\log N)^2} \\
&= \frac{(\log N)^2}{2\pi G_{D_N}(z,z)} + \frac{(\log N)^2 G_{D_N}(z,z')^2}{2\pi G_{D_N}(z,z)^2 G_{D_N\setminus\{z\}}(z',z')} \\
&= \frac{(\log N)^2 (G_{D_N}(z,z) G_{D_N\setminus\{z\}}(z',z') + G_{D_N}(z,z')^2)}{2\pi G_{D_N}(z,z)^2 G_{D_N\setminus\{z\}}(z',z')} \\
&= \frac{(\log N)^2}{2\pi G_{D_N}(z,z)} + \frac{(\log N)^2 G_{D_N}(z,z')^2}{2\pi G_{D_N}(z,z)^2 G_{D_N\setminus\{z\}}(z',z')} \\
&= \frac{(\log N)^2 G_{D_N}(z,z) G_{D_N}(z',z')}{2\pi G_{D_N}(z,z)^2 G_{D_N\setminus\{z\}}(z',z')} = \frac{(\log N)^2 G_{D_N}(z',z')}{2\pi G_{D_N}(z,z) G_{D_N\setminus\{z\}}(z',z')} \\
&= \frac{(\log N)^2}{2\pi G_{D_N\setminus\{z'\}}(z,z)} = \log N - \log(c_0) - \log(\mathrm{CR}_{N,z'}(z,D_N)).
\end{aligned}$$

This concludes. □

Given $z \neq z' \in D_N$, denote

$$q_{N,z}(z') := \frac{\log N}{2\pi G_{D_N\setminus\{z\}}(z',z')}, \tag{4.142}$$

$$\begin{aligned}
J_{N,K,z}(z') :&= \int \mathbf{1}_{\{\wp \text{ visits } z'\}} (1 - e^{-KT(\wp)}) \mu^{\mathrm{loop}}_{D_N\setminus\{z\}}(d\wp) \\
&= \int_0^{+\infty} \big(e^{-t/G_{D_N\setminus\{z\}}(z',z')} - e^{-t/G_{D_N\setminus\{z\},K}(z',z')}\big) \frac{\mathrm{d}t}{t},
\end{aligned}$$

$$J_N(z,z') := \int \mathbf{1}_{\{\wp \text{ visits } z \text{ and } z'\}} \mu^{\mathrm{loop}}_{D_N}(d\wp), \tag{4.143}$$
$$J_{N,K}(z,z') := \int \mathbf{1}_{\{\wp \text{ visits } z \text{ and } z'\}} (1 - e^{-KT(\wp)}) \mu^{\mathrm{loop}}_{D_N}(d\wp). \tag{4.144}$$

The following proposition is to be compared to Lemma 4.40 in the continuum setting.

Proposition 4.69. *Let $z, z' \in D_N$ such that the graph distance on $\mathbb{Z}_N^2$ between z and z' is at least 2, i.e. $|z' - z| > \frac{1}{N}$. Let $a, a' > 0$ and $K > 0$. Then for any bounded measurable admissible function F the following holds.*

1. *The case massless-massless :*

$$\mathbb{E}[F(\mathcal{L}^\theta_{D_N})\mathcal{M}^N_a(\{z\})\mathcal{M}^N_{a'}(\{z'\})] = \frac{q_{N,z'}(z)^\theta q_{N,z}(z')^\theta (\log N)^2}{N^4\Gamma(\theta)^2} e^{-\theta J_N(z,z')} \tag{4.145}$$
$$\int_{\substack{\rho,\tilde\rho>0\\ \rho+\tilde\rho\geq a}} \mathrm{d}\rho \tilde\rho^{\theta-1}\mathrm{d}\tilde\rho\, \mathrm{CR}_{N,z'}(z,D_N)^{\rho+\tilde\rho} \int_{\substack{\rho',\tilde\rho'>0\\ \rho'+\tilde\rho'\geq a'}} \mathrm{d}\rho' \tilde\rho'^{\theta-1}\mathrm{d}\tilde\rho'\, \mathrm{CR}_{N,z'}(z,D_N)^{\rho'+\tilde\rho'} \frac{c_0^{\rho+\tilde\rho+\rho'+\tilde\rho'}}{N^{\rho+\tilde\rho+\rho'+\tilde\rho'-a-a'}}$$
$$\times \sum_{l\geq 0}\frac{\theta^l}{l!}\int_{\substack{\mathsf{a}\in E(\rho,l)\\ \mathsf{a}'\in E(\rho',l)}} \frac{\mathrm{d}\mathsf{a}}{a_1\dots a_l}\frac{\mathrm{d}\mathsf{a}'}{a_1'\dots a_l'}\prod_{i=1}^l \mathsf{B}\left((2\pi)^2 a_i a_i' \widetilde{G}_{D_N}(z,z')^2\right)$$
$$\times \mathbb{E}\left[F\Big(\mathcal{L}^\theta_{D_N\setminus\{z,z'\}} \cup \{\Xi^{z,z'}_{N,a_i,a_i'}\wedge \Xi^z_{N,z',a_i}\wedge \Xi^{z'}_{N,z,a_i'}\}_{i=1}^l \cup \{\Xi^z_{N,z',\tilde a_i}, i\geq 1\}\cup\{\Xi^{z'}_{N,z,\tilde a_i'}, i\geq 1\}\Big)\right],$$

where on the right-hand side the four collections of loops $\mathcal{L}^\theta_{D_N\setminus\{z,z'\}}$, $\{\Xi^{z,z'}_{N,a_i,a_i'}\wedge \Xi^z_{N,z',a_i}\wedge \Xi^{z'}_{N,z,a_i'}\}_{i=1}^l$, $\{\Xi^z_{N,z',\tilde a_i}, i\geq 1\}$ *and* $\{\Xi^{z'}_{N,z,\tilde a_i'}, i\geq 1\}$ *are independent,* $(\tilde a_i)_{i\geq 1}$ *and* $(\tilde a_i')_{i\geq 1}$ *are two independent Poisson-Dirichlet partitions* $PD(0,\theta)$ *of respectively* $[0,\tilde\rho]$ *and* $[0,\tilde\rho']$*, the* $\Xi^z_{N,z',\tilde a_i}$*, respectively* $\Xi^{z'}_{N,z,\tilde a_i'}$*, are independent conditionally on* $(\tilde a_i)_{i\geq1}$*, respectively* $(\tilde a_i')_{i\geq 1}$*, and the* $\Xi^{z,z'}_{N,a_i,a_i'}$, Ξ^z_{N,z',a_i} *and* $\Xi^{z'}_{N,z,a_i'}$ *are all independent.*

2. *The case massless-massive :*

$$\mathbb{E}[F(\mathcal{L}^\theta_{D_N})\mathcal{M}^N_a(\{z\})\mathcal{M}^{N,K}_{a'}(\{z'\})] = \frac{q_{N,z'}(z)^\theta (\log N)^2}{N^4\Gamma(\theta)} e^{-\theta(J_{N,K,z}(z')+J_{N,K}(z,z'))} \tag{4.146}$$
$$\int_{\substack{\rho,\tilde\rho>0\\ \rho+\tilde\rho\geq a}} \mathrm{d}\rho \tilde\rho^{\theta-1}\mathrm{d}\tilde\rho\, \mathrm{CR}_{N,z'}(z,D_N)^{\rho+\tilde\rho} \int_{a'}^{+\infty} \mathrm{d}\rho'\, \mathrm{CR}_{N,z'}(z,D_N)^{\rho'} \frac{c_0^{\rho+\tilde\rho+\rho'}}{N^{\rho+\tilde\rho+\rho'-a-a'}}$$
$$\times \sum_{\substack{m\geq 1\\ 0\leq l\leq m}}\frac{\theta^m}{(m-l)!l!}\int_{\substack{\mathsf{a}\in E(\rho,l)\\ \mathsf{a}'\in E(\rho',m)}} \frac{\mathrm{d}\mathsf{a}}{a_1\dots a_l}\frac{\mathrm{d}\mathsf{a}'}{a_1'\dots a_m'}\prod_{i=1}^l \mathsf{B}\left((2\pi)^2 a_i a_i' \widetilde{G}_{D_N}(z,z')^2\right)$$
$$\times \mathbb{E}\Bigg[\prod_{i=1}^l\left(1-e^{-KT(\Xi^{z,z'}_{N,a_i,a_i'}\wedge\Xi^z_{N,z',a_i}\wedge\Xi^{z'}_{N,z,a_i'})}\right)\prod_{i=l+1}^m\left(1-e^{-KT(\Xi^{z'}_{N,z,a_i'})}\right)$$
$$F\Big(\mathcal{L}^\theta_{D_N\setminus\{z,z'\}} \cup \widetilde{\mathcal{L}}^\theta_{D_N,K,z'}\cup \{\Xi^{z,z'}_{N,a_i,a_i'}\wedge \Xi^z_{N,z',a_i}\wedge \Xi^{z'}_{N,z,a_i'}\}_{i=1}^l \cup \{\Xi^z_{N,z',\tilde a_i}, i\geq 1\}\cup\{\Xi^{z'}_{N,z,a_i'}\}_{i=l+1}^m\Big)\Bigg],$$

where on the right-hand side the five collections of loops $\mathcal{L}^\theta_{D_N\setminus\{z,z'\}}$, $\widetilde{\mathcal{L}}^\theta_{D_N,K,z'}$, $\{\Xi^{z,z'}_{N,a_i,a_i'}\wedge \Xi^z_{N,z',a_i}\wedge \Xi^{z'}_{N,z,a_i'}\}_{i=1}^l$, $\{\Xi^z_{N,z',\tilde a_i}, i\geq 1\}$ *and* $\{\Xi^{z'}_{N,z,a_i'}\}_{i=l+1}^m$ *are independent,* $(\tilde a_i)_{i\geq 1}$ *is a Poisson-Dirichlet partition* $PD(0,\theta)$ *of* $[0,\tilde\rho]$*, the* $\Xi^z_{N,z',\tilde a_i}$ *are independent conditionally on* $(\tilde a_i)_{i\geq1}$*, the* $\Xi^{z,z'}_{N,a_i,a_i'}$, Ξ^z_{N,z',a_i} *and* $\Xi^{z'}_{N,z,a_i'}$ *are all independent, and* $\widetilde{\mathcal{L}}^\theta_{D_N,K,z'}$ *is distributed as the loops in* $\mathcal{L}^\theta_{D_N}\setminus\mathcal{L}^\theta_{D_N}(K)$ *visiting* z'.

3. The case massive-massive :

$$\mathbb{E}[F(\mathcal{L}_{D_N}^{\theta})\mathcal{M}_a^{N,K}(\{z\})\mathcal{M}_{a'}^{N,K}(\{z'\})] = \frac{(\log N)^2}{N^4} e^{-\theta(J_{N,K,z'}(z)+J_{N,K,z}(z')+J_{N,K}(z,z'))} \tag{4.147}$$
$$\times \int_a^{+\infty} \mathrm{d}\rho \, \mathrm{CR}_{N,z'}(z, D_N)^{\rho} \int_{a'}^{+\infty} \mathrm{d}\rho' \, \mathrm{CR}_{N,z}(z', D_N)^{\rho'} \frac{c_0^{\rho+\rho'}}{N^{\rho+\rho'-a-a'}}$$
$$\times \sum_{\substack{n,m\geq 1\\ 0\leq l\leq n\wedge m}} \frac{\theta^{n+m-l}}{(n-l)!(m-l)!l!} \int_{\substack{\mathsf{a}\in E(\rho,n)\\ \mathsf{a}'\in E(\rho',m)}} \frac{\mathrm{d}\mathsf{a}}{a_1\dots a_n} \frac{\mathrm{d}\mathsf{a}'}{a_1'\dots a_m'} \prod_{i=1}^{l} \mathsf{B}\left((2\pi)^2 a_i a_i' \widetilde{G}_{D_N}(z,z')^2\right)$$
$$\times \mathbb{E}\Bigg[\prod_{i=1}^{l}\left(1-e^{-KT(\Xi^{z,z'}_{N,a_i,a_i'}\wedge \Xi^{z}_{N,z',a_i}\wedge\Xi^{z'}_{N,z,a_i'})}\right)\prod_{i=l+1}^{n}\left(1-e^{-KT(\Xi^{z}_{N,z',a_i})}\right)\prod_{i=l+1}^{m}\left(1-e^{-KT(\Xi^{z'}_{N,z,a_i'})}\right)$$
$$F\Big(\mathcal{L}^{\theta}_{D_N\setminus\{z,z'\}} \cup \widetilde{\mathcal{L}}^{\theta}_{D_N,K,z,z'} \cup \{\Xi^{z,z'}_{N,a_i,a_i'}\wedge\Xi^{z}_{N,z',a_i}\wedge\Xi^{z'}_{N,z,a_i'}\}_{i=1}^{l} \cup \{\Xi^{z}_{N,z',a_i}\}_{i=l+1}^{n} \cup \{\Xi^{z'}_{N,z,a_i'}\}_{i=l+1}^{m}\Big)\Bigg],$$

where on the right-hand side the five collections of loops $\mathcal{L}^{\theta}_{D_N\setminus\{z,z'\}}$, $\widetilde{\mathcal{L}}^{\theta}_{D_N,K,z,z'}$, $\{\Xi^{z,z'}_{N,a_i,a_i'}\wedge\Xi^{z}_{N,z',a_i}\wedge\Xi^{z'}_{N,z,a_i'}\}_{i=1}^{l}$, $\{\Xi^{z}_{N,z',a_i}\}_{i=l+1}^{n}$ *and* $\{\Xi^{z'}_{N,z,a_i'}\}_{i=l+1}^{m}$, *are independent, the* $\Xi^{z,z'}_{N,a_i,a_i'}$, Ξ^{z}_{N,z',a_i} *and* $\Xi^{z'}_{N,z,a_i'}$ *are all independent, and* $\widetilde{\mathcal{L}}^{\theta}_{D_N,K,z,z'}$ *is distributed as the loops in* $\mathcal{L}^{\theta}_{D_N}\setminus\mathcal{L}^{\theta}_{D_N}(K)$ *visiting* z *or* z'*.*

Proof. Let us first consider the case 1. massless-massless. We divide the random walk loop soup $\mathcal{L}^{\theta}_{D_N}$ into four independent Poisson point processes:

- The loops visiting neither z nor z'. These correspond to $\mathcal{L}^{\theta}_{D_N\setminus\{z,z'\}}$.
- The loops visiting z but not z'. We apply to these Proposition 4.61 in the domain $D_N\setminus\{z'\}$. These loops correspond to $\{\Xi^{z}_{N,z',\tilde{a}_i}, i\geq 1\}$.
- The loops visiting z' but not z. We apply to these Proposition 4.61 in the domain $D_N\setminus\{z\}$. These loops correspond to $\{\Xi^{z'}_{N,z,\tilde{a}_i'}, i\geq 1\}$.
- The loops visiting both z and z'. These form an a.s. finite Poisson point process. The corresponding intensity measure is described, up to the factor θ, by Lemma 4.68. The corresponding total mass is $\theta J_N(z,z')$. These loops correspond to $\{\Xi^{z,z'}_{N,a_i,a_i'}\wedge\Xi^{z}_{N,z',a_i}\wedge\Xi^{z'}_{N,z,a_i'}\}_{i=1}^{l}$.

By combining the above, we obtain our expression.

Now let us consider the case 3. massive-massive. We divide the random walk loop soup $\mathcal{L}^{\theta}_{D_N}$ into five independent Poisson point processes:

- The loops visiting neither z nor z'. These correspond to $\mathcal{L}^{\theta}_{D_N\setminus\{z,z'\}}$.
- The loops visiting z or z' and surviving to the killing rate K. These correspond to $\widetilde{\mathcal{L}}^{\theta}_{D_N,K,z,z'}$.
- The loops visiting z but not z', and killed by K. These form an a.s. finite Poisson point process. The total mass of the corresponding intensity measure is $\theta J_{N,K,z'}(z)$. We apply to these Proposition 4.63 in the domain $D_N\setminus\{z'\}$. These loops correspond to $\{\Xi^{z}_{N,z',a_i}\}_{i=l+1}^{n}$.

- The loops visiting z' but not z, and killed by K. These form an a.s. finite Poisson point process. The total mass of the corresponding intensity measure is $\theta J_{N,K,z}(z')$. We apply to these Proposition 4.63 in the domain $D_N \setminus \{z\}$. These loops correspond to $\{\Xi^{z'}_{N,z,a'_i}\}_{i=l+1}^m$.
- The loops visiting both z and z', and killed by K. These form an a.s. finite Poisson point process. The total mass of the corresponding intensity measure is $\theta J_{N,K}(z,z')$. We apply to these Lemma 4.68. These loops correspond to $\{\Xi^{z,z'}_{N,a_i,a'_i} \wedge \Xi^{z}_{N,z',a_i} \wedge \Xi^{z'}_{N,z,a'_i}\}_{i=1}^l$.

By combining the above, we obtain our expression.

The case 2. massless-massive is similar to and intermediate between the cases 1. and 3. We will not detail it. □

We finish this section with an elementary lemma that we state for ease of reference. We omit its proof since it can be easily checked.

Lemma 4.70. *Let $D \subset \mathbb{R}^2$ be a bounded simply connected domain, z, z' be two distinct points of D. Consider a discrete approximation $(D_N)_N$ of D in the sense of (4.9) and let z_N and z'_N be vertices of D_N which converge to z and z' respectively. Then*

$$1 - q_{N,z_N}(z'_N), \quad J_N(z_N, z'_N), \quad J_{N,K}(z_N, z'_N) \quad \text{and} \quad J_{N,K,z_N}(z'_N) \tag{4.148}$$

all converge to 0. Moreover,

$$C_{N,K,z_N}(z'_N) \to C_K(z'). \tag{4.149}$$

4.11.3 Convergence of excursion measures

The goal of this section is to prepare the proof of Proposition 4.59 by establishing the convergence of the various measures on discrete paths that appear in the formulas obtained in Sections 4.11.1 and 4.11.2 towards their continuum analogues.

Consider $D \subset \mathbb{C}$ an open bounded simply connected domain containing the origin and $(D_N)_N$ a discrete approximation of D, with $D_N \subset \mathbb{Z}_N^2$. See (4.9). First we deal with the convergence of probability measures $\tilde{\mu}^{z_N,w_N}_{D_N} / \widetilde{G}_{D_N}(z_N, w_N)$ with $w_N \neq z_N$.

Lemma 4.71. *Let $z, w \in D$ with $z \neq w$. Consider sequences $(z_N)_{N\geq 1}$ and $(w_N)_{N\geq 1}$, with $z_N, w_N \in D_N$ and*

$$\lim_{N\to+\infty} z_N = z, \qquad \lim_{N\to+\infty} w_N = w.$$

Then the probability measures $\mu^{z_N,w_N}_{D_N} / G_{D_N}(z_N, w_N)$ (4.31) converge weakly as $N \to +\infty$, for the metric d_{paths} (4.28), towards $\mu^{z,w}_D / G_D(z,w)$ (4.15).

Proof. Since D is bounded, by performing a translation we can reduce to the case when $D \subset \mathbb{H}$, where $\mathbb{H}$ is the upper half-plane

$$\mathbb{H} = \{z \in \mathbb{C} : \text{Im}(z) > 0\},$$

and that for every $N \geq 1$, $D_N \subset \mathbb{Z}_N^2 \cap \mathbb{H}$. First, we have that $\mu^{z_N,w_N}_{\mathbb{Z}_N^2 \cap \mathbb{H}} / G_{\mathbb{Z}_N^2 \cap \mathbb{H}}(z_N, w_N)$ converges weakly towards $\mu^{z,w}_{\mathbb{H}} / G_{\mathbb{H}}(z,w)$. This follows from the following two points:

- For every $t > 0$, the bridges probability measures $\mathbb{P}^{z_N,w_N}_{\mathbb{Z}^2_N \cap \mathbb{H},t}$ converges towards the Brownian bridge measure $\mathbb{P}^{z,w}_{\mathbb{H},t}$.
- The transition densities $p_{\mathbb{Z}^2_N \cap \mathbb{H}}(t, z_N, w_N)$ converge to $p_{\mathbb{H}}(t, z, w)$ uniformly in $t \in [0, +\infty)$ (local central limit theorem); see [LL10, Theorem 2.5.6].
- The discrete Green function $G_{\mathbb{Z}^2_N \cap \mathbb{H}}(z_N, w_N)$ converges to $G_{\mathbb{H}}(z, w)$. This follows from [LL10, Theorem 4.4.4] and the reflection principle.

Further, the measure $\mu^{z,w}_D / G_D(z, w)$ is obtained by conditioning a path under $\mu^{z,w}_{\mathbb{H}} / G_{\mathbb{H}}(z, w)$ to stay in D. Similarly, $\mu^{z_N,w_N}_{D_N} / G_{D_N}(z_N, w_N)$ is obtained by conditioning a path under $\mu^{z_N,w_N}_{\mathbb{Z}^2_N \cap \mathbb{H}} / G_{\mathbb{Z}^2_N \cap \mathbb{H}}(z_N, w_N)$ to stay in D_N. Moreover, on the event that the path under $\mu^{z,w}_{\mathbb{H}} / G_{\mathbb{H}}(z, w)$ exits D, a.s. there is $\varepsilon > 0$ such that any continuous deformation of the path of size less than ε also has to exit D. This is because a Brownian path exiting D will a.s. create a loop around the point where it first exits D. We refer to [Lup16, Lemma 2.6] for details. Thus, one gets the convergence of $\mu^{z_N,w_N}_{D_N} / G_{D_N}(z_N, w_N)$ towards $\mu^{z,w}_D / G_D(z, w)$. □

Proposition 4.72. *Let $z, w \in D$ with $z \neq w$. Consider sequences $(z_N)_{N \geq 1}$ and $(w_N)_{N \geq 1}$, with $z_N, w_N \in D_N$ and*

$$\lim_{N \to +\infty} z_N = z, \qquad \lim_{N \to +\infty} w_N = w.$$

Then the probability measures $\tilde{\mu}^{z_N,w_N}_{D_N} / \widetilde{G}_{D_N}(z_N, w_N)$ (4.134) converges weakly as $N \to +\infty$, for the metric d_{paths} (4.28), towards $\mu^{z,w}_D / G_D(z, w)$ (4.15).

