

Finding left inverses for operators on $\ell^p(\mathbf{Z}^d)$ with polynomial decay.

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Abstract

We study the left-invertibility of infinite matrices indexed by metric spaces satisfying the doubling property. Under different conditions on the rows and the columns, we prove that being bounded-below in ℓ^p for some $1 \leq p \leq \infty$, implies that there is a left-inverse which is bounded in ℓ^q , for all $1 \leq q \leq \infty$. In particular, this applies to matrices with polynomial decay, indexed by discrete groups of polynomial growth.

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1 Introduction

In this paper, we study the left-invertibility of certain classes of bounded linear operators $A : \ell^p(X) \rightarrow \ell^p(Y)$ where X is a metric space and Y is any set.

We say that such an operator is bounded below in ℓ^p if

$$\lambda_p(A) := \inf_{f \neq 0} \frac{\|Af\|_p}{\|f\|_p} > 0.$$

If A is left invertible in ℓ^p , i.e. if there exists $B : \ell^p(Y) \rightarrow \ell^p(X)$ such that $BA = I$, then A is clearly bounded below in ℓ^p . But unless $p = 2$, the converse is not true in general. Our main concern in this article will be to prove the converse in certain situations. This problem arises naturally in frame theory and in sampling theory (see [ABK] for a survey on the applications of such results).

Given a doubling metric space X (e.g. $X = \mathbf{Z}^d$ for some $d \in \mathbf{N}$) and a countable set Y , we consider an operator $A = (a_{y,x})_{(y,x) \in Y \times X}$, bounded on ℓ^2 , whose rows are supported in balls of bounded radius (i.e. are thin), and whose

columns have only a bounded number of non-zero entries (i.e. are sparse): such a matrix is called thin-sparse. Our main result states that if A is bounded below in ℓ^p for some $1 \leq p \leq \infty$, then, $B = (A^*A)^{-1}A^*$ defines a left-inverse for A , which is uniformly bounded on ℓ^q for $q \in [1, \infty]$.

We are able to extend this result to a class of operators with “strictly polynomial decay” from the diagonal: for instance, a matrix $A = (a_{y,x})_{(y,x) \in \mathbf{Z} \times \mathbf{Z}}$ is said to have strictly polynomial decay from the diagonal, if there exists $\delta > 0$ such that

$$\sum_{x \in \mathbf{Z}} a_\infty(x)(1 + |x|)^\delta < \infty,$$

where $a_\infty(x) = \sup_{y \in \mathbf{Z}} |a_{y,y+x}|$. Moreover in this case, we obtain that the left-inverse also have polynomial decay of degree δ .

We discuss various generalizations of these results (e.g. in the context of discrete groups), and give explicit uniform bounds on the norms of the left-inverses.

Unless specified, all the matrices considered have complex coefficients.

2 Statements of results

In all the results stated in this paper, ℓ^∞ can be replaced by c_0 , i.e. the subspace of ℓ^∞ of sequences going to 0 at infinity.

2.1 Left-inverses for operators with thin rows

Recall that a metric space X has the doubling property (with doubling constant at most D) if there exists $D < \infty$ such that $|B(x, 2r)| \leq D|B(x, r)|$ for all $x \in X$ and all $r > 0$, where $|\Omega|$ denotes the cardinality of the subset Ω , and $B(x, r)$ denotes the closed ball of radius r . Our main example will be finitely generated free abelian groups, i.e. $X = \mathbf{Z}^d$, equipped with its usual ℓ^1 -metric. One checks easily that \mathbf{Z}^d has a finite doubling constant (e.g. for $d = 1$, we have $D = 2$). More generally, one can consider any finitely generated nilpotent group.

Theorem 2.1. *Let X be a doubling metric space and let $A = (a_{y,x})_{(y,x) \in Y \times X}$ be an infinite matrix whose rows are supported in balls of bounded radius and such that columns have supports with bounded cardinality. Assume moreover that the coefficients of A are bounded. Then,*

- either

$$\lambda_p(A) = 0$$

for all $1 \leq p \leq \infty$,

- or there exists $c > 0$, such that $B = (A^*A)^{-1}A^*$ satisfies $BA = I$ and

$$\|B\|_p \leq c,$$

for all $1 \leq p \leq \infty$.

Remark 2.2. Note that for a matrix A whose rows have bounded support, a uniform bound on the coefficients is equivalent to the fact that A is bounded in ℓ^∞ . So, if A is bounded in ℓ^p for some $1 \leq p \leq \infty$, as in particular its coefficients are bounded, it is also bounded in ℓ^∞ . Hence by interpolation, it is bounded for all $p \leq q \leq \infty$.

We can actually drop the assumption of sparseness on the columns of A , and then obtain the following stronger statement (indeed Theorem 2.1 follows by taking $p < 1$ in the following theorem).

Theorem 2.3. *Let $A = (a_{y,x})_{(y,x) \in Y \times X}$ be an infinite matrix whose rows are supported in balls of bounded radii (such an matrix is called thin- \emptyset). Assume moreover that A is bounded as an operator $\ell^p(X) \rightarrow \ell^p(Y)$ for some $0 < p < \infty$ (equivalently bounded on ℓ^q for all $p \leq q \leq \infty$). Then,*

- either

$$\lambda_q(A) := \inf_{f \neq 0} \frac{\|Af\|_q}{\|f\|_q} = 0$$

as soon as $p < q \leq \infty$ and $q \geq 1$;

- or there exists $c > 0$, such that

$$\lambda_q(A) \geq c,$$

for all $\max(p, 1) \leq q \leq \infty$. In the latter case, if $p \leq 2$, then $B = (A^*A)^{-1}A^*$ defines a left-inverse for A , which is uniformly bounded on ℓ^q for

$$\max(p, 1) \leq q \leq p/(\max(p, 1) - 1).$$

The conclusion of Theorem 2.3 is optimal as one can easily construct for every $1 \leq p \leq \infty$ a matrix $A = (a_{y,x})_{y,x \in \mathbf{N}}$ with one non-zero coefficient in each row and such that

- A is bounded in ℓ^q , for $q \geq p$,
- $\lambda_p(A) > 0$,
- $\lambda_q(A) = 0$ for all $p < q \leq \infty$.

To see that, consider a matrix such that the n 'th column contains exactly n non-zero coefficients equal to $1/n^{1/p}$, and make sure that the columns are piecewise orthogonal.

Remark 2.4. Theorem 2.1 has been proved very recently [ABK] for slanted matrices: let $\alpha \in \mathbf{R}^*$, a matrix $(a_{y,z})_{y,z \in \mathbf{Z}^d}$ is called α -slanted if its support in $\mathbf{Z}^d \times \mathbf{Z}^d$ lies at bounded distance from the subspace of $\mathbf{R}^d \times \mathbf{R}^d$ defined by $\{(x, y) \in \mathbf{R}^d \times \mathbf{R}^d, y = \alpha z\}$.

Although our proof is clearly different from the one of [ABK], both approaches share an important idea which consists in restricting A to functions supported in balls of radius L , which reduces the problem to dimension $\approx L^d$, where we can use quantitative comparisons between ℓ^p -norms, before letting L go to infinity. The idea that we introduce to perform this reduction relies on a geometric property of the metric space X satisfied by \mathbf{Z}^d , but not for instance by the three-regular tree¹. Precisely, we prove the following fact (see Theorem 6.1 for a more general statement).

Theorem 2.5. *Let X be a doubling metric space, and let $A = (a_{y,x})_{(y,x) \in Y \times X}$ be a thin- \emptyset matrix. Assume that A is bounded as an operator $\ell^p(X) \rightarrow \ell^p(Y)$, for some $1 \leq p \leq \infty$. Then, there exist C_1 and C_2 such that for all $L \geq 1$, there is a non-zero function h supported in a ball of radius L such that for all $p \leq q \leq \infty$,*

$$\frac{\|Ah\|_q}{\|h\|_q} \leq C_1 \lambda_q(A) + \frac{C_2}{L}.$$

(C_1 only depends on the space X , and for $X = \mathbf{Z}$, we can take $C_1 = 6$. But C_2 also depends on A).

The estimate in $O(1/L)$ for the error term is optimal as one can easily check with A being the convolution by the characteristic function of $\{-1, 1\}$, acting on $\ell^p(\mathbf{Z})$.

2.2 Operators with strict polynomial decay from the diagonal and generalizations

We will assume here that the metric space X has some additional structure, namely is a finitely generated group. The reader who is not familiar with this level of generality can think of $G = \mathbf{Z}^d$, equipped with its ℓ^1 -distance. Let G

¹In fact, the conclusion of Theorem 2.1 does not hold for the three-regular tree T : take the diffusion operator P of the simple random walk on T . The operator $I - P$, usually called the discrete Laplacian on T is bounded below in $\ell^2(X)$, but not for $p = \infty, 1$. See [T] for more details.

be a finitely generated group, and let S be a finite symmetric generating set. The word length of an element $g \in G$, associated to S , is defined as follows: $|g| = |g|_S = \min\{n \in \mathbf{N}, g = s_1 s_2 \dots s_n, s_i \in S\}$. We define a distance on G which is invariant under left-translations by $d(g, h) = |g^{-1}h|$.

We say that G has polynomial growth if there exists a positive number D such that $v(r) = O(r^D)$, where $v(r)$ denotes the volume (i.e. cardinality) of balls of radius $r > 0$ in G . A classical result due to Guivarc'h [Gui] and Gromov [Gro1] assert that for such a group, there exists an integer $d \in \mathbf{N}$, such that $C^{-1}r^d \leq v(r) \leq Cr^d$, so in particular, G has the doubling property.

For us in this paper, a weight on G will denote a function $\omega : G \rightarrow [1, \infty)$ which is

- radial, i.e. $\omega(g)$ only depends on $|g|$,
- sub-multiplicative, i.e.

$$\omega(gg') \leq \omega(g)\omega(g'),$$

- sub-exponential, i.e.

$$\lim_{|g| \rightarrow \infty} \frac{\ln \omega(g)}{|g|} = 0.$$

Let Y be a countable set, and consider a map $T : Y \rightarrow G$ whose pre-images have bounded cardinality, i.e. such that

$$\sup_{x \in G} |T^{-1}(\{x\})| < \infty.$$

We define

$$a_{\infty, T}(x) = \sup_{y \in Y} |a_{y, (Ty)x}| \quad \forall x \in G.$$

If ω is a weight on G , we say that $A \in \Sigma_T^\omega(G, Y)$ if $a_\infty \in \ell^1(G, \omega)$. The space $\Sigma_T^\omega(G, Y)$, equipped with the norm $\|A\|_{T, \omega} = \sum_{x \in G} a_{\infty, T}(x)\omega(x)$, is a Banach space, and there is C such that $\|A\|_p \leq C\|A\|_{T, \omega}$ for all $1 \leq p \leq \infty$ (see Proposition 10.3). Our main result is that if G has polynomial growth, then the conclusion of Theorem 2.1 holds for any $A \in \Sigma_T^\omega(G, Y)$. Moreover, we obtain a more precise description of the left-inverse.

