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A HYPERBOLIC PROOF OF PASCAL'S THEOREM

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Abstract. We provide a simple proof of Pascal's Theorem on cyclic hexagons, as well as a generalization by Möbius, using hyperbolic geometry.

1. John Conway and Pascal's Theorem

Among the many eclectic interests of Conway figures the classical topic of incidence theorems in projective geometry. Together with Alex Ryba, Conway wrote two papers [1, 2] on what they called the *Pascal Mysticum*. The Pascal Mysticum stems from a family of 6 distinct points on an ellipse. Each pair A, B of points defines a line AB, two pairs of points A, B and C, D define two lines that intersect at a point $AB \cdot CD$. Pascal's theorem below indicates that if A, B, C, D, E and F are the 6 points considered on an ellipse, then $AB \cdot CD$, $AB \cdot EF$ and $CD \cdot EF$ are on a line. The different permutations of the 6 points therefore give rise in this manner to 60 different "Pascal line". But those Pascal lines themselves have remarkable incidence properties: Steiner proved in 1828 that the 60 Pascal lines intersect by group of three in 20 "Steiner nodes", and the next year Plücker proved that those Steiner nodes lie in groups of four on 15 "Plücker lines". The description of the incidence relations stemming from the Pascal lines was further expanded by contributions by Kirkman, Cayley and Salmon (see [1] for a complete description and suitable references). The *Pascal Mysticum*, or *mysticum hexagrammaticum*, is this family of 95 lines and 95 points exhibiting those intricate yet beautiful incidence relations.

Conway and Ryba provide in [1] a full description of the incidence relations of the 95 lines and 95 points associated to 6 distinct points on an ellipse. They introduce a beautifully crafted notation for those lines and points, together with short and self-contained proofs of their incidence properties. In [2], they further extend their analysis of the Pascal Mysticum and discover (or sometimes rediscover) additional striking properties, for instance subfamilies with pentagonal or heptagonal symmetry.

As already mentioned, the Pascal Mysticum arises from repeated applications of the Pascal Theorem, which we can now state more formally. Pascal called this theorem the *hexagrammum mysticum*, one of the motivations for the name *mysticum hexagrammaticum* given by Conway and Ryba to the whole configurations of lines and points it gives rise to.

Theorem A (Pascal's Theorem). *Let $ABCDEF$ be a cyclic hexagon. Let X be the intersection point of AB and DE, Y the intersection point of BC and EF and Z the intersection point of CD and FA. Then X, Y and Z are aligned.*

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Blaise Pascal (1623–1662) is a towering intellectual figure of the XVII$^{\text{th}}$ century. He is credited with inventing and building the first mechanical calculator, the *Pascaline*, and with laying the foundations of probability theory, in particular in his correspondence with Fermat – he came up for instance with Pascal's triangle. He is also known for his work on hydrostatics and Pascal's law (as well as the invention of the syringe), and for discovering the variation of air pressure with altitude – the SI unit of pressure is called the Pascal. However, he was most influential in his time as a philosopher and theologian, and well-known for "Pascal's wager".

Pascal was raised and educated by his father, Etienne Pascal, who had a strong interest in the intellectual developments of his time and was an active member of a group of scientists meeting around Martin Mersenne, including Desargues, Descartes and others. According to contemporary sources [8, p. 176], Blaise was extraordinarily precocious, and so passionate in studying mathematics that, when he was 11, his father forbid him to read any mathematics book before he turned 15 and knew latin and greek. Blaise therefore continued studying geometry by himself and in secret and, at 16, published his first article on projective geometry, the *Essay pour les coniques* [6], which contained Theorem A.

The main goal of this note is to provide a simple proof, based on hyperbolic geometry, of Pascal's Theorem, and of an extension of this theorem discovered by Möbius two centuries later (see Theorem E below).