Proof. According to the Markovian decomposition of Lemma 4.16, a path $\wp$ under $\mu^{z_N,w_N}_{D_N} / G_{D_N}(z_N, w_N)$ has the same law as a concatenation of three independent paths $\wp_1 \wedge \tilde{\wp} \wedge \wp_2$, with $\tilde{\wp}$ following the distribution $\tilde{\mu}^{z_N,w_N}_{D_N} / \widetilde{G}_{D_N}(z_N, w_N)$, $\wp_1$ following the distribution $\mu^{z_N,z_N}_{D_N} / G_{D_N}(z_N, z_N)$, and $\wp_2$ following the distribution $\mu^{w_N,w_N}_{D_N \setminus \{z_N\}} / G_{D_N \setminus \{z_N\}}(w_N, w_N)$. Moreover, it is easy to see that $\mathrm{diam}(\wp_1)$, $T(\wp_1)$, $\mathrm{diam}(\wp_2)$ and $T(\wp_2)$ converge in probability to 0 as $N \to +\infty$. Thus, the convergence of $\tilde{\mu}^{z_N,w_N}_{D_N} / \widetilde{G}_{D_N}(z_N, w_N)$ is equivalent to the convergence of $\mu^{z_N,w_N}_{D_N} / G_{D_N}(z_N, w_N)$, and the latter converge to $\mu^{z,w}_D / G_D(z, w)$ according to Lemma 4.71. □

Next we deal with the convergence of measures $\tilde{\mu}^{z_N,z_N}_{D_N}$. Given $z \in D$ and $r > 0$, let $E_{z,r}$ denote the event that a path goes at distance at least r from z. If $r < d(z, \partial D)$, then $\mu^{z,z}_D(E_{z,r}) < +\infty$.

Lemma 4.73. *Let $z \in D$ and $r \in (0, d(z, \partial D))$. Consider a sequence $(z_N)_{N \geq 1}$, $z_N \in D_N$, converging to z. Then*

$$\lim_{N \to +\infty} \mu^{z_N,z_N}_{D_N}(E_{z_N,r}) = \mu^{z,z}_D(E_{z,r}).$$

Moreover, the probability measures $\mathbf{1}_{\{E_{z_N,r}\}} \mu^{z_N,z_N}_{D_N} / \mu^{z_N,z_N}_{D_N}(E_{z_N,r})$ converge weakly as $N \to +\infty$, for the metric d_{paths}, towards $\mathbf{1}_{\{E_{z,r}\}} \mu^{z,z}_D / \mu^{z,z}_D(E_{z,r})$.

Proof. Since D is bounded, by performing a translation we can reduce to the case when $D \subset \mathbb{H}$ and that for every $N \geq 1$, $D_N \subset \mathbb{Z}^2_N \cap \mathbb{H}$. We only need to show that

$$\lim_{N \to +\infty} \mu^{z_N,z_N}_{\mathbb{Z}^2_N \cap \mathbb{H}}(E_{z_N,r}) = \mu^{z,z}_{\mathbb{H}}(E_{z,r}) \tag{4.150}$$

and that the probability measures $\mathbf{1}_{\{E_{z_N,r}\}}\mu^{z_N,z_N}_{\mathbb{Z}^2_N\cap\mathbb{H}}/\mu^{z_N,z_N}_{\mathbb{Z}^2_N\cap\mathbb{H}}(E_{z_N,r})$ converge weakly towards the measure $\mathbf{1}_{\{E_{z,r}\}}\mu^{z,z}_{\mathbb{H}}/\mu^{z,z}_{\mathbb{H}}(E_{z,r})$. Indeed, the measure $\mu^{z_N,z_N}_{D_N}$ is a restriction of $\mu^{z_N,z_N}_{\mathbb{Z}^2_N\cap\mathbb{H}}$ to the paths that stay in D_N and $\mu^{z,z}_D$ is a restriction of $\mu^{z,z}_{\mathbb{H}}$ to the paths that stay in D. Using that, one can conclude as in the proof of Lemma 4.71.

Now, consider $(B_t)_{t\geq 0}$ a Brownian motion starting from z and let τ_r be the stopping time

$$\tau_r := \min\{t \geq 0 : |B_t - z| = r\}.$$

Also consider $(X^{(N)}_t)_{t\geq 0}$ the Markov jump process on $\mathbb{Z}^2_N$ (see Section 4.2.2) starting from z_N and let $\tau_{N,r}$ be the stopping time

$$\tau_{N,r} := \min\{t \geq 0 : |X^{(N)}_t - z| \geq r\}.$$

The following holds:

$$\mu^{z_N,z_N}_{\mathbb{Z}^2_N\cap\mathbb{H}}(E_{z_N,r}) = \mathbb{E}^{z_N}\left[G_{\mathbb{Z}^2_N\cap\mathbb{H}}(X^{(N)}_{\tau_{N,r}}, z_N)\right], \qquad \mu^{z,z}_{\mathbb{H}}(E_{z,r}) = \mathbb{E}^{z}\left[G_{\mathbb{H}}(B_{\tau_r}, z)\right].$$

So (4.150) follows from the convergence in law of $X^{(N)}_{\tau_{N,r}}$ to B_{τ_r} and the convergence of $G_{\mathbb{Z}^2_N\cap\mathbb{H}}(w, z_N)$ to $G_{\mathbb{H}}(w, z)$ uniformly for w away from z. Further, a path $\wp$ under the probability $\mathbf{1}_{\{E_{z,r}\}}\mu^{z,z}_{\mathbb{H}}/\mu^{z,z}_{\mathbb{H}}(E_{z,r})$ can de decomposed as a concatenation $\wp_1 \wedge \wp_2$ with the following distribution. The distribution of $\wp_1$ is that of $(B_t)_{0\leq t\leq\tau_r}$ tilted by the density

$$\frac{G_{\mathbb{H}}(B_{\tau_r}, z)}{\mu^{z,z}_{\mathbb{H}}(E_{z,r})}.$$

Conditionally on $\wp_1$, $\wp_2$ follows the distribution $\mu^{w,z}_{\mathbb{H}}/G_{\mathbb{H}}(w, z)$, where w is the endpoint of $\wp_1$. A similar decomposition holds for a path under $\mathbf{1}_{\{E_{z_N,r}\}}\mu^{z_N,z_N}_{\mathbb{Z}^2_N\cap\mathbb{H}}/\mu^{z_N,z_N}_{\mathbb{Z}^2_N\cap\mathbb{H}}(E_{z_N,r})$, with $X^{(N)}_t$ instead of B_t, $G_{\mathbb{Z}^2_N\cap\mathbb{H}}$ instead of $G_{\mathbb{H}}$ and $\mu^{w,z_N}_{\mathbb{Z}^2_N\cap\mathbb{H}}$ instead of $\mu^{w,z}_{\mathbb{H}}$. So the desired convergence of measures follows from the convergence in law of $(X^{(N)}_t)_{0\leq t\leq\tau_{N,r}}$ to $(B_t)_{0\leq t\leq\tau_r}$, the convergence of the Green functions $G_{\mathbb{Z}^2_N\cap\mathbb{H}}$ to $G_{\mathbb{H}}$, and from Lemma 4.71. □

Proposition 4.74. *Let $z \in D$ and $r \in (0, d(z, \partial D))$. Consider a sequence $(z_N)_{N\geq 1}$, $z_N \in D_N$, converging to z. Then*

$$\lim_{N\to+\infty} \tilde{\mu}^{z_N,z_N}_{D_N}(E_{z_N,r}) = \mu^{z,z}_D(E_{z,r}).$$

Moreover, the probability measures $\mathbf{1}_{\{E_{z_N,r}\}}\tilde{\mu}^{z_N,z_N}_{D_N}/\tilde{\mu}^{z_N,z_N}_{D_N}(E_{z_N,r})$ converge weakly as $N \to +\infty$, for the metric d_{paths}, towards $\mathbf{1}_{\{E_{z,r}\}}\mu^{z,z}_D/\mu^{z,z}_D(E_{z,r})$.

Proof. Denote

$$B_N := \{w \in D_N : |w - z_n| < r\}.$$

According to the Markovian decomposition of Lemma 4.16, a path $\wp$ under $\mathbf{1}_{\{E_{z_N,r}\}}\mu^{z_N,z_N}_{D_N}/\mu^{z_N,z_N}_{D_N}(E_{z_N,r})$ has the same law as a concatenation of three independent paths $\wp_1\wedge\tilde{\wp}\wedge\wp_2$, with $\tilde{\wp}$ following the distribution $\mathbf{1}_{\{E_{z_N,r}\}}\tilde{\mu}^{z_N,z_N}_{D_N}/\tilde{\mu}^{z_N,z_N}_{D_N}(E_{z_N,r})$, $\wp_1$ following the distribution $\mu^{z_N,z_N}_{D_N}/G_{D_N}(z_N, z_N)$, and $\wp_2$ following the distribution $\mu^{z_N,z_N}_{B_N}/G_{B_N}(z_N, z_N)$. Further, $\mathrm{diam}(\wp_1)$, $T(\wp_1)$, $\mathrm{diam}(\wp_2)$ and $T(\wp_2)$ converge in probability to 0 as $N \to +\infty$. Thus, the convergence of $\mathbf{1}_{\{E_{z_N,r}\}}\tilde{\mu}^{z_N,z_N}_{D_N}/\tilde{\mu}^{z_N,z_N}_{D_N}(E_{z_N,r})$ is equivalent to the convergence of $\mathbf{1}_{\{E_{z_N,r}\}}\mu_{D_N}^{z_N,z_N}/\mu_{D_N}^{z_N,z_N}(E_{z_N,r})$, and the latter converge to $\mathbf{1}_{\{E_{z,r}\}}\mu_D^{z,z}/\mu_D^{z,z}(E_{z,r})$ according to Lemma 4.73. Moreover,

$$\mu_{D_N}^{z_N,z_N}(E_{z_N,r}) = \frac{G_{D_N}(z_N,z_N)G_{B_N}(z_N,z_N)}{\left(\frac{1}{2\pi}\log N\right)^2}\tilde{\mu}_{D_N}^{z_N,z_N}(E_{z_N,r}).$$

Thus, $\tilde{\mu}_{D_N}^{z_N,z_N}(E_{z_N,r})$ and $\mu_{D_N}^{z_N,z_N}(E_{z_N,r})$ have the same limit. □

Corollary 4.75. *Let $z \neq z' \in D$ and $r \in (0, d(z,\partial D))$. Consider sequences $(z_N)_{N\geq 1}$ and $(z'_N)_{N\geq 1}$, with $z_N, z'_N \in D_N$ and*

$$\lim_{N\to+\infty} z_N = z, \qquad \lim_{N\to+\infty} z'_N = z'.$$

Then

$$\lim_{N\to+\infty} \tilde{\mu}_{D_N\setminus\{z'_N\}}^{z_N,z_N}(E_{z_N,r}) = \mu_D^{z,z}(E_{z,r}).$$

Moreover, the probability measures $\mathbf{1}_{\{E_{z_N,r}\}}\tilde{\mu}_{D_N\setminus\{z'_N\}}^{z_N,z_N}/\tilde{\mu}_{D_N}^{z_N,z_N}(E_{z_N,r})$ converge weakly as $N\to+\infty$, for the metric d_{paths}, towards $\mathbf{1}_{\{E_{z,r}\}}\mu_D^{z,z}/\mu_D^{z,z}(E_{z,r})$.

Proof. The measure $\tilde{\mu}_{D_N\setminus\{z'_N\}}^{z_N,z_N}$ is obtained by restricting $\tilde{\mu}_{D_N}^{z_N,z_N}$ to the paths that do not visit z'_N. Given that almost every path under $\mu_D^{z,z}$ stays at positive distance from z', the result follows from Proposition 4.74. □

4.12 Controlling the effect of mass in discrete loop soup

The purpose of this section is to prove Proposition 4.59. As in the continuum, the proof relies on a careful analysis of truncated first and second moments. We start off by introducing the good events that we will work with.

Let $z \in N^{-1}\mathbb{Z}^2$ and $r > 0$. We will denote by $\partial\mathbb{D}_N(z,r)$ the discrete circle defined as the outer boundary of the discrete disc

$$\mathbb{D}_N(z,r) := z + \{y \in N^{-1}\mathbb{Z}^2 : |y| < r\}.$$

If $\wp$ is a discrete trajectory on $N^{-1}\mathbb{Z}^2$ and if $\mathcal{C}$ is a collection of such trajectories, we will denote by $N_{z,r}^{\wp}$ the number of upcrossings from $\partial\mathbb{D}_N(z,r)$ to $\partial\mathbb{D}_N(z,er)$ in $\wp$ and $N_{z,r}^{\mathcal{C}} = \sum_{\wp\in\mathcal{C}} N_{z,r}^{\wp}$. We will not keep track of the dependence in the mesh size N^{-1} in the notations of the number of crossings since it will be clear from the context.

Let $\eta \in (0, 1-a/2)$ be a small parameter, $b > a$ be close to a and define the discrete analogues of the good event $\mathcal{G}_K(z)$:

$$\mathcal{G}_N(z) := \left\{\forall r \in \{e^{-n}, n\geq 1\} \cap (N^{-1+\eta}, r_0) : N_{z,r}^{\mathcal{L}_{D_N}^\theta} \leq b|\log r|^2\right\} \tag{4.151}$$

and

$$\mathcal{G}_{N,K}(z) := \left\{\forall r \in \{e^{-n}, n\geq 1\} \cap (N^{-1+\eta}, r_0) : N_{z,r}^{\mathcal{L}_{D_N}^\theta(K)} \leq b|\log r|^2\right\}. \tag{4.152}$$

We emphasise here that we only restrict the number of crossings of annuli at scales $r > N^{-1+\eta}$. We will see that it is enough to turn the measure into a measure bounded in L^2 and it will simplify the analysis since we will always look at scales at least mesoscopic ($N^{-1+\beta}$, for some $\beta > 0$).

Once the good events are defined, we consider the modified versions of $\mathcal{M}_a^N$ and $\mathcal{M}_a^{N,K}$:

$$\tilde{\mathcal{M}}_a^N(dz) := \mathbf{1}_{\mathcal{G}_N(z)} \mathcal{M}_a^N(dz) \quad \text{and} \quad \tilde{\mathcal{M}}_a^{N,K}(dz) := \mathbf{1}_{\mathcal{G}_{N,K}(z)} \mathcal{M}_a^{N,K}(dz).$$

In the remaining of Section 4.12, we will fix a Borel set A compactly included in D and the constants underlying our estimates will implicitly be allowed to depend on A, η, a and b.

The proof of Proposition 4.59 relies on three lemmas that are the discrete analogues of Lemmas 4.43, 4.44 and 4.45. We first state these lemmas without proof and explain how the proof of Proposition 4.59 is obtained from them.

We will first need to show that the introduction of the good events almost does not change the first moment:

Lemma 4.76. *We have*

$$\lim_{r_0 \to 0} \limsup_{N \to \infty} \mathbb{E}\left[\left| \tilde{\mathcal{M}}_a^N(A) - \mathcal{M}_a^N(A) \right| \right] = 0 \tag{4.153}$$

and

$$\lim_{r_0 \to 0} \limsup_{K \to \infty} (\log K)^{-\theta} \limsup_{N \to \infty} \mathbb{E}\left[\left| \tilde{\mathcal{M}}_a^{N,K}(A) - \mathcal{M}_a^{N,K}(A) \right| \right] = 0. \tag{4.154}$$

Once the good events are introduced, the second moment becomes finite:

Lemma 4.77. *For $z \in D$ and $N \geq 1$, denote by z_N some element of D_N closest to z (with some arbitrary rule). If $b > a$ is close enough to a, then*

$$\int_{A \times A} \sup_{N \geq 1} N^4 \mathbb{E}\left[\tilde{\mathcal{M}}_a^N(\{z_N\}) \tilde{\mathcal{M}}_a^N(\{z'_N\}) \right] \mathrm{d}z \mathrm{d}z' < \infty \tag{4.155}$$

and

$$\int_{A \times A} \sup_{K \geq 1} (\log K)^{-2\theta} \sup_{N \geq 1} N^4 \mathbb{E}\left[\tilde{\mathcal{M}}_a^{N,K}(\{z_N\}) \tilde{\mathcal{M}}_a^{N,K}(\{z'_N\}) \right] \mathrm{d}z \mathrm{d}z' < \infty. \tag{4.156}$$

Finally,

Lemma 4.78. *If $b > a$ is close enough to a, then*

$$\limsup_{K \to \infty} \limsup_{N \to \infty} \mathbb{E}\left[\left(\tilde{\mathcal{M}}_a^N(A) - \frac{2^\theta}{(\log K)^\theta} \tilde{\mathcal{M}}_a^{N,K}(A) \right)^2 \right] = 0. \tag{4.157}$$

Proof of Proposition 4.59. Proposition 4.59 follows from Lemmas 4.76 and 4.78 in a very similar way as Proposition 4.26 follows from Lemmas 4.43 and 4.45; see below Lemma 4.45. Note that we can first restrict ourselves to a Borel set A compactly included in D since the contribution of points near the boundary to the measures is negligible. We omit the details. □

The remaining of Section 4.12 is organised as follows. We will start in Section 4.12.1 by analysing the lengthy formulas appearing in Proposition 4.69 in the same spirit as what we did in Lemma 4.41.

We will then study in Section 4.12.2 the number of crossings in the processes of excursions that appear in Propositions 4.61, 4.63 and 4.69. The proofs of Lemmas 4.76, 4.77 and 4.78 will then be given in Sections 4.12.3, 4.12.4 and 4.12.5 respectively.

4.12.1 Simplifying the second moment

Define for all $\lambda, \lambda' > 0$, $0 < v < \lambda^2 \wedge \lambda'^2$ and $u, u' \geq 1$,

$$\hat{\mathsf{H}}_a(\lambda, \lambda', v) := \int_{\substack{\rho,\tilde{\rho}>0 \\ \rho+\tilde{\rho}\geq a}} \mathrm{d}\rho \, \mathrm{d}\tilde{\rho} \; e^{-\lambda(\rho+\tilde{\rho})}\tilde{\rho}^{\theta-1} \int_{\substack{\rho',\tilde{\rho}'>0 \\ \rho'+\tilde{\rho}'\geq a}} \mathrm{d}\rho' \, \mathrm{d}\tilde{\rho}' e^{-\lambda'(\rho'+\tilde{\rho}')}(\tilde{\rho}')^{\theta-1} \times \sum_{l\geq 1} \frac{\theta^l}{l!} \int_{\substack{\mathsf{a}\in E(\rho,l) \\ \mathsf{a}'\in E(\rho',l)}} \mathrm{d}\mathsf{a}\mathrm{d}\mathsf{a}' \prod_{i=1}^{l} \frac{\mathsf{B}(a_i a_i' v)}{a_i a_i'} \tag{4.158}$$

and

$$\hat{\hat{\mathsf{H}}}_a(\lambda, \lambda', u, u', v) := \int_a^{+\infty} \mathrm{d}\rho \; e^{-\lambda\rho} \int_{a'}^{+\infty} \mathrm{d}\rho' e^{-\lambda'\rho'} \sum_{\substack{n,m\geq 1 \\ 0\leq l\leq n\wedge m}} \frac{\theta^{n+m-l}}{(n-l)!(m-l)!l!} \times \int_{\substack{\mathsf{a}\in E(\rho,n) \\ \mathsf{a}'\in E(\rho',m)}} \mathrm{d}\mathsf{a}\mathrm{d}\mathsf{a}' \prod_{i=1}^{l} \frac{\mathsf{B}\,(a_i a_i' v)}{a_i a_i'} \prod_{i=l+1}^{n} \frac{1-e^{-ua_i}}{a_i} \prod_{i=l+1}^{m} \frac{1-e^{-u'a_i'}}{a_i'}. \tag{4.159}$$

By Proposition 4.69, $\hat{\mathsf{H}}_a$ is related to the second moment of $\mathcal{M}_a^N$ as follows:

$$\mathbb{E}\left[\mathcal{M}_a^N(\{z\})\mathcal{M}_a^N(\{z'\})\right] = \frac{q_{N,z'}(z)^\theta q_{N,z}(z')^\theta (\log N)^2}{N^{4-2a}\Gamma(\theta)^2} e^{-\theta J_N(z,z')} \hat{\mathsf{H}}_a(\lambda, \lambda', v)$$

with

$$\lambda = \log N - \log \mathrm{CR}_{N,z'}(z, D_N) - \log c_0, \quad \lambda' = \log N - \log \mathrm{CR}_{N,z}(z', D_N) - \log c_0 \tag{4.160}$$

and $v = (2\pi \tilde{G}_{D_N}(z,z'))^2$. On the other hand, by neglecting the killing for loops that visit both z and z', we see that $\hat{\hat{\mathsf{H}}}_a$ provides a good upper bound on the second moment of $\mathcal{M}_a^{N,K}$ (see Proposition 4.69):

$$\mathbb{E}\left[\mathcal{M}_a^{N,K}(\{z\})\mathcal{M}_a^{N,K}(\{z'\})\right] \leq \frac{(\log N)^2}{N^{4-2a}} e^{-\theta(J_{N,K,z'}(z)+J_{N,K,z}(z')+J_{N,K}(z,z'))} \hat{\hat{\mathsf{H}}}_a(\lambda, \lambda', u, u', v)$$

where λ, λ' and v are as above and, recalling the definition (4.139) of $C_{N,K,z}(z')$,

$$u = C_{N,K,z'}(z) \quad \text{and} \quad u' = C_{N,K,z}(z').$$

In the following lemma, which is the discrete counterpart of Lemma 4.41, we give exact expressions for $\hat{\mathsf{H}}_a$ and $\hat{\hat{\mathsf{H}}}_a$ and deduce upper bounds. These upper bounds will be crucial for us in order to prove Lemma 4.77. Recall the definition (4.92) of $\mathsf{H}_{\rho,\rho'}$.