Theorem 2.6. *Let G be a finitely generated group with polynomial growth. Consider a weight $\omega(x) \geq (1 + |x|)^\delta$ on G , for some $\delta > 0$. Let $T : Y \rightarrow G$ be any application whose pre-images have bounded cardinality, and let $A \in \Sigma_T^\omega(G, Y)$. Then*

- (i) either $\lambda_p(A) = 0$ for all $1 \leq p \leq \infty$,

(ii) or A admits a left-inverse B whose adjoint lies in $\Sigma_T^\omega(G, Y)$. In particular, B is uniformly bounded on ℓ^p for $1 \leq p \leq \infty$

Remark 2.7. In [ABK], this theorem has been proved in the following particular case. Let α be some non-zero real number and let $T_\alpha : \mathbf{Z}^d \rightarrow \mathbf{Z}^d$ be the map defined by $T_\alpha y = [\alpha y]$ where $[\alpha y]$ is the vector of \mathbf{Z}^d whose coordinates are the integer parts of those of $\alpha y \in \mathbf{R}^d$. They prove Theorem 2.6 for $A \in \Sigma_{T_\alpha}^\omega(\mathbf{Z}^d, \mathbf{Z}^d)$, and under the additional condition that $\omega(x) \geq (1 + |x|)^\delta$ for some $\delta > (d + 1)^2$.

Let us finish this section by an interesting and simple particular case. Assume here that $X = Y = G$ and that T is the identity map. We simply denote $\Sigma_{\text{id}}^\omega(G, G) = \Sigma^\omega(G)$. We say that a matrix A has strictly polynomial decay from the diagonal if it belongs to $\Sigma^\omega(G)$ for $\omega(x) = (1 + |x|)^\delta$ on G , for some $\delta > 0$.

Corollary 2.8. *Assume that A has strictly polynomial decay from the diagonal. Then,*

- (i) either $\lambda_p(A) = 0$ for all $1 \leq p \leq \infty$,
- (ii) or A admits a left-inverse B with strictly polynomial decay from the diagonal (of same degree).

Remark 2.9. Note that (ii) should be compared to similar statements for the inverses of matrices with a certain decay from the diagonal [J, Bas]. Actually, to prove (ii), we make a crucial use of these results. Let us insist on the fact that the non-trivial part of our results consist in showing that $\lambda_p(A) > 0$ implies a uniform lower bound on $\lambda_q(A)$ for all $1 \leq p \leq \infty$, and in particular that $\lambda_2(A) > 0$.

2.3 Some trivial situations

Recall that a finite rectangular matrix $(a_{i,j})_{(i,j) \in I \times J}$ has a non-trivial kernel if $|J| > |I|$. Let Y be a set and let $T : Y \rightarrow G$ be a map. We will say that a matrix $A = (a_{y,x})_{(y,x) \in Y \times G}$ is concentrated about the graph of T if $a_{y,(Ty)x} = 0$ as soon as $|x| \geq r$ for some constant r . Assume that there exists a finite subset C of G such that

$$|T^{-1}([C]_r)| < |C|,$$

where $[C]_r = \{x \in G, d(x, C) \leq r\}$ (note that this condition roughly means that T is “far from being surjective”). Then the kernel of A contains non-zero functions supported on C (hence belonging to all ℓ^p). Assume now that T has bounded pre-images. As seen in the proof of Proposition 10.4, elements of $\Sigma_T^\omega(G, Y)$ are limits (for all ℓ^p -operator norms) of operators concentrated about the graph of T . Hence, we deduce the following proposition.

Proposition 1. *Let $A \in \Sigma_T^\omega(G, Y)$ (as in Theorem 2.6), and assume that for all $r > 0$, there exists a finite subset C of \mathbf{Z}^d such that*

$$|T^{-1}([C]_r)| < |C|.$$

Then $\lambda_p(A) = 0$ for all $1 \leq p \leq \infty$.

In particular [ABK, Lemma 2.2], we have $\lambda_p(A) = 0$ for all $1 \leq p \leq \infty$ if $A \in \Sigma_{T_\alpha}^\omega(G, G)$ with $\alpha > 1$.

2.4 The case of convolution operators on groups

Let G be a finitely generated group. Recall that the reduced C^* -algebra of a discrete group G is the completion of the group-algebra of G for the norm of operators on $\ell^2(G)$. Let $(A, \|\cdot\|_A)$ be a unital Banach subalgebra of a Banach $(B, \|\cdot\|_B)$. One says that A is spectral in B if every element a of A which is invertible in B is actually invertible in A (or, equivalently, if the spectrum of a in A coincides with the spectrum of a in B). As already mentioned, it is known that for any group G with polynomial growth, $\ell^1(G, \omega)$ is a spectral C^* -subalgebra of $C_r^*(G)$. But note that $\ell^1(G, \omega)$ is also a subalgebra of $C_r^p(G)$, the completion of the group-algebra of G for the norm of operators on $\ell^p(G)$, for all $1 \leq p \leq \infty$. Here, we prove

Theorem 2.10. *Assume that G has polynomial growth. Consider a weight $\omega(x) \geq (1 + |x|)^\delta$ on G with $\delta > 0$, and let $f \in \ell^1(G, \omega)$. Let A be the operator of convolution by f . If $\lambda_p(A) > 0$ for some $1 \leq p \leq \infty$, then f is invertible in $\ell^1(G, \omega)$.*

Proof: By Theorem 10.2, $\lambda_p(A) > 0$ implies that $\lambda_2(A) > 0$, and hence f is a left-invertible element of the reduced C^* -algebra $C_r^*(G)$. As $C_r^*(G)$ is tracial, f is actually invertible. So to conclude, we just need to know that $\ell^1(G, \omega)$ is a spectral in $C_r^*(G)$, which follows from [FGL1] and from the fact that in our definition of weight, we included the condition $\lim_{n \rightarrow \infty} (\omega(g^n))^{1/n} = 1$. ■

We can improve this result in the abelian case, using the fact that the elements of $C_r^*(G)$ are normal.

Theorem 2.11. *Let ω be a weight on $G = \mathbf{Z}^d$ and let $f \in \ell^1(G, \omega)$. If $f \in \ell^1(G, \omega)$ satisfies $\lambda_p(f) > 0$ for some $1 \leq p \leq \infty$, then f is actually invertible in $\ell^1(G, \omega)$.*

Proof: Let us start with a lemma.

Lemma 2.12. *Let G be a group with polynomial growth and let $1 \leq p \leq \infty$. Then for any weight ω on G , $\ell^1(G, \omega)$ is spectral in $C_r^p(G)$.*

Proof of the lemma. First, note that $\ell^1(G, \omega)$ is spectral in $\ell^1(G)$ holds for all groups. The main result of [Bar] states that a (locally compact) group G is amenable and $L^1(G)$ is symmetric if and only if $L^1(G)$ is spectral in $C_r^p(G)$. As a group with polynomial growth is amenable, the lemma follows from the fact [Lu] that $\ell^1(G)$ is symmetric. ■

Now, we assume that $G = \mathbf{Z}^n$. Thanks to the lemma, it is enough to show that f is invertible in $C_r^*(G)$, which is equivalent to saying that $\lambda_2(f) > 0$. Suppose on the contrary that $\lambda_2(f) = 0$ and let us prove that $\lambda_p(f) = 0$ for all $1 \leq p \leq \infty$. Let $f_n \in \mathbf{C}G$ be a sequence converging to f in $\ell^1(G)$. In particular, as $\|f_n - f\|_{2 \rightarrow 2} \rightarrow 0$, we have $\lambda_2(f_n) \rightarrow \lambda_2(f) = 0$. But since G is abelian, the f_n are normal operators on $\ell^2(G)$. Hence,

$$\lambda_2(f_n) = \min\{|\mu|, \mu \in Sp(f_n)\}.$$

In particular, this means that up to replacing f_n by $f_n - \mu_n$ for some sequence $\mu_n \rightarrow 0$ in \mathbf{C} , we can assume that $\lambda_2(f_n) = 0$. Now, by Theorem 2.10, $\lambda_p(f_n) = 0$ for all $1 \leq p \leq \infty$. So, as $\|f_n - f\|_{p \rightarrow p} \rightarrow 0$, we have $\lambda_p(f) = \lim_{n \rightarrow \infty} \lambda_p(f_n) = 0$. ■

Remark 2.13. Note that we do not use Fourier calculus in this proof. However, instead of using Theorem 2.10, one can show directly using Fourier calculus that the convolution in $\ell^p(\mathbf{Z}^d)$ by some non-zero $f \in \mathbf{C}\mathbf{Z}^d$ has a dense image. Hence, $\lambda_p(f) > 0$ implies that f is invertible as an operator on ℓ^p , and the conclusion follows from [Bar]. It would be interesting to generalize this argument to nilpotent groups.

2.5 About the proofs

The proofs of Theorems 2.1, 2.3 and 2.6 split into two main parts. First, we need to show that if A is bounded below for some p , then it is uniformly bounded below in ℓ^q for all q 's. The second part of the proof consists in showing that the left-inverse exists and is uniformly bounded in ℓ^p for all p 's. The second part of the proof follows easily from the following observation (left to reader).

