UNIVERSITÉ DE CERGY-PONTOISE

ET

UNIVERSITÉ DE NEUCHATEL

THÈSE DE DOCTORAT

Spécialité Mathématiques

Ecole doctorale Economie et Mathématiques de Paris-Ouest.
Présentée par **Romain Tessera**

Questions d'analyse et de géométrie sur les espaces métriques mesurés et les groupes

Soutenue le 28 Juin 2006

Jury :

Thierry Coulhon (Directeur)
Bruno Colbois (Examinateur)
Pierre de la Harpe (Examinateur)
Pierre Pansu (Rapporteur)
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Introduction

Cette thèse rassemble plusieurs travaux, certains réalisés sous l'impulsion de mes directeurs de thèse, d'autres provenant d'une démarche plus personnelle. Elle se répartit en sept articles, dont deux ont été écrits en collaboration avec Alain Valette et Yves de Cornulier. Ce mémoire s'organise en deux parties. La première traite d'actions par isométries et de plongements uniformes dans un espace de Banach ; la seconde d'isopérimétrie à grande échelle dans les espaces métriques mesurés, avec quelques applications au cas des groupes localement compacts et notamment un théorème ergodique. Notons que ces deux parties ne sont pas complètement étrangères l'une à l'autre, la notion d'isopérimétrie à grande échelle apparaissant comme un outil central dans la démonstration de plusieurs résultats de la première partie. Par ailleurs, la première partie comporte elle-même un résultat nouveau et optimal d'isopérimétrie dans les groupes de Lie moyennables.

Dans cette introduction et sauf exception, dans toute l'exposition qui suit, on désigne par comportement asymptotique (en l'infini) d'une fonction croissante positive *non nulle* $f : \mathbf{R}_+ \to \overline{\mathbf{R}}_+$ la classe de f modulo la relation d'équivalence :

$$f \approx g \Leftrightarrow \exists C < \infty,\ C^{-1}g(C^{-1}t) - C \leq f(t) \leq Cg(Ct) + C.$$

On écrit $f \preceq g$ (resp. $f \prec g$) s'il existe $C < \infty$ tel que $f(t) \leq g(Ct) + C$ (resp. si pour tout $c > 0$, $f(t) = o(g(ct))$.

Partie I : actions par isométries et plongements uniformes dans un espace de Banach

L'étude dynamique des actions par isométries sur un espace de Banach a fait l'objet d'une attention particulière depuis les années 70 avec notamment la découverte de la propriété (T) par Kazhdan [Kaz] et de la propriété de Haagerup [AW]. Les conséquences spectaculaires de ces deux propriétés, comme le fait qu'un groupe (T) est de type fini [Kaz], ou bien la validation de la conjecture de Baum-Connes pour les groupes ayant la propriété de Haagerup [HiKa], ont motivé leurs généralisations au contexte des espaces de Banach superréflexifs (voir par exemple [BFGM] et [Yu2]).

Rappelons [Del, Gu1] qu'un groupe localement compact σ-compact a la propriété (T) si toute action par isométries affines sur un espace de Hilbert a un point fixe. À l'inverse, un tel groupe satisfait la propriété de Haagerup s'il agit proprement par isométries sur un espace de Hilbert. Il découle immédiatement des définitions qu'un groupe est simultanément (T) et Haagerup si et seulement si il est compact.

L'objectif principal du présent travail est d'étudier la propriété de Haagerup généralisée à des espaces de Banach uniformément convexes, principalement les espaces L^p, sous un angle quantitatif. Étant donnée une action propre par isométries σ d'un groupe localement compact compactement engendré G sur un espace de Banach E, on peut s'intéresser à la "vitesse" à laquelle les orbites tendent vers l'infini dans E. Comme G est compactement engendré, on dispose d'une métrique de référence invariante à gauche : la métrique des mots associée à une partie génératrice symétrique compacte S de G, notée d_S. La longueur d'un élément g de G est définie par $|g|_S = d_S(1, g)$. On introduit la compression de l'action σ qui est une fonction croissante $\rho : \mathbf{R}_+ \to \mathbf{R}_+$ définie par :

$$\rho(t) = \inf_{|g|_S \geq t} \|\sigma(g) \cdot 0\|.$$

La compression croît au plus linéairement, i.e. $\rho(t) \preceq t$, et peut croître plus lentement, par exemple en t^a avec $a < 1$. Fixons $1 \leq p < \infty$. On peut par commodité[1] définir un taux de compression associé aux actions par isométries sur un espace L^p : ce taux, noté $B_p(G)$ est la borne supérieure des $0 \leq a \leq 1$ tels qu'il existe une action par isométrie de G sur un espace L^p avec compression $\rho(t) \succeq t^a$.

Il est connu [CCJJV, Proposition 6-1-5] que les groupes moyennables ont la propriété de Haagerup. La réciproque est fausse puisque par exemple, le groupe libre à deux générateurs est Haagerup. Une réciproque faible a néanmoins été démontrée par Guentner et Kaminker[2], fournissant ainsi un nouveau critère de moyennabilité : un groupe agissant proprement sur un espace de Hilbert avec une compression vérifiant $\rho(t) \succ t^{1/2}$ est moyennable. L'action par isométrie "classique" du groupe libre à deux générateurs ayant une compression $\rho(t) = t^{1/2}$, ce résultat est en un certain sens optimal.

Par ailleurs, si l'on se restreint aux groupes moyennables, le nombre $B_2(G)$ est un invariant de quasi-isométrie. Il devient alors intéressant de le comparer à certaines quantités géométriques bien connues comme l'asymptotique du profil isopérimétrique ou de la probabilité de retour d'une marche aléatoire réversible à support compact sur G.

Compression et isopérimétrie dans les boules

Dans ce travail, je montre que la connaissance des profils isopérimétriques-L^p dans les boules, pour $1 \leq p < \infty$, permettent de construire des actions par isométries sur des espaces L^p avec de bonnes compressions.

Définition. *Fixons $1 \leq p < \infty$. Soit G un groupe localement compact, compactement engendré et soit S une partie symétrique génératrice compacte de G.*

[1] Mais il est important de ne pas se limiter à l'étude de $B_p(G)$, qui ne représente qu'une approximation des comportements possibles de la compression : en particulier d'éventuels facteurs logarithmiques n'apparaissent pas.

[2] Ils prouvent le résultats pour un groupe de type fini dans [GuKa]. Dans le chapitre 3, nous étendons leur résultat à tout groupe localement compact compactement engendré à l'aide d'arguments plus directs.

Le profil isopérimétrique-L^p dans les boules de G est la fonction qui à tout réel positif r associe

$$J^b_{G,p}(r) = \sup_{\varphi \in L^p(B(1,r))} \frac{\|\varphi\|_p}{\sup_{s\in S} \|\lambda_{G,p}(s)\varphi - \varphi\|_p},$$

où $B(1,r)$ est la boule fermée centrée en l'élément neutre, de rayon r pour la métrique des mots associée à S ; et $\lambda_{G,p}$ désigne la représentation régulière gauche de G dans $L^p(G)$.

Signalons que pour $p = 1$,

$$J^b_{G,1}(r) \approx \sup_{A\subset B(1,r)} \frac{\mu(A)}{\mu(SA \vartriangle A)};$$

et que pour les groupes de Lie unimodulaires, $J^b_{G,2}$ correspond, à constante multiplicative près, à la première valeur propre du laplacien[3] dans $B(1,r)$. Notons enfin que $J^b_{G,p}(r) \preceq r$. Le lien entre $J^b_{G,p}$ et la compression des actions affines sur un espace L^p est résumé par le théorème suivant.

Théorème. *Fixons $1 \le p < \infty$. Soit G un groupe localement compact compactement engendré. Pour toute fonction croissante $f : \mathbf{R}_+ \to \mathbf{R}_+$ vérifiant la condition d'intégrabilité suivante*

$$\int_1^\infty \left(\frac{f(t)}{J^b_{G,p}(t)}\right)^p \frac{dt}{t} < \infty, \tag{0.0.1}$$

il existe une action par isométries affine de G sur $L^p(G)$ dont la partie linéaire est $\lambda_{G,p}$, et dont la compression vérifie $\rho \succeq f$.

Alors que le profil isopérimétrique-L^2 habituel (voir §6.1.3) est une quantité bien étudiée pour les groupes de type fini du fait de ses liens avec la probabilité de retour des marches aléatoires, le profil isopérimétrique-L^2 dans les boules lui, semble peu connu.

Un aspect important de mon travail a été de démontrer que $J^b_{G,p}(r) \approx r$ pour une classe de groupes moyennables incluant les groupes de Lie connexes moyennables, et pour tout $1 \le p < \infty$. On en déduit avec le théorème précédent que $B_p(G) = 1$ pour cette classe de groupes.

J'ai également relié le profil-L^2 dans les boules à la croissance du volume et à la probabilité de retour des marches aléatoires sur G. Pour certains groupes de type fini obtenu par produit en couronne, j'utilise alors des estimations connues sur cette probabilité de retour pour en déduire une borne inférieure du profil-L^2 dans les boules.

[3] C'est aussi vrai pour le laplacien discret dans les groupes de type fini ; et plus généralement, pour un groupe localement compact unimodulaire, avec la définition de laplacien à grande échelle proposée pour les espaces métriques mesurés en deuxième partie.

La compression peut se définir plus généralement pour toute application entre deux espaces métriques. En particulier, si $F:\ G \to E$ est un plongement uniforme d'un groupe localement compact, compactement engendré G dans un espace de Banach E, on définit la compression

$$\rho(t) = \inf_{d_S(g,h)\geq t} \|F(g) - F(h)\|.$$

En combinant le théorème précédent, une généralisation d'un résultat de Bourgain sur les plongements d'arbres dans les espaces de Banach superréflexifs et un résultat d'existence de semi-groupes plongés quasi-isométriquement dans les groupes résolubles à croissance exponentielle, obtenu avec Yves de Cornulier [CT], je parviens à caractériser les comportements asymptotiques possibles de la compression associée à des plongements uniformes d'une large classe de groupes dans un espace L^p, pour tout $1 \leq p < \infty$. Cette classe inclut en particulier les groupes de Lie connexes à croissance exponentielle et les groupes de type fini hyperboliques non-élémentaires. Les résultats obtenus sont nouveaux, y compris dans le cas hilbertien et pour le groupe libre à deux générateurs.

Non-existence de plongements quasi-isométriques dans un espace de Hilbert

Dans une étude plus spécifique au cas hilbertien, menée en collaboration avec Alain Valette et Yves de Cornulier, nous avons obtenu des démonstrations entièrement nouvelles et remarquablement peu techniques des deux théorèmes suivants.

Théorème. *[Bou] Un arbre 3-régulier ne se plonge pas quasi-isométriquement dans un espace de Hilbert.*

Théorème. *Soit G un groupe de Lie moyennable, ou bien un groupe polycyclique, se plongeant quasi-isométriquement dans un espace de Hilbert. Alors G agit proprement et co-compactement par isométries sur un espace euclidien. En particulier, si G est polycyclique, alors il est virtuellement abélien.*

On peut en déduire le corollaire suivant.

Corollaire. *[Pau] Un groupe de Lie nilpotent simplement connexe non abélien ne se plonge pas quasi-isométriquement dans un espace de Hilbert.*

Notons que ces résultats subsistent lorsque l'on remplace espace de Hilbert par espace de Banach superréflexif mais que nos arguments ne semblent pas s'adapter à une telle généralité.