Lemma 4.79. *We have*

$$\hat{\mathsf{H}}_a(\lambda, \lambda', v) = \Gamma(\theta) v^{(1-\theta)/2} \int_{[a,\infty)^2} (tt')^{(\theta-1)/2} e^{-\lambda t} e^{-\lambda' t'} I_{\theta-1}\left(2\sqrt{vtt'}\right) \mathrm{d}t\mathrm{d}t' \tag{4.161}$$

and

$$\hat{\hat{\mathsf{H}}}_a(\lambda, \lambda', u, u', v) = \int_{[a,\infty)^2} e^{-\lambda\rho} e^{-\lambda'\rho'} \mathsf{H}_{\rho,\rho'}(u, u', v) \mathrm{d}\rho \, \mathrm{d}\rho' . \tag{4.162}$$

Moreover, if $\lambda \wedge \lambda' \geq \sqrt{v} + 1$ *and if* $u, u' \geq 1$*, then*

$$\hat{\mathsf{H}}_a(\lambda, \lambda', v) \leq C v^{1/4-\theta/2} \frac{1}{(\lambda - \sqrt{v})(\lambda' - \sqrt{v})} e^{(2\sqrt{v} - \lambda - \lambda')a} \tag{4.163}$$

and

$$\hat{\hat{\mathsf{H}}}_a(\lambda, \lambda', u, u', v) \leq C (uu')^\theta v^{1/4-\theta/2} \frac{1}{(\lambda - \sqrt{v})(\lambda' - \sqrt{v})} e^{(2\sqrt{v} - \lambda - \lambda')a}. \tag{4.164}$$

Proof. In (4.97), we noticed that

$$\sum_{l \geq 1} \frac{\theta^l}{l!} \int_{\substack{\mathsf{a} \in E(\rho, l) \\ \mathsf{a}' \in E(\rho', l)}} \mathrm{d}\mathsf{a}\mathrm{d}\mathsf{a}' \prod_{i=1}^{l} \frac{\mathsf{B}(a_i a_i' v)}{a_i a_i'} = \sum_{k \geq 1} \frac{v^k (\rho\rho')^{k-1} \theta^{(k)}}{(k-1)!^2 k!}.$$

We can therefore rewrite

$$\hat{\mathsf{H}}_a(\lambda, \lambda', v) = \sum_{k \geq 1} \frac{v^k \theta^{(k)}}{k!} \left(\int_{\substack{\rho, \tilde\rho > 0 \\ \rho + \tilde\rho \geq a}} \mathrm{d}\rho \, \mathrm{d}\tilde\rho e^{-\lambda(\rho + \tilde\rho)} \frac{\tilde\rho^{\theta-1} \rho^{k-1}}{(k-1)!} \right) \left(\lambda \leftrightarrow \lambda' \right)$$

where the second term in parenthesis is equal to the first one with λ replaced by λ'. We can further compute (see (4.222))

$$\int_{\substack{\rho, \tilde\rho > 0 \\ \rho + \tilde\rho \geq a}} \mathrm{d}\rho \, \mathrm{d}\tilde\rho e^{-\lambda(\rho + \tilde\rho)} \frac{\tilde\rho^{\theta-1} \rho^{k-1}}{(k-1)!} = \int_a^\infty \mathrm{d}t \; e^{-\lambda t} \int_0^t \mathrm{d}\rho \frac{(t-\rho)^{\theta-1} \rho^{k-1}}{(k-1)!} = \frac{1}{\theta^{(k)}} \int_a^\infty t^{\theta+k-1} e^{-\lambda t} \mathrm{d}t.$$

We have obtained that

$$\hat{\mathsf{H}}_a(\lambda, \lambda', v) = \int_a^\infty \mathrm{d}t \; t^{\theta-1} e^{-\lambda t} \int_a^\infty \mathrm{d}t' \; t'^{\theta-1} e^{-\lambda' t'} \sum_{k \geq 1} \frac{(vtt')^k}{k! \theta^{(k)}}.$$

We recognise here a modified Bessel function (4.223) concluding the proof of (4.161).

To obtain the upper bound (4.163), we first bound (thanks to (4.225))

$$I_{\theta-1}(u) \leq C e^u / \sqrt{u}, \quad u > 0,$$

which gives

$$\hat{\mathsf{H}}_a(\lambda, \lambda', v) \leq C v^{1/4-\theta/2} \int_{[a,\infty)^2} (tt')^{\theta/2-3/4} e^{-\lambda t - \lambda' t'} e^{2\sqrt{vtt'}} \; \mathrm{d}t \; \mathrm{d}t'.$$

We next bound $2\sqrt{tt'} \leq t + t'$ and

$$\begin{aligned}\hat{\mathsf{H}}_a(\lambda, \lambda', v) &\leq C v^{1/4-\theta/2} \left(\int_a^\infty t^{\theta/2-3/4} e^{-\lambda t} e^{\sqrt{v t}} \mathrm{d}t \right) \left(\lambda \leftrightarrow \lambda' \right) \\ &\leq C v^{1/4-\theta/2} \frac{1}{(\lambda - \sqrt{v})(\lambda' - \sqrt{v})} e^{(2\sqrt{v} - \lambda - \lambda')a}\end{aligned}$$

where we used the assumption that $\lambda \wedge \lambda' \geq \sqrt{v} + 1$ to obtain the last inequality. This concludes the proof of (4.163).

(4.162) directly follows from the definition (4.92) of $\mathsf{H}_{\rho,\rho'}$. The bound (4.164) then follows from Lemma 4.41. This concludes the proof. □

Remark 4.80. We now state a generalisation of (4.161) that will be needed in the proof of Lemma 4.78. This generalisation is proven in a very similar way and we omit its proof. Let $p : [0, \infty)^2 \to [0, 1]$ be a measurable function. Then for all $k \geq 1$,

$$\begin{aligned}&\int_{\substack{\rho,\tilde{\rho}>0 \\ \rho+\tilde{\rho}\geq a}} \mathrm{d}\rho \, \mathrm{d}\tilde{\rho} \; e^{-\lambda(\rho+\tilde{\rho})} \tilde{\rho}^{\theta-1} \int_{\substack{\rho',\tilde{\rho}'>0 \\ \rho'+\tilde{\rho}'\geq a}} \mathrm{d}\rho' \, \mathrm{d}\tilde{\rho}' e^{-\lambda'(\rho'+\tilde{\rho}')} (\tilde{\rho}')^{\theta-1} p(\rho + \tilde{\rho}, \rho' + \tilde{\rho}') \\ &\times \sum_{l=1}^k \frac{\theta^l}{l!} \int_{\substack{\mathsf{a}\in E(\rho,l) \\ \mathsf{a}'\in E(\rho',l)}} \mathrm{d}\mathsf{a}\mathrm{d}\mathsf{a}' \prod_{i=1}^l \sum_{\substack{k_1,\dots,k_l\geq 1 \\ k_1+\dots+k_l=k}} \frac{v^{k_i}(a_i a_i')^{k_i-1}}{k_i!(k_i-1)!} \\ &= \frac{v^k}{\theta^{(k)} k!} \int_{(a,\infty)^2} e^{-\lambda t - \lambda' t'} p(t, t') (tt')^{\theta+k-1} \mathrm{d}t \, \mathrm{d}t'.\end{aligned} \tag{4.165}$$

To recover (4.161) from (4.165), one simply needs to take the function $p = 1$ and sum over $k \geq 1$. Similarly, one can prove that

$$\begin{aligned}&\int_{\substack{\rho,\tilde{\rho}>0 \\ \rho+\tilde{\rho}\geq a}} \mathrm{d}\rho \, \mathrm{d}\tilde{\rho} \; e^{-\lambda(\rho+\tilde{\rho})} \tilde{\rho}^{\theta-1} \int_a^\infty \mathrm{d}t' e^{-\lambda' t'} \sum_{\substack{m\geq 1 \\ 1\leq l\leq m}} \frac{\theta^m}{(m-l)!l!} \\ &\times \int_{\substack{\mathsf{a}\in E(\rho,l) \\ \mathsf{a}'\in E(t',m)}} \mathrm{d}\mathsf{a}\mathrm{d}\mathsf{a}' \sum_{k\geq l} v^k p(\rho + \tilde{\rho}, \rho', k) \sum_{\substack{k_1,\dots,k_l\geq 1 \\ k_1+\dots+k_l=k}} \prod_{i=1}^l \frac{(a_i a_i')^{k_i-1}}{k_i!(k_i-1)!} \prod_{i=l+1}^m \frac{1 - e^{-a_i' u'}}{a_i'} \\ &= \sum_{k\geq 1} v^k \int_{(a,\infty)^2} \mathrm{d}t \, \mathrm{d}t' \; e^{-\lambda t} e^{-\lambda' t'} \frac{t^{\theta+k-1}}{k!(k-1)!} p(t, t', k) \Big(\int_0^{t'} \mathrm{d}\rho' \rho'^{k-1} \frac{\mathsf{F}(u'(t' - \rho'))}{t' - \rho'} + t'^{k-1} \Big).\end{aligned} \tag{4.166}$$

This latter equality will be useful in the mixed case massless–massive.

4.12.2 Number of crossings in the processes of excursions

In addition to A, η and b, we will also fix some large $M > 0$ throughout Section 4.12.2.

Let $z, z' \in A$ and $r > 0$ be such that $D(z, er) \subset A$ and $1 \leq Mr/|z - z'| < e$. In view of Propositions 4.61, 4.63 and 4.69, we will need to study the number of crossings $N_{z,r}^{\mathcal{C}}$ for $\mathcal{C} = \Xi_{N,a}^z, \Xi_{N,z',a}^z$ and $\Xi_{N,z,a}^{z'}$.

By property of Poisson point processes, in each cases, we can decompose

$$N_{z,r}^{\mathcal{C}} = \sum_{i=1}^{P} G_i$$

where $G_i, i \geq 1$, are i.i.d. random variables independent of P with the following distributions: P is a Poisson random variable with mean

$$\begin{aligned}
\mathbb{E}[P] &= 2\pi a \tilde{\mu}_{D_N}^{z,z}(\tau_{\partial \mathbb{D}_N(z,er)} < \infty), \quad \mathcal{C} = \Xi_{N,a}^z, \\
\mathbb{E}[P] &= 2\pi a \tilde{\mu}_{D_N \setminus \{z'\}}^{z,z}(\tau_{\partial \mathbb{D}_N(z,er)} < \infty), \quad \mathcal{C} = \Xi_{N,z',a}^z, \\
\mathbb{E}[P] &= 2\pi a \tilde{\mu}_{D_N \setminus \{z\}}^{z',z'}(\tau_{\partial \mathbb{D}_N(z,er)} < \infty), \quad \mathcal{C} = \Xi_{N,z,a}^{z'}
\end{aligned}$$

and the common distribution of the G_i's is the law of $N_{z,r}^{\wp}$ where $\wp$ is distributed according to

$$\frac{\mathbf{1}_{\{\wp \text{ hits } \partial \mathbb{D}_N(z,er)\}} \tilde{\mu}_{D'_N}^{w,w}(d\wp)}{\tilde{\mu}_{D'_N}^{w,w}(\tau_{\partial \mathbb{D}_N(z,er)} < \infty)}$$

and

$$\begin{aligned}
&D'_N = D_N \quad \text{and} \quad w = z, && \mathcal{C} = \Xi_{N,a}^z, \\
&D'_N = D_N \setminus \{z'\} \quad \text{and} \quad w = z, && \mathcal{C} = \Xi_{N,z',a}^z, \\
&D'_N = D_N \setminus \{z\} \quad \text{and} \quad w = z', && \mathcal{C} = \Xi_{N,z,a}^{z'}.
\end{aligned} \tag{4.167}$$

In what follows, we will refer to the variables P and G_i in "**Cases 1, 2 and 3**" when we mean that we consider the number of excursions $N_{z,r}^{\mathcal{C}}$ in the cases $\mathcal{C} = \Xi_{N,a}^z, \Xi_{N,z',a}^z$ and $\Xi_{N,z,a}^{z'}$, respectively. In the upcoming Lemmas 4.81 and 4.82, we will respectively estimate the mean of P and show that the G_i's can be well approximated by geometric random variable. These lemmas are to be compared with Lemma 4.46 in the continuum, but we will see that the discrete setting leads to some technical difficulties.

Lemma 4.81. *Let $z \in A$ and $r \in \{e^{-n}, n \geq 1\}, r > N^{-1+\eta}$ be such that $D(z,er) \subset A$. We have, in Case 1,*

$$\mathbb{E}[P] = a \left(1 + O\left(\frac{1}{|\log r|}\right)\right) \left(\frac{1}{|\log r|} - \frac{1}{\log(N)}\right)^{-1}. \tag{4.168}$$

Let $z' \in A$ be such that $1 \leq Mr/|z-z'| < e$ and denote $\beta = 1 - \frac{|\log r|}{\log N}$, so that $r = N^{-1+\beta}$. Then, in Cases 2 and 3,

$$\mathbb{E}[P] = a \left(1 + O\left(\frac{1}{|\log r|}\right)\right) \frac{1}{1-(1-\beta)^2} |\log r|. \tag{4.169}$$

Proof of Lemma 4.81. We will show upper bounds on $\mathbb{E}[P]$ as stated in the lemma. The matching lower bounds will follow from the same proof: one simply has to replace maxima by minima below.

Let us first start by showing the following intermediate result: in Case 1,

$$\mathbb{E}[P] \leq a \frac{(\log N)^2}{2\pi G_{D_N}(z,z) G_{D(z,er)}(z,z)} \max_{y \in \partial \mathbb{D}_N(z,er)} G_{D_N}(y,z), \tag{4.170}$$

in Case 2,

$$\mathbb{E}[P] \leq a \frac{(\log N)^2}{2\pi G_{D_N \setminus \{z'\}}(z,z) G_{D(z,er)}(z,z)} \max_{y \in \partial \mathbb{D}_N(z,er)} G_{D_N \setminus \{z'\}}(y,z) \tag{4.171}$$

and in Case 3,

$$\mathbb{E}[P] \leq a \frac{(\log N)^2}{2\pi G_{D_N \setminus \{z\}}(z',z')} 4\mathbb{P}^{z'} \left(\tau_{\partial \mathbb{D}_N(z,er)} < \tau_{z'}^+ \wedge \tau_{\partial D_N} \right) \max_{y \in \partial \mathbb{D}_N(z,er)} G_{D_N \setminus \{z\}}(y,z'). \tag{4.172}$$

We will show (4.170) and we will then explain what needs to be changed in order to have (4.171) and (4.172). First of all, the total mass of $2\pi \tilde{\mu}_{D_N}^{z,z}$ is given by

$$\begin{aligned} &\frac{1}{2\pi} (\log N)^2 \sum_{w_1, w_2 \sim z} G_{D_N \setminus \{z\}}(w_1, w_2) \\ &= \frac{1}{2\pi} (\log N)^2 \sum_{w_1, w_2 \sim z} \left(G_{D_N}(w_1, w_2) - \frac{G_{D_N}(w_1, z) G_{D_N}(z, w_2)}{G_{D_N}(z,z)} \right) \end{aligned}$$

where we used (4.137) to obtain the last equality. For w_1 fixed, $G_{D_N}(w_1, \cdot)$ is harmonic outside of w_1 which implies that

$$\sum_{w_2 \sim z} G_{D_N}(w_1, w_2) = 4 G_{D_N}(w_1, z).$$

Since

$$\sum_{w_1 \sim z} G_{D_N}(w_1, z) = 4(G_{D_N}(z,z) - 1/4),$$

we obtain that the total mass of $2\pi \tilde{\mu}_{D_N}^{z,z}$ is equal to

$$\begin{aligned} &\frac{1}{2\pi} (\log N)^2 \left(4 \sum_{w_1 \sim z} G_{D_N}(w_1, z) - \frac{1}{G_{D_N}(z,z)} \left(\sum_{w_1 \sim z} G_{D_N}(w_1, z) \right)^2 \right) \\ &= \frac{1}{2\pi} (\log N)^2 \left(16(G_{D_N}(z,z) - 1/4) - \frac{16}{G_{D_N}(z,z)} (G_{D_N}(z,z) - 1/4)^2 \right) \\ &= \frac{1}{2\pi} (\log N)^2 \left(4 - \frac{1}{G_{D_N}(z,z)} \right). \end{aligned}$$

Moreover, $\tilde{\mu}_{D_N}^{z,z}$ normalised by its total mass is the law of a random walk $(X_t)_{0 \leq t \leq \tau_z^+}$ starting at z, killed upon returning at z for the first time:

$$\tau_z^+ := \inf\{t > 0 : X_t = z, \exists s \in (0,t), X_s \neq z\}$$

and conditioned to stay in D_N. We wish to compute the probability for such a walk to visit $\partial \mathbb{D}_N(z, er)$.

By strong Markov property, we have

$$\mathbb{P}^z\left(\tau_{\partial\mathbb{D}_N(z,er)} < \tau_z^+ | \tau_z^+ < \tau_{\partial\mathbb{D}_N}\right) = \frac{\mathbb{P}^z\left(\tau_{\partial\mathbb{D}_N(z,er)} < \tau_z^+ < \tau_{\partial\mathbb{D}_N}\right)}{\mathbb{P}^z\left(\tau_z^+ < \tau_{\partial\mathbb{D}_N}\right)}$$
$$\leq \frac{\mathbb{P}^z\left(\tau_{\partial\mathbb{D}_N(z,er)} < \tau_z^+\right)}{\mathbb{P}^z\left(\tau_z^+ < \tau_{\partial\mathbb{D}_N}\right)} \max_{y\in\partial\mathbb{D}_N(z,er)} \mathbb{P}^y\left(\tau_z < \tau_{\partial\mathbb{D}_N}\right).$$

We can express these probabilities in terms of Green functions as follows:

$$\mathbb{P}^z\left(\tau_z^+ < \tau_{\partial\mathbb{D}_N}\right) = 1 - \frac{1}{4G_{D_N}(z,z)} = \frac{G_{D_N}(z,z) - 1/4}{G_{D_N}(z,z)},$$

$$\max_{y\in\partial\mathbb{D}_N(z,er)} \mathbb{P}^y\left(\tau_z < \tau_{\partial\mathbb{D}_N}\right) = \frac{\max_{y\in\partial\mathbb{D}_N(z,er)} G_{D_N}(y,z)}{G_{D_N}(z,z)}$$

and

$$\mathbb{P}^z\left(\tau_{\partial\mathbb{D}_N(z,er)} < \tau_z^+\right) = \frac{1}{4G_{D(z,er)}(z,z)}. \tag{4.173}$$

Overall, we have shown that

$$2\pi a\tilde{\mu}_{D_N}^{z,z}(\tau_{\partial\mathbb{D}_N(z,er)} < \infty) \leq \frac{1}{2\pi}(\log N)^2\left(4 - \frac{1}{G_{D_N}(z,z)}\right)\frac{\max_{y\in\partial\mathbb{D}_N(z,er)} G_{D_N}(y,z)}{4G_{D(z,er)}(z,z)(G_{D_N}(z,z) - 1/4)}$$
$$= \frac{(\log N)^2}{2\pi G_{D_N}(z,z)G_{D(z,er)}(z,z)} \max_{y\in\partial\mathbb{D}_N(z,er)} G_{D_N}(y,z)$$

which is the desired upper bound (4.170). The proof of (4.171) follows along the exact same lines. To prove (4.172), the only thing that needs to be changed is that now, instead of (4.173), we have

$$\mathbb{P}^{z'}\left(\tau_{\partial\mathbb{D}_N(z,er)} < \tau_{z'}^+ \wedge \tau_{\partial D_N}\right).$$

We leave it as it is and directly obtain (4.172).

We now move on to explaining how (4.168) and (4.169) follow from (4.170), (4.171) and (4.172). We start with (4.168). By Lemma 4.93, we have

$$2\pi G_{D_N}(z,z) = \log N + O(1), \quad 2\pi G_{D(z,er)}(z,z) = \log(Nr) + O(1)$$

and

$$2\pi \max_{y\in\partial\mathbb{D}_N(z,er)} G_{D_N}(y,z) = |\log r| + O(1)$$

and therefore, in Case 1,

$$\mathbb{E}[P] \leq a\left(1 + O\left(\frac{1}{\log r}\right)\right)\frac{\log N|\log r|}{\log(Nr)} = a\left(1 + O\left(\frac{1}{\log r}\right)\right)\left(\frac{1}{|\log r|} - \frac{1}{\log N}\right)^{-1}.$$

This concludes the proof of (4.168). We now prove (4.169) in Case 2. Recall that $\beta = 1 - |\log r|/\log N$.

Using the expression (4.137) of the Green function in $D_N \setminus \{z'\}$ and then Lemma 4.93, we see that

$$G_{D_N \setminus \{z'\}}(z,z) = G_{D_N}(z,z) - \frac{G_{D_N}(z,z')^2}{G_{D_N}(z',z')} = \frac{1}{2\pi}\left(1-(1-\beta)^2\right)\log N + O(1),$$

and if $y \in \partial \mathbb{D}_N(z,er)$,

$$\begin{aligned} G_{D_N \setminus \{z'\}}(y,z) &= G_{D_N}(y,z) - \frac{G_{D_N}(y,z')G_{D_N}(z,z')}{G_{D_N}(z',z')} \\ &= \frac{1}{2\pi}\left((1-\beta)-(1-\beta)^2\right)\log N + O(1) = \frac{\beta}{2\pi}|\log r| + O(1). \end{aligned}$$

Recall also that

$$G_{D(z,er)}(z,z) = \frac{1}{2\pi}\log(Nr) + O(1) = \frac{\beta}{2\pi}\log N + O(1).$$

Plugging these three estimates in (4.171) concludes the proof of (4.169) in Case 2.