Proposition 2.14. *Let X and Y be two sets, and let A be an operator $\ell^2(X) \rightarrow \ell^2(Y)$ such that A and A^* are uniformly bounded in ℓ^p for all $1 \leq p \leq \infty$. We have*

- $\lambda_2(A^*A) = \lambda_2(A)^2$,
- if A is self-adjoint and $\lambda_p(A) > 0$, then A is invertible in ℓ^p , and $\|A^{-1}\|_p = 1/\lambda_p(A)$. ■

Indeed, if $\lambda_p(A) \geq c > 0$ for all $1 \leq p \leq \infty$, then in particular, this is true for $p = 2$. So we have $\lambda_2(A^*A) \geq c^2$, and as A^*A is self-adjoint, it is invertible. Now, if A is thin- \emptyset and if A^*A exists, then A^*A is thin-thin (see Proposition 5.2). Similarly, if $A \in \Sigma_T^\omega(G, Y)$, then $A^*A \in \Sigma^\omega(G)$ (see Proposition 10.7). Hence, as $\lambda_2(A^*A) > c^2$, there exists $c' > 0$ such that $\lambda_p(A^*A) \geq c'$ for all $1 \leq p \leq \infty$. Finally as $\|(A^*A)^{-1}\|_p = 1/\lambda_p(A^*A) \leq 1/c'$, we conclude that $B = (A^*A)^{-1}A^*$ satisfies

$$\|B\|_p \leq \|A^*\|_p/c'.$$

Let us now summarize the structure of the first part of the proof of Theorems 2.1, 2.3 and 2.6.

1. The first step, Theorem 2.5, is by far the central part of this paper (see Section 6). We show that the doubling property can be used to approximate the ℓ^p -norm of a function f by taking the norm of its projection over a subset consisting of a union of distant balls of fixed radius. However, the naive idea consisting in applying A directly to this projection would only yield an error term in $L^{1/p}$, which would not enable us to obtain a uniform lower bound for $\lambda_p(A)$ (or even that $\lambda_\infty(A) > 0$). Instead, we multiply f by a certain Lipschitz function which is also supported on a union of distant balls.
2. To obtain the uniform lower bound for $\lambda_q(A)$, using Theorem 2.5 is quite technical but the general idea is easy to understand: Theorem 2.5 says that we can approximate $\lambda_q(A)$ by quotients of the form $\frac{\|Ah\|_q}{\|h\|_q}$, where h are supported in balls of radius L (hence, restricting to subspaces of dimension $\approx v(L)$ which is less than L^d for some d), and the error that we make is roughly in $1/L$. Comparing these quotients for different values of q (and the same function h), we multiply our error term by $L^{d|1/p-1/q|}$. The resulting error term will therefore go to zero if p and q are close enough, namely if $d|1/p-1/q| < 1$. Then, we just need to “propagate” the comparison that we get between $\lambda_p(A)$ and $\lambda_q(A)$ to obtain a uniform lower bound.
3. Then, we extend Theorem 2.1 to operators that are somehow “polynomially approximated” by thin-sparse operators: we call them almost thin-sparse operators (see Section 8). The idea of the proof is very similar to the above (see Lemma 8.3).

4. The proof of Theorem 2.3 consists essentially in showing that a thin- \emptyset operator which is bounded in ℓ^p , is almost thin-sparse in ℓ^q for all $q > p$.
5. The proof of Theorem 2.6 also consists in showing that elements of $\Sigma_T^\omega(G, Y)$ (see Proposition 10.4) are almost thin-sparse. To prove that the adjoint of the left-inverse lies in $\Sigma_T^\omega(G, Y)$, we make a crucial use of the following result, which implies that $(A^*A)^{-1} \in \Sigma^\omega(G)$.

Theorem 2.15. *[FGL2, Theorem 1](see [Bas, Theorem 2] for the particular case $G = \mathbf{Z}^d$, and [J] for results of the same kind) Let ω be a weight on G . If $A \in \Sigma^\omega(G)$ is invertible, then its inverse also lies in $\Sigma^\omega(G)$.*

An alternative proof of Theorem 2.1 (compare [ABK, Corollary 2.9].

It is actually possible to deduce Theorem 2.1 from a very particular case: when $A = (a_{y,x})_{(y,x) \in X^2}$ is concentrated about the diagonal, i.e. a particular case of slanted matrices corresponding to $\alpha = 1$ (and hence this also reduces the main result of [ABK] to this case). However, let us emphasize that this reduction does not simplify the proof and actually hides the main reason why the proof is working. As the reduction requires an extension of the results to matrices with operator coefficients (see Section 11), it will be explained in Section 11.2.

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3 Thin-sparse operators

In all the sequel, X and Y are discrete metric spaces with bounded geometry (balls of radius r have less than $v(r)$ elements, for a given function v). However, in the definition of thin-sparse operators, only X needs a structure of metric space (Y can be any set).

Let $C_c(X)$ be the space of finitely supported real-valued functions on X . Let A be a linear morphism from $C_c(X)$ to R^Y . The kernel (also called the matrix) of A , $(a_{y,x})_{(y,x) \in Y \times X}$ is defined by the relation

$$Af(y) = \sum_{x \in X} a_{y,x} f(x),$$

for every $f \in C_c(X)$. Conversely a matrix, i.e. a family of reals $(a_{y,x})_{(y,x) \in Y \times X}$ defines a linear morphism by the same formula.

The row of index $y \in Y$ of A is the vector $(a_{y,x})_{x \in X}$ of \mathbf{R}^X . The column of index $x \in X$ of A is the vector $(a_{y,x})_{y \in Y}$ of \mathbf{R}^Y . The support of A is the subset of $Y \times X$ on which $a_{y,x} \neq 0$. We define similarly the support of a row or of a column of A .

Definition 3.1. If the rows of a matrix $A = (a_{y,x})_{y \in Y}$ satisfy the property P, and if its columns satisfy the property Q, we will say that A is P-Q. If we make no assumption on the columns, we will say that A is P- \emptyset , and so on. We will consider two properties for the rows or the columns:

- We say that the rows (or the column) of A are thin, of thickness at most r if their support are contained in balls of radius r .
- We say that the rows (or the columns) are sparse, of sparseness at most v if their support has cardinality at most v .
- We denote by $TS(X, Y)$ (resp. $ST(X, Y)$, $T(X, Y)$, $\emptyset T(X, Y)$ and $T\emptyset(X, Y)$) the space of thin-sparse (resp. sparse-thin, thin-thin, \emptyset -thin and thin- \emptyset) operators.

As the spaces have bounded geometry, sparse is a weaker condition than thin. Hence sparse-sparse is weaker than thin-sparse, which is weaker than thin-thin, etc.

Remark 3.2. A particular case of thin-thin matrices (when $X = Y$) are matrices which are concentrated about the diagonal: i.e. for which the support is contained in $\{(y, x) \in X^2, d(x, y) \leq r\}$ for some $r > 0$. Such matrices are sometimes called banded, or with finite propagation.

Definition 3.3.

- For all $1 \leq p \leq \infty$, the norm of an operator $A : \ell^p(X) \rightarrow \ell^p(Y)$ is called the ℓ^p -norm of A and is denoted by $\|A\|_{p \rightarrow p}$.
- Let $A = (a_{y,x})_{(y,x) \in Y \times X}$. The absolute value of A is operator $|A| = (|a_{y,x}|)_{(y,x) \in Y \times X}$.
- We say that A is absolutely uniformly bounded if

$$\sup_{1 \leq p \leq \infty} \| |A| \|_{p \rightarrow p} < \infty.$$

4 More precise results for thin-sparse operators

Here are the general versions of Theorems 2.1, 2.3 and 2.5, with some precisions that we omitted in the introduction.

Theorem 4.1. *Let X be a metric space of doubling constant $D < \infty$ and let Y be any set. Fix some $r, v > 0$. Let $A \in TS(X, Y)$ be of thickness at most r , sparseness at most v . Assume moreover that the coefficients of A have absolute value at most 1. Then there is $c = c(r, v, D) > 0$ and $\delta = \delta(D) > 0$ for all $1 \leq p, q \leq \infty$,*

$$\lambda_p(A) \geq c\lambda_q(A)^\delta.$$

Theorem 4.2. *Let X be a metric space of doubling constant $D < \infty$ and let Y be any set. Let $A = (a_{y,x})_{(y,x) \in Y \times X}$ be a thin- \emptyset matrix. Assume moreover that A is bounded as an operator $\ell^{p_0}(X) \rightarrow \ell^{p_0}(Y)$ for some $1 \leq p_0 \leq \infty$. Then for every $p_1 > p_0$, there exists $c = c(p_1 - p_0, r, D) > 0$ and $\delta = \delta(p_1 - p_0, D) > 0$ such that for all $p_1 \leq p, q \leq \infty$,*

$$\lambda_p(A) \geq c\lambda_q(A)^\delta.$$

In the spirit of the example explained in the introduction, one can show that for $r = 1$ and $X = Y = \mathbf{Z}$, we can find a sequence of thin-sparse operators $A_n = (a_{y,x})_{(y,x) \in Y \times X}$ of thickness 1, sparseness n , and such that

- $\|A_n\|_{p_0 \rightarrow p_0} = \lambda_{p_0}(A_n) = 1$ for all $n \in \mathbf{N}$,
- and $\lambda_p(A_n) \rightarrow 0$ when $n \rightarrow \infty$ for all $p > p_0$.

In particular, one cannot expect c and δ to depend only on D and r , for all $p_0 \leq p \leq \infty$. However, we can prove that there exists $c' = c'(r, D) > 0$ and $\delta' = \delta'(D) > 0$ such that for all $p_0 \leq p \leq q \leq \infty$

$$\lambda_p(A) \geq c'\lambda_q(A)^{\delta'}.$$

For more precise statements, see Theorems 7.4 and 7.5.

In Section 8, we prove that Theorems 4.1 and 4.2 are true for more general operators which are “well” approximated by thin-sparse and thin- \emptyset operators respectively. In Section 10, we give concrete and still very general classes of such operators: see Corollary 10.5.

5 Preliminary remarks about thin-sparse operators

5.1 Combinatorial properties

The following easy fact is a crucial property of TS operators. We say that two subsets U and V of a metric space are t -disjoint if $d(x, y) \geq t$ for all $(x, y) \in U \times V$.

Proposition 5.1. *Let X and Y be metric spaces with bounded geometry. Let A be a thin-sparse operator of thickness r and let v and u be two functions on X whose supports are $2r$ -disjoint. Then, Au and Av (which are well defined functions) have disjoint support.*

Proof: We just have to consider a row L of A and to prove that $Lu \neq 0$ implies $Lv = 0$. But this is a trivial consequence of the fact that L is supported in a ball of radius r and that the supports of u and v are at distance at least $2r$. ■

The following proposition is straightforward and left as an exercise.