Ces deux théorèmes découlent d'une étude générale de la croissance des 1-cocycles à valeurs dans une représentation unitaire. Étant donnée une représentation unitaire continue $(\pi, \mathcal{H})$ d'un groupe localement compact G, on appelle 1-cocycle toute application continue de G vers $\mathcal{H}$ vérifiant la relation de cocycle $b(gh) = \pi(g)b(h) + b(g)$. Rappelons qu'à un tel 1-cocycle b est toujours associée une unique action par isométries σ de G sur $\mathcal{H}$ de partie linéaire π telle que

$b(g) = \sigma(g)0$. La démarche employée pour démontrer les théorèmes précédents fait appel à un résultat de Guichardet [Gu1] sur la cohomologie réduite des représentations unitaires de certains groupes moyennables incluant les groupes de Lie moyennables. Ce résultat, assez simple dans le cas des groupes de Lie nilpotents, nous permet de montrer qu'il n'existe pas pour ces groupes, d'action par isométries dont les orbites sont plongées quasi-isométriquement. Pour en déduire que ces groupes ne se plongent pas quasi-isométriquement dans un espace de Hilbert, nous utilisons un argument de moyenne assez standard (mais néanmoins non trivial si le groupe n'est pas discret) qui nous ramène au cas d'une action par isométries affines.

Non-existence d'orbites denses

Dans un article plus court également écrit avec Alain Valette et Yves de Cornulier, nous étudions la structure des orbites pour les actions par isométries affines. Nous introduisons la notion de représentation fortement cohomologique qui est une représentation unitaire telle que toute sous-représentation possède de la 1-cohomologie. Nous déduisons des propriétés générales de ces représentations un théorème de structure des orbites pour les groupes nilpotents.

Théorème. *Soit G un groupe nilpotent agissant isométriquement sur un espace de Hilbert $\mathcal{H}$. Soit π la partie linéaire de cette action. Soit $\mathcal{O}$ une orbite sous cette action. Alors il existe*
- *un sous-espace fermé T de $\mathcal{H}$ (la "partie translation"), contenu dans le sous-espace des vecteurs invariants de π, et*
- *un convexe localement fermé U inclus dans orthogonal de T,*

tel que $\mathcal{O}$ est contenu dans $T \times U$.

Nous en déduisons qu'un groupe nilpotent localement compact, compactement engendré, agissant par isométries sur un espace de Hilbert de dimension infinie, n'admet pas d'orbite dense. Par contre, nous montrons qu'il existe une action isométrique d'un groupe métabélien à trois générateurs sur $\ell^2(\mathbf{Z})$, dont toutes les orbites sont denses.

Annulation de la L^p-cohomologie réduite.

Dans le cas $p = 1$, le comportement asymptotique linéaire de $J^b_{G,p}$ pour les groupes de Lie moyennables peut se formuler de la manière suivante.

Théorème. *Soit G un groupe de Lie connexe moyennable et S une partie compacte génératrice de G. Alors G possède une suite de Følner $(F_n)_{n\in\mathbf{N}}$ vérifiant les conditions suivantes :*
(i) il existe une constante $c > 0$ tel que pour tout $n \in \mathbf{N}$ et tout $s \in S$,

$$\mu(sF_n \vartriangle F_n) \leq c\mu(F_n)/n;$$

(ii) pour tout $n \in \mathbf{N}$,

$$F_n \subset B(1, n).$$

Ce théorème améliore une construction similaire due à Pittet [Pit2] au sens où, d'une part, on ne demande pas au groupe d'être unimodulaire, et d'autre part, le contrôle du diamètre des F_n est une propriété qui n'est pas vérifiée en général par les suites construites[4] dans [Pit2].

Ce résultat joue un rôle clé dans la démonstration du théorème suivant. Soit (X, m) un espace mesuré. On dira qu'une action mesurable d'un groupe localement compact G sur (X, m) est mélangeante si elle préserve la mesure et si pour toute partie mesurable de mesure finie A de X,

$$m(gA \cap A) \to 0$$

lorsque g sort de tout compact de G. Pour tout $p \geq 1$, G agit alors par isométries vectorielles sur $L^p(X, m)$ et on note π_p la représentation correspondante.

Théorème. *Fixons $1 < p < \infty$. Soit G un groupe de Lie moyennable agissant de manière mélangeante sur un espace mesuré X. Alors toute action affine σ de G sur $L^p(X, m)$, de partie linéaire π_p, admet une suite de points presque fixes, i.e. une suite $u_n \in L^p(X, m)$ telle que pour tout $g \in G$, $\|\sigma(g)u_n - u_n\| \to 0$. En d'autres termes, $\overline{H^1}(G, \pi_p) = 0$.*

Pour obtenir une suite de points presque fixes[5], on moyenne l'action sur une suite de Følner. Il reste alors à démontrer, en utilisant le fait que l'action est mélangeante, que la croissance du 1-cocycle est strictement sous-linéaire, i.e. que $\|\sigma(g)0\| = o(|g|)$. Ce dernier point, bien que non-trivial, est quant à lui valable pour tout groupe localement compact, compactement engendré.

En particulier, ceci s'applique à la représentation régulière $\lambda_{G,p}$. J'en déduis le corollaire suivant, conjecturé par Pansu dans [Pa2]. Rappelons qu'une variété de dimension d est dite fermée à l'infini s'il existe une suite exhaustive de parties compactes à bord régulier (A_n) telle que $\mu_{d-1}(\partial A_n)/\mu_d(A_n) \to 0$, où ∂A_n est le bord de A_n et μ_k désigne la mesure riemannienne d'une sous-variété de M de dimension k.

Corollaire. *Une variété riemannienne homogène fermée à l'infini n'a pas de cohomologie réduite dans L^p pour tout $1 < p < \infty$.*

On dit qu'une variété riemannienne M vérifie une propriété de Liouville-D_p si toute fonction p-harmonique sur M dont le gradient est dans L^p est constante. On peut alors reformuler le corollaire précédent.

Corollaire. *Une variété riemannienne homogène fermée à l'infini vérifie la propriété de Liouville-D_p pour tout $1 < p < \infty$.*

[4] Ce contrôle est remplacé dans [Pit2] par un contrôle exponentiel du volume de F_n. Comme le volume des boules est au plus exponentiel, cette condition est donc plus faible.

[5] Cette idée est déjà présente dans notre premier article commun avec Cornulier et Valette ; voir aussi chapitre 3.

Partie II : isopérimétrie dans espaces métriques mesurés et les groupes localement compacts

Dans cette partie, nous abordons un sujet très large aux multiples applications : l'isopérimétrie. Le point de vue adopté est celui de la géométrie à grande échelle sur un espace métrique mesuré. Une partie de mon travail consiste à définir un cadre général permettant de définir des propriétés isopérimétriques d'un espace métrique mesuré ne tenant compte que de sa géométrie à grande échelle. Une façon de traduire cette exigence est de demander que les "propriétés isopérimétriques" définies de cette façon soient invariantes par une classe d'applications entre espaces métriques mesurés qui généralisent les quasi-isométries.

Isopérimétrie à grande échelle, probabilité de retour des marches aléatoires.

L'étude asymptotique du profil isopérimétrique dans les variétés non compactes a été très étudié depuis la fin des années 80 avec comme principale motivation d'obtenir des estimations sur le comportement en grand temps du noyau de la chaleur. Une étude similaire s'est développée dans les graphes où les processus de diffusion à temps continu font place aux marches aléatoires, plus naturellement définies dans ce contexte. Il existe toutefois un contexte dans lequel on n'est en présence ni d'une variété, ni d'un graphe, et où pourtant, les processus markoviens ne manquent pas : ce sont les groupes localement compacts[6]. Il devient alors utile de disposer d'une notion de profil isopérimétrique dans un contexte d'espace métrique mesuré général. Grâce aux travaux fondateurs de Varopoulos dans le contexte des groupes de Lie puis de divers auteurs parmi lesquels Coulhon, Saloff-Coste, Grigor'yan, etc. il est devenu clair que le comportement diagonal en grand temps, soit du noyau de la chaleur sur les variétés, soit des marches aléatoires réversibles sur les graphes, est essentiellement régi par le profil isopérimétrique-L^2.

Parmi les 3 travaux exposés dans cette partie, le plus récent se consacre à définir une notion de gradient à grande échelle pour les fonctions définies sur un espace métrique mesuré général. Ceci fournit alors une notion d'inégalités de Sobolev et de profil isopérimétrique-L^p à grande échelle pour tout $1 \leq p \leq \infty$ dont le comportement asymptotique est invariant par une classe d'applications très générales que j'appelle équivalences à grande échelle. Les résultats que j'obtiens dans ce contexte permettent de généraliser ceux connus pour les graphes et les variétés [CouSa1].

L'idée principale est de construire le gradient à partir d'un opérateur markovien de largeur bornée. J'ai donc été amené à définir la notion de point de vue à échelle $h > 0$:

Définition. *Soit (X, d, μ) un espace métrique mesuré. Un point de vue (ou une marche aléatoire) à échelle h sur X est une famille de probabilités $(P_x)_{x \in X}$ telle*

[6] Comme par exemple, les groupes algébriques sur un corps local.

que
- $P_x \ll \mu$;
- $p_x = dP_x/d\mu$ *est supporté par la boule* $B(x, Ah)$*, où* $A \geq 1$ *est une constante ;*
- p_x *est plus grand qu'une constante* $c > 0$ *dans* $B(x, h)$*.*

En pratique, il n'est pas nécessaire de définir le gradient d'une fonction[7], mais seulement une norme locale du gradient. Étant donné P, un point de vue à échelle h et f, une fonction mesurable définie sur X, on définit la norme-L^p locale en $x \in X$ du gradient de f relativement à P par

$$|\nabla f|_{P,p}(x) = \|f - f(x)\|_{P_x,p} = \left(\int |f(y) - f(x)|^p dP_x(y) \right)^{1/p},$$

si $p < \infty$; et pour $p = \infty$, on pose

$$|\nabla f|_{P,\infty}(x) = \|f - f(x)\|_{P_x,\infty} = \sup\{|f(y) - f(x)|,\ y \in Supp(P_x)\}.$$

Un choix de (norme locale de) gradient à échelle h est donc la donnée d'un point de vue à échelle h et d'un nombre $1 \leq p \leq \infty$. On peut alors définir un profil isopérimétrique-L^p et des inégalités isopérimétriques-L^p, lesquelles correspondent à une certaine formulation des inégalités de Sobolev. Donnons-nous une fonction croissante $\varphi : \mathbf{R}_+ \to \mathbf{R}_+$ et un nombre $p \in [1, \infty]$. Les inégalités de Sobolev sont introduites sous la forme suivante dans [Cou2].

Définition. *On dit que X vérifie une inégalité de Sobolev (S^p_φ) pour un gradient $|\nabla|_{P,q}$ à échelle h s'il existe une constante positive C ne dépendant que de h et q telles que*

$$\|f\|_p \leq C\varphi(C\mu(\Omega))\||\nabla f|_{P,q}\|_p$$

où Ω parcourt les parties de mesure finie de X, et f, les fonctions lipschitziennes à support dans Ω.

Définition. *On dit que X satisfait une inégalité de Sobolev (S^p_φ) à grande échelle s'il existe $h > 0$ et un gradient à échelle h, $|\nabla|_{P,q}$ pour lequel l'inégalité (S^p_φ) est vérifiée.*

On peut également définir un opérateur laplacien associé à un point de vue $P = (P_x)_{x \in X}$ à échelle h en posant simplement $\Delta_P = \mathrm{id} - P$. Lorsque P est auto-adjoint par rapport à μ, on obtient les formules habituelles permettant de relier la forme de Dirichlet au laplacien (voir la remarque 6.1.6). Notons que le laplacien ainsi défini apparaît naturellement comme générateur "infinitésimal" de la marche aléatoire réversible associée à P. On peut alors appliquer formellement les théorèmes connus reliant inégalités de Sobolev (S^2_φ) et bornes supérieures du comportement diagonal en grand temps de la marche aléatoire.