To conclude the proof of Lemma 4.81, it remains to prove (4.169) in Case 3. We need to work a bit more and we need to estimate precisely $\mathbb{P}^{z'}\left(\tau_{\partial\mathbb{D}_N(z,er)} < \tau_{z'}^+ \wedge \tau_{\partial D_N}\right)$. In view of what we did, in order to conclude, it is enough to show that

$$\mathbb{P}^{z'}\left(\tau_{\partial\mathbb{D}_N(z,er)} < \tau_{z'}^+ \wedge \tau_{\partial D_N}\right) = \left(1 + O\left(\frac{1}{\log r}\right)\right)\frac{\pi}{2\beta \log N}. \tag{4.174}$$

The rest of the proof is dedicated to this estimate. We claim that the probability on the left hand side of (4.174) is at most equal to

$$\frac{1-\mathbb{P}^{z'}\left(\tau_{z'}^+ < \tau_{\partial D_N}\right)}{1-\mathbb{P}^{z'}\left(\tau_{\partial\mathbb{D}_N(z,er)} < \tau_{\partial D_N}\right)\max_{y\in\partial\mathbb{D}_N(z,er)}\mathbb{P}^y\left(\tau_{z'} < \tau_{\partial D_N}\right)}\mathbb{P}^{z'}\left(\tau_{\partial\mathbb{D}_N(z,er)} < \tau_{\partial D_N}\right). \tag{4.175}$$

Indeed, if we denote by

$$p = \mathbb{P}^{z'}\left(\tau_{\partial\mathbb{D}_N(z,er)} < \tau_{z'}^+ \wedge \tau_{\partial D_N}\right) \quad \text{and} \quad q = \mathbb{P}^{z'}\left(\tau_{z'}^+ < \tau_{\partial\mathbb{D}_N(z,er)} \wedge \tau_{\partial D_N}\right),$$

the strong Markov property shows that

$$\mathbb{P}^{z'}\left(\tau_{\partial\mathbb{D}_N(z,er)} < \tau_{\partial D_N}\right) = p + q\mathbb{P}^{z'}\left(\tau_{\partial\mathbb{D}_N(z,er)} < \tau_{\partial D_N}\right)$$

and also

$$\begin{aligned} \mathbb{P}^{z'}\left(\tau_{z'}^+ < \tau_{\partial D_N}\right) &= q + p\mathbb{P}^{z'}\left(\tau_{z'}^+ < \tau_{\partial D_N} | \tau_{\partial\mathbb{D}_N(z,er)} < \tau_{z'}^+ \wedge \tau_{\partial D_N}\right) \\ &\leq q + p \max_{y\in\partial\mathbb{D}_N(z,er)} \mathbb{P}^y\left(\tau_{z'} < \tau_{\partial D_N}\right). \end{aligned}$$

Combining the two above estimates yields the claim (4.175). Now, by [LL10, Proposition 6.4.1],

$$\mathbb{P}^{z'}\left(\tau_{\partial\mathbb{D}_N(z,er)} < \tau_{\partial \mathbb{D}_N}\right) = 1 - \frac{\log|z'-z|/(er) + O(1/\log r)}{|\log r| + O(1)} = 1 + O\left(\frac{1}{\log r}\right).$$

Moreover, for all $y \in \partial \mathbb{D}_N(z, er)$,

$$\mathbb{P}^y \left(\tau_{z'} < \tau_{\partial D_N} \right) = \frac{G_{D_N}(y, z')}{G_{D_N}(z', z')} = 1 - \beta + O\left(\frac{1}{\log N} \right)$$

and

$$1 - \mathbb{P}^{z'} \left(\tau_{z'}^+ < \tau_{\partial D_N} \right) = \frac{1}{4 G_{D_N}(z', z')} = \frac{\pi}{2 \log N} \left(1 + O\left(\frac{1}{\log r} \right) \right).$$

Plugging those three estimates into (4.175) shows (4.174) (or more precisely, the upper bound, but the lower bound is similar). This concludes the proof. □

We now turn to the study of the variables G_i.

Lemma 4.82. *Let $z \in A$ and $r \in \{e^{-n}, n \geq 1\}, r > N^{-1+\eta}$ be such that $D(z, er) \subset A$. In Case 1, we have for all $k \geq 1$,*

$$\mathbb{P}\left(G_i \geq k\right) = \left(1 + O\left(\frac{1}{\log r}\right)\right) \left(1 - \frac{1 + O(1/\log r)}{|\log r|} - \frac{1 + O(1/\log(Nr))}{\log(Nr)}\right)^{k-1} \tag{4.176}$$

Let $z' \in A$ be such that $1 \leq Mr/|z - z'| < e$ and denote $\beta = 1 - \frac{|\log r|}{\log N}$, so that $r = N^{-1+\beta}$. There exists $M_\eta > 0$ such that if $M > M_\eta$, then we have in Cases 2 and 3, for all $k \geq 1$,

$$\mathbb{P}\left(G_i \geq k\right) = \left(1 + O\left(\frac{1}{|\log r|}\right)\right) \left(1 - \frac{2 - \beta \pm \eta^2 + O(1/\log r)}{\beta |\log r|}\right)^{k-1}. \tag{4.177}$$

Proof. We start with the following claim: let D'_N be a finite subset of $\frac{1}{N}\mathbb{Z}^2$, $w \in D'_N \setminus \mathbb{D}_N(z, er)$ and let $\wp$ be distributed according to

$$\frac{\mathbf{1}_{\{\wp \text{ hits } \partial \mathbb{D}_N(z,er)\}} \tilde{\mu}^{w,w}_{D'_N}(d\wp)}{\tilde{\mu}^{w,w}_{D'_N}(\tau_{\partial \mathbb{D}_N(z,er)} < \infty)}.$$

Then for all $k \geq 1$, $\mathbb{P}\left(N^{\wp}_{z,r} \geq k\right)$ is at most

$$\max_{y_1, y_2 \in \partial \mathbb{D}_N(z,er)} \frac{G_{D'_N}(y_1, w)}{G_{D'_N}(y_2, w)} \Big(\max_{y \in \partial \mathbb{D}_N(z,er)} \mathbb{P}^y \left(\tau_{\partial D(z,r)} < \tau_{\partial D'_N} \wedge \tau_w \right) \times \max_{y \in \partial \mathbb{D}_N(z,r)} \mathbb{P}^y \left(\tau_{\partial \mathbb{D}_N(z,er)} < \tau_{\partial D'_N} \wedge \tau_w \right) \Big)^{k-1} \tag{4.178}$$

and at least the same quantity with maxima replaced by minima. We will apply this with D'_N and w given as in (4.167). The proof of this claim is a quick consequence of strong Markov property. Indeed, the trajectory $\wp$, after hitting for the first time $\partial \mathbb{D}_N(z, er)$, has the law of a random walk starting at some vertex of $\partial \mathbb{D}_N(z, er)$ (with some law that is irrelevant to us), stopped upon reaching w and conditioned to hit w before exiting D'_N; and we wish to estimate the probability for such a trajectory to cross the annulus at least $k - 1$ times. We omit the details.

We now explain how the proof of Lemma 4.82 follows from (4.178). Recall that we will apply the above claim with D'_N and w given as in (4.167). In all cases, one can show that the ratio of the Green functions equals $1 + O(1/\log r)$. In all cases, we also have that the second probability in (4.178) is equal to

$$\max_{y \in \partial \mathbb{D}_N(z,r)} \mathbb{P}^y \left(\tau_{\partial \mathbb{D}_N(z,er)} < \tau_z \right) = \max_{y \in \partial \mathbb{D}_N(z,r)} \left(1 - \frac{G_{D(z,er)}(y,z)}{G_{D(z,er)}(z,z)} \right).$$

By [Law13, Propositions 1.6.6 and 1.6.7], we deduce that

$$\max_{y \in \partial \mathbb{D}_N(z,r)} \mathbb{P}^y \left(\tau_{\partial \mathbb{D}_N(z,er)} < \tau_z \right) = 1 - \frac{1 + O((Nr)^{-1})}{\log(Nr) + O(1)} = 1 - \frac{1 + O(1/\log(Nr))}{\log(Nr)}.$$

Now, in Case 1, the first probability in (4.178) is equal to

$$\max_{y \in \partial \mathbb{D}_N(z,er)} \mathbb{P}^y \left(\tau_{\partial D(z,r)} < \tau_{\partial D_N} \right)$$

which is estimated in [LL10, Proposition 6.4.1] and is equal to

$$1 - \frac{1 + O(1/\log r)}{|\log r| + O(1)} = 1 - \frac{1 + O(1/\log r)}{|\log r|}.$$

This concludes the upper bound (4.176). The lower bound is similar. In Cases 2 and 3, the first probability in (4.178) is equal to

$$\max_{y \in \partial \mathbb{D}_N(z,er)} \mathbb{P}^y \left(\tau_{\partial D(z,r)} < \tau_{\partial D_N} \wedge \tau_{z'} \right).$$

To conclude the proof of (4.177), a small computation shows that it is sufficient to prove that for all $y \in \partial \mathbb{D}_N(z, er)$,

$$p := \mathbb{P}^y \left(\tau_{\partial D(z,r)} < \tau_{\partial D_N} \wedge \tau_{z'} \right) = 1 - \frac{1 + O(1/M) + O(1/\log r)}{\beta |\log r|}. \tag{4.179}$$

The rest of the proof is dedicated to this estimate. The strategy is very similar to the one we used to prove (4.174). Let us denote

$$q = \mathbb{P}^y \left(\tau_{z'} < \tau_{\partial D_N} \wedge \tau_{\partial D(z,r)} \right).$$

By the strong Markov property, we have

$$\mathbb{P}^y \left(\tau_{\partial \mathbb{D}_N(z,r)} < \tau_{\partial D_N} \right) = p + q \mathbb{P}^{z'} \left(\tau_{\partial \mathbb{D}_N(z,r)} < \tau_{\partial D_N} \right)$$

and also

$$\mathbb{P}^y \left(\tau_{z'} < \tau_{\partial D_N} \right) = q + p \mathbb{P}^y \left(\tau_{z'} < \tau_{\partial D_N} | \tau_{\partial \mathbb{D}_N(z,r)} < \tau_{z'} \wedge \tau_{\partial D_N} \right).$$

Combining these two equalities yields

$$\begin{aligned} p &= \frac{\mathbb{P}^y\left(\tau_{\partial \mathbb{D}_N(z,r)} < \tau_{\partial D_N}\right) - \mathbb{P}^y\left(\tau_{z'} < \tau_{\partial D_N}\right)\mathbb{P}^{z'}\left(\tau_{\partial \mathbb{D}_N(z,r)} < \tau_{\partial D_N}\right)}{1 - \mathbb{P}^{z'}\left(\tau_{\partial \mathbb{D}_N(z,r)} < \tau_{\partial D_N}\right)\mathbb{P}^y\left(\tau_{z'} < \tau_{\partial D_N} | \tau_{\partial \mathbb{D}_N(z,r)} < \tau_{z'} \wedge \tau_{\partial D_N}\right)} \\ &= 1 - \frac{1 - \mathbb{P}^y\left(\tau_{\partial \mathbb{D}_N(z,r)} < \tau_{\partial D_N}\right)}{***} \\ &+ \frac{\left(\mathbb{P}^y\left(\tau_{z'} < \tau_{\partial D_N} | \tau_{\partial \mathbb{D}_N(z,r)} < \tau_{z'} \wedge \tau_{\partial D_N}\right) - \mathbb{P}^y\left(\tau_{z'} < \tau_{\partial D_N}\right)\right)\mathbb{P}^{z'}\left(\tau_{\partial \mathbb{D}_N(z,r)} < \tau_{\partial D_N}\right)}{***} \end{aligned}$$

where the denominator did not change from the first identity to the second one. The probability p increases with the domain D_N. By including a macroscopic disc centred at z inside D_N (z is in the bulk of D), we will obtain a lower bound on p and by including D_N in a disc centred at z (D is bounded) we will obtain an upper bound. Therefore, assume that $D = D(z, R)$ for some $R > 0$. Now, by [LL10, Proposition 6.4.1],

$$\mathbb{P}^{z'}\left(\tau_{\partial \mathbb{D}_N(z,r)} < \tau_{\partial D_N}\right) = 1 - \frac{\log |z' - z|/r + O(1/\log r)}{\log(R/r)}$$

and

$$\mathbb{P}^{y}\left(\tau_{\partial \mathbb{D}_N(z,r)} < \tau_{\partial D_N}\right) = 1 - \frac{1 + O(1/\log r)}{\log(R/r)}.$$

Moreover,

$$\mathbb{P}^y\left(\tau_{z'} < \tau_{\partial D_N} | \tau_{\partial \mathbb{D}_N(z,r)} < \tau_{z'} \wedge \tau_{\partial D_N}\right) \geq \min_{x \in \partial \mathbb{D}_N(z,r)} \frac{G_{D_N}(x, z')}{G_{D_N}(z', z')} = 1 - \beta + O(1/\log N).$$

This shows that the denominator is equal to $\beta + O(1/\log r)$. Since for all $x \in \partial D(z, r)$, we can bound

$$1 - \frac{C}{M} \leq \frac{|x - z'|}{|y - z'|} \leq 1 + \frac{C}{M},$$

we have

$$\begin{aligned} &\left|\mathbb{P}^y\left(\tau_{z'} < \tau_{\partial D_N} | \tau_{\partial \mathbb{D}_N(z,r)} < \tau_{z'} \wedge \tau_{\partial D_N}\right) - \mathbb{P}^y\left(\tau_{z'} < \tau_{\partial D_N}\right)\right| \\ &\leq \max_{x \in \partial \mathbb{D}_N(z,r)} \frac{|G_{D_N}(y, z') - G_{D_N}(x, z')|}{G_{D_N}(z', z')} \leq O(1) \frac{1}{M \log N} \end{aligned}$$

We obtain that

$$p = 1 - \frac{1 + O(1/M) + O(1/\log r)}{\beta |\log r|}$$

which concludes the proof of (4.179). This finishes the proof of Lemma 4.82. □

From Lemmas 4.81 and 4.82, we obtain the discrete analogues of Corollaries 4.47 and 4.49 that we state below. We provide them without proofs since they follow from Lemmas 4.81 and 4.82 in the same way as the two aforementioned Corollaries in the continuum follow from Lemma 4.46.

Note that in Case 1, although $\mathbb{E}[P]/a$ and $\mathbb{E}[G_i]$ differ from $(1 + o(1))|\log r|$ which contrasts the continuous setting, the product $\mathbb{E}[P]\mathbb{E}[G_i]$ is still equal to $a(1+o(1))|\log r|^2$ like in the continuum.

Corollary 4.83. *Let $u \in (0,1/2)$. There exists $C(u)>0$, $r_0>0$ and $c>0$ (which may depend on a,b,η,A) such that for all $z \in A$ and $r = N^{-1+\beta} \in (N^{-1+\eta}, r_0)$,*

$$\mathbb{P}\left(N_{z,r}^{\Xi_{N,a}^z} > (a+(b-a)/2)|\log r|^2 \right) \leq r^c \tag{4.180}$$

and

$$\mathbb{E}\left[\left(1 - e^{-KT(\Xi_{N,a}^z)}\right) e^{\frac{u}{|\log r|} N_{z,r}^{\Xi_{N,a}^z}} \right] \leq \left(1 - e^{-a(3/2 q_N(z) q_{N,K}(z) C_{N,K}(z) + C(u)|\log r|)}\right) \times \exp\left(a \frac{u}{1-u\beta}(1+o(1))|\log r| \right) \tag{4.181}$$

To quickly see why we have $\frac{u}{1-u\beta}$ instead of $\frac{u}{1-u}$ as in Corollary 4.47, we compute

$$\begin{aligned}\mathbb{E}\left[e^{\frac{u}{|\log r|} N_{z,r}^{\Xi_{N,a}^z}} \right] &= \exp\left(\mathbb{E}[P]\mathbb{E}\left[e^{\frac{u}{|\log r} G_i} - 1 \right] \right) \\ &= \exp\left((1+o(1)) \frac{a}{\beta} |\log r| \Big(\frac{1}{1-u\beta} - 1 \Big) \right) = \exp\left((1+o(1)) a \frac{u}{1-u\beta} |\log r| \right).\end{aligned}$$

Corollary 4.84. *Let $u>0$. There exists $M_\eta > 0$ such that for all $M \geq M_\eta$, for all $z,z' \in D_N \cap A$ and $r = N^{-1+\beta} > N^{-1+\eta}$ being such that $1 \leq Mr/|z-z'| < e$, we have*

$$\mathbb{E}\left[\exp\left(-\frac{u}{|\log r|} N_{z,r}^{\Xi_{N,z',a}^z} \right) \right] \leq \exp\left(-a \frac{u}{(2-\beta)(2-\beta+\beta u+\eta^2)} (1+o(1))|\log r| \right) \tag{4.182}$$

and

$$\mathbb{E}\left[\exp\left(-\frac{u}{|\log r|} N_{z,r}^{\Xi_{N,z,a}^{z'}} \right) \right] \leq \exp\left(-a \frac{u}{(2-\beta)(2-\beta+\beta u+\eta^2)} (1+o(1))|\log r| \right). \tag{4.183}$$

Finally, we will need a control on the number of excursions in the process $\Xi_{N,a,a'}^{z,z'}$ (4.141). The following lemma is to be compared with Lemma 4.50.

Lemma 4.85. *Let $u>0$. There exists $M = M(\eta)$ large enough, so that if $z,z' \in D_N \cap A$ and $r = N^{-1+\beta} > 0$ are such that $1 \leq Mr/|z-z'| < e$, then*

$$\mathbb{E}\left[\exp\left(-\frac{u}{|\log r|} N_{z,r}^{\Xi_{N,a,a'}^{z,z'}} \right) \right] \leq \frac{\mathsf{B}\left((2\pi)^2 aa' \widetilde{G}_{D_N}(z,z')^2 (1+o(1)) \left(1 + \frac{\beta u}{2-\beta+\eta^2}\right)^{-2} \right)}{\mathsf{B}\left((2\pi)^2 aa' \widetilde{G}_{D_N}(z,z')^2 \right)}$$

Proof. The proof follows from the definition (4.141) of $\Xi_{N,a,a'}^{z,z'}$ and from Lemma 4.82 in a very similar way as Lemma 4.50 was a consequence of Lemma 4.46. We omit the details. □

4.12.3 Proof of Lemma 4.76 and localised KMT coupling

We remind the reader that Lemma 4.76 shows that the restriction to good events comes essentially for free in an L^1 sense. To do this, a crucial argument is that a typical (deterministic) point z is not thick for the discrete loop soups. In the continuum, the corresponding large deviation estimate followed from Lemma 4.51. The proof of that lemma could probably be adapted with some tedious but ultimately superficial difficulties coming from the fact that we cannot easily condition on the maximum modulus of a loop when the space is discrete. However, we find it more instructive to deduce Lemma 4.76 from a coupling argument between discrete (random walk) loops and continuous (Brownian) loops. This coupling is a relatively simple modification of an argument put forward by Lawler and Trujillo-Ferreras [LTF07], in which discrete random walks loop soups were in fact first introduced, with however one major difference. Indeed, [LTF07] shows that discrete and continuous loops are in one-to-one correspondence provided that they are not too small (essentially, of discrete duration at least N^κ with $\kappa > 2/3$, corresponding to loops of mesoscopic diameter $N^{\kappa/2}/N = N^{-2/3}$ when we scale the lattice so that the mesh size is $1/N$). In this correspondence, Lawler and Trujillo-Ferreras show furthermore that such loops are then not more than $\log N/N$ apart from one another with overwhelming probability, similar to a KMT approximation rate from which the result of [LTF07] follows.

While the KMT approximation is excellent (we in fact do not need the full power of the logarithmic KMT rate), the restriction to mesoscopic loops of sufficiently large polynomial diameter is problematic for us. It would indeed prevent us from getting any meaningful estimate concerning the crossings of annuli of diameter $r \ll N^{-2/3}$. This would place a restriction on the thickness parameter a or equivalently γ; in order to treat the whole range of values $\gamma \in (0,2)$ we need to be able to consider crossings of annuli of any polynomial diameter $r \geq N^{-1+\eta}$, with $\eta > 0$ arbitrarily small (depending on $\gamma < 2$).

On the other hand, it is fairly clear from the proof of [LTF07] that their result is sharp, and that the coupling described above cannot hold without the restriction $\kappa > 2/3$; that is, at all scales smaller than $N^{-2/3}$ some discrete and continuous loops *somewhere* will be quite different from one another. The lemma below shows however that if one is interested in the behaviour of small mesoscopic loops *locally* (close to a given point z) then discrete and continuous loops *at all polynomial scales* may be coupled to be close to one another. In this sense, Lemma 4.86 below is a localised strengthening of Theorem 1.1 of [LTF07].

This lemma may be of independent interest, and we state it now. Let $\tilde{\mathcal{L}}^\theta_{D_N}$ denote the discrete skeleton of $\mathcal{L}^\theta_{D_N}$, which is formed by turning the continuous-time loops of $\mathcal{L}^\theta_{D_N}$ into discrete-time ones, which consist of the ordered (rooted) sequence of successive vertices visited by each loop. If $\wp \in \mathcal{L}^\theta_D \cup \tilde{\mathcal{L}}^\theta_{D_N}$, let $T(\wp)$ denote the lifetime of $\wp$ (which is an integer if $\wp \in \tilde{\mathcal{L}}^\theta_{D_N}$). With a small abuse of notation, we will consider a path $\tilde{\wp} \in \tilde{\mathcal{L}}^\theta_{D_N}$ as being defined over the entire interval of time $[0, T(\tilde{\wp})]$ via linear interpolation. Note that with our conventions, the time variable $T(\tilde{\wp})$ is typically of order N^2 for a macroscopic discrete random walk loop $\tilde{\wp}$, while its space variable is of order 1 (i.e., the mesh size is $1/N$ and $\tilde{\wp}$ takes values in $(\mathbb{Z}/N)^2$). The following will be applied with r of order $N^{-1+\eta}$ for some $\eta > 0$.