Proposition 5.2. *Let X be a metric space and let Y be a set. If $A \in T\mathcal{O}(X, Y)$ then A^*A (when it exists) is concentrated about the diagonal. ■*

5.2 Norms of sparse-sparse operators are equivalent

Proposition 5.3. *A sparse-sparse operator A is absolutely uniformly bounded, if and only if it is bounded in ℓ^p for some $1 \leq p \leq \infty$, if and only if it has bounded coefficients.*

Proof: Let X and Y be two sets and let $A = (a_{y,x})_{(x,y) \in X \times Y}$ be a sparse-sparse operator of sparseness v . Note that the norm $\|a\|_\infty = \sup_{(y,x) \in Y \times X} |a(y,x)|$ is trivially less than all operator norms. Hence it is enough to prove that for every $1 \leq p \leq \infty$, $\|A\|_{p \rightarrow p} \leq C\|a\|_\infty$ for some C depending only on v . Fix $y \in Y$, and let S_y be the support of the corresponding row $(a_{y,x})_{x \in X}$. For every $f \in C_c(X)$,

$$|Af(y)| = \left| \sum_{x \in X} a_{y,x} f(x) \right| \leq \|a\|_\infty \sum_{x \in S_y} |f(x)|.$$

Hence, using Hölder's inequality and the majoration $|S_y| \leq v$ for all $y \in Y$, we

obtain

$$\begin{aligned} \|Af\|_p^p &\leq \|a\|_\infty^p \sum_{y \in Y} \left(\sum_{x \in S_y} |f(x)| \right)^p \\ &\leq \|a\|_\infty^p \sum_{y \in Y} v^{p-1} \sum_{x \in S_y} |f(x)|^p \end{aligned}$$

Now, note that for every $x \in X$ and every $k \in \mathbf{N}$, $f(x)$ appears k times in the sum above if there are k distinct elements of Y , y_1, \dots, y_k such that $x \in S_{y_1} \cap \dots \cap S_{y_k}$, hence if y_1, \dots, y_k lie in the support of the column $(a_{y,x})_{y \in Y}$. But as the sparseness of A is at most v , this implies that $k \leq v$. Therefore, we have

$$\begin{aligned} \|a\|_\infty^p \sum_{y \in Y} v^{p-1} \sum_{x \in S_y} |f(x)|^p &\leq v^p \|a\|_\infty^p \sum_{x \in X} |f(x)|^p \\ &= v^p \|a\|_\infty^p \|f\|_p^p. \quad \blacksquare \end{aligned}$$

6 Proof of the approximation property

Our purpose in this section is to prove the following theorem

Theorem 6.1. *Assume that X is a doubling metric space and let $A \in T\mathcal{O}(X, Y)$ of thickness r , such that $\|A\|_{p \rightarrow p} \leq 1$ for some $1 \leq p < \infty$. There exists C such that for every $f \in L^p(X)$, and every $L \geq r$, there exists a function $h \in L^p(X)$ supported in a ball of radius $2L$ such that*

$$\frac{\|Ah\|_p}{\|h\|_p} \leq C \left(\frac{\|Af\|_p}{\|f\|_p} + \frac{r}{L} \right).$$

The quantity C only depends on the doubling constant of X .

6.1 d -coloring of a family of balls

Recall that a d -coloring of a set \mathcal{P} of subsets of X is a map

$$j : \mathcal{P} \rightarrow \{1, \dots, d, d+1\}$$

such that every two elements in \mathcal{P} with the same color (i.e. same image by j) are disjoint.

Also classical is the notion of coloring of a graph: a d -coloring of a graph \mathcal{G} is a map

$$j : V(\mathcal{G}) \rightarrow \{1, \dots, d, d+1\},$$

where $V(\mathcal{G})$ is the vertex set of \mathcal{G} , such that any two adjacent vertices have distinct colors. A classical result of graph theory, known as Brooks' theorem says that any graph of degree at most d admits a d -coloring.

A link between these two definitions of coloring appears with the notion of dual graph. Recall that the dual graph \mathcal{G} of \mathcal{P} is defined as follows: the set of vertices $V(\mathcal{G})$ is \mathcal{P} , and two vertices are adjacent if and only if they have a non-empty intersection. Clearly, a d -coloring of \mathcal{G} yields a d -coloring of \mathcal{P} and conversely.

We will need the following lemma. Recall that a discrete space X is said to be doubling if there exists a constant $C < \infty$ such that for all $x \in X$ and every $r > 0$,

$$|B(x, 2r)| \leq C|B(x, r)|.$$

Lemma 6.2. *Let X be a doubling metric space and let $\alpha \geq 1$. There exists an integer d such that for every $L > 0$, there exists a covering of X by balls of radius L admitting a d -coloring such that the centers of two balls of same color are at distance $\geq \alpha L$ from one another.*

Proof: Consider a minimal covering $\mathcal{B} = (B(x_i, L))_i$ of radius n of X (exists as X is doubling). By minimality, the balls $B(x_i, L/4)$ are piecewise disjoint.

Now, consider the covering $\mathcal{B}' = (B(x_i, \alpha L))_i$. It is classical and easy to see that the doubling property implies that the dual graph of \mathcal{B}' has degree less than a certain constant d . Hence, by Brooks' theorem, this graph admits a d -coloring. Going back to X , this means that \mathcal{B}' has a d -coloring. Inducing this coloring to \mathcal{B} yields the desired d -coloring. ■

6.2 Approximating a function by a function supported by a disjoint union of balls of fixed radius.

In the following lemma we characterize the doubling condition in terms of approximation of functions by functions supported by disjoint unions of balls of fixed radius.

For every subset Ω of a metric space X and every $L > 0$, we denote

$$[\Omega]_L = \{x \in X, d(x, \Omega) \leq L\}.$$

We also denote the characteristic function of a subset Ω by 1_Ω .

Lemma 6.3. *A metric space X is doubling if and only if for every $\alpha \geq 1$, there exists a constant $c > 0$ such that for every $1 \leq p \leq \infty$, every $f \in \ell^p(X)$ and every $L > 0$, one can find an αL -separated subset P of X such that*

$$\|1_{[P]_L} f\|_p \geq c \|f\|_p.$$

Proof: Consider the covering \mathcal{B} of the previous lemma and for every $1 \leq k \leq d+1$, let P_k be the set of centers of balls of \mathcal{B} with same color k . Since $X = \bigcup_{k=1}^{d+1} [P_k]_L$, we have

$$\|f\|_p \leq \left\| \sum_k 1_{[P_k]_L} |f| \right\|_p \leq \sum_k \|1_{[P_k]_L} f\|_p \leq (d+1) \max_k \|1_{[P_k]_L} f\|_p.$$

So Lemma 6.3 follows taking $P = P_k$ with a k for which the max is attained. The converse follows by taking f to be the characteristic function of a ball of radius $2L$ and $\alpha \geq 7$, so that the intersection between $[P]_L$ and our ball of radius $2L$ is contained in a single ball of radius L . ■

In the sequel, we fix $\alpha = 6$.

The following lemma is trivial and left to the reader.

Lemma 6.4. *For each P like in the previous lemma, the function Δ_P , defined by*

$$\Delta_P(x) = \max\{0, 1 - d(x, P)/(2L)\},$$

satisfies

1. $\Delta_P = 0$ outside of $[P]_{2L}$
2. $\Delta_P \geq 1/2$ on $[P]_L$.
3. Δ_P is $1/(2L)$ -Lipschitz.
4. $0 \leq \Delta_P \leq 1$.

■

Remark 6.5. Keeping the notation of the previous lemmas, the function $g = \Delta_P f$ satisfies, thanks to the second property of Δ_P and to Lemma 6.3,

$$\|g\|_p \geq c \|f\|_p.$$

On the other hand, the support of g is contained in a union of $4L$ -disjoint balls of radius $2L$. Write $g = \sum_i g_i$, where each g_i is supported in one of those balls. Assume that $4L \geq 2r$. Then by Proposition 5.1,

$$\|Ag\|_p^p = \sum_i \|Ag_i\|_p^p.$$

So we have

$$\inf_i \frac{\|Ag_i\|_p}{\|g_i\|_p} \leq \frac{\|Ag\|_p}{\|g\|_p}.$$

Proof of Theorem 6.1. Thanks to the previous remark, we just need to prove the theorem replacing h by a function g supported in a union of $2r$ -disjoint balls of radius $2L$. We consider $g = \Delta_P f$, which has this property since $L \geq r$. Let us start with a pointwise estimate. Let $y \in Y$. For every $x, z \in X$,

$$g(x) = \Delta_P(x)f(x) = \Delta_P(z)f(x) + (\Delta_P(x) - \Delta_P(z))f(x).$$

We take $z = \bar{y}$, such that the support of $w \rightarrow a_{y,w}$ is contained in $B(\bar{y}, r)$. We have

$$Ag(y) = \Delta_P(\bar{y}) \sum_x a_{y,x} f(x) + \sum_x a_{y,x} (\Delta_P(x) - \Delta_P(\bar{y})) f(x).$$

So by Property (4) of Δ_P ,

$$|Ag(y)| \leq |Af(y)| + \sum_x |a_{y,x}| |\Delta_P(x) - \Delta_P(\bar{y})| |f(x)|.$$

By Property (3) of Δ_P ,

$$|Ag(y)| \leq |Af(y)| + \frac{r\|A\| |f|(y)}{L}.$$

Now, taking the ℓ^p norm, we obtain

$$\|Ag\|_p \leq \|Af\|_p + \frac{r\|A\| \|f\|_p}{L}.$$

We conclude thanks to the inequality $\|g\|_p \geq c\|f\|_p$. ■

Remark 6.6. For $X = \mathbf{Z}$, we have $v(k) = 2k + 1$, and the doubling constant is less than 2. Note that we can take $P = \{x_0 + 6kL, k \in \mathbf{Z}\}$ for some x_0 . Moreover, one checks easily that a good choice of x_0 gives

$$\|1_P f\|_p \geq \|f\|_p / 3.$$

Now assume that A is thin-thin of thickness $\leq r$. By the proof of Proposition 5.3, we have $\|A\| \leq v(r)|a|_\infty$. Hence, we obtain that there exists a function h supported on a ball of radius r such that

$$\frac{\|Ah\|_p}{\|h\|_p} \leq 3 \left(\frac{\|Af\|_p}{\|f\|_p} + \frac{3r^2|a|_\infty}{L} \right).$$

7 Proof of Theorem 4.1

Lemma 7.1. Fix some $1 \leq p_0 < \infty$. Let $A \in T\mathcal{O}(X, Y)$ of thickness r and such that $\|A\|_{p \rightarrow p} \leq 1$ for all $p_0 \leq p \leq \infty$.