Mon résultat principal consiste à démontrer que si deux espaces métriques mesurés X et Y sont équivalents à grande échelle (voir définition 6.1.21), alors si l'un vérifie une inégalité de Sobolev (S^p_φ) à grande échelle, l'autre aussi.

[7]C'est néanmoins possible, voir la remarque 6.1.5.

Ceci permet de définir une notion *naturelle* d'inégalités de Sobolev à grande échelle pour les groupes localement compacts σ-compacts[8], autrement dit qui ne dépend pas de la métrique propre invariante à gauche choisie sur le groupe. En guise d'applications de ces nouvelles notions je démontre le fait général suivant (voir [Kest, Bro, Salv, SoW, Pit2, SW] pour des résultats[9] semblables dans des cas particuliers). Soit (X, d, μ) un espace métrique mesuré et G un groupe localement compact. On dit que X est quasi-G-transitif si G agit proprement et co-compactement sur X, par isométries préservant la mesure.

Théorème. *Soit G un groupe localement compact et (X, d, μ) un espace métrique mesuré quasi-G-transitif. Le groupe G est unimodulaire et moyennable si et seulement toute marche aléatoire reversible à échelle suffisamment grande sur (X, d, μ) a un rayon spectral $\rho(P) = 1$, ou en d'autres termes, si le laplacien $\Delta = I - P$ n'a pas de trou spectral.*

Je montre aussi grâce à ces outils le résultat suivant, qui généralise le cas des groupes de type fini [Er, Lemme 4].

Théorème. *Soit H un sous-groupe fermé d'un groupe localement compact, σ-compact G. On suppose que le quotient G/H admet une mesure borélienne G-invariante. Si H vérifie une inégalité de Sobolev à grande échelle (S^p_φ), alors G aussi.*

Enfin, je discute des relations entre inégalités de Sobolev à grande échelle, inégalité de Sobolev à échelle donnée, et sur une variété riemannienne, inégalité de Sobolev pour le gradient habituel. Je compare également mon approche avec la notion classique de discrétisation d'une variété riemannienne.

Isopérimétrie asymptotique des boules dans les espaces mesurés doublants

Dans ce travail, on étudie l'isopérimétrie à grande échelle sous un angle purement géométrique. Deux problèmes se posent naturellement dans ce contexte : d'une part, déterminer le comportement asymptotique du profil isopérimétrique ; d'autre part, trouver des familles de parties, de volume tendant vers l'infini, qui optimisent, en général à constante près, le profil isopérimétrique. Une variante consiste également à se donner une famille particulière de parties, par exemple les boules, et à étudier leurs propriétés isopérimétriques. Je tente ici de répondre à la question suivante, que m'a posé Thierry Coulhon : dans un graphe à croissance polynomiale, le profil isopérimétrique est-il optimisé à constante près par des boules ? Cette question m'a amené à étudier de manière assez systématique les propriétés isopérimétriques à grande échelle des espaces métriques mesurés doublants, et plus particulièrement du rôle joué par

[8] On ne demande pas au groupe d'être compactement engendré.

[9] Bien que ma méthode ait le mérite de donner une interprétation géométrique simple de ce phénomène, ces auteurs obtiennent parfois des informations plus précises, notamment en reliant les marches aléatoires G-équivariantes sur X à des marches aléatoires sur G.

les boules. Rappelons qu'un espace métrique mesuré (X, d, μ) est dit doublant s'il existe une constante $C < \infty$ telle que pour tout $r > 0$ et tout $x \in X$, $V(x, 2r) \leq CV(x, r)$ où $V(x, r)$ désigne la mesure de la boule fermée de centre x et de rayon r. On dit que l'espace est à croissance strictement polynomiale de degré $d > 0$ s'il existe une constante $C < \infty$ telle que

$$C^{-1}r^d \leq V(x, r) \leq Cr^d.$$

Clairement, un espace à croissance strictement polynomiale est doublant. Rappelons que le 0-squelette d'un graphe simplicial connexe est muni d'une structure évidente d'espace métrique mesuré. On appelle simplement "graphe" un tel espace. Définissons le bord ∂A d'une partie A comme étant l'ensemble des sommets de A voisins d'un sommet situé à l'extérieur de A. Il existe plusieurs définitions non équivalentes du profil isopérimétrique. Pour cette étude, j'ai choisi la suivante qui me semblait bien adaptée à la question.

Définition. *Le profil isopérimétrique d'un graphe X est la fonction croissante définie sur $\mathbf{R}_+$ par :*

$$I(t) = \inf_{\mu(A) \geq t} \mu(\partial A),$$

où A parcourt les parties de mesure finie de X.

On définit de même deux notions de profils restreints à une famille de parties :

Définition. *Soit $\mathcal{A}$ une famille de parties finies de volume non borné. On définit un profil restreint à $\mathcal{A}$*

$$I^{\downarrow}_{\mathcal{A}}(t) = \inf_{\mu(A) \geq t, A \in \mathcal{A}} \mu(\partial A),$$

et un profil supérieur restreint à $\mathcal{A}$:

$$I^{\uparrow}_{\mathcal{A}}(t) = \sup_{\mu(A) \leq t, A \in \mathcal{A}} \mu(\partial A)$$

Si l'ensemble des volumes de parties de $\mathcal{A}$ n'est pas trop lacunaire[10], on a $I^{\downarrow}_{\mathcal{A}} \preceq I^{\uparrow}_{\mathcal{A}}$. D'autre part, on a trivialement $I^{\downarrow}_{\mathcal{A}} \leq I$.

Définition. *On dit que la famille $\mathcal{A}$ est asymptotiquement isopérimétrique si $I^{\downarrow}_{\mathcal{A}} \preceq I$ et qu'elle est fortement asymptotiquement isopérimétrique si $I^{\uparrow}_{\mathcal{A}} \preceq I$.*

Être asymptotiquement isopérimétrique signifie que pour tout $t > 0$, on peut trouver une partie dans $\mathcal{A}$ qui optimise à constante près le profil ; alors qu'être fortement asymptotiquement isopérimétrique signifie que *tout* élément de $\mathcal{A}$ est presque optimal. Cette dernière propriété est donc plus forte (à condition encore une fois que $\mathcal{A}$ ne soit pas trop lacunaire).

[10]Il suffit qu'il existe $\alpha > 1$ tel que pour tout $n \in \mathbf{N}$, on puisse trouver une partie A dans $\mathcal{A}$ vérifiant $\alpha^n \leq \mu(A) \leq \alpha^{n+1}$.

Définition. *Soit un graphe X à croissance strictement polynomiale de degré d. On dit que X vérifie une inégalité isopérimétrique forte si pour toute partie A de X,*

$$\mu(\partial A) \geq c\mu(A)^{(d-1)/d}.$$

Notons que cette inégalité isopérimétrique est celle qui est vérifiée sur $\mathbf{Z}^d$. On montre facilement que dans un graphe satisfaisant une inégalité isopérimétrique forte, les boules sont asymptotiquement isopérimétriques. Ceci est en particulier vrai dans un groupe à croissance polynomiale. Rappelons que l'on sait grâce à Gromov qu'un groupe de type fini à croissance polynomiale est virtuellement nilpotent, ce qui entraîne notamment qu'il est en fait à croissance strictement polynomiale de degré entier. A part dans le cas abélien, on ne sait pas si dans un groupe à croissance polynomiale, les boules sont fortement isopérimétriques. Autrement dit, il pourrait exister une suite (r_n) tendant vers l'infini telle que les sphères de rayon r_n ont un volume grand devant r_n^{d-1}, où d est l'exposant de croissance. C'est ce qui arrive dans l'exemple suivant qui n'est pas un graphe homogène.

Théorème. *Il existe un graphe quasi-isométrique à Z^2 tel que la mesure des sphères de rayon r centrées en 0 n'est pas dominé par $r^{\log 3/\log 2}$.*

Ce résultat est à comparer au théorème 8.2.4.

Lorsque X ne satisfait pas d'inégalité isopérimétrique forte, on voit apparaître de nombreuses pathologies mais aussi quelques rares faits généraux. Par exemple, je construis deux graphes quasi-isométriques, à croissance polynomiale de degré entier quelconque, tel que les boules sont asymptotiquement isopérimétriques dans l'un mais pas dans l'autre. J'en construit un autre tel que dans tout graphe qui lui est quasi-isométrique, les boules ne sont jamais asymptotiquement isopérimétriques. Par contre, si un graphe a un profil isopérimétrique borné, autrement dit s'il possède une suite (A_n) de parties de volume non-borné dont le bord est de volume borné, alors il existe[11] une constante $C < \infty$ et des suites r_n de rayons et x_n de sommets de X tels que

$$B(x_n, r_n) \subset A_n \subset B(x_n, Cr_n).$$

Mais les boules elles-même peuvent ne pas être asymptotiquement isopérimétriques.

Pour terminer je prouve un résultat général concernant l'isopérimétrie des parties connexes. Rappelons qu'une partie A d'un graphe X est dite connexe si l'on ne peut pas la partitioner en deux parties non vides A_1 et A_2 telles que $d(A_1, A_2) \geq 2$. On note $I_C^{\downarrow}$ le profil restreint aux parties connexes de X. La seconde assertion du résultat suivant dit que même dans un graphe à croissance strictement polynomiale, il faut parfois des parties[12] non connexes pour optimiser le profil à constante près !

[11] Ici le graphe n'est pas supposé à croissance polynomiale.

[12] En particulier, le nombre de composantes connexes de ces parties n'est pas borné.