Lemma 4.86. *Fix $\theta > 0$ and let $\eta > 0$. There exists $c > 0$ (depending on the intensity θ and on η)*

such that the following holds. Let $z \in D$*. For all* $N^{-1+\eta} \leq r \leq \mathrm{diam}(D)$ *we can define on the same probability space* $\mathcal{L}_D^\theta$ *and* $\tilde{\mathcal{L}}_{D_N}^\theta$ *in such a way that :*

$$\mathcal{A}_{r,z} = \{\wp \in \mathcal{L}_D^\theta; T(\wp) \geq \tfrac{r^2}{(\log N)^2}; |\wp(0) - z| \leq \sqrt{T(\wp)}\log N; \}$$
$$\text{and } \tilde{\mathcal{A}}_{r,z,N} = \{\tilde{\wp} \in \tilde{\mathcal{L}}_{D_N}^\theta; T(\tilde{\wp}) \geq \tfrac{r^2 N^2}{(\log N)^2}; |\tilde{\wp}(0) - z| \leq \sqrt{\tfrac{T(\tilde{\wp})}{N^2}}\log N\}$$

are in one-to-one correspondence with probability at least $1 - c(\log N)^6/(rN) \geq 1 - cN^{-\eta/2}$*. Furthermore, if* $\wp$ *and* $\tilde{\wp}$ *are paired in this correspondence,*

$$|\frac{T(\tilde{\wp})}{N^2} - T(\wp)| \leq (5/8)N^{-2}; \tag{4.184}$$

$$\sup_{0\leq s\leq 1} |\wp(sT(\wp)) - \tilde{\wp}(sT(\tilde{\wp}))| \leq cN^{-1}\log N \tag{4.185}$$

on an event of probability at least $1 - cN^{-4}$*.*

Proof. We observe that the law of $\tilde{\mathcal{L}}_{D_N}^\theta$ is that of a discrete random walk loop soup (in the sense of [LTF07], i.e., in discrete time) with intensity θ. Using the notations from [LTF07], let $\tilde{q}_n$ denote the mass of discrete random walk loops with duration exactly n (rooted at a specific point), and let q_n denote the total mass of Brownian loops whose duration falls in the interval $[n - 3/8, n + 5/8]$ starting from a region of unit area (see top of p. 773 in [LTF07]). These constants are chosen so that the length of this interval is 1 (needed for coupling) and q_n and $\tilde{q}_n$ are as close as possible: that is, they coincide not only in their first but also their second order, so that

$$|q_n - \tilde{q}_n| \leq Cn^{-4}.$$

To do the coupling it is easier to start with a random walk loop soup on the usual (unscaled) lattice $\mathbb{Z}^2$ and then apply Brownian scaling. That is, the Poisson processes of discrete loops emanating from each possible $x \in \mathbb{Z}^2$ and of duration $n \geq (rN)^2/(\log N)^2$ with $|Nz - x|_{\mathbb{Z}^2} \leq \sqrt{n}(\log N)$, can then be put in one-to-one correspondence for each $n \geq (rN)^2$ with a Poisson point processes of continuous Brownian loops of duration $t \in [n - 3/8, n + 5/8]$ starting in a unit square centered at x. This coupling fails with a probability at most

$$\begin{aligned}
&\leq C \sum_{n\geq r^2N^2/(\log N)^2} \sum_{\substack{x\in\mathbb{Z}^2;\\ |x-Nz|_{\mathbb{Z}^2}\leq\sqrt{n}\log N}} |q_n - \tilde{q}_n| \\
&\leq C \sum_{n\geq(rN)^2/(\log N)^2} n(\log N)^2 n^{-4} \\
&= C(\log N)^6/(rN)^4.
\end{aligned}$$

We then apply Brownian scaling to the above Brownian loops (this leaves the Brownian loop soup invariant in law), and scale the space variable of the discrete random walk loops, which provides the desired correspondence between $\mathcal{A}_{r,z}$ and $\tilde{\mathcal{A}}_{r,z,N}$.

By definition, the loops in this correspondence satisfy (4.184). We now finish the argument in a similar manner to [LTF07], coupling the discrete random walk and continuous Brownian loops of a given duration and starting point in the manner of Corollary 3.3 in [LTF07], but with exponent n^{-k} instead of n^{-30} (as remarked in Corollary 3.2, the exponent 30 was arbitrary, and can be replaced with any number k with a suitably chosen constant $c = c_k$). Let A be the event that in this coupling,

$$A = \{ \sup_{0\le s\le 1} |\wp(sT(\wp)) - \tilde{\wp}(sT(\tilde{\wp}))| \ge c_k \frac{\log(N^6)}{N} \text{ for some } \wp \in \mathcal{A}_{r,z}, \tilde{\wp} \in \tilde{\mathcal{A}}_{r,z,N} \}.$$

Then we get (similar to [LTF07], except we cannot take advantage of the fact that the duration of loops is at least $N^{2/3}$, and we use an error bound on the coupling which is $O(\text{duration})^{-k}$ instead of $O(\text{duration})^{-30}$):

$$\begin{aligned} \mathbb{P}(A) &\le c\theta r^2 N^{-4} + r^2 N^2 N^6 N^5 c_k \Big(\frac{r^2N^2}{(\log N)^2}\Big)^{-k} \\ &\le c\theta r^2 N^{-4} + c_k N^{11} (\log N)^{2k} (r^2N^2)^{1-k} \\ &\le c(N^{-4} + N^{11+2\eta(1-k)} (\log N)^{2k}) \end{aligned}$$

where c depends on k and θ. If we choose k large enough that $2\eta(1-k) + 11 < -4$, we obtain

$$\mathbb{P}(A) \le cN^{-4},$$

where c depends on θ and η, as desired. □

Lemma 4.87. *Fix $u > 0$ and $\eta > 0$. There exists $c > 0$, such that for all $r \ge N^{-1+\eta}$ (and $r \le \operatorname{diam}(D)$, say), for all $z \in D_N$,*

$$\mathbb{P}\left(N_{z,r}^{\mathcal{L}^\theta_{D_N\setminus\{z\}}} > u|\log r|^2 \right) \le r^c.$$

Proof. We first dominate $N_{z,r}^{\mathcal{L}^\theta_{D_N\setminus\{z\}}}$ by $N_{z,r}^{\mathcal{L}^\theta_{D_N}}$; that is, we forget about the restriction that the loops must not visit z itself. We then apply the coupling of Lemma 4.86. Note that to each crossing of $A(z, r, er)$ by a discrete loop must correspond a crossing of the slightly smaller annulus $A' = A(z, 1.01r, 0.99er)$ by a continuous Brownian loop to which it is paired; let $N_{z,r'}^{\mathcal{L}_D}$ denote the number of crossing of the annulus A' by the Brownian loop soup $\mathcal{L}_D$.

We now show that with overwhelming probability all possible loops that cross the annulus $A(z, r, er)$ are accounted for in the one-to-one correspondence of Lemma 4.86. To see this, observe that in order for a loop $\tilde{\wp}$ to cross the annulus $A(z, r, er)$ and not to be accounted for in the set $\tilde{\mathcal{A}}_{z,r,N}$, the loop $\tilde{\wp}$ must either be extremely short or start far away from z: more precisely, its duration $T(\tilde{\wp})$ should be less than

$$T(\tilde{\wp}) \le \tfrac{r^2N^2}{(\log N)^2}, \tag{4.186}$$

or its starting point should be at a distance at least

$$|\tilde{\wp}(0) - z| \ge \sqrt{\tfrac{T(\tilde{\wp})}{N^2}} (\log N) \tag{4.187}$$

from z. Either possibility is of course very unlikely since it requires the loop to travel a great distance in a short span of time. Let $\tilde{\mathcal{B}}_1$ (resp. $\tilde{\mathcal{B}}_2$) denote the set of (discrete) loops which verify (4.186) (resp. (4.187)) and cross the annulus $A(z, r, er)$.

Let us show first $\mathbb{E}(\tilde{\mathcal{B}}_1)$ decays faster than any polynomial. Fix $n \le r^2 N^2/(\log N)^2$ and a starting point x. For a discrete random walk loop $\tilde{\wp}$ of duration $T(\tilde{\wp}) = n$ and started at x, the probability to cross an annulus of width r in time n is bounded by

$$Cn \exp(-c\tfrac{r^2N^2}{n}) \le Cn \exp(-c(\log N)^2),$$

for some universal constants $c, C > 0$. The exponential term above is obtained from elementary large deviation estimates (e.g. Hoeffding inequality) for discrete unconditioned random walk via a maximal inequality, and the factor n in front accounts for the conditioning to return to the starting point in time n. Summing over $n \le r^2 N^2$, and multiplying by the intensity of loops of duration n (which is at most polynomial) we see that $\mathbb{E}(\tilde{\mathcal{B}}_1) \le N^C \exp(-c(\log N)^2)$ and so decays faster than any polynomial.

Let us turn to $\tilde{\mathcal{B}}_2$, which we can handle similarly. Fix $n \ge r^2N^2/(\log N)^2$, and a starting point $x \in D \cap (\mathbb{Z}/N)^2$ such that $|x - z| \ge \sqrt{n} \log N/N$ (note that this means $n \le \mathrm{diam}(D)(N/\log N)^2 \le CN^2$. In order for a random walk loop $\tilde{\wp}$ starting from x and of duration n to cross A, it must touch A and so travel a distance at least $\sqrt{n}\log N/(2N)$ in time n. This is also bounded by

$$Cn \exp(-c\tfrac{n(\log N)^2}{n}) \le Cn \exp(-c(\log N)^2).$$

Summing again over all possible values of x and $n \le CN^2$, we get $\mathbb{E}(\tilde{\mathcal{B}}_2) \le N^C \exp(-c(\log N)^2)$ and so also decays faster than any polynomial.

Thus, except on an event of probability at most $CN^{-\eta/2}$, $N_{z,r}^{\mathcal{L}_{D_N \setminus \{z\}}} \le N_{z,r'}^{\mathcal{L}_D}$. We can now use Lemma 4.51 to bound the probability that the continuous loop soup has many crossing of the annulus $A' = A(z, 1.01r, 0.99er)$. Since the right hand side of the bound in Lemma 4.51 is of the desired form (in fact, is more precise), we deduce

$$\mathbb{P}\left(N_{z,r}^{\mathcal{L}^\theta_{D_N \setminus \{z\}}} > u|\log r|^2 \right) \le CN^{-\eta/2} + r^c,$$

for some $c > 0$. Since $r \ge N^{-1+\eta}$, the right hand side above is at most r^c for some (possibly different) value of c (depending on η and u only). ☐

We now have all the ingredients we need to prove Lemma 4.76.

Proof of Lemma 4.76. By Proposition 4.61, we have

$$\begin{aligned}
&\mathbb{E}\left[\left|\tilde{\mathcal{M}}_a^N(A) - \mathcal{M}_a^N(A)\right|\right] = \mathbb{E}\left[\int_A \mathbf{1}_{\mathcal{G}_N(z)^c} \mathcal{M}_a^N(dz)\right] \\
&= \frac{\log N}{\Gamma(\theta)N^2} \sum_{z \in D_N \cap A} q_N(z)^\theta \int_a^\infty \mathrm{d}\rho \, \frac{c_0^\rho \rho^{\theta-1}}{N^{\rho - a}} \mathrm{CR}_N(z, D_N)^\rho \\
&\times \mathbb{P}\left(\exists r \in \{e^{-n}, n \ge 1\} \cap (N^{-1+\eta}, r_0), N_{z,r}^{\mathcal{L}^\theta_{D_N \setminus \{z\}} \cup \Xi^z_{N,\rho}} > b|\log r|^2 \right)
\end{aligned}$$

Let $z \in D_N \cap A$. By Lemma 4.93, we can bound $q_N(z) \leq C$ and $\mathrm{CR}_N(z, D_N) \leq C$ for some constant $C > 0$. We divide the integral over $\rho \in (a, \infty)$ into two parts corresponding to the integrals from a to $a + (b-a)/2$ and from $a + (b-a)/2$ to infinity respectively. To bound the latter contribution, we simply bound the probability in the integrand by 1 and observe that

$$\int_{a+(b-a)/2}^{\infty} \frac{C^\rho \rho^{\theta-1}}{N^{\rho-a}} \mathrm{d}\rho \leq \frac{C}{N^{(b-a)/2} \log N}.$$

To bound the contribution of the integral for $\rho \in (a, a + (b-a)/2)$, we notice that the probability in the integrand can be bounded by its value at $\rho = a + (b-a)/2$. Because

$$\int_a^{a+(b-a)/2} \mathrm{d}\rho \, \frac{C^\rho \rho^{\theta-1}}{N^{\rho-a}} \leq \frac{C}{\log N},$$

this leads to

$$\int_a^{a+(b-a)/2} \mathrm{d}\rho \, \frac{C^\rho \rho^{\theta-1}}{N^{\rho-a}} \mathbb{P}\left(\exists r \in \{e^{-n}, n \geq 1\} \cap (N^{-1+\eta}, r_0), N_{z,r}^{\mathcal{L}^\theta_{D_N \setminus \{z\}} \cup \Xi^z_{N,\rho}} > b |\log r|^2 \right)$$
$$\leq \frac{C}{\log N} \mathbb{P}\left(\exists r \in \{e^{-n}, n \geq 1\} \cap (N^{-1+\eta}, r_0), N_{z,r}^{\mathcal{L}^\theta_{D_N \setminus \{z\}} \cup \Xi^z_{N,a+(b-a)/2}} > b |\log r|^2 \right)$$

A union bound, Corollary 4.83 and Lemma 4.87 show that the above probability is bounded by $C r_0^c$ for some $C, c > 0$. This concludes the proof of (4.153). The proof of (4.154) is an interpolation of the proofs of (4.153) and Lemma 4.43. Note that we use (4.181) instead of (4.180). We leave the details to the reader. □

4.12.4 Proof of Lemma 4.77 (truncated L^2 bound)

Proof of Lemma 4.77. Let $z, z' \in A \cap D_N$. Assume for now that $|z - z'| < N^{-1+\eta}$. By forgetting the good events and the requirement that z' is a-thick, we can simply bound

$$\mathbb{E}\left[\tilde{\mathcal{M}}_a^N(\{z\}) \tilde{\mathcal{M}}_a^N(\{z'\}) \right] \leq \frac{(\log N)^{2-2\theta}}{N^{4-2a}} \mathbb{P}\left(z \in \mathcal{T}_N(a) \right) \leq C \frac{(\log N)^{1-\theta}}{N^{4-a}}.$$

Since $|z - z'| < N^{1-\eta}$, we can further bound

$$(\log N)^{1-\theta} N^a \leq \log(N) N^a \leq \frac{1}{1-\eta} \log\left(\frac{1}{|z-z'|} \right) \frac{1}{|z-z'|^{a/(1-\eta)}}.$$

Since η is smaller than $1 - a/2$, $a/(1-\eta)$ is smaller than 2 which guarantees that

$$\int_{A \times A} \log\left(\frac{1}{|z-z'|} \right) \frac{1}{|z-z'|^{a/(1-\eta)}} \mathrm{d}z \mathrm{d}z' < \infty.$$

The remaining of the proof consists in controlling the contribution when $|z - z'| \geq N^{-1+\eta}$. We will denote $|z - z'| = N^{-1+\beta}$ and β is therefore at least η. Let $M > 0$ be a large parameter. Let

$r \in \{e^{-n}, n \geq 1\} \cap (0, r_0)$ be such that

$$\frac{|z-z'|}{M} \leq r < e\frac{|z-z'|}{M}.$$

We choose M large enough to ensure that $r < r_0$, but it will be also important to take M large enough to ensure that we can use Corollary 4.84 and Lemma 4.85. For any collection $\mathcal{C}$ of discrete loops, define

$$F(\mathcal{C}) := \mathbf{1}_{\left\{N_{z,r}^{\mathcal{C}} < b|\log r|^2\right\}}.$$

By only keeping the requirement on the number of crossings of $\mathbb{D}_N(z, er) \setminus \mathbb{D}_N(z, r)$, we can bound

$$\mathbb{E}\left[\tilde{\mathcal{M}}_a^N(\{z\})\tilde{\mathcal{M}}_a^N(\{z'\})\right] \leq \mathbb{E}\left[F(\mathcal{L}_{D_N}^\theta)\mathcal{M}_a^N(\{z\})\mathcal{M}_a^N(\{z'\})\right].$$

As in the proof of Lemma 4.44, we will bound F in the spirit of an exponential Markov inequality: define

$$F_1(\mathcal{C}) := r^{-b}\exp\left(-\frac{1}{|\log r|}N_{z,r}^{\mathcal{C}}\right).$$

We have $F \leq F_1$. We use Proposition 4.69 and the notations therein to bound the expectation of $F_1(\mathcal{L}_{D_N}^\theta)\mathcal{M}_a^N(\{z\})\mathcal{M}_a^N(\{z'\})$. We end up with the following expectation to bound:

$$\mathbb{E}\left[F_1\Big(\mathcal{L}_{D_N\setminus\{z,z'\}}^\theta \cup \{\Xi_{N,a_i,a_i'}^{z,z'} \wedge \Xi_{N,z',a_i}^{z} \wedge \Xi_{N,z,a_i'}^{z'}\}_{i=1}^l \cup \{\Xi_{N,z',\tilde{a}_i}^{z}, i \geq 1\} \cup \{\Xi_{N,z,\tilde{a}_i'}^{z'}, i \geq 1\}\Big)\right]. \quad (4.188)$$

This expectation does not increase when one forgets $\mathcal{L}_{D_N\setminus\{z,z'\}}^\theta$ above and we bound it by

$$\begin{aligned} & r^{-b}\prod_{i=1}^l \mathbb{E}\left[\exp\left(-\frac{1}{|\log r|}N_{z,r}^{\Xi_{N,a_i,a_i'}^{z,z'}}\right)\right] \\ & \times \mathbb{E}\left[\mathbb{E}\left[\exp\left(-\frac{1}{|\log r|}\left(\sum_{i=1}^l N_{z,r}^{\Xi_{N,z',a_i}^{z}} + \sum_{i\geq 1} N_{z,r}^{\Xi_{N,z',\tilde{a}_i}^{z}}\right)\right)\middle| \tilde{a}_i, i \geq 1\right]\right] \\ & \times (z \leftrightarrow z') \end{aligned}$$

where in the above, we wrote informally that the last line corresponds to the second line with the processes of excursions around z replaced by the corresponding processes of excursions around z'. By superposition property of Poisson point processes and because $\sum_{i\geq 1}\tilde{a}_i = \tilde{\rho}$ and $\sum_{i=1}^l a_i = \rho$,

$$\bigcup_{i=1}^l \Xi_{N,z',a_i}^z \cup \bigcup_{i\geq 1} \Xi_{N,z',\tilde{a}_i}^z \overset{(d)}{=} \Xi_{N,z',\rho+\tilde{\rho}}^z$$

and a similar result for z'. By Corollary 4.84 and by taking M large enough (depending on η), the expectation in the second line is bounded by

$$\exp\left(-\frac{\rho+\tilde{\rho}}{(2-\beta)(2+\eta^2)}(1+o(1))|\log r|\right)$$

The expectation in the third line can be bounded by the same quantity with $\rho + \tilde{\rho}$ replaced by $\rho' + \tilde{\rho}'$ (see (4.183)). Lemma 4.85 allows us to bound the expectation in the first line by

$$\prod_{i=1}^{l} \frac{\mathsf{B}\left((2\pi)^2 a_i a_i' \widetilde{G}_{D_N}(z,z')^2 (1+o(1)) \left(\frac{2-\beta+\eta^2}{2+\eta^2}\right)^2\right)}{\mathsf{B}\left((2\pi)^2 a_i a_i' \widetilde{G}_{D_N}(z,z')^2\right)}.$$

To wrap things up, we have obtained that (4.188) is at most

$$r^{-b} \prod_{i=1}^{l} \frac{\mathsf{B}\left((2\pi)^2 a_i a_i' \widetilde{G}_{D_N}(z,z')^2 (1+o(1)) \left(\frac{2-\beta+\eta^2}{2+\eta^2}\right)^2\right)}{\mathsf{B}\left((2\pi)^2 a_i a_i' \widetilde{G}_{D_N}(z,z')^2\right)} \exp\left(-\frac{\rho + \tilde{\rho} + \rho' + \tilde{\rho}'}{(2-\beta)(2+\eta^2)}(1+o(1))|\log r|\right)$$

Plugging this into Proposition 4.69 and using the function $\hat{\mathsf{H}}$ defined in (4.158), we obtain that

$$\mathbb{E}\left[\tilde{\mathcal{M}}_a^N(\{z\})\tilde{\mathcal{M}}_a^N(\{z'\})\right] \le \frac{q_{N,z'}(z)^\theta q_{N,z}(z')^\theta (\log N)^2}{N^{4-2a}\Gamma(\theta)^2} e^{-\theta J_N(z,z')} r^{-b} \hat{\mathsf{H}}_a(\lambda, \lambda', v)$$

where

$$\lambda = \log N - \log \mathrm{CR}_{N,z'}(z, D_N) - \log c_0 + \frac{1}{(2-\beta)(2+\eta^2)}(1+o(1))|\log r|,$$

$$\lambda' = \log N - \log \mathrm{CR}_{N,z}(z', D_N) - \log c_0 + \frac{1}{(2-\beta)(2+\eta^2)}(1+o(1))|\log r|,$$

and

$$v = (2\pi)^2 \widetilde{G}_{D_N}(z,z')^2 (1+o(1)) \left(\frac{2-\beta+\eta^2}{2+\eta^2}\right)^2.$$

Since $J_N(z,z')$ (4.143) is nonnegative and $q_{N,z}(z')$ (4.142) is bounded from above (this follows from (4.137) and Lemma 4.93), we further bound

$$\mathbb{E}\left[\tilde{\mathcal{M}}_a^N(\{z\})\tilde{\mathcal{M}}_a^N(\{z'\})\right] \le C N^{-4+2a} (\log N)^2 r^{-b} \hat{\mathsf{H}}_a(\lambda, \lambda', v) \tag{4.189}$$

and it remains to estimate $\hat{\mathsf{H}}_a(\lambda, \lambda', v)$. We have

$$\begin{aligned}\lambda &= \frac{(\log N)^2}{2\pi G_{D_N \setminus \{z'\}}(z,z)} + \frac{1}{(2-\beta)(2+\eta^2)}(1+o(1))|\log r| \\ &= \frac{1}{\beta(2-\beta)} \log N + (1+o(1)) \frac{1}{(2-\beta)(2+\eta^2)} |\log r| = (1+o(1))\lambda'\end{aligned}$$

and

$$\sqrt{v} = (1+o(1)) \frac{2-\beta+\eta^2}{(2+\eta^2)\beta(2-\beta)} |\log r|.$$

We see that $\lambda - \sqrt{v}$ is always of order $\log N$. In particular, $\lambda > \sqrt{v} + 1$ so that we can use (4.163) and bound

$$\hat{\mathsf{H}}_a(\lambda, \lambda', v) \leq C v^{1/4-\theta/2} \frac{1}{(\lambda - \sqrt{v})(\lambda' - \sqrt{v})} e^{(2\sqrt{v} - \lambda - \lambda')a} \leq \frac{C}{(\log N)^2} r^{o(1)} e^{(2\sqrt{v} - \lambda - \lambda')a}.$$

Coming back to (4.189), we have obtained that

$$\mathbb{E}\left[\tilde{\mathcal{M}}_a^N(\{z\}) \tilde{\mathcal{M}}_a^N(\{z'\}) \right] \leq N^{-4} r^{-b+o(1)} \exp\left(a(2\sqrt{v} - \lambda - \lambda' + 2\log N) \right)$$

An elementary computation shows that

$$\begin{aligned} \sqrt{v} - \lambda + \log N &= (1+o(1)) \frac{1}{\beta(2-\beta)} \left(\frac{2 - 2\beta + \eta^2}{2 + \eta^2} - (1-\beta) \right) |\log r| \\ &\leq (1+o(1)) \frac{\eta^2}{\beta(2-\beta)} |\log r| \leq (1+o(1)) \eta |\log r| \end{aligned}$$

where we use the fact that $\beta \in [\eta, 1]$ to obtain the last inequality. By choosing η and $b - a$ small enough, we can therefore ensure that

$$b|\log r| + a(2\sqrt{v} - \lambda - \lambda' + 2\log N) \leq c|\log r|$$

for some constant c smaller than 2. To conclude, we have proven that

$$\mathbb{E}\left[\tilde{\mathcal{M}}_a^N(\{z\}) \tilde{\mathcal{M}}_a^N(\{z'\}) \right] \leq C N^{-4} |z - z'|^{-c}$$

for some $c < 2$. This provides an integrable domination as stated in (4.155).