(i) there exist $d > 0$ and C' (depending on the doubling constant) such that for all $p_0 \leq p \leq q \leq \infty$ and all $L \geq r$

$$\lambda_q(A) \leq C' \left(L^{\left(\frac{d}{p} - \frac{d}{q}\right)} \lambda_p(A) + 2rL^{\left(\frac{d}{p} - \frac{d}{q}\right) - 1} \right).$$

(ii) if moreover, $A \in TS(X, Y)$ of sparseness v , then for all $p_0 \leq p, q \leq \infty$,

$$\lambda_q(A) \leq C' v^{\left|\frac{1}{p} - \frac{1}{q}\right|} \left(L^{\left|\frac{d}{p} - \frac{d}{q}\right|} \lambda_p(A) + 2rL^{\left|\frac{d}{p} - \frac{d}{q}\right| - 1} \right).$$

Proof: Theorem 6.1 implies

$$\inf_{\text{Supp}(h) \subset B(x, 2L)} \frac{\|Ah\|_p}{\|h\|_p} \leq C \left(\lambda_p(A) + \frac{r}{L} \right).$$

On the other hand, if h is supported in a subset of size N , then for $p \leq q$,

$$\|h\|_q \leq \|h\|_p \leq N^{\left|\frac{1}{p} - \frac{1}{q}\right|} \|h\|_q. \quad (7.1)$$

Finally, note that since X is doubling, the volume of balls grows polynomially, so there exist d and C'' such that $V(x, L) \leq C'' L^d$.

If $p \leq q$, then we obtain (i) applying the left inequality of (7.1) to Ah (where the support of Ah does not play any role) and the right inequality to h , whose support has cardinality at most $C'' L^d$. We have $C' = CC''$.

If $p \geq q$, then the right inequality of (7.1) concerns Ah for which we control the support thanks to the sparseness of A 's columns. Namely, the cardinality of the support of Ah is at most v times the cardinality of h 's support. So this yields (ii). ■

Lemma 7.2. *Let $A \in T\mathcal{O}(X, Y)$ of thickness r and such that $\|A\|_{p \rightarrow p} \leq 1$ for all $p_0 \leq p \leq \infty$.*

(i) for all $p_0 \leq p \leq \infty$, $\lambda_p(A) = 0$ implies $\lambda_q(A) = 0$ for all $q \geq p$.

(ii) Let K be twice the constant C' of Lemma 7.1. Then, for all $p_0 \leq p \leq q \leq \infty$,

$$\lambda_q(A) \leq Kr^{\left|\frac{d}{p} - \frac{d}{q}\right|} \lambda_p(A)^{1 - \left|\frac{d}{p} - \frac{d}{q}\right|}.$$

Lemma 7.3. *Let $A \in TS(X, Y)$ of sparseness v and thickness r , and such that $\|A\|_{p \rightarrow p} \leq 1$. Then, For all $p_0 \leq p \leq \infty$,*

(i) For every $p_0 \leq p, q \leq \infty$, $\lambda_p(A) = 0$ if and only $\lambda_q(A) = 0$.

(ii) Let K be twice the constant C' of Lemma 7.1. For all $p_0 \leq p, q \leq \infty$,

$$\lambda_q(A) \leq Kv^{|\frac{1}{p}-\frac{1}{q}|} r^{|\frac{d}{p}-\frac{d}{q}|} \lambda_p(A)^{1-|\frac{d}{p}-\frac{d}{q}|}.$$

Proof: Both lemmas are proved in the same way: so let us show Lemma 7.3. To obtain (ii), take $L = r/\lambda_p(A)$ in Lemma 7.1. To prove (i), we just have to note that the vanishing of $\lambda_p(A)$ “propagates” thanks to Lemma 7.1: $\lambda_p(A) = 0 \Rightarrow \lambda_q(A) = 0$ if $|\frac{d}{q} - \frac{d}{p}| \leq 1/2$ (let $L \rightarrow \infty$). ■

Theorem 4.1 and the remark following Theorem 4.2 follow from the following more precise results. Let

$$\lambda = \min_{p_0 \leq p \leq \infty} \lambda_p(A),$$

attained for $p = p_m$; and

$$\Lambda = \max_{p_0 \leq p \leq \infty} \lambda_p(A),$$

attained for $p = p_M$.

Theorem 7.4. Let $A \in TS(X, Y)$ of thickness r , sparseness v and such that $\|A\|_{p \rightarrow p} \leq 1$ for all $p_0 \leq p \leq \infty$. Then there exists a $k = k(v, r, d) > 0$ such that

$$\lambda \geq k\Lambda^{4d}.$$

Theorem 7.5. Let $A \in T\mathcal{O}(X, Y)$ of thickness r and such that $\|A\|_{p \rightarrow p} \leq 1$ for all $p_0 \leq p \leq \infty$. Then there exists $k = k(r, d) > 0$ such that for all $p_0 \leq p \leq q \leq \infty$,

$$\lambda_p \geq k\lambda_q^{4d}.$$

Proof: To show Theorems 7.4 and 7.5. we “propagate” the inequalities (ii) of Lemmas 7.2 and 7.3. As the proofs are the same for both theorems, let us focus on the first one. If $|\frac{d}{p} - \frac{d}{q}| \leq 1/2$, the inequality (ii) of Lemma 7.3 yields

$$\lambda_p(A) \leq C(v, r, d)\lambda_q(A)^2.$$

Now, as $|\frac{d}{p_m} - \frac{d}{p_M}| \leq d$, we just need to iterate this $2d$ times, which gives the theorem. ■

Remark 7.6. Here, assume that $X = Y = \mathbf{Z}$, and that A is thin-thin of thickness r . Instead of assuming that $\|A\| = 1$, we prefer to write Lemma 7.3 with respect to $|a|_\infty$ (which is easier to compute in practice): a consequence is that we have to replace r by $3r^3|a|_\infty$. From Remark 6.6 that we can take $C' = 9$ in Lemma 7.1 (as $v(r) \leq 3r$). Hence we can take $K = 18$. Directly from Lemma 7.3 (ii), we obtain that

$$\lambda_2(A) \geq \frac{\Lambda^2}{162r^3|a|_\infty}.$$

8 Extension to (t, s) -almost thin-sparse operators

Definition 8.1. Fix some $t, s > 0$ and some $1 \leq p \leq \infty$. An operator is (t, s) -almost thin-sparse for in ℓ^q for all $q \geq p$ if there exists $K < \infty$ such that for all $r, v > 0$, there is an element $A_{r,v} \in TS(X, Y)$ of thickness $\leq r$ and sparseness $\leq v$ such that $\| \|A - A_{r,v}\| \|_{q \rightarrow q} \leq K(r^{-t} + v^{-s})$ for all $q \geq p$.

This section is devoted to the proof of the following result (see also the more precise result: Theorem 8.5.)

Theorem 8.2. Fix some $t, s > 0$ and some $1 \leq p_0 \leq \infty$. Let X be a metric space with the doubling property, Y be any set. Let A be (t, s) -almost thin-sparse in ℓ^p for all $p \geq p_0$. Then either $\lambda_p(A) = 0$ for all $1 \leq p_0 \leq p \leq \infty$, or there is a positive real $c = c(A) > 0$ such that $\lambda_p(A) > c$ for all $p_0 \leq p \leq \infty$.

Proof: First, we need to prove analogues of Theorem 6.1.

Lemma 8.3. For every $r, v > 0$, there exists C, K' such that for every $p_0 \leq p \leq \infty$, every $f \in L^p(X)$, and every $L \geq 1$, there exists a function $h \in L^p(X)$ supported in a ball of radius $2L$ such that

$$\frac{\|Ah\|_p}{\|h\|_p} \leq C \left(\frac{\|Af\|_p}{\|f\|_p} + \frac{2r}{L} \right) + K'(r^{-t} + v^{-s}).$$

Proof: This is immediate, writing $A = A_{r,v} + (A - A_{r,v})$ where $A_{r,v}$ is thin-sparse of thickness r and sparseness v , and using $\| \|A - A_{r,v}\| \|_p \leq K(r^{-t} + v^{-s})$. (we have $K' = (1 + C)K$). ■

Lemma 8.4. There exists C' such that for all $p_0 \leq p, q \leq \infty$, and for all $L \geq 1$

$$\lambda_q(A) \leq C' v^{|\frac{1}{p} - \frac{1}{q}|} \left(L^{|\frac{d}{p} - \frac{d}{q}|} \lambda_p(A) + 2rv^{|\frac{1}{p} - \frac{1}{q}|} L^{|\frac{d}{p} - \frac{d}{q}| - 1} \right) + K' v^{|\frac{1}{p} - \frac{1}{q}|} r^{-t} L^{|\frac{d}{p} - \frac{d}{q}|} + K' v^{|\frac{1}{p} - \frac{1}{q}| - s} L^{|\frac{d}{p} - \frac{d}{q}|}.$$

Proof: This is proved exactly as we proved Lemma 7.1. ■

Now, to prove Theorem 8.2, assume that $\lambda_p(A) = 0$ for some $p_0 \leq p \leq \infty$. Then for every q , we have

$$\lambda_q(A) \leq 2C' r L^{|\frac{d}{p} - \frac{d}{q}| - 1} + K' r^{-t} v^{|\frac{1}{p} - \frac{1}{q}|} L^{|\frac{d}{p} - \frac{d}{q}|} + K' v^{|\frac{1}{p} - \frac{1}{q}| - s} L^{|\frac{d}{p} - \frac{d}{q}|}.$$

Putting for example $r = L^{1/2}$ and $v = L^d$, it becomes

$$\lambda_q(A) \leq 2C' L^{|\frac{d}{p} - \frac{d}{q}| - 1/2} + K' L^{2|\frac{d}{p} - \frac{d}{q}| - \frac{t}{2}} + K' L^{2|\frac{d}{p} - \frac{d}{q}| - ds}.$$

Hence, if $|\frac{d}{p} - \frac{d}{q}| < \min\{1/2, t/2, ds/2\}$, letting $L \rightarrow \infty$, we obtain

$$\lambda_p(A) = 0 \Rightarrow \lambda_q(A) = 0,$$

and we conclude, propagating this to all $p_0 \leq q \leq \infty$. The proof of the uniform lower bound for $\lambda_p(A)$ when it is non-zero uses the same argument of propagation (and yields a similar result as Theorem 7.4). ■

Let us finish with an analogue of Theorem 7.4 for (t, s) -almost thin-sparse operator. As the proof is very similar to the one of Theorem 7.4, we leave it to the reader. This result will be useful to prove Theorem 4.2.