Théorème. *Soit X un graphe.*

- *Supposons que X soit fermé à l'infini, i.e. que le profil vérifie $I(t)/t \to 0$. Alors il existe une suite (t_n) tendant vers l'infini tel que $I_C^{\downarrow}(t_n) = I(t_n)$.*
- *Cependant, pour tout entier $d \geq 2$, il existe un graphe à croissance strictement polynomiale de degré d et une suite s_n tendant vers l'infini tel que $I_C^{\downarrow}(s_n)/I(s_n) \to \infty$.*

Mesure des sphères dans un espace doublant et application à la théorie ergodique

Soit G un groupe localement compact muni d'une mesure de Haar à gauche μ. Rappelons qu'une suite (A_n) de parties compactes de G est dite de Følner si pour tout compact K de G, on a :

$$\frac{\mu(KA_n \triangle A_n)}{\mu(A_n)} \to 0.$$

Une motivation importante pour chercher des suites de Følner provient de la théorie ergodique. Considérons un espace probabilisé (X, m) sur lequel un groupe localement compact G agit mesurablement en préservant la mesure. Cette action induit une représentation fortement continue π de G comme groupe d'isométries de l'espace de Banach $L^p(X)$, pour tout $1 \leq p < \infty$:

$$\pi(g)f(x) = f(g^{-1}x).$$

Pour toute mesure de probabilité β sur G et tout $1 \leq p < \infty$, on considère l'opérateur de moyenne défini par

$$\pi(\beta)f(x) = \int_G f(g^{-1}x)d\beta(g), \quad \forall f \in L^p(X).$$

Soit (β_n) une suite de probabilités sur G. On dit que (β_n) vérifie un théorème ergodique point par point dans $L^p(X)$ si

$$\lim_{n\to\infty} \pi(\beta)f(x) = \int_X f dm$$

pour presque tout $x \in X$, et en norme L^p, pour tout $f \in L^p(X)$. Des efforts important ont été déployés depuis les années 50 pour démontrer de tels théorèmes ergodiques, en particulier dans le cas où β_n est la moyenne sur une partie de mesure finie. Notons qu'étant donnée une suite (β_n) (par exemple, de moyennes sur les boules de rayon n), il est en général plus facile et moins intéressant d'obtenir un théorème ergodique pour une sous-suite (β_{i_n}) que pour la suite elle-même. En particulier, lorsque le groupe est moyennable, on peut obtenir des théorèmes ergodiques pour des moyennes sur certaines suites de Følner dont la mesure croît généralement très vite [Li][13]. Il est beaucoup plus exigeant d'avoir un théorème ergodique pour une suite de Følner dont la croissance du

[13] Voir aussi dans l'excellent survey d'Amos Nevo : [N, Theorem 6-10].

volume est par exemple, au plus exponentielle. Pour traiter ce cas, on dispose du résultat général suivant. Une suite croissante de parties de mesure finie N_k est dite régulière s'il existe une constante $C < \infty$ telle que

$$\mu(N_k^{-1}N_k) \leq C\mu(N_k).$$

Théorème. *[Bew][Chat][Em][Tem] Supposons que le groupe G possède une suite régulière de Følner $(N_n)_{n\in\mathbf{N}}$, avec $\cup_{n\in\mathbf{N}}N_n = G$. Alors la suite $(\beta_n)_{n\in\mathbf{N}}$ des moyennes sur (N_k) satisfait le théorème ergodique point par point dans $L^p(X)$ pour tout $1 \leq p < \infty$.*

Notons qu'on ne connaît de suites régulières de Følner que dans les groupes à croissance polynomiale. Dans ce cas, des candidats naturels sont les puissances d'un compact générateur de G. On sait en effet qu'un groupe à croissance polynomiale est quasi-isométrique à un groupe de Lie connexe à croissance polynomiale (voir l'annexe C). On connaît alors exactement son exposant de croissance et en particulier, G est un espace doublant. Autrement dit, si K est un compact générateur, alors les puissances de K forment une suite régulière. Pour obtenir un théorème ergodique pour les moyennes sur (K^n), il reste à montrer que (K^n) est une suite de Følner. Ce problème d'apparence simple a été conjecturé par Greenleaf en 1969, puis démontré par Pansu [Pa1] en 1983 dans le cas d'un groupe de type fini nilpotent. Sa preuve, qui utilise pleinement la structure des groupes nilpotents a été généralisée très récemment à tout groupe à croissance polynomiale par Breuillard [Bre]. Précisément, ils prouvent que $\mu(K^n) \sim Cn^d$ pour une constante C dépendant de K, ce qui implique directement que (K^n) est de Følner. En utilisant seulement la propriété de doublement des groupes à croissance polynomiale, je démontre le résultat suivant :

Théorème. *Soit G un groupe à croissance polynomiale, $K \subset K'$ des compacts générateurs et (K_n) une suite de parties compactes telles que pour tout $n \in \mathbf{N}$, $K \subset K_n \subset K'$. Alors, il existe des constantes $\delta > 0$ et $C < \infty$ ne dépendant que de K et K', telles que la suite $N_n = K_n \cdot K_{n-1} \dots K_0$ vérifie :*

$$\frac{\mu(N_{n+1} \smallsetminus N_n)}{\mu(N_n)} \leq Cn^{-\delta}.$$

Ce théorème implique donc la conjecture de Greenleaf pour les groupes à croissance polynomiale. En outre, la suite des moyennes β_n sur N_n vérifie le théorème ergodique.

On peut en fait obtenir un énoncé semblable dans le cadre bien plus général des espaces métriques mesurés doublants (voir théorème 8.2.4).

Plan

Comme son introduction, cette thèse comporte deux parties. La première partie comporte quatre articles. Le premier contient mon travail le plus important, où je relie la compression dans L^p au profil isopérimétrique-L^p dans les boules ainsi qu'à d'autres quantités géométriques comme la croissance du volume ou la probabilité de retour des marches aléatoires. Il contient également le calcule de ce profil isopérimétrique pour une classe de groupes moyennables incluant les groupes de Lie connexes moyennables.

Viennent ensuite deux articles co-écrits avec Alain Valette et Yves de Cornulier où nous étudions différentes propriétés dynamiques des actions par isométries sur un espace de Hilbert. Le premier article met l'accent sur la croissance des cocycles. Dans le second article, nous étudions la structure des orbites d'un point de vue dynamique. Nous y démontrons un résultat de non-existence d'orbites denses pour les actions de groupes nilpotents.

Enfin, nous terminons par un article où nous montrons un résultat d'annulation de la cohomologie réduite à valeur dans une représentation mélangeante dans un espace L^p pour une classe de groupes moyennables incluant les groupes de Lie connexes moyennables.

La deuxième partie se compose de trois articles. Dans le premier, nous présentons un cadre général pour définir les inégalités de Sobolev à grande échelle. Le second contient une étude des propriétés isopérimétriques asymptotiques des boules dans les graphes à croissance polynomiale. Enfin, nous terminons par un article où nous démontrons que les boules dans un espace doublant forment une famille de Følner, ce qui nous permet de prouver un théorème ergodique pour les groupes à croissance polynomiale.

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Première partie

Actions par isométries et plongements uniformes dans les espaces de Banach

Chapitre 1

Une brève introduction à l'étude quantitative des plongements uniformes d'espaces métriques dans les espaces de Banach

Ce chapitre est destiné à compléter les introductions des deux chapitres suivants. On y trouvera des notions, définitions de base ainsi que des résultats généraux connus concernant l'étude quantitative des plongements uniformes d'espaces métriques. En revanche, nos résultats ne sont exposés qu'à partir du chapitre suivant.

1.1 Compression des plongements uniformes

L'objectif est ici de comparer la géométrie à grande échelle de deux espaces métriques. De ce point de vue, les notions que nous introduisons ne sont pertinentes que pour des espaces métriques non bornés.

Comme dans l'introduction, on désigne par comportement asymptotique d'une fonction croissante *non nulle* $f : \mathbf{R}_+ \to \overline{\mathbf{R}}_+$ la classe de f modulo la relation d'équivalence :

$$f \approx g \Leftrightarrow \exists C < \infty,\ C^{-1}g(C^{-1}t) - C \leq f(t) \leq Cg(Ct) + C.$$

On écrit $f \preceq g$ (resp. $f \prec g$) s'il existe $C > 0$ tel que $f(t) \leq g(Ct) + C$ (resp. si pour tout $c > 0$, $f(t) = o(g(ct))$.

Definition 1.1.1. Une application $F : (X, d) \to (X', d')$ entre deux espaces métriques est un plongement uniforme si pour toute suite $(x_n, y_n) \in (X \times X)^{\mathbf{N}}$,

$$d(x_n, y_n) \to \infty \iff d'(F(x_n), F(y_n)) \to \infty;$$

ou de manière équivalente s'il existe deux fonctions croissantes propres ρ_1, ρ_2 : $[0, \infty) \to \overline{\mathbf{R}}_+$ vérifiant

$$\rho_1(d(x,y)) \leq d'(F(x), F(y)) \leq \rho_2(d(x,y)),$$

pour tout x et y dans X. En prenant la borne supérieure des fonctions ρ_1, on obtient la compression de F, notée ρ_F, aussi définie par

$$\rho_F(t) = \inf_{d(x,y)\geq t} d'(F(x), F(y));$$

De même, en prenant la borne inférieure des fonctions ρ_2, on obtient la dilatation de F, notée δ_F, aussi donnée par

$$\delta_F(t) = \sup_{d(x,y)\leq t} d'(F(x), F(y)).$$

Example 1.1.2. L'application identité $F : (R, |\cdot|) \mapsto (\mathbf{R}, |\cdot|^{1/2})$, ainsi que son inverse F^{-1} sont des plongements uniformes : leurs compressions/dilatations valent respectivement : $\rho_F(t) = \delta_F(t) = t^{1/2}$ et $\rho_{F^{-1}}(t) = \delta_{F^{-1}}(t) = t^2$.

Rappelons qu'un espace métrique (X, d) est dit quasi-géodésique s'il existe des constantes $C < \infty$ et $\delta > 0$ telles que tout couple $(x, y) \in X^2$ est joignable par une chaine $x = x_0, \ldots, x_n = y$ de pas inférieur à δ et vérifiant

$$\sum_{i=1}^{n} d(x_{i-1}, x_i) \leq Cd(x,y).$$

Si l'espace X est quasi-géodésique, il est facile de voir que tout plongement uniforme $F : X \mapsto Y$ est lipschitzien pour les grandes distances. En d'autres termes, la pseudo-métrique $d_F = d(F(\cdot), F(\cdot))$ induite par F sur X est dominée par d. Dans ce cas, on a $\delta_F(t) \approx t$. Par la suite, les espaces X considérés : graphes, variétés riemanniennes et groupes localement compacts compactement engendrés sont quasi-géodésiques, de sorte que seul le comportement asymptotique de la compression présente un intérêt.

Notons enfin que $\rho(t) \approx t$ si et seulement si F est un plongement quasi-isométrique, c'est à dire une application bilipschitzienne pour les grandes distances.

1.2 Exemples d'espaces métriques

Il y a de nombreuses motivations pour étudier les plongements uniformes d'espaces métriques dans les espaces de Banach. Citons par exemple le lien avec les conjectures de Novikov et de Baum-Connes pour les groupes de type fini. Une telle étude peut aussi se révéler fructueuse dans l'analyse des propriétés géométriques des espaces de Banach. On a par exemple le résultat suivant dû à Bourgain [Bou] : un espace de Banach admet un arbre 3-régulier plongé quasi-isométriquement si et seulement s'il n'est pas superreflexif. D'autre motivations proviennent de l'étude algorithmique des espaces métriques : dans ce cadre, on s'intéresse à plonger un espace métrique dans un espace L^p, particulièrement lorsque $p = 1$ ou 2.

Les exemples d'espaces sources auxquels on s'intéressera au cours des chapitres suivants sont les graphes, les variétés riemanniennes et surtout les groupes localement compact, compactement engendrés. Rappelons brièvement ce que l'on entend par "graphe". Étant donné un graphe simplicial connexe X, on appelle simplement "graphe", l'espace métrique (X_0, d), où X_0 est le 0-squelette de X, d est la distance induite par la distance géodésique usuelle sur le 1-squelette[1]. Il est clair qu'un tel espace métrique est quasi-géodésique. Une famille particulièrement intéressante de graphes est la famille des arbres de valence ≥ 3. Notons qu'il existe, pour tout p, un plongement standard d'un tel arbre[2] dans un espace ℓ^p, à savoir l'espace $\ell^p(\text{arêtes})$: on fixe un sommet x_0 de l'arbre et on applique tout sommet x sur la somme des Diracs des arêtes séparant x_0 de x. Pour cette fonction F,

$$\rho_F(t) = \delta_F(t) = t^{1/p};$$

de sorte que ce plongement est quasi-isométrique (seulement) pour $p = 1$. Lorsque p est différent de 1, Guentner et Kaminker [GuKa] on montré que l'on peut "déformer" ces plongements de manière à obtenir des compressions plus grandes que t^a pour tout $a < 1$. Le résultat de Bourgain mentionné plus haut implique que dans le cas où la valence est au moins 3, un tel arbre ne se plonge pas quasi-isométriquement dans un espace L^p pour $1 < p < \infty$. On peut dès lors en déduire que tout espace métrique possédant une partie quasi-isométrique à un arbre 3-régulier ne se plonge pas quasi-isométriquement dans un espace L^p pour $1 < p < \infty$. L'un des résultats du chapitre suivant consiste à donner une version quantitative optimale de cette impossibilité en terme de compression.