Theorem A has a natural setting in the projective plane. Instead of considering a cyclic hexagon, one then considers a hexagon with vertices on a conic. Any non-degenerate conic is projectively equivalent to a circle, while the statements for degenerate conics can be obtained by a limiting argument where a hexagon with vertices on a degenerate conic is obtained as a limit of hexagons with vertices on non-degenerate conics.

There are many proofs of Theorem A, using a wide variety of tools. A proof using algebraic geometry is sketched by Conway and Ryba in [1]. Other proofs involve cross-ratios and symmetries of the projective plane, as in [7] or Euclidean lengths and Menalaus's theorem like in [3, p.77].

The proof given here, based on hyperbolic geometry, is not really novel (it can be deduced easily from [7, page 436]) but it brings to light a striking link between Pascal's theorem and elementary hyperbolic geometry. A recent and similar link is exhibited by Drach and Schwarz in [4], where they revisit the seven-circles theorem, in Euclidean geometry, in terms of hyperbolic geometry. Even if those links between hyperbolic geometry and results on projective or Euclidean geometry do not lead to novel results, they give a beautiful perspective to them.

2. A hyperbolic statement of Pascal's Theorem

We are going to use the Klein model of the hyperbolic plane. Consider an open disk Δ on the projective plane, bounded by a circle Γ. The *hyperbolic plane* is defined as Δ, endowed with the *Hilbert distance*, defined as follows

Definition. Let $P, Q \in \Delta$, and A, B be the intersection points of the line PQ with Γ. Then

$$d(P,Q) = \frac{1}{2}\log\left(\frac{BP}{BQ}\cdot\frac{AQ}{AP}\right).$$

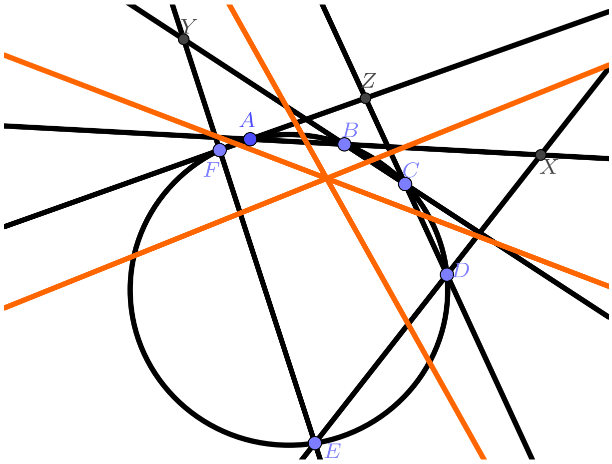

FIGURE 1. The configuration of Pascal's theorem (Theorem A) and Proposition B.

This distance induces a complete Riemannian metric on Δ, and thus notions of angles and lengths. A key property of this Hilbert distance is that it is invariant under projective transformations that leave Δ invariant. The geodesics are precisely the straight lines, but the angles are not the Euclidean ones. The circle Γ is then the *boundary at infinity of the hyperbolic plane*, and its points are called *ideal points*. The model is very rich, but we focus here only on a few points related to the polarity in the projective plane and orthogonality in the hyperbolic plane.

Given a circle in the projective plane, one can define a polarity relation between points and lines: there is a *polar line* for each point, and a *pole* for each line. By definition, given points $P \in \Delta$ and $Q \notin \bar{\Delta}$, Q is in the polar line of P, and conversely, if and only if

$$\frac{BP}{BQ} \cdot \frac{AQ}{AP} = -1 \ ,$$

where A and B are again the intersections of the line PQ with Γ. We will denote the polar line of P by P^*. The polarity relation is also invariant under projective transformations that leave Δ invariant.

It follows from the definition that if $P, Q \in \Delta$, then P and Q are both in the polar line of $P^* \cap Q^*$, the intersection point of the polar lines P^* and Q^*. It follows that the line PQ is the polar line of $P^* \cap Q^*$. As a consequence, the polarity relation preserves the incidence: three points are aligned if and only if their polar lines are concurrent.