Theorem 8.5. *Fix some $t, s > 0$ and some $1 \leq p_0 \leq \infty$. Let X be a doubling metric space with the doubling constant D , and let Y be any set. Let A be (t, s) -almost thin-sparse in ℓ^p for all $p \geq p_0$. Then there is $c = c(r, D, t, s) > 0$, and $\delta = \delta(D, t, s) > 0$ such that for all $p_0 \leq p, q \leq \infty$,*

$$\lambda_p(A) \geq c\lambda_q(A)^\delta.$$

9 Proof of Theorem 4.2

Theorem 4.2 results from Theorem 8.5 and from the fact that thin- \emptyset operators that are bounded in ℓ^p are $(1, 1/p - 1/q)$ -almost thin-sparse for all $q > p$. This is a consequence of the following proposition.

Proposition 9.1. *Let $X = (X, d)$ be a metric space such that balls of radius r have cardinality at most $v(r)$, and let Y be a set. Fix some $\varepsilon > 0$ and some $r \geq 1$. Let $A = (a_{y,x})_{(y,x) \in Y \times X}$ be a thin- \emptyset operator of thickness $\leq r$ such that $\|A\|_{p \rightarrow p} = 1$ for some $1 \leq p < \infty$. Then, there is $C = C(\varepsilon)$ such that for every $q \geq p + \varepsilon$ and every $m \in \mathbf{N}$, there exists a thin-sparse operator A_m of thickness $\leq r$, sparseness $\leq m$ such that*

$$\| \|A - A_m\| \| \leq \frac{Cv(r)^{1-1/q}}{m^{1/p-1/q}}.$$

Proof: First, let us prove the following lemma.

Lemma 9.2. *Let n be a positive integer, and $0 < a_n \leq \dots \leq a_1$ such that $\sum_{i=1}^n a_i^p = 1$, then for all $0 \leq m \leq n$,*

$$\left(\sum_{i=m-n}^n a_i^q \right)^{1/q} \leq \frac{(p/q)^{1/q} (1 - p/q)^{1/p-1/q}}{m^{1/p-1/q}}. \quad (9.1)$$

In particular, for every $\varepsilon > 0$ there exists $C = C(\varepsilon)$ such that for all $q \geq p + \varepsilon$,

$$\left(\sum_{i=m-n}^n a_i^q \right)^{1/q} \leq \frac{C}{m^{1/p-1/q}}.$$

Proof of the lemma. Let us find the maximum of the function

$$\theta_{m,q}(a_1, \dots, a_n) = \sum_{m+1}^n a_i^q,$$

under the conditions

$$\sum_{i=1}^n a_i^p = 1,$$

and for all $1 \leq i \leq n-1$,

$$a_{i+1} - a_i \leq 0.$$

Claim 9.3. *The maximum of $\theta_{m,q}$ is attained at (a_1, \dots, a_n) such that $a_i = 0$ for $i \leq k$ and $a_i = 1/k^p$ for $i < k$, where k is an integer $\geq m+1$.*

Proof of the claim. First, note that for $1 \leq i \leq m$, a_i does not appear in the sum $\sum_{i=m+1}^n a_i^q$. Hence, as (a_i) is non-increasing, the maximum will be attained when $a_i = a_j$ for all $i \leq j \leq m$.

On the other hand, a straightforward application of Lagrange multipliers shows that if $0 < a_{i+1} < a_i < 1$, then $i > m$, and $a_{i+1} = a_i$. Hence there exists $k \geq m$ such that the sequence $a_i = 0$ for $i \leq k$ and $a_i = 1/k^{1/p}$ for $i < k$. Hence,

$$\sum_{m+1}^n a_i^q = \frac{k-m}{k^{q/p}}. \quad \blacksquare \tag{9.2}$$

To finish the proof of the Lemma, note that the derivative of $\frac{k-m}{k^{q/p}}$ with respect to k vanishes exactly at the value $m/(1-p/q)$, which corresponds to a maximum. Replacing k by this value in (9.2) yields (9.1). \blacksquare

Now, let us prove the proposition. As $\|A\|_{p \rightarrow p} = 1$, for every $x \in X$, the column $C_x = (a_{y,x})_{y \in Y}$ has ℓ^p -norm at most 1. By Lemma 9.2, there exists a subset S_x of Y of cardinality $\leq m$ such that

$$\sum_{y \in Y \setminus S_x} |a_{y,x}|^q \leq C^q / m^{q/p-1}.$$

Now, we define A_m from A by replacing the coefficient $a_{y,x}$ by 0 whenever $y \in Y \setminus S_x$. By construction, A_m is thin-sparse of thickness $\leq r$ and sparseness $\leq m$.

Let $f \in \ell^q(X)$. Denote by $C_m = |A - A_m| = (c_{y,x})_{(y,x) \in Y \times X}$. We have

$$\begin{aligned}
\| |A - A_m| f \|_q^q &= \sum_{y \in Y} \left(\sum_{x \in X} c_{y,x} f(x) \right)^q \\
&\leq \sum_{y \in Y} v(r)^{q-1} \left(\sum_{x \in X} c_{y,x}^q |f(x)|^q \right) \\
&= v(r)^{q-1} \sum_{x \in X} |f(x)|^q \sum_{y \in Y \setminus S_x} |a_{y,x}|^q \\
&\leq \frac{C^q v(r)^{q-1}}{m^{q/p-1}} \|f\|_q^q. \blacksquare
\end{aligned}$$

10 Proof of Theorem 2.6

10.1 The algebra of weighted operators indexed by a group G .

Let G be a finitely generated group, and let S be a finite symmetric generating set. Let ω be a weight on G . The space $\Sigma^\omega(G)$ is easily seen to be a Banach algebra. Moreover, by Theorem 5, if $A \in \Sigma^\omega(G)$, and A is invertible as an operator on $\ell^2(G)$, then A^{-1} also belongs to $\Sigma^\omega(G)$.

Proposition 10.1. *Consider a weight $\omega(x) \geq (1 + |x|)^\delta$ for some $\delta > 0$. Then, $\Sigma^\omega(G) \subset T_\delta(G)$.*

Proof: Let A be in $\Sigma^\omega(G)$. It suffices to prove that for all $r > 0$,

$$\| |A_{>r}| \| \leq r^{-\delta} \|A\|_{(1+|x|)^\delta}$$

where $A_{>r}$ is the matrix obtained from A by replacing $a_{x,y}$ by 0 whenever $d(x, y) \leq r$.

$$\begin{aligned}
\|A_{>r}\| &\leq \|A_{>r}\|_1 \\
&\leq r^{-\delta} \sum_{x \in \mathbf{Z}^d, |x| > r} |a_\infty(x)| (1 + |x|)^\delta \\
&\leq r^{-\delta} \|A\|_{(1+|x|)^\delta} \blacksquare
\end{aligned}$$

Using Proposition 10.1, Theorem 8.2 and Theorem 5, we obtain the following particular case of Theorem 2.5.

Theorem 10.2. *Assume that G has polynomial growth. Consider a weight $\omega(x) \geq (1 + |g|)^\delta$ on G with $\delta > 0$ and let $A \in \Sigma^\omega(G)$. Then*

- either $\lambda_p(A) = 0$ for all $1 \leq p \leq \infty$, or there exists $c > 0$ such that $\lambda_p(A) \geq c$ for all $1 \leq p \leq \infty$.
- If $\lambda_p(A) > 0$, then A admits a left-inverse whose adjoint lies in $\Sigma^\omega(G)$.

10.2 The space $\Sigma_T^\omega(G, Y)$

We keep the notation of the previous section.

Let $T : Y \rightarrow G$ be any map. We defined

$$a_{\infty, T}(x) = \sup_z |a_{z, (Tz)x}| \quad \forall x \in G.$$

If ω is a weight on G , we say that $A \in \Sigma_T^\omega(G, Y)$ if $\|a_{\infty, T}\| \in \ell^1(G, \omega)$, and we have

$$\|A\|_{\omega, T} = \|a_{\infty, T}\|_{\ell^1(G, \omega)}$$

Proposition 10.3. *Assume that T is such that pre-images have cardinality at most v . Then, for all weight ω , all $A \in \Sigma_T^\omega(G, Y)$, and all $1 \leq p \leq \infty$,*

$$\|A\|_{p \rightarrow p} \leq v \|A\|_{\omega, T}.$$

Proof: It suffices to prove it for $\omega = 1$. We will prove that

$$\|A\|_{\infty \rightarrow \infty} \leq \|A\|_{\omega, T},$$

and

$$\|A\|_{1 \rightarrow 1} \leq v \|A\|_{\omega, T},$$

and the proposition will follow by interpolation.