Soit G un groupe localement compact, compactement engendré. Soit S une partie génératrice compacte symétrique ($S = S^{-1}$) de G. La longueur des mots d'un élément $g \in G$ par rapport à S est l'entier $|g|_S = \inf\{n \in \mathbf{N},\ g \in S^n\}$. On en déduit une distance invariante à gauche $d_S(g,h) = |g^{-1}h|_S$, appelée la distance des mots par rapport à S. Les groupes qui nous intéresseront principalement sont les groupes de Lie connexes et les groupes de type fini.

Notons que lorsque G est de type fini, alors l'espace métrique ainsi obtenu est en fait un graphe[3], appelé graphe de Cayley associé à la partie génératrice S et noté (G,S). Par exemple, si $\langle x_1, \ldots, x_k\rangle$ est le groupe libre à k générateurs et si $S = \{x_1^{\pm}, \ldots, x_k^{\pm}\}$, alors (G,S) est l'arbre $2k$-régulier.

[1] Une arête a pour longueur 1.

[2] Ici aucune condition n'est requise sur la valence.

[3] G est l'ensemble des sommets, et deux sommets g et g' sont reliés par une arête s'il existe $s \in S$ tel que $g' = gs$.

1.3 Plongements uniformes d'un groupe associés à une action par isométries

Soit G un groupe localement compact, compactement engendré et soit S une partie compacte, symétrique, génératrice de G. Soit σ une action continue, isométrique de G sur un espace métrique Y. Chaque orbite $F_x(g) = \sigma(g)x$ définie une application Lipschitzienne de (G, d_S) vers Y. De plus comme l'action est par isométries,

$$d(F_x(g), d(F_y(g)) = d(x, y) \quad \forall x, y \in Y, \forall g \in G,$$

de sorte que

$$\rho_{F_x} \approx \rho_{F_y} \quad \forall x, y \in Y.$$

Une action par isométries σ est dite métriquement propre (par abus de langage, on dit propre) si pour tout $x \in Y$, l'image réciproque de toute partie bornée de Y par F_x est compacte dans G. Notons que σ est propre si et seulement si pour tout $x \in Y$, F_x est un plongement uniforme.

Dans le cas où $Y = E$ est un espace de Banach, on appelle compression de l'action σ la compression de ρ_{F_0}.

Une particularité des plongements définis de cette manière est que la distance induite sur G : $d_{F_x}(g, g') = d(F_x(g), F_x(g'))$ est invariante à gauche. Lorsque $Y = \mathcal{H}$ est un espace de Hilbert, ce fait a une réciproque remarquable. On appelle distance hilbertienne sur G une pseudo-métrique induite par une application vers un espace de Hilbert. On montre grâce à la construction GNS [HV, 5.b] que pour toute distance hilbertienne d, invariante à gauche sur G, il existe une action par isométries σ de G sur un espace de Hilbert $\mathcal{H}$ telle que $d(g, g') = d(\sigma(g).0, \sigma(g')0)$.

Ce fait a un corollaire important remarqué par Gromov. Supposons que le groupe G est moyennable et que G admet un plongement uniforme dans un espace de Hilbert de compression ρ. Soit d la distance hilbertienne induite par ce plongement uniforme. Les carrés de distances hilbertiennes[4] formant un cône convexe, on peut par un procédé de moyenne obtenir une distance hilbertienne $\tilde{d}$ invariante à gauche et ayant la même compression que d. Les obstructions à plonger quasi-isométriquement un groupe moyennable dans un Hilbert se ramène alors à une obstruction à agir isométriquement sur un espace de Hilbert avec une compression linéaire. L'intérêt est qu'on est alors ramené à étudier la 1-cohomologie des représentations unitaires de G (voir le chapitre 3 pour plus de détails).

Enfin, on dit qu'un groupe localement compact agissant proprement par isométries sur un espace de Hilbert possède la propriété de Haagerup.

[4] Aussi appelées fonctions conditionnellement de type négatif.

1.4 Quelques questions qualitatives

Question 1.4.1. Quels sont les groupes qui se plongent uniformément dans un espace de Banach ? Un espace L^p ? Un espace de Hilbert ?

Réponse : Il est connu et standard que tout espace métrique X se plonge isométriquement dans $L^\infty(X)$. Pour les groupes, on a aussi le résultat mentionné à la question suivante. En revanche, on sait construire des graphes à degré borné appelés expanseurs qui ne se plongent uniformément dans aucun espace superreflexif, à fortiori dans aucun L^p, pour $1 < p < \infty$, ni dans L^1 qui se plonge uniformément [BrDaKr] dans L^2. Gromov [Gro6] a récemment proposé une méthode de construction aléatoire de groupes G de type fini tel qu'un expanseur se plonge uniformément dans G. Ce résultat, qui attend encore à l'heure qu'il est une preuve complète fournirait le premier exemple connu de groupe de type fini non plongeable uniformément dans un espace de Hilbert (ni aucun espace superreflexif). On sait par ailleurs plonger uniformément la plupart des groupes connus dans un Hilbert : par exemple, les groupes hyperboliques au sens de Gromov, les groupes de Lie et bien sûr les groupes ayant la propriété de Haagerup (voir la question suivante), ce qui constitue une très large classe de groupes. Notons que l'ensemble des $p \geq 1$ tels qu'un espace métrique X se plonge uniformément dans un espace L^p forme un intervalle du type[5] (p_0, ∞) ou $[p_0, \infty)$: ceci découle du fait [BrDaKr] que pour $p \leq q$, L^p équipé de la distance $\|x-y\|_p^{p/q}$ est isométrique à une partie de L^q. De plus si 2 appartient à l'intervalle, alors celui-ci vaut $[1, \infty)$ car L^2 est isomorphe [Wo] à un sous-espace de L^p, pour tout $p \leq 2$.

Question 1.4.2. Quels sont les groupes qui agissent proprement sur un espace de Banach ? Un espace L^p ? Un espace de Hilbert ?

Réponse : Tout groupe localement compact séparable admet une action propre isométrique [BrGu, HaPr] sur l'espace de Banach strictement convexe

$$\oplus_{n\in\mathbf{N}}^{\ell^2} L^{2n}(G),$$

où $\oplus^{\ell^2}$ est une somme directe ℓ^2.

Les groupes ayant la propriété de Haagerup forment une importante classe contenant les groupes moyennables, les groupes de Coxeter, les groupes agissant proprement sur un complexe cubique CAT(0), les groupes de Lie simples $SO(n, 1)$, $SU(n, 1)$ etc. Toutefois, une classe de groupes largement étudiée, contenant les groupes de Lie simples de rang supérieur, possède une propriété en un certain sens opposée à la propriété de Haagerup : la propriété (T) de Kazhdan. Cette propriété se caractérise par le fait que toute métrique hilbertienne invariante à gauche est bornée. Il existe des groupes de type fini Gromov-hyperboliques ayant propriété (T). Pourtant, G. Yu a récemment montré que de tels groupes admettent toujours des actions propres isométriques sur un espace

[5] On ne connait que des exemples où cet intervalle est soit vide, soit tout $[1, \infty)$.

ℓ^p pour p assez grand. Signalons enfin qu'il est fort probable que l'ensemble des p pour lesquels un groupe admet une action propre par isométries sur un espace L^p est de la forme (p_0, ∞) ou $[p_0, \infty)$ mais aucune preuve de ce fait n'existe pour l'instant à notre connaissance, même dans le cas des groupes hyperboliques.

Question 1.4.3. Quels groupes se plongent quasi-isométriquement dans un espace de Banach ? Un espace L^p ? Un Hilbert ?

Réponse : Le plus ancien résultat concernant cette question est le théorème de Bourgain déjà cité. Cette question fait en fait l'objet d'une étude approfondie dans les deux chapitres suivants. Disons en résumé que parmi une très large classe de groupes, seuls les groupes virtuellement abéliens admettent de tels plongements.

Chapitre 2

Isopérimétrie asymptotique dans les groupes et plongements uniformes dans des espaces de Banach

Résumé

We characterize the possible asymptotic behaviors of the compression associated to a uniform embedding into some L^p-space, with $1 < p < \infty$, for a large class of groups including connected Lie groups with exponential growth and word-hyperbolic finitely generated groups. In particular, the Hilbert compression rate of these groups is equal to 1. This also provides new and optimal estimates for the compression of a uniform embedding of the infinite 3-regular tree into some L^p-space. The main part of the paper is devoted to the explicit construction of affine isometric actions of amenable connected Lie groups on L^p-spaces whose compressions are asymptotically optimal. These constructions are based on an asymptotic lower bound of the L^p-isoperimetric profile inside balls. We compute the asymptotic of this profile for all amenable connected Lie groups and for all $1 \leq p < \infty$, providing new geometric invariants of these groups. We also relate the Hilbert compression rate with other asymptotic quantities such as volume growth and probability of return of random walks.

2.1 Introduction

The study of uniform embeddings of locally compact groups into Banach spaces and especially of those associated to proper affine isometric actions plays a crucial role in various fields of mathematics ranging from K-theory to geometric group theory. Recall that a locally compact group is called a-T-menable if it admits a proper affine action by isometries on a Hilbert space (for short : a proper isometric Hilbert action). An amenable σ-compact locally compact group is always a-T-menable [CCJJV] ; but the converse is false since for instance non-amenable free groups are a-T-menable. However, if a locally compact, compactly generated group G admits a proper isometric Hilbert action whose compression ρ satisfies

$$\rho(t) \succ t^{1/2},$$

then G is amenable[1]. On the other hand, in [CTV], we prove that non-virtually abelian polycyclic groups cannot have proper isometric Hilbert actions with linear compression. These results motivate a systematic study of the possible asymptotic behaviors of compression functions, especially for amenable groups.

In this paper, we "characterize" the asymptotic behavior of the L^p-compression, with $1 < p < \infty$, for a large class of groups including all connected Lie groups with exponential growth. Some partial results in this direction for $p = 2$ had been obtained in [GuKa] and [BrSo] by completely different methods.

2.1.1 L^p-compression : optimal estimates

Let us recall some basic definitions. Let G be some locally compact compactly generated group. Equip G with the word length function $|\cdot|_S$ associated to a compact symmetric generating subset S and consider a uniform embedding F of G into some Banach space. The compression ρ of F is the nondecreasing function defined by

$$\rho(t) = \inf_{|g^{-1}h|_S \geq t} \|F(g) - F(h)\|.$$

Let $f, g : \mathbf{R}_+ \to \overline{\mathbf{R}}_+$ be nondecreasing, nonzero functions. We write respectively $f \preceq g$, $f \prec g$ if there exists $C < \infty$ such that $f(t) = O(g(Ct))$, resp. for all $c > 0$, $f(t) = o(g(ct))$ when $t \to \infty$. We write $f \approx g$ if both $f \preceq g$ and $g \preceq f$. The asymptotic behavior of f is its class modulo the equivalence relation $\approx$.

Note that the asymptotic behavior of the compression of a uniform embedding does not depend on the choice of S.