This last fact allows to have *dual* statements. For example, the dual statement of Pascal's theorem is Brianchon's theorem: if a conic is inscribed in a hexagon, then the three diagonals joining the opposite vertices of the hexagon are concurrent.

Back to hyperbolic geometry, the link between polarity and the Klein model that we are going to use is the following:

Proposition. *Two lines l_1 and l_2 in the hyperbolic plane are orthogonal if and only if the pole of l_1 is contained in the extension of l_2 to the projective plane.*

In particular, if l_1 and l_2 are two lines that do not intersect in the hyperbolic plane, they have a unique common perpendicular, that is the polar line of their intersection point in the projective plane.

The proposition follows from the projective invariance of the hyperbolic metric and of the polarity relation under projective transformations leaving Δ invariant, since one can always find such a projective transformation leaving Δ invariant and bringing the intersection of l_1 and l_2 to the center of Δ – in this case the proposition is easy to check.

We can now restate Pascal's theorem in terms of hyperbolic geometry. Considering the polar lines l_1, l_2 and l_3 of the points X, Y and Z of the statement of Theorem A, we obtain the following equivalent statement.

Proposition B. *Let $ABCDEF$ be an ideal hyperbolic hexagon. Let l_1 be the common perpendicular to AB and DE, l_2 the common perpendicular to BC and EF and l_3 the common perpendicular to CD and FA. Then l_1, l_2 and l_3 are concurrent.*

3. Proof of Proposition B

The proof of Proposition B is based on the hyperbolic version of an elementary statement on triangles.

Theorem C. *Let PQR be a triangle. Then the angle bisectors of PQR are concurrent.*

Remark. Theorem C holds for Euclidean, spherical and hyperbolic triangles. The proof is, in all three cases, elementary. A point of a triangle is in the bisector of an angle if and only if it is at equal distance from the two corresponding edges. Therefore, the intersection point of two angle bisectors is at equal distance from all three edges, and is therefore contained in the third bisector.

Lemma D. *Let $ABDE$ be an ideal hyperbolic quadrilateral. Then, the common orthogonal to AB and DE is the angle bisector of the lines AD and BE.*

Proof. Let l be the angle bisector of the lines AD and BE. The hyperbolic reflection on l exchanges A and B, so the line AB is preserved by the reflection. Thus, the angle bisector of the two lines is orthogonal to AB. Similarly, l is also perpendicular to DE, so it is precisely the common perpendicular to AB and DE. □

Proof of Proposition B. If the diagonals AD, BE and CF are concurrent at a point P, then Lemma D implies that l_1, l_2 and l_3 contain P, and are therefore concurrent.

Now, suppose that the diagonals AD, BE and CF are not concurrent. We call P the intersection point of BE and CF, Q the intersection point of AD and CF, and R the intersection point of AD and BE.

It follows from Lemma D that the angle bisector of PQR at P is the common perpendicular l_1 to AB and DE, while the angle bisector of PQR at Q is the common perpendicular l_2 to BC and EF, and the angle bisector of PQR at R is the common perpendicular l_3 to CD and FA.

By Theorem C applied to the triangle PQR, the lines l_1, l_2 and l_3 are concurrent, and the result follows. □

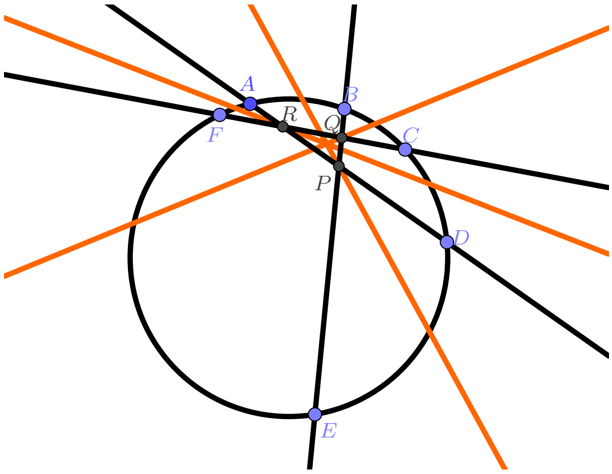

FIGURE 2. The triangle PQR and its hyperbolic angle bisectors.