The proof of (4.156) is very similar. Note that we use (4.164) instead of (4.163) and, as in the proof of Lemma 4.44 (specifically (4.118)), we use FKG-inequality for Poisson point processes (see [Jan84, Lemma 2.1]) in order to decouple, on the one hand, the killing associated to the mass and, on the other hand, the negative exponential of the number of crossings. We do not give more details. □

4.12.5 Proof of Lemma 4.78 (convergence)

In this section, we assume that the parameter b, used in the definitions (4.151) and (4.152) of the good events, is close enough to a so that the conclusions of Lemma 4.77 hold. By developing the product, we have

$$\begin{aligned} &\mathbb{E}\left[\left(\tilde{\mathcal{M}}_a^N(A) - \frac{2^\theta}{(\log K)^\theta} \tilde{\mathcal{M}}_a^{N,K}(A) \right)^2 \right] \\ &= \int_{A \times A} N^4 \mathbb{E}\left[\tilde{\mathcal{M}}_a^N(z) \left(\tilde{\mathcal{M}}_a^N(z') - \frac{2^\theta}{(\log K)^\theta} \tilde{\mathcal{M}}_a^{N,K}(z') \right) \right] \mathrm{d}z \mathrm{d}z' \\ &+ \int_{A \times A} N^4 \mathbb{E}\left[\frac{2^\theta}{(\log K)^\theta} \tilde{\mathcal{M}}_a^{N,K}(z) \left(\frac{2^\theta}{(\log K)^\theta} \tilde{\mathcal{M}}_a^{N,K}(z') - \tilde{\mathcal{M}}_a^N(z') \right) \right] \mathrm{d}z \mathrm{d}z'. \end{aligned}$$

Lemma 4.77 provides the domination we need in order to apply dominated convergence theorem and it only remains to show that for *fixed* distinct points $z, z' \in A$,

$$\limsup_{K\to\infty} \limsup_{N\to\infty} N^4 \mathbb{E}\left[\tilde{\mathcal{M}}_a^N(z) \left(\tilde{\mathcal{M}}_a^N(z') - \frac{2^\theta}{(\log K)^\theta} \tilde{\mathcal{M}}_a^{N,K}(z') \right) \right] \leq 0 \tag{4.190}$$

and

$$\limsup_{K\to\infty} \limsup_{N\to\infty} N^4 \mathbb{E}\left[\frac{2^\theta}{(\log K)^\theta} \tilde{\mathcal{M}}_a^{N,K}(z) \left(\frac{2^\theta}{(\log K)^\theta} \tilde{\mathcal{M}}_a^{N,K}(z') - \tilde{\mathcal{M}}_a^N(z') \right) \right] \leq 0. \tag{4.191}$$

We emphasise that, since z and z' are fixed points of the continuous set A, they are at a macroscopic distance from each other. We will sketch the proof of (4.190). Since the proof of (4.191) is very similar, we will omit it. Let $r_1 > 0$ be much smaller than $|z - z'| \vee r_0$ and consider the good events $\mathcal{G}_N'(z)$ and $\mathcal{G}_{N,K}'(z)$ defined in the same way as $\mathcal{G}_N(z)$ and $\mathcal{G}_{N,K}(z)$ (see (4.151) and (4.152)) except that the restriction on the number of crossings of annuli is only on radii $r \in (r_1, r_0)$ instead of $(N^{-1+\eta}, r_0)$. The advantage of the event $\mathcal{G}_N'(z)$, compared to $\mathcal{G}_N(z)$, is that it is a macroscopic event which is well suited to study asymptotics as the mesh size goes to zero (see (4.196)). Since z and z' are at a distance much larger than r_1, one can show that

$$\begin{aligned} &\liminf_{K\to\infty} \liminf_{N\to\infty} \frac{N^4}{(\log K)^\theta} \mathbb{E}\left[\tilde{\mathcal{M}}_a^N(z) \tilde{\mathcal{M}}_a^{N,K}(z') \right] \\ &\qquad \geq -o_{r_1\to 0}(1) + \liminf_{K\to\infty} \liminf_{N\to\infty} \frac{N^4}{(\log K)^\theta} \mathbb{E}\left[\mathcal{M}_a^N(z) \mathcal{M}_a^{N,K}(z') \mathbf{1}_{\mathcal{G}_N'(z)\cap\mathcal{G}_{N,K}'(z')} \right] \end{aligned}$$

where $o_{r_1\to 0}(1) \to 0$ as $r_1 \to 0$ and may depend on z, z', a, b, η, r_0. This estimate is in the same spirit as Lemma 4.76 and we omit the details. We can therefore bound the left hand side of (4.190) by

$$o_{r_1\to 0}(1) + \limsup_{K\to\infty} \limsup_{N\to\infty} N^4 \mathbb{E}\Big[\mathcal{M}_a^N(z) \mathbf{1}_{\mathcal{G}_N'(z)} \Big(\mathcal{M}_a^N(z') \mathbf{1}_{\mathcal{G}_N'(z')} - \frac{2^\theta}{(\log K)^\theta} \mathcal{M}_a^{N,K}(z') \mathbf{1}_{\mathcal{G}_{N,K}'(z')} \Big) \Big]. \tag{4.192}$$

The rest of the proof is dedicated to showing that the second term above vanishes. Letting $r_1 \to 0$ will conclude the proof of (4.190).

Proposition 4.69 gives an exact expression for the expectation in (4.192). We use the notations therein that we recall for the reader's convenience. The loops visiting z are divided into two collections of loops: the ones that also visit z' and the ones that do not. $l \geq 0$ corresponds to the number of loops in the first collections and a_i, $i = 1 \dots l$, are the thicknesses at z of each individual loop in that collection. $\tilde{a}_i$, $i \geq 1$, are the thicknesses at z of the loops which visit z but not z'. Finally, $\rho = \sum_{i=1}^l a_i$ and $\tilde{\rho} = \sum_{i\geq 1} \tilde{a}_i$ are the overall thicknesses of the two above sets of loops. Similar notations are used for the point z'. We define $E_N(a_i, a_i', i = 1 \dots l, \tilde{\rho}, \tilde{\rho}')$ the event that for all $r \in \{e^{-n}, n \geq 1\} \cap (r_1, r_0)$ and $w \in \{z, z'\}$, the number $N_{w,r}^{\mathcal{C}}$ of discrete crossings in the collection

$$\mathcal{C} := \mathcal{L}_{D_N \setminus \{z,z'\}}^\theta \cup \{\Xi_{N,a_i,a_i'}^{z,z'} \wedge \Xi_{N,z',a_i}^{z} \wedge \Xi_{N,z,a_i'}^{z'}\}_{i=1\dots l} \cup \{\Xi_{N,z',\tilde{a}_i}^{z}\}_{i\geq 1} \cup \{\Xi_{N,z,\tilde{a}_i'}^{z'}\}_{i\geq 1}$$

is at most $b(\log r)^2$. We also define $p_N(a_i, a_i', i = 1 \dots l, \tilde{\rho}, \tilde{\rho}')$ the probability of the event $E_N(a_i, a_i', i =$

$1 \dots l, \tilde{\rho}, \tilde{\rho}')$. Note that, by superposition property of Poisson point processes, this probability only depends on the $\tilde{a}_i$ via their sum $\sum \tilde{a}_i = \tilde{\rho}$. When $l = 0$, this probability degenerates to the probability $p'_N(\tilde{\rho}, \tilde{\rho}', 0)$ where the restriction concerns the number of crossings of $\mathcal{L}^\theta_{D_N \setminus \{z,z'\}} \cup \{\Xi^z_{N,z',\tilde{a}_i}\}_{i \geq 1} \cup \{\Xi^{z'}_{N,z,\tilde{a}_i'}\}_{i \geq 1}$. The notation $p'_N(\tilde{\rho}, \tilde{\rho}', 0)$ is justified by the fact that it corresponds to the case $k = 0$ of the probability $p'_N(\tilde{\rho}, \tilde{\rho}', k)$ that will be defined in (4.194) below. By Proposition 4.69, the expectation $\mathbb{E}\left[\mathcal{M}^N_a(z)\mathbf{1}_{\mathcal{G}'_N(z)}\mathcal{M}^N_a(z')\mathbf{1}_{\mathcal{G}'_N(z')}\right]$ is then equal to (4.145) where the expectation of the function F has to be replaced by $p_N(a_i, a_i', i = 1 \dots l, \tilde{\rho}, \tilde{\rho}')$. In the display below, we develop this last probability according to the number $2k_i$ of trajectories that were used to form the i-th loop $\Xi^{z,z'}_{N,a_i,a_i'}$. By superposition of Poisson point processes and by definition of $\Xi^{z,z'}_{N,a_i,a_i'}$ (see (4.141)), we can rewrite $p_N(a_i, a_i', i = 1 \dots l, \tilde{\rho}, \tilde{\rho}')$ as

$$\prod_{i=1}^l \frac{1}{\mathsf{B}\left((2\pi)^2 a_i a_i' \tilde{G}_{D_N}(z,z')^2\right)} \sum_{k \geq l} \left(2\pi \tilde{G}_{D_N}(z,z')\right)^{2k} \sum_{\substack{k_1,\dots,k_l \geq 1 \\ k_1 + \dots + k_l = k}} \prod_{i=1}^l \frac{(a_i a_i')^{k_i}}{k_i!(k_i-1)!} p'_N(\rho + \tilde{\rho}, \rho' + \tilde{\rho}', k) \tag{4.193}$$

where

$$p'_N(\rho + \tilde{\rho}, \rho' + \tilde{\rho}', k) := \mathbb{P}\Big(\forall r \in \{e^{-n}, n \geq 1\} \cap (r_1, r_0), \forall w \in \{z, z'\}, \\ N_{w,r}^{\mathcal{L}^\theta_{D_N \setminus \{z,z'\}}} + \sum_{i=1}^{2k} N_{w,r}^{\wp_i} + N_{w,r}^{\Xi^z_{N,z',\rho+\tilde{\rho}}} + N_{w,r}^{\Xi^{z'}_{N,z,\rho'+\tilde{\rho}'}} \leq b(\log r)^2\Big) \tag{4.194}$$

and where $\wp_i, i = 1 \dots 2k$, are i.i.d. trajectories with common law $\tilde{\mu}^{z,z'}_{D_N}/\tilde{G}_{D_N}(z,z')$. When one plugs this in (4.145), the products of the functions B cancel out and, by using the notations λ, λ' and v as in (4.160), we deduce that

$$\begin{aligned} N^4 \mathbb{E}\left[\mathcal{M}^N_a(z)\mathbf{1}_{\mathcal{G}'_N(z)}\mathcal{M}^N_a(z')\mathbf{1}_{\mathcal{G}'_N(z')}\right] &= \frac{q_{N,z'}(z)^\theta q_{N,z}(z')^\theta (\log N)^2}{\Gamma(\theta)^2} e^{-\theta J_N(z,z')} N^{2a} \\ &\times \int_{\substack{\rho,\tilde{\rho}>0 \\ \rho+\tilde{\rho}\geq a}} \mathrm{d}\rho \, \mathrm{d}\tilde{\rho} \, e^{-\lambda(\rho+\tilde{\rho})} \tilde{\rho}^{\theta-1} \int_{\substack{\rho',\tilde{\rho}'>0 \\ \rho'+\tilde{\rho}'\geq a}} \mathrm{d}\rho' \, \mathrm{d}\tilde{\rho}' e^{-\lambda'(\rho'+\tilde{\rho}')} (\tilde{\rho}')^{\theta-1} \\ &\times \sum_{l \geq 1} \frac{\theta^l}{l!} \int_{\substack{\mathsf{a} \in E(\rho,l) \\ \mathsf{a}' \in E(\rho',l)}} \mathrm{d}\mathsf{a}\mathrm{d}\mathsf{a}' \sum_{k \geq l} v^k p'_N(\rho+\tilde{\rho}, \rho'+\tilde{\rho}', k) \sum_{\substack{k_1,\dots,k_l \geq 1 \\ k_1+\dots+k_l = k}} \prod_{i=1}^l \frac{(a_i a_i')^{k_i-1}}{k_i!(k_i-1)!} \end{aligned} \tag{4.195}$$

plus the following term which corresponds to the case $l = 0$:

$$\frac{q_{N,z'}(z)^\theta q_{N,z}(z')^\theta (\log N)^2}{\Gamma(\theta)^2} e^{-\theta J_N(z,z')} N^{2a} \int_{(a,\infty)^2} e^{-\lambda t - \lambda' t'} p'_N(t,t',0)(tt')^{\theta-1} \mathrm{d}t\mathrm{d}t'.$$

By Lemma 4.70, the multiplicative factor in front of the first integral in (4.195) is asymptotic to $(\log N)^2 N^{2a} \Gamma(\theta)^{-2}$. (4.165) gives a simple expression for the remaining part of the right hand side of

(4.195) and

$$N^4 \mathbb{E}\left[\mathcal{M}_a^N(z)\mathbf{1}_{\mathcal{G}'_N(z)}\mathcal{M}_a^N(z')\mathbf{1}_{\mathcal{G}'_N(z')}\right] \sim \frac{(\log N)^2 N^{2a}}{\Gamma(\theta)^2} \sum_{k\geq 0} \frac{v^k}{\theta^{(k)}k!} \int_{(a,\infty)^2} e^{-\lambda t - \lambda' t'} p'_N(t,t',k)(tt')^{\theta+k-1} \mathrm{d}t\mathrm{d}t'.$$

We now argue that for any fixed $k \geq 0$, $t, t' \geq a$,

$$p'_N(t,t',k) \xrightarrow[N\to\infty]{} p'(t,t',k) := \mathbb{P}\left(\forall r \in \{e^{-n}, n\geq 1\} \cap (r_1, r_0), \forall w \in \{z,z'\}, N^{\mathcal{C}}_{w,r} \leq b(\log r)^2\right) \tag{4.196}$$

where

$$\mathcal{C} := \mathcal{L}_D^\theta \cup \{\wp_i\}_{i=1\ldots 2k} \cup \{\Xi_t^z, \Xi_{t'}^{z'}\}$$

with $\wp_i, i = 1\ldots 2k$, i.i.d. trajectories distributed according to $\mu_D^{z,z'}/G_D(z,z')$ and the above collections of trajectories are all independent. This follows from 1) the convergence of $\tilde{\mu}_{D_N}^{z,z'}/\tilde{G}_{D_N}(z,z')$ towards $\mu_D^{z,z'}/G_D(z,z')$ established in Proposition 4.72, 2) the convergence of $\mathcal{L}^\theta_{D_N\setminus\{z,z'\}}$ towards $\mathcal{L}_D^\theta$ [LTF07], and 3) the convergence of $\tilde{\mu}^{z,z}_{D_N\setminus\{z'\}}$ towards $\mu_D^{z,z}$ stated in Corollary 4.75. It is then a simple verification that the integral concentrates around $t = t' = a$ as $N\to\infty$ (recall that λ and λ' are defined in (4.160) and go to infinity) and

$$N^4 \mathbb{E}\left[\mathcal{M}_a^N(z)\mathbf{1}_{\mathcal{G}'_N(z)}\mathcal{M}_a^N(z')\mathbf{1}_{\mathcal{G}'_N(z')}\right] \sim \frac{1}{\Gamma(\theta)^2}\frac{(\log N)^2}{\lambda\lambda'} N^{2a} e^{-a(\lambda+\lambda')} \sum_{k\geq 0} \frac{v^k a^{2\theta+2k-2}}{\theta^{(k)}k!} p'(a,a,k) \tag{4.197}$$

$$\sim \frac{(c_0)^{2a}}{\Gamma(\theta)^2} \mathrm{CR}(z,D)^a \,\mathrm{CR}(z',D)^a \sum_{k\geq 0} \frac{(2\pi G_D(z,z'))^{2k} a^{2\theta+2k-2}}{\theta^{(k)}k!} p'(a,a,k).$$

For the mixed case, the situation is slightly different. Because of the killing, the expectation of $\mathcal{M}_a^N(z)\mathbf{1}_{\mathcal{G}'_N(z)}\mathcal{M}_a^{N,K}(z')\mathbf{1}_{\mathcal{G}'_{N,K}(z')}$ is expressed in terms of (see Proposition 4.69)

$$\mathbb{E}\left[\prod_{i=1}^l \left(1 - e^{-KT(\Xi^{z,z'}_{N,a_i,a'_i} \wedge \Xi^z_{N,z',a_i} \wedge \Xi^{z'}_{N,z,a'_i})}\right) \prod_{i=l+1}^m \left(1 - e^{-KT(\Xi^{z'}_{N,z,a'_i})}\right) \mathbf{1}_{E_N(a_i,a'_i,i=1\ldots l,\tilde{\rho},\tilde{\rho}')}\right].$$

Since the points z and z' are macroscopically far apart, the durations of the loops $\Xi^{z,z'}_{N,a_i,a'_i}$, $i = 1\ldots l$, are macroscopic and one can show that the first product is very close to 1. With an argument very similar to what was done in Corollary 4.47, one can show that the expectation of the second product times the indicator function is well approximated by

$$\mathbb{E}\left[\prod_{i=l+1}^m \left(1 - e^{-KT(\Xi^{z'}_{N,z,a'_i})}\right)\right] \mathbb{P}\left(E_N(a_i,a'_i,i=1\ldots l,\tilde{\rho},\tilde{\rho}')\right)$$
$$= \prod_{i=l+1}^m \left(1 - e^{-a'_i C_{N,K,z}(z')}\right) p_N(a_i,a'_i,i=1\ldots l,\tilde{\rho},\tilde{\rho}')$$

where $C_{N,K,z}(z')$ is defined in (4.139). Using (4.146) together with (4.193), we obtain that the expectation $N^4 \mathbb{E}\left[\mathcal{M}_a^N(z) \mathbf{1}_{\mathcal{G}'_N(z)} \mathcal{M}_a^{N,K}(z') \mathbf{1}_{\mathcal{G}'_{N,K}(z')} \right]$ has the same asymptotics as

$$\frac{q_{N,z'}(z)^\theta (\log N)^2}{\Gamma(\theta)} e^{-\theta(J_{N,K,z}(z') + J_{N,K}(z,z'))} N^{2a}$$
$$\times \Bigg(\int_{\substack{\rho,\tilde\rho>0 \\ \rho+\tilde\rho \geq a}} \mathrm{d}\,\rho \,\mathrm{d}\tilde\rho \; e^{-\lambda(\rho+\tilde\rho)} \tilde\rho^{\theta-1} \int_a^\infty \mathrm{d}\,\rho' \, e^{-\lambda'\rho'} \sum_{\substack{m \geq 1 \\ 1 \leq l \leq m}} \frac{\theta^m}{(m-l)! l!} \int_{\substack{\mathsf{a} \in E(\rho,l) \\ \mathsf{a}' \in E(\rho',m)}} \mathrm{d}\mathsf{a} \mathrm{d}\mathsf{a}'$$
$$\sum_{k \geq l} v^k p'_N(\rho+\tilde\rho, \rho', k) \sum_{\substack{k_1,\dots,k_l \geq 1 \\ k_1+\dots+k_l = k}} \prod_{i=1}^l \frac{(a_i a_i')^{k_i-1}}{k_i!(k_i-1)!} \prod_{i=l+1}^m \frac{1-e^{-a_i' C_{N,K,z}(z')}}{a_i'}$$
$$+ \int_a^\infty \mathrm{d}\tilde\rho \tilde\rho^{\theta-1} e^{-\lambda\tilde\rho} \int_a^\infty \mathrm{d}\rho' e^{-\lambda'\rho'} p'_N(\tilde\rho, \rho', 0) \sum_{m\geq 1} \frac{\theta^m}{m!} \int_{\mathsf{a}' \in E(\rho',m)} \mathrm{d}\mathsf{a}' \prod_{i=1}^m \frac{1-e^{-a_i' C_{N,K,z}(z')}}{a_i'} \Bigg).$$

The second term of the sum in parenthesis corresponds to the case $l = 0$. The front factor is asymptotic to $\Gamma(\theta)^{-1} (\log N)^2 N^{2a}$, whereas the first term in parenthesis can be simplified thanks to (4.166) and the second term in parenthesis can be directly expressed in terms of the function F (see (4.44)). Overall, we obtain that $N^4 \mathbb{E}\left[\mathcal{M}_a^N(z) \mathbf{1}_{\mathcal{G}'_N(z)} \mathcal{M}_a^{N,K}(z') \mathbf{1}_{\mathcal{G}'_{N,K}(z')} \right]$ has the same asymptotics as

$$\frac{(\log N)^2}{\Gamma(\theta)} N^{2a} \Bigg(\sum_{k\geq 1} v^k \int_{(a,\infty)^2} \mathrm{d}t \, \mathrm{d}t' \; e^{-\lambda t} e^{-\lambda' t'} \frac{t^{\theta+k-1}}{k!(k-1)!} p'_N(t,t',k)$$
$$\times \Big(\int_0^{t'} \mathrm{d}\rho' \rho'^{k-1} \frac{\mathsf{F}(C_{N,K,z}(z')(t'-\rho'))}{t'-\rho'} + t'^{k-1} \Big)$$
$$+ \int_{(a,\infty)^2} \mathrm{d}t \, \mathrm{d}t' \; e^{-\lambda t} e^{-\lambda' t'} t^{\theta-1} p'_N(t,t',0) \frac{\mathsf{F}\left(C_{N,K,z}(z') t'\right)}{t'} \Bigg).$$

By dominated convergence theorem, Lemma 4.23 and the convergence (4.149) of $C_{N,K,z}(z')$ towards $C_K(z')$, we have

$$\lim_{K\to\infty} \lim_{N\to\infty} \frac{2^\theta}{(\log K)^\theta} \int_0^{t'} \mathrm{d}\rho' \rho'^{k-1} \frac{\mathsf{F}(C_{N,K,z}(z')(t'-\rho'))}{t'-\rho'} = \frac{1}{\Gamma(\theta)} \int_0^{t'} \mathrm{d}\rho' \rho'^{k-1} (t'-\rho')^{\theta-1}.$$

The right hand side term can be computed thanks to (4.222) and is equal to $\frac{(k-1)!}{\Gamma(\theta)\theta^{(k)}} t'^{k+\theta-1}$. From this and the asymptotic behaviour (4.196) of $p'_N(t,t',k)$, one can easily deduce that the asymptotics of $2^\theta (\log K)^{-\theta} N^4 \mathbb{E}\left[\mathcal{M}_a^N(z) \mathbf{1}_{\mathcal{G}'_N(z)} \mathcal{M}_a^{N,K}(z') \mathbf{1}_{\mathcal{G}'_{N,K}(z')} \right]$ is given by

$$\frac{1}{\Gamma(\theta)^2} \frac{(\log N)^2}{\lambda\lambda'} N^{2a} e^{-a(\lambda+\lambda')} \sum_{k\geq 0} \frac{v^k a^{2\theta+2k-2}}{k! \theta^{(k)}} p'(a,a,k)$$
$$\sim \frac{(c_0)^{2a}}{\Gamma(\theta)^2} \mathrm{CR}(z,D)^a \, \mathrm{CR}(z',D)^a \sum_{k\geq 0} \frac{(2\pi G_D(z,z'))^{2k} a^{2\theta+2k-2}}{\theta^{(k)} k!} p'(a,a,k).$$

Since we obtain the same limit as in (4.197), it concludes the proof of (4.190). The proof of (4.191) follows from a very similar line of argument. This concludes the proof of Lemma 4.78.