For every $x \in G$, consider the row $L_x = (a_{x,y})_{y \in G}$. Let $f \in \ell^\infty(G)$. For every $x \in X$,

$$|Af(x)| = |\langle L_x, f \rangle| \leq \|L_x\|_{\ell^1(G)} \|f\|_\infty \leq \|a_{\infty, T}\|_{\ell^1(G)} \|f\|_\infty.$$

Hence, our first claim follows. Now, let $f \in \ell^1(G)$, $f \geq 0$. We have

$$\begin{aligned} \sum_{x \in \mathbf{Z}^d} |A|f(x) &\leq \sum_{x, z \in \mathbf{Z}^d} |a_{z, (Tz)x}| f((Tz)x) \\ &\leq \sum_{x, z \in \mathbf{Z}^d} a_{\infty, T}(x) f((Tz)x) \\ &\leq \sum_{x \in \mathbf{Z}^d} a_{\infty, T}(x) \sum_{z \in \mathbf{Z}^d} f((Tz)x) \\ &\leq v \|a_{\infty, T}\|_{\ell^1(\mathbf{Z}^d)} \|f\|_{\ell^1(\mathbf{Z}^d)} \end{aligned}$$

In the last inequality, we used the fact that in the sum $\sum_{z \in \mathbf{Z}^d} f((Tz)x)$, each term is repeated at most v times. ■

Proposition 10.4. *Let G be a finitely generated group with polynomial growth of degree d , i.e. $v(r) \leq Kr^d$. Assume that $\omega(x) \geq (1 + |x|)^\delta$ for some $\delta > 0$, and that T has pre-images with bounded cardinality. Then A is $(\delta, \delta/d)$ -almost thin-sparse in ℓ^p for all $1 \leq p \leq \infty$.*

Proof: The proof is very similar to the one of Proposition 10.1. Let v be an upper bound on the cardinality of the pre-images of T . The matrix $A_{\leq r, T}$ (resp. $A_{> r, T}$) is obtained from A by annihilating the coefficients $a_{z, (Tz)x}$ whenever $|x| > r$ (resp. $|x| \leq r$). Note that $A_{\leq r, T}$ is thin-sparse with thickness at most r and sparseness at most $Kr^d v$. By Proposition 10.3, for all $1 \leq p \leq \infty$,

$$\begin{aligned} \|A_{> r, T}\|_{p \rightarrow p} &\leq C \|A_{> r, T}\|_1 \\ &\leq Cr^{-\delta} \sum_{x \in G, |x| > r} a_{\infty, T}(x) (1 + |x|)^\delta \\ &\leq Cr^{-\delta} \|A\|_{\omega, T}. \blacksquare \end{aligned}$$

The following result, follows immediately from the previous proposition and Theorem 8.2. It is a first (the main) step towards Theorem 2.6. The next step will be to prove that the adjoint of the left inverse lies in the same space.

Corollary 10.5. *Suppose that G has polynomial growth, that $\omega(x) \geq (1 + |x|)^\delta$ for some $\delta > 0$, and that T has pre-images with bounded cardinality. Let $A \in \Sigma_T^\omega(G, Y)$. Then either $\lambda_p(A) = 0$ for all $1 \leq p \leq \infty$, or there exists $c > 0$ such that $\lambda_p(A) \geq c$ for all $1 \leq p \leq \infty$.*

10.3 The adjoint of the left-inverse belongs to $\Sigma_T^\omega(G, Y)$

The purpose of this subsection is to complete the proof Theorem 2.5. A first step was achieved by Corollary 10.5. The remaining statement of the theorem is contained in the following proposition.

Proposition 10.6. *Let ω be a weight on \mathbf{Z}^d and let $T : Y \rightarrow G$ be a application whose pre-images have bounded cardinality. Let $A \in \Sigma_T^\omega(G, Y)$ such that $\lambda_2(A) > 0$. Then A admits a left-inverse whose adjoint lies in $\Sigma_T^\omega(G, Y)$.*

To prove this proposition, we first need to show that $A^*A \in \Sigma^\omega(G)$. Hence [FGL2, Theorem 1] implies that $(A^*A)^{-1} \in \Sigma^\omega(G)$. Then, we will have to prove that $((A^*A)^{-1}A^*)^* = A(A^*A)^{-1}$ belongs to $\Sigma_T^\omega(G)$. These two facts follow from the following proposition, which is actually true for any Cayley graph (G, S) equipped with a weight ω .

Proposition 10.7. *Consider a weight ω on a group G . Let $T : Y \rightarrow G$ be a application whose pre-images have bounded cardinality.*

(i) Let $A, B \in \Sigma_T^\omega(G, Y)$. Then $A^*B \in \Sigma^\omega(G)$.

(ii) let $A \in \Sigma_T^\omega(G, Y)$ and $B \in \Sigma^\omega(G)$. Then $AB \in \Sigma_T^\omega(G)$.

Proof: Let v be an upper bound on the cardinality of pre-images of T .

Let us prove (i). For every $x \in G$, let A_x be the matrix obtained from A by replacing all the coefficients by zero except for those indexed by $(z, (Tz)x)$ for all $z \in Y$. Note that $\|A_x\|_{\omega, T} = a_\infty(x)\omega(x)$. Hence,

$$\|A\|_{\omega, T} = \sum_{x \in G} \|A_x\|_{\omega, T}.$$

First, let us prove two lemmas.

Lemma 10.8. Let $x, y \in G$. Then the support of $(A_x)^*B_y$ is included in $\{((Tz)x, (Tz)y), z \in Y\}$.

Proof: The general non-zero coefficient of $(A_x)^*$ is $a_{z, (Tz)x}$, and has position $((Tz)x, z)$. The general non-zero coefficient of B_y is $b_{z', (Tz')y}$ and has position $(z', (Tz')y)$. Hence, the general non-zero coefficient of the product has position $((Tz)x, (Tz)y)$, in which case it equals $\sum_{z' \in Y, Tz' = Tz} a_{z', (Tz')x} b_{z', (Tz')y}$, where the sum has at most v elements. ■

Lemma 10.9. Let $x, y \in G$. Then there exists $K < \infty$ such that $\|(A_x)^*B_y\|_\omega \leq K\|A_x\|_{\omega, T}\|B_y\|_{\omega, T}$.

Proof: First, note that there exists C such that

$$\omega(x - y) \leq C\omega(x)\omega(y).$$

Using this observation and the previous lemma, we have

$$\begin{aligned} \|(A_x)^*B_y\|_\omega &\leq \sup_{z \in Y} \sum_{z' \in Y, Tz' = Tz} |a_{z', (Tz')x} b_{z', (Tz')y}| \omega(x - y) \\ &\leq v a_{\infty, T}(x) b_{\infty, T}(y) \omega(x - y) \\ &= v C a_{\infty, T}(x) b_{\infty, T}(y) \omega(x) \omega(y) \\ &= v C \|A_x\|_{\omega, T} \|B_y\|_{\omega, T} \quad \blacksquare \end{aligned}$$

Using the last lemma, we can finish the proof of the proposition.

$$\begin{aligned}
\|A^*B\|_\omega &= \left\| \left(\sum_{x \in G} (A_x)^* \right) \left(\sum_{y \in G} B_y \right) \right\|_\omega \\
&= \left\| \sum_{w \in G} \sum_{xy=w} (A_x)^* B_y \right\|_\omega \\
&\leq \sum_{w \in G} \sum_{xy=w} \|(A_x)^* B_y\|_\omega \\
&\leq K \sum_{w \in G} \sum_{xy=w} \|A_x\|_{\omega, T} \|B_y\|_{\omega, T} \\
&= K \|A\|_{\omega, T} \|B\|_{\omega, T}. \blacksquare
\end{aligned}$$

The proof of (ii) is similar, and relies on the following statement (left to the reader), that plays the same role of Lemma 10.8. Recall that here, $A \in \Sigma_T^\omega(G, Y)$ and $B \in \Sigma^\omega(G)$.

Lemma 10.10. *Let $x, y \in G$. Then the support of $A_x B_y$ is included in $\{(z, (Tz)xy), z \in Y\}$. \blacksquare*

11 Further generalizations.

In this section, we discuss generalizations of the above results that do not affect the proofs.

11.1 Matrices with operator coefficients

It is interesting to note that (as also remarked in [ABK]), except for the results using Theorem 5, we can allow the matrix coefficients to be bounded operators between Banach spaces. To be precise, let X and Y be two sets, let $(E_x)_{x \in X}$ and $(F_y)_{y \in Y}$ be two families of Banach spaces. For every $1 \leq p \leq \infty$, we denote $\ell^p(X, (E_x))$ the Banach space of sequences $e = (e_x)_{x \in X}$, equipped with the norm

$$\|e\|_p = \left(\sum_{x \in X} \|e_x\|^p \right)^{1/p}$$

when $p > \infty$ and

$$\|e\|_\infty = \sup_{x \in X} \|e_x\|.$$

Let $c_0(X, (E_x)_x)$ be the subspace of $\ell^\infty(X, (E_x))$ of sequences such that $\|e_x\| \rightarrow 0$ when x leaves eventually every finite subset. Now, for every $(y, x) \in Y \times X$, let

$a_{y,x} : E_x \rightarrow F_y$ be a bounded operator. We denote by A the corresponding matrix, and by $\|A\|_{p \rightarrow p}$ its norm of operator when it induces a bounded operator $\ell^p(X, (E_x)) \rightarrow \ell^p(Y, (F_y))$.

Claim 11.1. *Theorems 4.1, 4.2, 6.1, 8.2, and Corollary 10.5 are true under the assumption that the coefficients of A are bounded operators between Banach spaces. And as Theorem 5 actually holds for matrices whose entries are bounded operators between Hilbert spaces, Theorem 2.6 can be generalized to matrices whose coefficients are bounded operators from Hilbert spaces to Banach spaces.*

11.2 Reduction of Theorem 2.1 to matrices which are concentrated about the diagonal

The reduction is an immediate consequence of the following easy fact.

Claim 11.2. *Let X be a metric space. Fix some $r, v \in \mathbf{N}$ and assume that the balls of radius r in X have cardinality at most n for some $n \in \mathbf{N}$. Let $A = (a_{y,x})_{(y,x) \in Y \times X}$ be an infinite matrix whose rows are supported in balls of radius r and such that columns have supports with cardinality at most v . Then there is a map $j : Y \rightarrow X$ whose pre-images have cardinality at most nv and such that the coefficient $a_{y,x} = 0$ if $d(j(y), x) > r$. In particular, for all $1 \leq p \leq \infty$, the space $\ell^p(Y)$ is isometric to a subspace of $\ell^p(X, \ell^p(\{1, \dots, nv\}))$ such that the operator $\tilde{A} : \ell^p(X) \rightarrow \ell^p(X, \ell^p(\{1, \dots, nv\}))$ induced by $A : \ell^p(X) \rightarrow \ell^p(Y)$ is concentrated about the diagonal. Its coefficients are operators $\tilde{a}_{j(y),x} : \mathbf{R} \rightarrow \ell^p(\{1, \dots, nv\})$.*

Proof: Define the map $j : Y \rightarrow X$ as follows: for every $y \in Y$, let $j(y)$ be an element of X such that the row L_y is supported in $B(j(y), r)$. The rest of proof is straightforward. ■

11.3 Matrices indexed by more general measure spaces

One of the main interests of Claim 11.1 is that it will enable us to replace the spaces X and Y by more general measure spaces. The doubling condition that we need in the proofs is purely metric, so in order to generalize to non-discrete metric spaces, we need to reformulate it as follows.

Definition 11.3. A metric space (X, d) is doubling (at large scale) if there exists $N \in \mathbf{N}$ such that for all $r \geq 1$, balls of radius $2r$ can contain at least N disjoint balls of radius r .