In the sequel, a L^p-space denotes a Banach space of the form $L^p(X, m)$ where (X, m) is a measure space. A L^p-representation of G is a continuous linear G-action on some L^p-space. Let π be a isometric L^p-representation of G and consider a 1-cocycle $b \in Z^1(G, \pi)$, or equivalently an affine isometric action of G with linear part π : see the preliminaries for more details. The compression

[1] This was proved for finitely generated groups in [GuKa]. In [CTV], we give a shorter argument that applies to all locally compact compactly generated groups.

of b is defined by

$$\rho(t) = \inf_{|g|_S \geq t} \|b(g)\|_p.$$

In this paper, we mainly focus our attention on groups in the two following classes.
Denote $(\mathcal{L})$ the class of groups including

1. polycyclic groups and connected amenable Lie groups ;
2. semidirect products $\mathbf{Z}[\frac{1}{mn}] \rtimes_{\frac{m}{n}} \mathbf{Z}$, with m, n co-prime integers with[2] $|mn| \geq 2$ (if $n = 1$ this is the Baumslag-Solitar group $BS(1, m)$) ; semidirect products $\left(\mathbf{R} \oplus \bigoplus_{p \in P} \mathbf{Q}_p\right) \rtimes_{\frac{m}{n}} \mathbf{Z}$ with m, n coprime integers and P a finite set of primes (possibly infinite, $\mathbf{Q}_\infty = \mathbf{R}$) dividing mn ;
3. wreath products $F \wr \mathbf{Z}$ for F a finite group.

Denote $(\mathcal{L}')$ the class of groups including groups in the class $(\mathcal{L})$ and

1. connected Lie groups and their cocompact lattices ;
2. irreducible lattices in semisimple groups of rank ≥ 2 ;
3. hyperbolic finitely generated groups.

Let μ be a left Haar measure on the locally compact group G and write $L^p(G) = L^p(G, \mu)$. The group G acts by isometry on $L^p(G)$ via the left regular representation $\lambda_{G,p}$ defined by

$$\lambda_{G,p}(g)\varphi = \varphi(g^{-1}\cdot).$$

Theorem 1. Fix some $1 \leq p < \infty$. Let G be a group of the class $(\mathcal{L})$ and let f be an increasing function $f : \mathbf{R}_+ \to \mathbf{R}_+$ satisfying

$$\int_1^\infty \left(\frac{f(t)}{t}\right)^p \frac{dt}{t} < \infty. \qquad (C_p)$$

Then there exists a 1-cocycle $b \in Z^1(G, \lambda_{G,p})$ whose compression ρ satisfies

$$\rho \succeq f.$$

Corollary 2. Fix some $1 \leq p < \infty$. Let G be a group of the class $(\mathcal{L}')$ and let f be an increasing function $f : \mathbf{R}_+ \to \mathbf{R}_+$ satisfying Property (C_q), with $q = \max\{p, 2\}$. Then there exists a uniform embedding of G into some L^p-space whose compression ρ satisfies

$$\rho \succeq f.$$

Let us sketch the proof of the corollary. First, recall [Wo, III.A.6] that for $1 \leq p \leq 2$, $L^2([0,1])$ is isomorphic to a subspace of $L^p([0,1])$. It is thus enough to prove the theorem for $2 \leq p < \infty$. This is an easy consequence of Theorem 1 since every group of class $(\mathcal{L}')$ quasi-isometrically embeds into a group of $(\mathcal{L})$. Indeed, any connected Lie group admits a closed cocompact connected solvable

[2] This condition garanties that the group is compactly generated.

subgroup. On the other hand, irreducible lattices in semisimple groups of rank ≥ 2 are quasi-isometrically embedded [LMR]. Finally, any hyperbolic finitely generated group quasi-isometrically embeds into the real hyperbolic space $\mathbf{H}^n$ for n large enough [BoS] which is itself quasi-isometric to $\mathrm{SO}(n,1)$.

The particular case of nonabelian free groups, which are quasi-isometric to 3-regular trees, can also be treated by a more direct method. More generally that method applies to any simplicial[3] tree with possibly infinite degree.

Theorem 3. (see Theorem 2.7.3) Let T be a simplicial tree. For every increasing function $f : \mathbf{R}_+ \to \mathbf{R}_+$ satisfying

$$\int_1^\infty \left(\frac{f(t)}{t}\right)^p \frac{dt}{t} < \infty, \qquad (C_p)$$

there exists a uniform embedding F of T into $\ell^p(T)$ with compression $\rho \succeq f$.

Remark 2.1.1. In [BuSc, BuSc'], it is shown that real hyperbolic spaces and word hyperbolic groups quasi-isometrically embed into finite products of (simplicial) trees. Thus the restriction of Corollary 2 to word hyperbolic groups and to simple Lie groups of rank 1 can be deduced from Proposition 2.7.3. Nevertheless, not every connected Lie group quasi-isometrically embeds into a finite product of trees. Namely, a finite product of trees is a CAT(0) space, and in [Pau] it is proved that a non-abelian simply connected nilpotent Lie group cannot quasi-isometrically embed into any CAT(0) space.

Theorem 4. Let T_N be the binary rooted tree of depth N. Let ρ be the compression of some 1-Lipschitz map from T_N to some L^p-space for $1 < p < \infty$. Then there exists $C < \infty$, depending only on p, such that

$$\int_1^{2N} \left(\frac{\rho(t)}{t}\right)^q \frac{dt}{t} \leq C,$$

where $q = \max\{p, 2\}$.

This result is a strengthening of [Bou, Theorem 1] ; see also Corollary 2.6.3. As a consequence, we have

Corollary 5. Assume that the 3-regular tree quasi-isometrically embeds into some metric space X. Then, the compression ρ of any uniform embedding of X into any L^p-space for $1 < p < \infty$ satisfies (C_q) for $q = \max\{p, 2\}$.

In [BeSc, Theorem 1.5], it is proved that the 3-regular tree quasi-isometrically embeds into any graph with bounded degree and positive Cheeger constant (e.g. any non-amenable finitely generated group). On the other hand, in a work in preparation with Cornulier [CT], we prove that finitely generated linear groups with exponential growth, and finitely generated solvable groups with exponential growth admit quasi-isometrically embedded free non-abelian subsemigroups. Together with the above corollary, they lead to the optimality of Theorem 1 (resp. Corollary 2) when the group has exponential growth and when $2 \leq p < \infty$ (resp. $1 < p < \infty$).

[3]By simplicial, we mean that every edge has length 1.

Corollary 6. Let G be a finitely generated group with exponential growth which is either virtually solvable or non-amenable. Let φ be a uniform embedding of G into some L^p-space for $1 < p < \infty$. Then its compression ρ satisfies Condition (C_q) for $q = \max\{p, 2\}$.

Corollary 7. Let G be a group of class $(\mathcal{L}')$ with exponential growth. Consider an increasing map f and some $1 < p < \infty$; then f satisfies Condition (C_q) with $q = \max\{p, 2\}$ if and only if there exists a uniform embedding of G into some L^p-space whose compression ρ satisfies $\rho \succeq f$.

Note that the 3-regular tree cannot uniformly embed into a group with subexponential growth. So the question of the optimality of Theorem 1 for non-abelian nilpotent connected Lie groups remains open.

About Condition (C_p). First, note that if $p \leq q$, then (C_p) implies (C_q) : this immediately follows from the fact that a nondecreasing function f satisfying (C^p) also satisfies $f(t)/t = O(1)$.

Let us give examples of functions f satisfying Condition (C_p). Clearly, if f and h are two increasing functions such that $f \preceq h$ and h satisfies (C_p), then f satisfies (C_p). The function $f(t) = t^a$ satisfies (C_p) for every $a < 1$ but not for $a = 1$. More precisely, the function

$$f(t) = \frac{t}{(\log t)^{1/p}}$$

does not satisfy (C_p) but

$$f(t) = \frac{t}{((\log t)(\log\log t)^a)^{1/p}}$$

satisfies (C_p) for every $a > 1$. In comparison, in [BrSo], the authors construct a uniform embedding of the free group of rank 2 into a Hilbert space with compression larger than

$$\frac{t}{((\log t)(\log\log t)^2)^{1/2}}.$$

As $t/(\log t)^{1/p}$ does not satisfy (C_p), one may wonder if (C_p) implies

$$\rho(t) \preceq \frac{t}{(\log t)^{1/p}}.$$

The following proposition answers negatively to this question. We say that a function f is sublinear if $f(t)/t \to 0$ when $t \to \infty$.

Proposition 8. (See Proposition 2.7.5) For any increasing sublinear function $h : \mathbf{R}_+ \to \mathbf{R}_+$ and every $1 \leq p < \infty$, there exists a nondecreasing function f satisfying (C_p), a constant $c > 0$ and a increasing sequence of integers (n_i) such that

$$f(n_i) \geq ch(n_i), \quad \forall i \in \mathbf{N}.$$

In particular, it follows from Theorem 1 that the compression ρ of a uniform embedding of a 3-regular tree in a Hilbert space does not satisfy any *a priori* majoration by any sublinear function.

2.1.2 Isoperimetry and compression

To prove Theorem 1, we observe a general relation between the L^p-isoperimetry inside balls and the L^p-compression. Let G be a locally compact compactly generated group and consider some compact symmetric generating subset S. For every $g \in G$, write[4]

$$|\tilde{\nabla}\varphi|(g) = \sup_{s\in S} |\varphi(sg) - \varphi(g)|.$$

Let $2 \leq p < \infty$ and let us call the L^p-isoperimetric profile inside balls the nondecreasing function $\tilde{J^b}_{G,p}$ defined by

$$\tilde{J^b}_{G,p}(t) = \sup_{\varphi} \frac{\|\varphi\|_p}{\|\tilde{\nabla}\varphi\|_p},$$

where the supremum is taken over all measurable functions in $L^p(G)$ with support in the ball $B(1,t)$. Note that the group G is amenable if and only if $\lim_{t\to\infty} \tilde{J^b}_{G,p}(t) = \infty$. Theorem 1 results from the two following theorems.

Theorem 9. (see Theorem 2.5.1) Let G be a group of class $(\mathcal{L})$. Then $\tilde{J^b}_{G,p}(t) \approx t$.

Theorem 10. (see Corollary 2.4.2) Let G be a locally compact compactly generated group and let f be a nondecreasing function satisfying

$$\int_1^\infty \left(\frac{f(t)}{\tilde{J^b}_{G,p}(t)}\right)^p \frac{dt}{t} < \infty \tag{CJ_p}$$

for some $1 < p < \infty$. Then there exists a 1-cocycle $b \in Z^1(G, \lambda_{G,p})$ whose compression ρ satisfies $\rho \succeq f$.

Theorem 9 may sound as a "functional" property of groups of class $(\mathcal{L})$. Nevertheless, our proof of this result is based on a purely geometric construction. Namely, we prove that these groups admit controlled Følner pairs (see Definition 2.4.4). In particular, when $p = 1$ we obtain the following corollary of Theorem 9, which has its own interest.

Theorem 11. (See Remark 2.4.6 and Theorem 2.5.1) Let G be a group of class $(\mathcal{L})$ and let S be some compact generating subset of G. Then G admits a sequence of compact subsets $(F_n)_{n\in\mathbf{N}}$ satisfying the two following conditions
(i) there is a constant $c > 0$ such that

$$\mu(sF_n \vartriangle F_n) \leq c\mu(F_n)/n \quad \forall s \in S, \forall n \in \mathbf{N};$$

(ii) for every $n \in \mathbf{N}$, F_n is contained [5] in S^n.
In particular, G admits a controlled Følner sequence in the sense of [CTV].