4.13 Scaling limit of massive random walk loop soup thick points

The goal of this section is to prove Proposition 4.58. As already alluded to, it relies heavily on an analogous statement from [Jeg19] about thick points of finitely many random walk trajectories running from internal to boundary points that we state now.

Let $(D_i, x_i, z_i), i \in I$, be a finite collection of bounded simply connected domains $D_i \subset \mathbb{C}$ with internal points $x_i \in D_i$ and boundary points $z_i \in \partial D_i$. Assume that the boundary points z_i are pairwise distinct $(i \neq j \implies z_i \neq z_j)$ and that for all $i \in I$, the boundary of D_i is locally analytic near z_i (below we will apply this result to boundaries that are locally flat at z_i). Let $\wp_i, i \in I$, be independent Brownian trajectories that start at x_i and are conditioned to exit D_i at z_i, i.e. $\wp_i \sim \mu_{D_i}^{x_i,z_i}/H_{D_i}(x_i,z_i)$; see (4.17). Let $D_{i,N}$ be a discrete approximation of D_i by a portion of the square lattice with mesh size $1/N$ as in (4.9) (take x_i as a reference point instead of the origin) and let $x_{i,N} \in D_{i,N}$ and $z_{i,N} \in \partial D_{i,N}$ be such that $x_{i,N} \to x_i$ and $z_{i,N} \to z_i$ as $N \to \infty$. Let $\wp_{i,N}, i \in I$, be independent random walk trajectories starting at $x_{i,N}$ and conditioned to exit $D_{i,N}$ at $z_{i,N}$.

For all subset J of the set of indices I, let $\mathcal{M}_a^{\cap_{j\in J}\wp_{j,N}}$ be the measure supported on a-thick points coming from the interaction of all the trajectories $\wp_{j,N}, j \in J$: for all Borel set $A \subset \mathbb{C}$,

$$\mathcal{M}_a^{\cap_{j\in J}\wp_{j,N}}(A) := \frac{\log N}{N^{2-a}} \sum_{x \in \cap_{j\in D_{j,N}}} \mathbf{1}_{\{x\in A\}} \mathbf{1}_{\left\{\sum_{j\in J} \ell_x(\wp_{j,N}) \geq \frac{a}{2\pi}(\log N)^2\right\}} \mathbf{1}_{\left\{\forall j\in J, \ell_x(\wp_{j,N})>0\right\}}. \tag{4.198}$$

Recall also that $\mathcal{M}_a^{\cap_{j\in J}\wp_j}$ denotes the Brownian chaos associated to $\wp_j, j \in J$, where each trajectory is required to contribute to the thickness; see Section 4.2.3. Of course, when $\cap_{j\in J} D_j = \varnothing$, these measures degenerate to zero. [Jeg19] shows that:

Theorem 4.88 (Theorem 5.1 of [Jeg19])**.** *As $N \to \infty$, the joint convergence*

$$\left(\mathcal{M}_a^{\cap_{j\in J}\wp_{j,N}}, J \subset I, \wp_{i,N}, i \in I\right) \to \left(c_0^a \mathcal{M}_a^{\cap_{j\in J}\wp_j}, J \subset I, \wp_i, i \in I\right)$$

holds in distribution where the topology associated to $\mathcal{M}_a^{\cap_{j\in J}\wp_{j,N}}$ is the topology of vague convergence on $\cap_{j\in J} D_j$ and the topology associated to $\wp_{i,N}$ is the one induced by d_{paths} (4.28).

To use this result, we will first need to describe a decomposition of the loop soup similar to the one described in Lemma 4.28 that holds in the discrete setting.

4.13.1 Decomposition of random walk loop soup

Let $D_N \subset \mathbb{Z}_N^2$ be such that both D_N and $\mathbb{Z}_N^2 \setminus D_N$ are non-empty. Denote

$$\mathrm{mi}(D_N) := \inf\{\mathrm{Im}(z) : z \in D_N\} \quad \text{and} \quad \mathrm{Mi}(D_N) := \sup\{\mathrm{Im}(z) : z \in D_N\}.$$

Consider the random walk loop soup $\mathcal{L}_{D_N}^\theta$. For $\wp \in \mathcal{L}_{D_N}^\theta$, we will use the same notations $\mathrm{mi}(\wp)$, $\mathrm{Mi}(\wp)$ (4.54) and $h(\wp)$ (4.55) as in the continuum case. Unlike in the continuum case, a loop $\wp \in \mathcal{L}_{D_N}^\theta$ can travel several times back and forth between $\mathbb{R} + i\,\mathrm{mi}(\wp)$ and $\mathbb{R} + i\,\mathrm{Mi}(\wp)$. So we will restrict to loops $\wp \in \mathcal{L}_{D_N}^\theta$ that do this only once in each direction. We will root such a loop at the first time (for the circular order) it visits $\mathbb{R} + i\,\mathrm{mi}(\wp)$ after having visited $\mathbb{R} + i\,\mathrm{Mi}(\wp)$ (see Figure 4.1 for an illustration in the continuum setting). This time is well defined provided $\wp$ travels only once back and forth between $\mathbb{R} + i\,\mathrm{mi}(\wp)$ and $\mathbb{R} + i\,\mathrm{Mi}(\wp)$, and after rerooting it is set to 0. We will denote by $z_\perp$ the position of $\wp$ at this time, as in the continuum case. We have that $z_\perp \in \mathbb{Z}_N + i\,\mathrm{mi}(\wp)$. Note however that in discrete $\wp$ may also visit other points in $\mathbb{Z}_N + i\,\mathrm{mi}(\wp)$. Given $\varepsilon > 0$, we will denote

$$\begin{aligned}\mathcal{L}^\theta_{D_N,\varepsilon} := \{\wp \in \mathcal{L}^\theta_{D_N} : h(\wp) \geq \varepsilon \text{ and}\\ \wp \text{ travels only once back and forth between } \mathbb{R} + i\,\mathrm{mi}(\wp) \text{ and } \mathbb{R} + i(\mathrm{mi}(\wp) + \lceil \varepsilon \rceil_N)\},\end{aligned} \tag{4.199}$$

where $\lceil \varepsilon \rceil_N := N^{-1}\lceil N\varepsilon \rceil$. Note that in discrete we add this condition of a single round trip between $\mathbb{R} + i\,\mathrm{mi}(\wp)$ and $\mathbb{R} + i(\mathrm{mi}(\wp) + \lceil \varepsilon \rceil_N)$. Recall that we root the loops $\wp \in \mathcal{L}^\theta_{D_N,\varepsilon}$ at $z_\perp$. Denote

$$\tau_\varepsilon(\wp) := \inf\{t \in [0, T(\wp)] : \mathrm{Im}(\wp(t)) \geq \mathrm{mi}(\wp) + \lceil \varepsilon \rceil_N\}$$

for $\wp \in \mathcal{L}^\theta_{D_N,\varepsilon}$. As in the continuum, we decompose the loop into two parts

$$\wp_{\varepsilon,1} := (\wp(t))_{0 \leq t \leq \tau_\varepsilon} \quad \text{and} \quad \wp_{\varepsilon,2} := (\wp(t))_{\tau_\varepsilon \leq t \leq T(\wp)}. \tag{4.200}$$

Denote $z_\varepsilon := \wp(\tau_\varepsilon)$. Recall the notations $\mathbb{H}_y$ and $S_{y,y'}$ for upper half planes and horizontal strips (4.56).

Lemma 4.89. *$\#\mathcal{L}^\theta_{D_N,\varepsilon}$ is a Poisson random variable with mean given by*

$$\theta \frac{1}{N} \sum_{\mathrm{mi}(D_N) \leq m \leq \mathrm{Mi}(D_N)} \frac{1}{N} \sum_{z_1 \in D_N \cap (\mathbb{R} + im)} \frac{1}{N} \sum_{z_2 \in D_N \cap (\mathbb{R} + i(\mathrm{mi} + \lceil \varepsilon \rceil_N))} (N H_{D_N \cap S_{m-N^{-1}, m+\lceil \varepsilon \rceil_N}}(z_1, z_2)) H_{D_N \cap \mathbb{H}_m}(z_2, z_1),$$

where $H_{D_N \cap S_{m-N^{-1}, m+\lceil \varepsilon \rceil_N}}(z_1, z_2)$ and $H_{D_N \cap \mathbb{H}_m}(z_2, z_1)$ are the discrete Poisson kernels (4.33) in $D_N \cap S_{m-N^{-1}, m+\lceil \varepsilon \rceil_N}$, respectively $D_N \cap \mathbb{H}_m$. Conditionally on $\#\mathcal{L}^\theta_{D_N,\varepsilon}$, the loops in $\mathcal{L}^\theta_{D_N,\varepsilon}$ are i.i.d. Moreover, for each $\wp \in \mathcal{L}^\theta_{D_N,\varepsilon}$, the joint law of $(z_\perp, z_\varepsilon, \wp_{\varepsilon,1}, \wp_{\varepsilon,2})$ can be described as follows:

1. *Conditionally on $(z_\perp, z_\varepsilon)$, $\wp_{\varepsilon,1}$ and $\wp_{\varepsilon,2}$ are two independent trajectories distributed according to*

$$\mu^{z_\perp, z_\varepsilon}_{D_N \cap S_{m-N^{-1}, m+\lceil \varepsilon \rceil_N}} / H_{D_N \cap S_{m-N^{-1}, m+\lceil \varepsilon \rceil_N}}(z_\perp, z_\varepsilon) \quad \text{and} \quad \mu^{z_\varepsilon, z_\perp}_{D_N \cap \mathbb{H}_m} / H_{D_N \cap \mathbb{H}_m}(z_\varepsilon, z_\perp) \tag{4.201}$$

   respectively, where $\mu^{z_\perp, z_\varepsilon}_{D_N \cap S_{m-N^{-1}, m+\lceil \varepsilon \rceil_N}}$ and $\mu^{z_\varepsilon, z_\perp}_{D_N \cap \mathbb{H}_m}$ follow the definition (4.32).

2. *The joint law of $(z_\perp, z_\varepsilon)$ is given by: for all $z_1, z_2 \in D_N$, $\mathbb{P}((z_\perp, z_\varepsilon) = (z_1, z_2))$ is equal to*

$$\frac{1}{Z} \mathbf{1}_{\{z_1, z_2 \in D_N, \mathrm{Im}(z_2) = \mathrm{Im}(z_1) + \lceil \varepsilon \rceil_N\}} (N H_{D_N \cap S_{m-N^{-1}, m+\lceil \varepsilon \rceil_N}}(z_1, z_2)) H_{D_N \cap \mathbb{H}_m}(z_2, z_1), \tag{4.202}$$

   with $m = \mathrm{Im}(z_1)$.

Proof. This is equivalent to saying that the concatenation $\wp_1 \wedge \wp_2$ under the measure

$$\frac{1}{N^2} \sum_{\mathrm{mi}(D_N) \leq m \leq \mathrm{Mi}(D_N)} \sum_{\substack{z_1 \in D_N \cap (\mathbb{R}+im) \\ z_2 \in D_N \cap (\mathbb{R}+i(\mathrm{mi}+\lceil \varepsilon \rceil_N))}} \mu^{z_1,z_2}_{D_N \cap S_{m-N^{-1}, m+\lceil \varepsilon \rceil_N}}(d\wp_1) \mu^{z_2,z_1}_{D_N \cap \mathbb{H}_m}(d\wp_2) \tag{4.203}$$

corresponds, up to rerooting of loops, to the measure on loops $\mu^{\mathrm{loop}}_{D_N}$ restricted to the loops γ with $h(\wp) \geq \varepsilon$ and that travel only once back and forth between $\mathbb{R} + i\,\mathrm{mi}(\wp)$ and $\mathbb{R} + i(\mathrm{mi}(\wp) + \lceil \varepsilon \rceil_N)$. For this, it is enough to check that the weights of the discrete skeletons of unrooted loops under this two measures coincide. Indeed, in both cases, the holding times conditionally on the discrete skeletons are i.i.d. exponential r.v.s with mean $\frac{1}{4N^2}$. Given $k \geq 2$, k even, the weight of a discrete-time nearest neighbour rooted loop of length k in D_N under $\mu^{\mathrm{loop}}_{D_N}$ is $\frac{1}{k} 4^{-k}$. So the weight of the corresponding discrete-time unrooted loop is 4^{-k}, provided the loop is aperiodic, that is to say its smallest period is k. This is simply because then the unrooted loop corresponds to k different rooted loops. Moreover, a loop that travels only once back and forth between $\mathbb{R} + i\,\mathrm{mi}(\wp)$ and $\mathbb{R} + i(\mathrm{mi}(\wp) + \lceil \varepsilon \rceil_N)$ is necessarily aperiodic. Further, the weight of a possible discrete-time path with k_1 jumps under $\mu^{z_1,z_2}_{D_N \cap S_{m-N^{-1}, m+\lceil \varepsilon \rceil_N}}$ is $N4^{-k_1}$. Similarly, the weight of a possible discrete-time path with k_2 jumps under $\mu^{z_2,z_1}_{D_N \cap \mathbb{H}_m}$ is $N4^{-k_2}$. Thus, the weight of the couple is $N^2 4^{-(k_1+k_2)}$, and $k_1 + k_2$ is the length of the loop created by concatenation. The N^2 is compensated by the N^{-2} factor in (4.203). So the weights of the discrete skeletons coincide. □

We conclude this section with a result about the convergence of the quantities appearing in Lemma 4.89 towards the quantities appearing in Lemma 4.28. In the following result, we assume that D is a bounded simply connected domain and that $(D_N)_N$ is the associated discrete approximations as in (4.9).

Lemma 4.90. *1. For all $n \geq 0$, we have*

$$\lim_{N \to \infty} \mathbb{P}\left(\# \mathcal{L}^\theta_{D_N, \varepsilon} = n \right) = \mathbb{P}\left(\# \mathcal{L}^\theta_{D, \varepsilon} = n \right). \tag{4.204}$$

2. Let $(z^N_\perp, z^N_\varepsilon)$ and $(z_\perp, z_\varepsilon)$ be distributed according to the laws (4.202) and (4.60), respectively. Then

$$(z^N_\perp, z^N_\varepsilon) \xrightarrow[N \to \infty]{(d)} (z_\perp, z_\varepsilon). \tag{4.205}$$

3. Let $\wp^N$ and $\wp$ be distributed according to the laws described in Lemmas 4.89 and 4.28, respectively. Then

$$T(\wp^N) \xrightarrow[N \to \infty]{(d)} T(\wp). \tag{4.206}$$

Proof. (4.204) and (4.205) follow from Lemmas 4.89 and 4.28 and from the convergence of the discrete Poisson kernel towards its continuum analogue. Alternatively, these two claims follow from the convergence in distribution of $\mathcal{L}^\theta_{D_N,\varepsilon}$ towards $\mathcal{L}^\theta_{D,\varepsilon}$ for the topology induced by $d_{\mathfrak{L}}$ (4.29). This latter fact is a direct consequence of the coupling of [LTF07] between random walk loop soup and Brownian loop soup. We omit the details. To prove (4.206), one only needs to notice that the law of $T(\wp^N)$ is given by the law of the total duration of $\mathcal{L}^\theta_{D_N,\varepsilon}$ conditioned on $\# \mathcal{L}^\theta_{D_N,\varepsilon} = 1$. The same holds for the

Brownian loop soup. Therefore, (4.206) follows from the joint convergence of $\#\mathcal{L}^\theta_{D_N,\varepsilon}$ and the total duration of $\mathcal{L}^\theta_{D_N,\varepsilon}$ which is again a consequence of [LTF07]. $\square$

4.13.2 Proof of Proposition 4.58

We now have all the ingredients for the proof of Proposition 4.58.

Proof of Proposition 4.58. We will focus on the convergence of the measure $\mathcal{M}_a^{N,K}$ towards its continuum analogue $\mathcal{M}_a^K$. Indeed, since Theorem 4.88 also takes care of the joint convergence of the trajectories, it is not difficult to extend our proof to the joint convergence of the measure $\mathcal{M}_a^{N,K}$ together with the killed loops $\mathcal{L}^\theta_{D_N}(K)$.

Let $\varepsilon > 0$. We first restrict $\mathcal{L}^\theta_{D_N}(K)$ to the loops with height larger than ε: recall the definition (4.199) of $\mathcal{L}^\theta_{D_N,\varepsilon}$ and recall that loops $\wp$ in $\mathcal{L}^\theta_{D_N,\varepsilon}$ are naturally split into two trajectories $\wp_{\varepsilon,1}$ and $\wp_{\varepsilon,2}$ (see (4.200)). The first part $\wp_{\varepsilon,1}$ becomes negligible as $\varepsilon \to 0$. Therefore, we will not loose much by only looking at the second part and we define the following measure: for all Borel set A,

$$\mathcal{M}_a^{N,K,\varepsilon}(A) := \frac{\log N}{N^{2-a}} \sum_{z \in D_N} \mathbf{1}_{\{z\in A\}} \mathbf{1}_{\left\{ \sum_{\wp \in \mathcal{L}^\theta_{D_N,\varepsilon} \cap \mathcal{L}^\theta_{D_N}(K)} \ell_z(\wp_{\varepsilon,2}) \geq \frac{a}{2\pi} (\log N)^2 \right\}}.$$

This definition is very close to the one without the restriction on the height; see (4.132) and (4.133). In (4.61) we define an analogous measure $\mathcal{M}_a^{K,\varepsilon}$ in the continuum. The main part of the proof is to show that for any nondecreasing bounded continuous function $g : [0,\infty) \to \mathbb{R}$ and any nonnegative bounded continuous function $f : D \to [0,\infty)$,

$$\liminf_{N\to\infty} \mathbb{E}\left[g\left(\left\langle \mathcal{M}_a^{N,K,\varepsilon}, f \right\rangle \right) \right] \geq \mathbb{E}\left[g\left(c_0^a \left\langle \mathcal{M}_a^{K,\varepsilon}, f \right\rangle \right) \right]. \tag{4.207}$$

Let us assume that (4.207) holds and let us explain how Proposition 4.58 follows. Firstly, Corollary 4.64 shows that

$$\sup_{N\geq 1} \mathbb{E}\left[\mathcal{M}_a^{N,K}(D) \right] < \infty$$

implying tightness of $\left(\mathcal{M}_a^{N,K}, N \geq 1 \right)$ for the topology of weak convergence (see e.g. [Kal73, Lemma 1.2] for an analogous statement concerning the topology of vague convergence). Let $\mathcal{M}_a^{\infty,K}$ be any subsequential limit. By first extracting a subsequence, we can assume without loss of generality that $\left(\mathcal{M}_a^{N,K}, N \geq 1 \right)$ converges in distribution towards $\mathcal{M}_a^{\infty,K}$. To conclude, we need to show that $\mathcal{M}_a^{\infty,K} \overset{(d)}{=} c_0^a \mathcal{M}_a^K$. To this end, it is enough to show that, for any nonnegative bounded continuous function $f : D \to [0,\infty)$, $\left\langle \mathcal{M}_a^{\infty,K}, f \right\rangle$ and $c_0^a \left\langle \mathcal{M}_a^K, f \right\rangle$ have the same distribution (see e.g. [Kal73, Lemma 1.1] for a similar statement for the topology of vague convergence). Let f be such a function, $g : [0,\infty) \to \mathbb{R}$ be a bounded nondecreasing function and let $\varepsilon > 0$. By first using the convergence in distribution of $\left\langle \mathcal{M}_a^{N,K}, f \right\rangle$ towards $\left\langle \mathcal{M}_a^{\infty,K}, f \right\rangle$, then by using monotonicity of g and finally by exploiting (4.207), we have

$$\mathbb{E}\left[g\left(\left\langle \mathcal{M}_a^{\infty,K}, f \right\rangle \right) \right] = \lim_{N\to\infty} \mathbb{E}\left[g\left(\left\langle \mathcal{M}_a^{N,K}, f \right\rangle \right) \right] \geq \liminf_{N\to\infty} \mathbb{E}\left[g\left(\left\langle \mathcal{M}_a^{N,K,\varepsilon}, f \right\rangle \right) \right] \geq \mathbb{E}\left[g\left(c_0^a \left\langle \mathcal{M}_a^{K,\varepsilon}, f \right\rangle \right) \right].$$

By definition of $\mathcal{M}_a^K$ (see Definition 4.29), $\left\langle \mathcal{M}_a^{K,\varepsilon}, f \right\rangle$ converges a.s. to $\left\langle \mathcal{M}_a^K, f \right\rangle$ as $\varepsilon \to 0$. Hence

$$\mathbb{E}\left[g\left(\left\langle \mathcal{M}_a^{\infty,K}, f \right\rangle \right) \right] \geq \mathbb{E}\left[g\left(c_0^a \left\langle \mathcal{M}_a^K, f \right\rangle \right) \right].$$

Since this is valid for all nondecreasing bounded continuous function g, we deduce that $\left\langle \mathcal{M}_a^{\infty,K}, f \right\rangle$ stochastically dominates $c_0^a \left\langle \mathcal{M}_a^K, f \right\rangle$. Because their expectations agree (Corollary 4.64 and Proposition 4.21), they must have the same distribution. This shows the expected convergence $\mathcal{M}_a^{N,K} \to c_0^a \mathcal{M}_a^K$.