4. The Möbius generalization

In 1847, Möbius proved a generalization of Pascal's Theorem for $(4n+2)$-gons [5].

Theorem E (Möbius, 1847)**.** *Let $A_1A_2\cdots A_{4n+2}$ be a cyclic $(4n+2)$-gon. Let $X_1,\dots,X_{2n+1}$ be the intersection points of the pairs of opposite sides of $A_1A_2\cdots A_{4n+2}$. If $X_1,\dots,X_{2n}$ are aligned, then X_{2n+1} lies in the same line as $X_1,\dots,X_{2n}$.*

By considering the polar lines $l_1,\dots,l_{2n+1}$ of $X_1,\dots,X_{2n+1}$, we obtain the corresponding hyperbolic statement, for which the proof of Proposition B extends easily.

Proposition F. *Let $A_1A_2\cdots A_{4n+2}$ be a hyperbolic ideal $(4n+2)$-gon. Let $l_1,\dots,l_{2n+1}$ be the common perpendicular to the pairs of opposite sides of $A_1A_2\cdots A_{4n+2}$. If $l_1,\dots,l_{2n}$ are concurrent, then the common intersection point belongs also to l_{2n+1}.*

For each $i\in\{1,\dots,2n+1\}$, let m_i be the line joining A_i and A_{i+2n+1}. Let R_i be the be the union of the two quarters defined by m_i and m_{i+1} that contain the sides A_iA_{i+1} and $A_{i+2n+1}A_{i+2n+2}$, as in Figure 3, where the indices are taken modulo $2n+1$. Observe that, by Lemma D, the line l_i is the set of points of R_i that is at the same distance from m_i and m_{i+1}. If $l_1,\dots,l_{2n}$ are concurrent at a point P, then P is at the same distance from all the lines m_i, so, in particular, it is at the same distance from m_{2n+1} and m_1. The only remaining point to complete the proof (and the reason the statement is false for $4n$-gons) is given by the following lemma.

Lemma G. *If $P\in R_1\cap\cdots\cap R_{2n}$, then $P\in R_1\cap\cdots\cap R_{2n+1}$*

Proof. Consider Cartesian equations for the lines m_i, that we still denote by m_i, so $m_i(A_i)=m_i(A_{i+2n+1})=0$. Up to changing signs, we can suppose that $m_i(A_{i+1})>0$. Thus, for each $i\in\{1,\dots,2n\}$ the region R_i is defined by the inequality $m_im_{i+1}<0$, but the region R_{2n+1}, bounded by m_{2n+1} and m_1, is defined by $m_{2n+1}m_1>0$. Now, if $P\in R_1\cap\cdots\cap R_{2n}$, then $m_1(P)m_2(P)<0,\ \dots\ ,m_{2n}(P)m_{2n+1}(P)<0$. By multiplying this even number of inequalities, we obtain $m_{2n+1}(P)m_1(P)>0$, so $P\in R_{2n+1}$. □

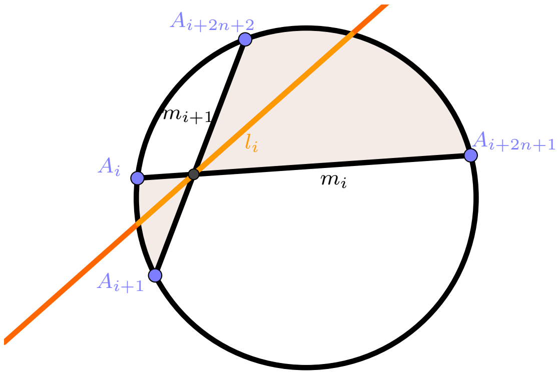

FIGURE 3. The region R_i.

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