[4] We write $\tilde{\nabla}$ instead of ∇ because this is not a "metric" gradient. The gradient associated to the metric structure would be the right gradient : $|\nabla\varphi|(g) = \sup_{s\in S} |\varphi(gs) - \varphi(g)|$. This distinction is only important when the group is non-unimodular.

[5] Actually, they also satisfy $S^{[cn]} \subset F_n$ for a constant $c > 0$.

This theorem is a strengthening of the well-known construction by Pittet [Pit2]. It is stronger first because it does not require the group to be unimodular, second because the control (ii) of the diameter is really a new property that was not satisfied in general by the sequences constructed in [Pit2].

2.1.3 Further results

Let π be a isometric L^p-representation of G. Denote by $B_\pi(G)$ the supremum of all α such that there exists a 1-cocycle $b \in Z^1(G,\pi)$ whose compression ρ satisfies $\rho(t) \succeq t^\alpha$. Denote by $B_p(G)$ the supremum of $B_\pi(G)$ over all isometric L^p-representations π. For $p = 2$, $B_2(G) = B(G)$ has been introduced in [GuKa] where it was called the equivariant Hilbert compression rate. On the other hand, define

$$\alpha_{G,p} = \liminf_{t\to\infty} \frac{\log \tilde{J}^b{}_{G,p}(t)}{\log t}.$$

As a corollary of Theorem 1, we have

Corollary 12. For every $1 \leq p < \infty$, and every group G of the class $(\mathcal{L})$, we have $B_p(G) = 1$.

The following result is a corollary of Theorem 10.

Corollary 13. (see Corollary 2.4.2) Let G be a locally compact compactly generated group. For every $0 < p < \infty$, we have

$$B_{\lambda_{G,p}}(G) \geq \alpha_{G,p}.$$

The interest of this corollary is illustrated by the two following propositions. Recall the volume growth of G is the $\approx$ equivalence class V_G of the function $r \mapsto \mu(B(1,r))$.

Proposition 14. (see Proposition 2.7.1) Assume that there exists $\beta < 1$ such that $V_G(r) \preceq e^{r^\beta}$. Then

$$\alpha_{G,p} \geq 1 - \beta.$$

As an example we obtain that $B(G) \geq 0,19$ for the first Grigorchuk's group (see [Ba] for the best known upper bound of the growth function of this group).

Let G be a finitely generated group and let ν be a symmetric finitely supported probability measure on G. Write $\nu^{(n)} = \nu * \ldots * \nu$ (n times). Recall that $\nu^{(n)}(1)$ is the probability of return of the random walk starting at 1 whose probability transition is given by ν.

Proposition 15. (see Proposition 2.7.2) Assume that there exists $\gamma < 1$ such that $\nu^{(n)}(1) \succeq e^{-n^\gamma}$. Then

$$\alpha_{G,2} \geq 1 - \gamma.$$

In [AGS], it is also proved that $B(\mathbf{Z} \wr \mathbf{Z}) \in [1/2, 3/4]$. Proposition 15 and Corollary 13 together with the lower bounds for $\nu^{(n)}(1)$ obtained in [PS, Theorem 3.15] and in [CGP] provide new lower bounds for $B(G)$.

Corollary 16. We have $B(H \wr \mathbf{Z}) \geq 2/3$ if H has polynomial growth and $B(H \wr \mathbf{Z}) \geq 1/2$ if H is a discrete group of class $(\mathcal{L})$. In particular, $B(\mathbf{Z} \wr \mathbf{Z}) \geq 2/3$.

In [PS], it is proved that if G is a finitely generated extension

$$1 \to K \to G \to N \to 1$$

where K is abelian and N is abelian with $\mathbf{Q}$-rank d. Then

$$\limsup_n \log(-\log(\nu^{(n)}(1))) \leq 1 - 2/(d+2)$$

for any symmetric finitely supported probability on G.

Corollary 17. Assume that G is a finitely generated extension $1 \to K \to G \to N \to 1$ where K is abelian and N is abelian with $\mathbf{Q}$-rank d. Then

$$B(G) \geq 2/(d+2).$$

In particular, $B(G) > 0$ for any finitely generated metabelian group G.

2.1.4 Questions

Question 2.1.2. (Condition (C_p) for nilpotent connected Lie groups.) Let N be a simply connected non-abelian nilpotent Lie group and let ρ be the compression of a 1-cocycle with values in some L^p-space (resp. of a uniform embedding into some L^p-space) for $2 \leq p < \infty$. Does ρ always satisfies Condition (C_p)?

A positive answer would lead to the optimality of Theorem 1. On the contrary, one should wonder if it is possible, for any increasing sublinear function f, to find a 1-cocycle (resp. a uniform embedding) in L^p with compression $\rho \succeq f$. This would also be optimal since we know [Pau] that N cannot quasi-isometrically embed into any uniformly convex Banach space. Namely, the main theorem in [Pau] states that such a group cannot quasi-isometrically embed into any CAT(0)-space. So this only directly applies to Hilbert spaces, but the key argument, consisting in a comparison between the large scale behavior of geodesics (not exactly in the original spaces but in tangent cones of ultra-products of them) is still valid if the target space is a Banach space with unique geodesics, a property satisfied by uniformly convex Banach spaces.

Question 2.1.3. (Quasi-isometric embeddings into L^1-spaces.) Which connected Lie groups quasi-isometrically embed into some L^1-space?

It is easy to quasi-isometrically embed a simplicial tree T into ℓ^1 (see for instance [GuKa]). In [BuSc, BuSc'], it is proved that every semisimple Lie group of rank 1 quasi-isometrically embeds into a finite product of simplicial trees, hence into a ℓ^1-space. The above question is of particular interest for simply-connected non-abelian nilpotent Lie groups since they do not quasi-isometrically embed into any finite product of trees. Kleiner and Cheeger recently announced a proof that the Heisenberg group cannot quasi-isometrically embed into any L^1-space.

Question 2.1.4. If G is an amenable group, is it true that

$$B_p(G) = \alpha_{G,p}?$$

We conjecture that this is true for $\mathbf{Z} \wr \mathbf{Z}$, i.e. that $B(\mathbf{Z} \wr \mathbf{Z}) = 2/3$. A first step to prove this is done by Proposition 2.3.9 which, applied to $G = \mathbf{Z} \wr \mathbf{Z}$ says that

$$B(\mathbf{Z} \wr \mathbf{Z}) = B_{\lambda_{G,2}}(\mathbf{Z} \wr \mathbf{Z}).$$

As a variant of the above question, we may wonder if the weaker equality $B_{\lambda_{G,p}}(G) = \alpha_{G,p}$ holds, in other words if Corollary 13 is optimal for all amenable groups. Possible counterexamples would be wreath products of the form $G = \mathbf{Z} \wr H$ where H has non-linear growth (e.g. $H = \mathbf{Z}^2$).

Question 2.1.5. Does there exist an amenable group G with $B(G) = 0$?

A candidate would be the wreath product $\mathbf{Z} \wr (\mathbf{Z} \wr \mathbf{Z})$ since the probability of return of any non-degenerate random walk in this group satisfies

$$\nu^{(n)}(1) \prec e^{-n^{\gamma}}$$

for every $\gamma < 1$ [Er', Theorem 2]. It is proved in [AGS] that $B(\mathbf{Z} \wr (\mathbf{Z} \wr \mathbf{Z})) \leq 1/2$.

Question 2.1.6. Let G be a compactly generated locally compact group. If G admits an isometric action on some L^p-space, $p \geq 2$, with compression $\rho(t) \succ t^{1/p}$, does it imply that G is amenable?

Recall that this was proved in [GuKa, CTV] for $p = 2$. The generalization to every $p \geq 2$ would be of great interest. For instance, this would prove the optimality of a recent result of Yu [Yu2] saying that every finitely generated hyperbolic group admits a proper isometric action on some ℓ^p-space for large p enough, with[6] compression $\rho(t) \approx t^{1/p}$.

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2.2 Preliminaries

2.2.1 Compression

Let us recall some definitions. Let (X, d_X) and (Y, d_Y) be metric spaces. A map $F : X \to Y$ is called a uniform embedding of X into Y if

$$d_X(x, y) \to \infty \iff d_Y(F(x), F(y)) \to \infty.$$

[6] This is clear in the proof.

Note that this property only concerns the large-scale geometry. A metric space (X,d) is called *quasi-geodesic* if there exist $\delta>0$ and $\gamma\geq 1$ such that for all $x,y\in X$, there exists a chain $x=x_0,x_1,\dots,x_n=y$ satisfying :

$$\sum_{k=1}^{n} d(x_{k-1},x_k)\leq \gamma d(x,y),$$

$$\forall k=1,\dots,n,\quad d(x_{k-1},x_k)\leq\delta.$$

If X is quasi-geodesic and if $F:X\to Y$ is a uniform embedding, then it is easy to see that F is large-scale Lipschitz, i.e. there exists $C\geq 1$ such that

$$\forall x,y\in X,\quad d_Y(F(x),F(y))\leq Cd_X(x,y)+C.$$

Nevertheless, such a map is not necessarily large scale bi-Lipschitz (in other words, quasi-isometric).

Definition 2.2.1. We define the compression $\rho:\mathbf{R}_+\to[0,\infty]$ of a map $F:X\to Y$ by

$$\forall t>0,\quad \rho(t)=\inf_{d_X(x,y)\geq t} d_Y(F(x),F(y)).$$

Clearly, if F is large-scale Lipschitz, then $\rho(t)\preceq t$.

2.2.2 Length functions on a group

Now, let G be a group. A length function on G is a function $L:G\to\mathbf{R}_+$ satisfying $L(1)=0$, $L(gh)\leq L(g)+L(h)$, and $L(g)=L(g^{-1})$. If L is a length function, then $d(g,h)=L(g^{-1}h)$ defines a left-invariant pseudo-metric on G. Conversely, if d is a left-invariant pseudo-metric on G, then $L(g)=d(1,g)$ defines a length function on G.

Let G be a locally compact compactly generated group and let S be some compact symmetric generating subset of G. Equip G with a proper, quasi-geodesic length function by

$$|g|_S=\inf\{n\in\mathbf{N}:g\in S^n\}.$$

Denote d_S the associated left-invariant distance. Note that any proper, quasi-geodesic left-invariant metric is quasi-isometric to d_S, and so belongs to the same "asymptotic class".

2.2.3 Affine isometric actions and first cohomology

Let G be a locally compact group, and π a isometric representation (always assumed continuous) on a Banach space $E=E_\pi$. The space $Z^1(G,\pi)$ is defined as the set of continuous functions $b:G\to E$ satisfying, for all g,h in G, the 1-cocycle condition $b(gh)=\pi(g)b(h)+b(g)$. Observe that, given a continuous function $b:G\to\mathcal{H}$, the condition $b\in Z^1(G,\pi)$ is equivalent to saying that G

acts by affine isometries on $\mathcal{H}$ by $\alpha(g)v = \pi(g)v + b(g)$. The space $Z^1(G,\pi)$ is endowed with the topology of uniform convergence on compact subsets.

The subspace of coboundaries $B^1(G,\pi)$ is the subspace (not necessarily closed) of $Z^1(G,\pi)$ consisting of functions of the form $g \mapsto v - \pi(g)v$ for some $v \in E$. In terms of affine actions, $B^1(G,\pi)$ is the subspace of affine actions fixing a point.

The first cohomology space of π is defined as the quotient space

$$H^1(G,\pi) = Z^1(G,\pi)/B^1(G,\pi).$$

Note that if $b \in Z^1(G,\pi)$, the map $(g,h) \to \|b(g) - b(h)\|$ defines a left-invariant pseudo-distance on G. Therefore the compression of a 1-cocycle $b : (G, d_S) \to E$ is simply given by

$$\rho(t) = \inf_{|g|_S \geq t} \|b(g)\|.$$

The compression of an affine isometric action is defined as the compression of the corresponding 1-cocycle.