Discrete setting: The first obvious generalization is to equip X and Y with discrete measures μ and ν . We will assume that these measures are non-degenerate, i.e. that for all $x \in X$ $\mu(x) > 0$, and for all $y \in Y$, $\nu(y) > 0$. Then for every $x \in X$, the space E_x is just \mathbf{R} , equip with the norm $|t|_\mu = |t|\mu(t)$ (note that this is a Hilbert space). Theorems 4.1, 4.2, 6.1, 8.2 are true under this generality. In order to generalize Theorem 2.6 to this setting using the last section, we need to view the coefficients as operators (between one dimensional Banach spaces). Namely, we will say that A belongs to $\Sigma_T^\omega((\mathbf{Z}^d, \mu), (Y, \nu))$ if $a_{\infty, T}$, defined as

$$a_{\infty, T}(x) = \sup_{y \in \mathbf{Z}^d} \frac{|a_{y, Ty+x}| \nu(y)}{\mu(Ty+x)}$$

lies in $\ell^1(\mathbf{Z}^d, \omega)$. With this definition of the space $\Sigma_T^\omega((\mathbf{Z}^d, \mu), (Y, \nu))$, Theorem 2.6 is true for any non-degenerate measures μ on \mathbf{Z}^d and ν on Y .

Continuous setting: First, let us start with an informal discussion about how our results can be generalized to a continuous setting. Then, we will formulate precise statements in two concrete situations.

The next step in the generalization is to take, for X , a metric measure space (X, d, μ) such that (X, d) satisfies the doubling condition. For Y , one can consider any measure space (Y, ν) . Note that to fully extend our results to this setting, a first step would be to prove them in the case of compact spaces: i.e. when $X = Y = [0, 1]$. To avoid this difficulty, we will not work with L^p -spaces, but with spaces of functions which are roughly speaking, locally in L^2 , and in ℓ^p at “large-scale”.

To apply our results in that setting, we need to be able to “discretize” these spaces, i.e. to find a discrete doubling subspace \tilde{X} of X , and a countable subset \tilde{Y} of Y together with isomorphisms

$$\ell^p L_{loc}^2(X, \mu) \simeq \ell^p \left(\tilde{X}, (E_x)_{x \in \tilde{X}} \right),$$

and

$$\ell^p L_{loc}^2(Y, \nu) \simeq \ell^p \left(\tilde{Y}, (F_y)_{y \in \tilde{Y}} \right),$$

where E_x and F_y are spaces of functions supported “about” x and y . The above isomorphisms allow us to replace an operator $A : \ell^p L_{loc}^2(X, \mu) \rightarrow \ell^p L_{loc}^2(X, \nu)$ by an operator $\tilde{A} : \ell^p(\tilde{X}, (E_x)) \rightarrow \ell^p(\tilde{Y}, (F_y))$. In order to apply Claim 11.1 to A , we need to the coefficients of \tilde{A} to be bounded operators.

Instead of working in the most possible general setting, let us describe two examples that can be useful for applications. In both situations, $X = \mathbf{R}^d$ will be equipped with its ℓ^∞ -norm and with the Lebesgue measure (more generally,

one can consider a locally compact group equipped with a left invariant metric which is proper and finite on compact subsets, but we leave this straightforward generalization to the reader). For all $1 \leq p \leq \infty$, all continuous, compactly supported continuous f , and all $x \in \mathbf{R}^d$ we denote by

$$\|f\|_{p,B(x,1/2)} = \|f \cdot 1_{B(x,1/2)}\|_p,$$

where 1_Ω denotes the characteristic function of the subset Ω .

We will consider the Banach space $\ell^p L_{loc}^2(\mathbf{R}^d)$ obtained by taking the completion of continuous, compactly supported functions on \mathbf{R}^d under the norm

$$\left(\sum_{x \in \mathbf{Z}^d} \|f\|_{2,B(x,1/2)}^p \right)^{1/p}.$$

As the family $(B(x,1/2))_{x \in \mathbf{Z}^d}$ forms an almost-partition of \mathbf{R}^d it follows that $\ell^p L_{loc}^2(\mathbf{R}^d)$ is isometric to $\ell^p(\mathbf{Z}^d, L^2(B(x,1/2))_{x \in \mathbf{Z}^d})$.

First example. Here, we suppose that (Y, ν) is any discrete space equipped with a measure ν . Consider an application $T : Y \rightarrow \mathbf{R}^d$ such that

$$\sup_{x \in X} |T^{-1}(\{x\})| < \infty.$$

Let ω be a weight on \mathbf{R}^d . Let $A : L^2(\mathbf{R}^d) \rightarrow \ell^2(Y, \nu)$ be a bounded operator. For all $(x, y) \in \mathbf{Z}^d \times Y$ define the ‘‘coefficient $A_{y,x}$ of A ’’ to be the operator

$$A_{y,x} = P_{\{y\}} A P_{B(x,1/2)} : L^2(B(x,1/2)) \rightarrow \mathbf{R},$$

where P_Z denotes the projector on functions supported on Z .

Let us say that an operator A belongs to the space $\Sigma_T^\omega(\mathbf{R}^d, Y)$ if $a_{\infty,T} \in \ell^1(\mathbf{Z}^d)$, where

$$a_{\infty,T}(x) = \sup_{y \in \mathbf{Z}^d} \|A_{y,Ty+z}\| \nu(y).$$

This space is a Banach space for the norm

$$\|A\|_{\Sigma_T^\omega(\mathbf{R}^d, Y)} = \|a_{\infty,T}\|_{\ell^1(\mathbf{Z}^d, \omega)}.$$

Now, we can reformulate Theorem 2.6 as follows.

Theorem 11.4. *Consider a weight $\omega(x) \geq (1 + |x|)^\delta$ on \mathbf{R}^d , for some $\delta > 0$. Let $A \in \Sigma_T^\omega(\mathbf{R}^d, Y)$. The operator $A : \ell^p L_{loc}^2(\mathbf{R}^d, \mu) \rightarrow \ell^p(Y, \nu)$ is bounded below for some $1 \leq p \leq \infty$ if and only if it is bounded below for all p . Moreover, in the latter case, A admits a left-inverse whose adjoint lies in $\Sigma_T^\omega(\mathbf{Z}^d, Y)$.*

Second example. Now, let us assume that $Y = \mathbf{R}^d$. Let $T : \mathbf{R}^d \rightarrow \mathbf{R}^d$ be any application whose pre-images intersect only a bounded number of balls of radius 1. Let $A : L^2(\mathbf{R}^d) \rightarrow L^2(\mathbf{R}^d)$ be an bounded operator. For all $x, y \in \mathbf{Z}^d$, define the coefficient $A_{y,x}$ to be the operator

$$A_{y,x} = P_{B(y,1/2)} A P_{B(x,1/2)} : L^2(B(x,1/2)) \rightarrow L^2(B(y,1/2)).$$

Let us say that an operator A belongs to the space $\Sigma_T^\omega(\mathbf{R}^d, \mathbf{R}^d)$ if $a_{\infty,T} \in \ell^1(\mathbf{Z}^d)$, where

$$a_{\infty,T}(x) = \sup_{y \in \mathbf{Z}^d} \|A_{y, Ty+z}\|_{2 \rightarrow 2}.$$

This space is a Banach space for the norm

$$\|A\|_{\Sigma_T^\omega(\mathbf{R}^d, Y)} = \|a_{\infty,T}\|_{\ell^1(\mathbf{Z}^d, \omega)}.$$

We can formulate Theorem 2.6 as follows.

Theorem 11.5. *Consider a weight $\omega(x) \geq (1 + |x|)^\delta$ on \mathbf{R}^d , for some $\delta > 0$. Let $A \in \Sigma_T^\omega(\mathbf{R}^d, \mathbf{R}^d)$. The operator $A : \ell^p L_{loc}^2(\mathbf{R}^d) \rightarrow \ell^p L_{loc}^2(\mathbf{R}^d)$ is bounded below for some $1 \leq p \leq \infty$ if and only if it is bounded below for all p . Moreover, in the latter case, A admits a left-inverse whose adjoint lies in $\Sigma_T^\omega(\mathbf{R}^d, \mathbf{R}^d)$.*

12 Concluding remarks, applications and questions

We did not discuss the possible applications of our main results to frame theory, and sampling theory as these applications are already mentioned in [ABK]. On the other hand, note that we explained in Section 11.3 how to extend Theorem 2.6 to a continuous setting (i.e. operators on \mathbf{R}^n), which was not done in [ABK].

For the sake of clarity, we chose not to mention our most precise results in the introduction, and to postpone them to Section 4. However we believe that these precisions (concerning the dependance of the uniform lower bound of $\lambda_p(A)$) can be of some interest. For instance, we can deduce an explicit bound for the left-inverse of A in the following particular case. Recall that $|a|_\infty = \sup_{(y,x) \in Y \times X} |a_{y,x}|$.

Theorem 12.1. *Let $A = (a_{y,x})_{(y,x) \in \mathbf{Z}^2}$ be a thin-thin operator of thickness $\leq r$. Assume that $\lambda_p(A) > 0$ for some $p \in [1, \infty]$. Then for all $q \in [1, \infty]$, we have*

$$\|B\|_q \leq \frac{6 \cdot 162^5 |a|_\infty^7 r^{17}}{\lambda_p(A)^8},$$

where $B = (A^* A)^{-1} A^*$.

Proof: Make quantitative the discussion following Proposition 2.14, using Remark 7.6 (two times: to go from $\lambda_p(A)$ to $\lambda_2(A)$, and then from $\lambda_2(A^*A)$ to $\lambda_p(A^*A) = 1/\|(A^*A)^{-1}\|_p$), plus the fact that, if $C = (A^*A)$, we have $|c|_\infty \leq v(r)|a|_\infty^2 \leq 3r|a|_\infty^2$. ■

Finally, let us invoke some possible generalizations of Theorem 2.6.

- First, as mentioned in the introduction, a particular case of doubling metric spaces are nilpotent groups, and hence it would be interesting to generalize Theorem 2.6 to this non-abelian setting.
- Another possible generalization that was suggested to us by Catalin Badea would be to replace $\lambda_p(A)$ by $\gamma_p(A) = \inf_f \frac{\|Af\|_p}{d_{\ell^p}(f, \text{Ker}A)}$, and hence in Theorem 2.6, left-inverses by Moore-Penrose pseudo inverses.

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