Next, we move on to the proof of (4.207). By conditioning on the number of loops in $\mathcal{L}^\theta_{D_N,\varepsilon}$ and by Fatou's lemma, we have

$$\liminf_{N\to\infty} \mathbb{E}\left[g\left(\left\langle \mathcal{M}_a^{N,K,\varepsilon}, f \right\rangle \right) \right] \geq \sum_{n=0}^\infty \liminf_{N\to\infty} \mathbb{P}\left(\#\mathcal{L}^\theta_{D_N,\varepsilon} = n \right) \mathbb{E}\left[g\left(\left\langle \mathcal{M}_a^{N,K,\varepsilon}, f \right\rangle \right) | \#\mathcal{L}^\theta_{D_N,\varepsilon} = n \right].$$

The claim (4.204) in Lemma 4.90 shows that for all $n \geq 0$, $\mathbb{P}\left(\#\mathcal{L}^\theta_{D_N,\varepsilon} = n \right)$ converges as $N \to \infty$ to its analogue in the continuum and it remains to show that

$$\liminf_{N\to\infty} \mathbb{E}\left[g\left(\left\langle \mathcal{M}_a^{N,K,\varepsilon}, f \right\rangle \right) | \#\mathcal{L}^\theta_{D_N,\varepsilon} = n \right] \geq \mathbb{E}\left[g\left(c_0^a \left\langle \mathcal{M}_a^{K,\varepsilon}, f \right\rangle \right) | \#\mathcal{L}^\theta_{D,\varepsilon} = n \right]. \tag{4.208}$$

Fix $n \geq 1$. Let $\wp^{i,N}, i = 1 \dots n$, be i.i.d. loops so that $\mathcal{L}^\theta_{D_N,\varepsilon}$, conditioned on $\#\mathcal{L}^\theta_{D_N,\varepsilon} = n$, has the same distribution as $\{\wp^{1,N}, \dots, \wp^{n,N}\}$ (see Lemma 4.89). We split these loops into two pieces $\wp^{i,N}_{\varepsilon,1}$ and $\wp^{i,N}_{\varepsilon,2}$ as in (4.200). Let $U_i, i = 1 \dots n$, be i.i.d. uniform random variables on $[0,1]$ that are independent of the loops above. By checking which loops are killed (in the next display, I corresponds to the set of indices of killed loops), we can rewrite the expectation on the left hand side of (4.208) as

$$\begin{aligned} &\sum_{I \subset \{1,\dots,n\}} \prod_{i \notin I} \mathbb{E}\left[e^{-KT(\wp^{i,N})} \right] \mathbb{E}\left[g\left(\left\langle \mathcal{M}_a^{\wp^{i,N}_{\varepsilon,2}, i \in I}, f \right\rangle \right) \mathbf{1}_{\left\{ \forall i \in I, U_i < 1 - e^{-KT(\wp^{i,N})} \right\}} \right] \\ &= \sum_{k=0}^n \binom{n}{k} \mathbb{E}\left[e^{-KT(\wp^{1,N})} \right]^{n-k} \mathbb{E}\left[g\left(\left\langle \mathcal{M}_a^{\wp^{1,N}_{\varepsilon,2}, \dots, \wp^{k,N}_{\varepsilon,2}}, f \right\rangle \right) \mathbf{1}_{\left\{ \forall i = 1 \dots k, U_i < 1 - e^{-KT(\wp^{i,N})} \right\}} \right] \end{aligned} \tag{4.209}$$

with the convention that, when $k = 0$, the last expectation equals 1 and with, for all $k = 1 \dots n$,

$$\mathcal{M}_a^{\wp^{1,N}_{\varepsilon,2}, \dots, \wp^{k,N}_{\varepsilon,2}}(A) := \frac{\log N}{N^{2-a}} \sum_{z \in D_N \cap A} \mathbf{1}_{\left\{ \sum_{i=1}^k \ell_z(\wp^{i,N}_{\varepsilon,2}) \geq \frac{a}{2\pi} (\log N)^2 \right\}}, \qquad A \text{ Borel set.} \tag{4.210}$$

The measure above differs from the measures introduced in (4.198) since it does not require all the trajectories to visit the point z. By looking at the subset $I \subset \{1, \dots, k\}$ of loops that actually contribute to the thickness, we see that they are related by

$$\mathcal{M}_a^{\wp^{1,N}_{\varepsilon,2}, \dots, \wp^{k,N}_{\varepsilon,2}} = \sum_{I \subset \{1,\dots,k\}} \mathcal{M}_a^{\cap_{i \in I} \wp^{i,N}_{\varepsilon,2}}. \tag{4.211}$$

Let us come back to the analysis of the asymptotics of (4.209). By (4.206) we already have the convergence of $\mathbb{E}\left[e^{-KT(\wp^{1,N})} \right]$ towards $\mathbb{E}\left[e^{-KT(\wp)} \right]$ where $\wp$ is distributed according to (4.59). In Lemma 4.91 below, we show that a consequence of Theorem 4.88 is that the liminf of the second expectation in (4.209) is at least

$$\mathbb{E}\left[g\left(c_0^a \left\langle \mathcal{M}_a^{\wp_{2,\varepsilon}^1,\dots,\wp_{2,\varepsilon}^k}, f \right\rangle \right) \mathbf{1}_{\left\{\forall i=1\dots k, U_i<1-e^{-KT(\wp^i)}\right\}} \right].$$

Here $\wp^i, i=1\dots k$, and $\mathcal{M}_a^{\wp_{2,\varepsilon}^1,\dots,\wp_{2,\varepsilon}^k}$ are the continuum analogues of the notations we introduced above. More precisely, $\wp^i, i=1\dots k$, are i.i.d. loops distributed according to (4.59) and

$$\mathcal{M}_a^{\wp_{\varepsilon,2}^1,\dots,\wp_{\varepsilon,2}^k} := \sum_{I\subset\{1,\dots,k\}} \mathcal{M}_a^{\cap_{i\in I}\wp_{\varepsilon,2}^i} \tag{4.212}$$

where $\mathcal{M}_a^{\cap_{i\in I}\wp_{\varepsilon,2}^i}$ is the Brownian chaos associated to $\wp_{\varepsilon,2}^i, i\in I$; see Section 4.A. Wrapping things up, we have obtained that the liminf of the left hand side of (4.208) is at least

$$\sum_{k=0}^n \binom{n}{k} \mathbb{E}\left[e^{-KT(\wp^1)}\right]^{n-k} \mathbb{E}\left[g\left(c_0^a \left\langle \mathcal{M}_a^{\wp_{2,\varepsilon}^1,\dots,\wp_{2,\varepsilon}^k}, f \right\rangle \right) \mathbf{1}_{\left\{\forall i=1\dots k, U_i<1-e^{-KT(\wp^i)}\right\}} \right].$$

By reversing the above line of argument (which is possible thanks to Lemma 4.28), we see that this is exactly the right hand side of (4.208). It concludes the proof. □

We finish this section by stating and proving Lemma 4.91. As in the proof of Proposition 4.58, we will consider two sets of i.i.d. loops $\wp^{i,N}, i=1\dots n$, and $\wp^i, i=1\dots n$, in the discrete and in the continuum respectively, as well as their associated measures $\mathcal{M}_a^{\wp_{\varepsilon,2}^{1,N},\dots,\wp_{\varepsilon,2}^{n,N}}$ and $\mathcal{M}_a^{\wp_{\varepsilon,2}^1,\dots,\wp_{\varepsilon,2}^n}$ defined respectively in (4.210) and (4.212). Let also $U_i, i=1\dots n$, be i.i.d. uniform random variables on $[0,1]$ that are independent of the loops above.

Lemma 4.91. *Let $f: D\to[0,\infty)$ be a nonnegative continuous function and $g:[0,\infty)\to\mathbb{R}$ be a nondecreasing bounded continuous function. Then,*

$$\liminf_{N\to\infty} \mathbb{E}\left[g\left(\left\langle \mathcal{M}_a^{\wp_{2,\varepsilon}^{1,N},\dots,\wp_{2,\varepsilon}^{n,N}}, f \right\rangle \right) \mathbf{1}_{\left\{\forall i=1\dots n, U_i<1-e^{-KT(\wp^{i,N})}\right\}} \right] \tag{4.213}$$

$$\geq \mathbb{E}\left[g\left(c_0^a \left\langle \mathcal{M}_a^{\wp_{2,\varepsilon}^1,\dots,\wp_{2,\varepsilon}^n}, f \right\rangle \right) \mathbf{1}_{\left\{\forall i=1\dots n, U_i<1-e^{-KT(\wp^i)}\right\}} \right]. \tag{4.214}$$

Proof of Lemma 4.91. To ease notations, we will assume that $n=1$. The general case follows from similar arguments. In particular, note that the convergence of the Brownian chaos measures in Theorem 4.88 holds jointly for any number of trajectories. In what follows, we will denote $(z_\perp^N, z_\varepsilon^N, \wp_{\varepsilon,1}^N, \wp_{\varepsilon,2}^N)$, resp. $(z_\perp, z_\varepsilon, \wp_{\varepsilon,1}, \wp_{\varepsilon,2})$, a random element whose law is described in Lemma 4.89, resp. in Lemma 4.28. We also consider a uniform random variable U on $[0,1]$ independent of all the variables above.

The expectation in (4.213) is equal to

$$\sum_{\tilde z_\perp^N, \tilde z_\varepsilon^N \in D_N} \mathbb{P}\left((z_\perp^N, z_\varepsilon^N) = (\tilde z_\perp^N, \tilde z_\varepsilon^N) \right) \mathbb{E}\left[g\left(\left\langle \mathcal{M}_a^{\wp_{\varepsilon,2}^N}, f \right\rangle \right) \mathbf{1}_{\left\{U<1-e^{-KT(\wp^N)}\right\}} \middle| (z_\perp^N, z_\varepsilon^N) = (\tilde z_\perp^N, \tilde z_\varepsilon^N) \right]. \tag{4.215}$$

Let us fix $\tilde z_\perp, \tilde z_\varepsilon \in D$ and denote $\tilde z_\perp^N = N^{-1}\lfloor N\tilde z_\perp \rfloor$ and $\tilde z_\varepsilon^N = N^{-1}\lfloor N\tilde z_\varepsilon \rfloor$. Assume that the event

$E_N := \{(z_\perp^N, z_\varepsilon^N) = (\tilde z_\perp^N, \tilde z_\varepsilon^N)\}$ has positive probability. By Lemma 4.89, conditioned on this event, $\wp_{1,\varepsilon}^N$ and $\wp_{2,\varepsilon}^N$ are independent random walk trajectories distributed according to (4.201). By Theorem 4.88, the joint law of $(\mathcal{M}_a^{\wp_{\varepsilon,2}^N}, T(\wp_{\varepsilon,2}^N))$ conditioned on E_N converges weakly towards the joint law of $(c_0^a \mathcal{M}_a^{\wp_{\varepsilon,2}}, T(\wp_{\varepsilon,2}))$ conditioned on $E := \{(z_\perp, z_\varepsilon) = (\tilde z_\perp, \tilde z_\varepsilon)\}$. The topology considered is the product topology with, on the one hand, the topology of vague convergence of measures on $D(\tilde z_\perp) := \{z \in D : \mathrm{Im}(z) > \mathrm{Im}(\tilde z_\perp)\}$ and, on the other hand, the standard Euclidean topology on $\mathbb{R}$. Because of this topology, we introduce for any $\delta > 0$ a bounded continuous function $f_\delta : D \to [0,\infty)$ which coincide with f on $\{z \in D(\tilde z) : \mathrm{dist}(z, \mathbb{C} \setminus D(\tilde z_\perp)) > \delta\}$ and which has a support compactly included in $D(\tilde z_\perp)$. We choose f_δ in such a way that $f \geq f_\delta$. Since the support of f_δ is a compact subset of $D(\tilde z_\perp)$, we will be able to use the convergence of the measures integrated against f_δ.

By conditional independence of $\wp_{\varepsilon,1}^N$ and $\wp_{\varepsilon,2}^N$ (and of $\wp_{\varepsilon,1}$ and $\wp_{\varepsilon,2}$), we can add a third component and we have the joint convergence of $(\mathcal{M}_a^{\wp_{\varepsilon,2}^N}, T(\wp_{\varepsilon,2}^N), T(\wp_{\varepsilon,1}^N))$. We add this third component because we are interested in the total duration $T(\wp^N) = T(\wp_{\varepsilon,1}^N) + T(\wp_{\varepsilon,2}^N)$. Overall, this shows that for all $\delta > 0$,

$$\begin{aligned}
&\liminf_{N\to\infty} \mathbb{E}\left[g\left(\left\langle \mathcal{M}_a^{\wp_{\varepsilon,2}^N}, f \right\rangle\right) \mathbf{1}_{\left\{U < 1 - e^{-KT(\wp^N)}\right\}} \middle| (z_\perp^N, z_\varepsilon^N) = (\tilde z_\perp^N, \tilde z_\varepsilon^N) \right] \\
&\geq \liminf_{N\to\infty} \mathbb{E}\left[g\left(\left\langle \mathcal{M}_a^{\wp_{\varepsilon,2}^N}, f_\delta \right\rangle\right) \mathbf{1}_{\left\{U < 1 - e^{-KT(\wp^N)}\right\}} \middle| (z_\perp^N, z_\varepsilon^N) = (\tilde z_\perp^N, \tilde z_\varepsilon^N) \right] \\
&= \mathbb{E}\left[g\left(c_0^a \left\langle \mathcal{M}_a^{\wp_{\varepsilon,2}}, f_\delta \right\rangle\right) \mathbf{1}_{\left\{U < 1 - e^{-KT(\wp)}\right\}} \middle| (z_\perp, z_\varepsilon) = (\tilde z_\perp, \tilde z_\varepsilon) \right].
\end{aligned}$$

Since $\left\langle \mathcal{M}_a^{\wp_{\varepsilon,2}}, f_\delta \right\rangle \to \left\langle \mathcal{M}_a^{\wp_{\varepsilon,2}}, f \right\rangle$ as $\delta \to 0$ in L^1 (see Remark 4.92 below), we have obtained

$$\begin{aligned}
&\liminf_{N\to\infty} \mathbb{E}\left[g\left(\left\langle \mathcal{M}_a^{\wp_{\varepsilon,2}^N}, f \right\rangle\right) \mathbf{1}_{\left\{U < 1 - e^{-KT(\wp^N)}\right\}} \middle| (z_\perp^N, z_\varepsilon^N) = (\tilde z_\perp^N, \tilde z_\varepsilon^N) \right] \\
&\geq \mathbb{E}\left[g\left(c_0^a \left\langle \mathcal{M}_a^{\wp_{\varepsilon,2}}, f \right\rangle\right) \mathbf{1}_{\left\{U < 1 - e^{-KT(\wp)}\right\}} \middle| (z_\perp, z_\varepsilon) = (\tilde z_\perp, \tilde z_\varepsilon) \right].
\end{aligned}$$

Moreover, by (4.205), $(z_\perp^N, z_\varepsilon^N)$ converges in distribution towards $(z_\perp, z_\varepsilon)$. One can then use an approach similar to the one used in [Jeg19] (see especially Lemma 3.6 therein) to deduce that the liminf of (4.215) is at least

$$\int_{D\times D} \mathbb{P}\left((z_\perp, z_\varepsilon) = (d\tilde z_\perp, d\tilde z_\varepsilon)\right) \mathbb{E}\left[g\left(c_0^a \left\langle \mathcal{M}_a^{\wp_{\varepsilon,2}}, f \right\rangle\right) \mathbf{1}_{\left\{U < 1 - e^{-KT(\wp)}\right\}} \middle| (z_\perp, z_\varepsilon) = (\tilde z_\perp, \tilde z_\varepsilon) \right].$$

We omit the details. This concludes the proof since the last display is equal to the expectation in (4.214). □

Remark 4.92. In the above proof, we had to consider a function f_δ whose support was compactly included in the underlying domain. We then got rid of this function by letting $\delta \to 0$ and arguing that $\left\langle \mathcal{M}_a^{\wp_{\varepsilon,2}}, f_\delta \right\rangle \to \left\langle \mathcal{M}_a^{\wp_{\varepsilon,2}}, f \right\rangle$ in L^1. This is justified by the simple fact that the first moment of the measure (see (1.4) in [Jeg19]), evaluated against a set located at a distance at most δ from the boundary of the domain, vanishes as $\delta \to 0$. In the discrete, because of poorer estimates on the discrete Poisson kernel, these estimates near the boundary are not as clear and this is why the convergence obtained in [Jeg19] is stated for the topology of vague (instead of weak) convergence. We mention nevertheless that these difficulties might very well be overcome for a flat portion of the boundary, which is the case in the setting of the current article. But our point is that this is not needed.

Appendix 4.B Green function

In this section, we briefly recall the behaviour of the Green function in the discrete setting. The Euler–Mascheroni constant γ_{EM} will appear in the asymptotics of the discrete Green function and we recall that it is defined by

$$\gamma_{\mathrm{EM}} = \lim_{n \to +\infty} \Big(-\log(n) + \sum_{1 \le k \le n} \frac{1}{k} \Big). \tag{4.216}$$

Lemma 4.93. *There exists $C > 0$ such that for all $z, z' \in D_N$,*

$$G_{D_N}(z, z') \le \frac{1}{2\pi} \log \max \left(N, \frac{1}{|z - z'|} \right) + C. \tag{4.217}$$

For all set A compactly included in D, there exists $C = C(A) > 0$ such that for all $z, z' \in A \cap D_N$,

$$G_{D_N}(z, z') \ge \frac{1}{2\pi} \log \max \left(N, \frac{1}{|z - z'|} \right) - C. \tag{4.218}$$

For all $z \in D$, if we denote z_N a point in D_N closest to z, then

$$\lim_{N \to \infty} G_{D_N}(z_N, z_N) - \frac{1}{2\pi} \log N = \frac{1}{2\pi} \log \mathrm{CR}(z, D) + \frac{1}{2\pi} \left(\gamma_{\mathrm{EM}} + \frac{1}{2} \log 8 \right). \tag{4.219}$$

Proof. (4.217) and (4.218) are direct consequences of Theorem 4.4.4 and Proposition 4.6.2 of [LL10]. (4.219) can be found for instance in Theorem 1.17 of [Bis20]. Note that the constant $\frac{1}{2\pi} \left(\gamma_{\mathrm{EM}} + \frac{1}{2} \log 8 \right)$ is the constant order term in the expansion of the 0-potential on $\mathbb{Z}^2$; see [LL10, Theorem 4.4.4]. We emphasise that in the current paper $G_{D_N}(z, z)$ blows up like $\frac{1}{2\pi} \log N$ whereas in [LL10] and [Bis20], $G_{D_N}(z, z)$ blows up like $\frac{2}{\pi} \log N$, hence the difference of factor 4 between our setting and theirs. ☐

Appendix 4.C Special functions

In this section, we recall the definition and list a few properties of some special functions that appear in the current paper.

- Gamma function:

$$\Gamma(x) = \int_0^\infty \frac{1}{t^{1-x}} e^{-t} \mathrm{d}t, \quad x > 0. \tag{4.220}$$

When $x = 1/2$,

$$\Gamma(1/2) = \sqrt{\pi}. \tag{4.221}$$

- The Beta function is related to the Gamma function as follows:

$$\int_0^1 t^{x-1} (1 - t)^{y-1} \mathrm{d}t = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x + y)}, \quad x, y > 0. \tag{4.222}$$

- Modified Bessel function of the first kind:

$$I_\alpha(x) = \sum_{n=0}^{\infty} \frac{1}{n!\Gamma(n+\alpha+1)} \left(\frac{x}{2}\right)^{2n+\alpha}, \quad x > 0,\ \alpha > -1. \tag{4.223}$$

Using Legendre duplication formula $\Gamma(x)\Gamma(x+1/2) = 2^{1-2x}\sqrt{\pi}\Gamma(2x)$, we see that when $\alpha = -1/2$,

$$I_{-1/2}(x) = \sqrt{\frac{2}{\pi}} \frac{1}{\sqrt{x}} \cosh(x). \tag{4.224}$$

In general, for all $\alpha > -1$,

$$I_\alpha(x) \sim \frac{e^x}{\sqrt{2\pi x}} \quad \text{as } x \to \infty. \tag{4.225}$$

- Kummer's confluent hypergeometric function:

$${}_1F_1(\theta, 1, x) = 1 + \sum_{n \geq 1} \frac{\theta(\theta+1)\dots(\theta+n-1)}{n!^2} x^n, \quad x \geq 0,\ \theta > 0. \tag{4.226}$$

For any $\theta > 0$,

$${}_1F_1(\theta, 1, x) \sim \frac{1}{\Gamma(\theta)} e^x x^{\theta-1} \quad \text{as } x \to \infty. \tag{4.227}$$

See [AS84, Section 13.5].

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