Remark 2.2.2. When the space E is a Hilbert space[7], it is well-known [HV, §4.a] that $b \in B^1(G,\pi)$ if and only if b is bounded on G.

2.3 The maximal L^p-compression functions $M\rho_{G,p}$ and $M\rho_{\lambda_{G,p}}$

2.3.1 Definitions and general results

Let (G, d_S, μ) be a locally compact compactly generated group, generated by some compact symmetric subset S and equipped with a left Haar measure μ. Denote by $Z^1(G,p)$ the collection of all 1-cocycles with values in any L^p-representation of G. Denote by ρ_b the compression function of a 1-cocycle $b \in Z^1(G,p)$.

Definition 2.3.1. We call maximal L^p-compression function of G the nondecreasing function $M\rho_{G,p}$ defined by

$$M\rho_{G,p}(t) = \sup\left\{\rho_b(t) : \ b \in Z^1(G,p), \ \sup_{s\in S}\|b(s)\| \leq 1\right\}.$$

We call maximal regular L^p-compression function of G the nondecreasing function $M\rho_{\lambda_{G,p}}$ defined by

$$M\rho_{\lambda_{G,p}} = \sup\left\{\rho_b(t) : \ b \in Z^1(G,\lambda_{G,p}), \ \sup_{s\in S}\|b(s)\| \leq 1\right\}.$$

[7] The same proof holds for uniformly convex Banach spaces.

Note that the asymptotic behaviors of both $M\rho_{G,p}$ and $M\rho_{\lambda_{G,p}}$ do not depend on the choice of the compact generating set S. Moreover, we have

$$M\rho_{\lambda_{G,p}}(t) \leq M\rho_{G,p}(t) \preceq t.$$

Let φ be a measurable function on G such that $\varphi - \lambda(s)\varphi \in L^p(G)$ for every $s \in S$. For every $t > 0$, define

$$\mathrm{Var}_p(\varphi, t) = \inf_{|g|_S \geq t} \|\varphi - \lambda(g)\varphi\|_p$$

The function φ and p being fixed, the map $t \mapsto \mathrm{Var}_p(\varphi, t)$ is nondecreasing.

Proposition 2.3.2. We have

$$M\rho_{\lambda_{G,p}}(t) = \sup_{\|\tilde{\nabla}\varphi\|_p \leq 1} \mathrm{Var}_p(\varphi, t).$$

Proof : We trivially have

$$M\rho_{\lambda_{G,p}}(t) \geq \sup_{\|\tilde{\nabla}\varphi\|_p \leq 1} \mathrm{Var}_p(\varphi, t).$$

Let b be an element of $Z^1(G, \lambda_{G,p})$. By a standard argument of convolution[8], one can approximate b by a cocycle b' such that $x \to b'(g)(x)$ is continuous for every g in G. Hence, we can assume that $b(g)$ is continuous for every g in G. Now, setting $\varphi(g) = b(g)(g)$, we define a measurable function satisfying

$$b(g) = \varphi - \lambda(g)\varphi.$$

So we have

$$\rho(t) = \mathrm{Var}_p(\varphi, t) \leq M\rho_{\lambda_{G,p}}(t)$$

where ρ is the compression of b. ■

Remark 2.3.3. It is not difficult to prove that the asymptotic behavior of $M\rho_{\lambda_{G,p}}$ is invariant under quasi-isometry between finitely generated groups.

Proposition 2.3.4. The group G admits a proper[9] 1-cocycle with values in some L^p-representation if and only if $M\rho_{G,p}(t)$ goes to infinity when $t \to \infty$.

Proof : The "only if" part is trivial. Assume that $M\rho_{G,p}(t)$ goes to infinity. Let (t_k) be an increasing sequence growing fast enough so that

$$\sum_{k \in \mathbf{N}} \frac{1}{t_k^p} < \infty.$$

---

[8] One can convolute $b(g)$, for every g, on the right by a Dirac approximation.

[9] For $p = 2$, this means that G is a-T-menable if and only if $M\rho_{G,2}$ goes to infinity. It should be compared to the role played by the H-metric (see [Co, § 2.6] and § 7.4) for Property (T).

For every $k \in \mathbf{N}$, choose some $b_k \in Z^1(G,p)$ whose compression ρ_k satisfies

$$\rho_k(t_k) \geq \frac{M\rho_{G,p}(t_k)}{2}$$

and such that

$$\sup_{s\in S} \|b_k(s)\| \leq 1.$$

Clearly, we can define a 1-cocycle $b \in Z^1(G,p)$ by

$$b = \oplus_k^{\ell^p} \frac{1}{t_k} b_k.$$

That is, if for every k, b_k takes values in the representation π_k, then b takes values in the direct sum $\oplus_k^{\ell^p} \pi_k$. Now, observe that for $|g| \geq t_k$ and $j \leq k$, we have $\|b_j(g)\| \geq 1/2$, so that

$$\|b(g)\|^p \geq k/2^p.$$

Thus the cocycle b is proper. ■

The following proposition, which is a quantitative version of the previous one, plays a crucial role in the sequel.

Proposition 2.3.5. Let $f : \mathbf{R}_+ \to \mathbf{R}_+$ be a nondecreasing map satisfying

$$\int_1^\infty \left(\frac{f(t)}{M\rho_{G,p}(t)}\right)^p \frac{dt}{t} < \infty, \qquad (CM_p)$$

Then,
(1) there exists a 1-cocycle $b \in Z^1(G,p)$ such that

$$\rho \succeq f;$$

(2) if one replace $M\rho_{G,p}$ by $M\rho_{\lambda_{G,p}}$ in Condition (CM_p), then b can be chosen in $Z^1(G,\lambda_{G,p})$.

Proof of (1) : For every $k \in \mathbf{N}$, choose some $b_k \in Z^1(G,p)$ (for (2), we take $b_k \in Z^1(G,\lambda_{G,p})$) whose compression ρ_k satisfies

$$\rho_k(2^{k+1}) \geq \frac{M\rho_{G,p}(2^{k+1})}{2}$$

and such that

$$\sup_{s\in S} \|b_k(s)\| \leq 1.$$

Then define another sequence of cocycles $\tilde{b}_k \in Z^1(G,p)$ by

$$\tilde{b}_k = \frac{f(2^k)}{M\rho_{G,p}(2^{k+1})} b_k.$$

Since $M\rho_{G,p}$ and f are nondecreasing, for any $2^k \leq t \leq 2^{k+1}$, we have

$$\frac{f(2^k)}{M\rho_{G,p}(2^{k+1})} \leq \frac{f(t)}{M\rho_{G,p}(t)}.$$

Hence, for $s \in S$,

$$\begin{aligned} \sum_k \|\tilde{b}_k(s)\|_p^p &\leq \sum_k \left(\frac{f(2^k)}{M\rho_{G,p}(2^{k+1})}\right)^p \\ &\leq 2\int_1^\infty \left(\frac{f(t)}{M\rho_{G,p}(t)}\right)^p \frac{dt}{t} < \infty \end{aligned}$$

So we can define a 1-cocycle on $b \in Z^1(G,p)$ by

$$b = \oplus_k \tilde{b}_k. \tag{2.3.1}$$

On the other hand, if $|g|_S \geq 2^{k+1}$, then

$$\begin{aligned} \|b(g)\|_p &\geq \|\tilde{b}_k(g)\|_p \\ &\geq \frac{f(2^k)}{M\rho_{\lambda_{G,p}}(2^{k+1})}\rho_k(2^{k+1}) \\ &\geq f(2^k) \end{aligned}$$

So if ρ is the compression of the 1-cocycle b, we have $\rho \succeq f$.

Proof of (2) : We keep the previous notation. Assume that f satisfies

$$\int_1^\infty \left(\frac{f(t)}{M\rho_{\lambda_{G,p}}(t)}\right)^p \frac{dt}{t} < \infty.$$

The cocycle b provided by the proof of (1) has the expected compression but it takes values in an infinite direct sum of regular representation $\lambda_{G,p}$. Now, we would like to replace the direct sum $b = \oplus_k b_k$ by a mere sum, in order to obtain a cocycle in $Z^1(G, \lambda_{G,p})$. Since G is not assumed unimodular, the measure μ is not necessarily right-invariant. However, one can define a isometric representation $r_{G,p}$ on $L^p(G)$, called the right regular representation by

$$r_{G,p}(g)\varphi = \Delta(g)^{-1}\varphi(\cdot g) \quad \forall \varphi \in L^p(G),$$

where Δ is the modular function of G. We will use the following well-known property of the representation $r_{G,p}$, for $p > 1$. To simplify, let us write $r(g)$ instead of $r_{G,p}(g)$. For every $(\varphi, \psi) \in L^p(G) \times L^p(G)$, we have

$$\lim_{|g|\to\infty} \|r(g)\varphi + \psi\|_p^p = \|\varphi\|_p^p + \|\psi\|_p^p. \tag{2.3.2}$$

Moreover, this limit is uniform on compact subsets of $(L^p(G))^2$. As $r_{G,p}$ and $\lambda_{G,p}$ commute, $r_{G,p}$ acts by isometries on $Z^1(G, \lambda_{G,p})$.

Lemma 2.3.6. There exists a sequence (g_k) of elements of G such that $b' = \sum r(g_k)b_k$ defines a cocycle in $Z^1(G, \lambda_{G,p})$ and such that

$$\left| \|b'(g)\|_p^p - \left\| \sum_{j=0}^{k-1} r(g_j)b_j(g) \right\|_p^p - \sum_{j\geq k} \|b_j(g)\|_p^p \right| \leq 1 \qquad (2.3.3)$$

for any k large enough and every $g \in B(1, 2^{k+2})$.

Proof of Lemma 2.3.6. By an immediate induction, using (2.3.2), we construct a sequence $(g_k) \in G^{\mathbf{N}}$ satisfying, for every $K \geq 0, s \in S$,

$$\| \sum_{k=0}^{K} r(g_k)b_k(s)\|_p^p \leq \sum_{k=0}^{K} \|b_k(s)\|_p^p + \sum_{k=0}^{K} 2^{-k-1} \leq 1,$$

which implies that b' is a well-defined 1-cocycle in $Z^1(G, \lambda_{G,p})$. Similarly, one can choose (g_k) satisfying the additional property that, for every $k \in \mathbf{N}$, $|g| \leq 2^{k+2}$,

$$\left| \| \sum_{j=0}^{k} r(g_j)b_j(g)\|_p^p - \| \sum_{j=0}^{k-1} r(g_j)b_j(g)\|_p^p - \|b_k(g)\|_p^p \right| \leq 2^{-k-1}.$$

Fixing $k \in \mathbf{N}$, an immediate induction over K shows that for every $|g| \leq 2^{k+2}$ and every $K \geq k$,

$$\left| \| \sum_{j=0}^{K} r(g_j)b_j(g)\|_p^p - \| \sum_{j=0}^{k-1} r(g_j)b_j(g)\|_p^p - \sum_{j=k}^{K} \|b_j(g)\|_p^p \right| \leq \sum_{j=k}^{K} 2^{-j-1}.$$

This proves (8.3.3). ■

By the lemma, for $|g| \leq 2^{k+2}$,

$$\|b'(g)\|_p^p \quad \geq \quad \|b_k(g)\|_p^p - 1.$$

Then, for $2^{k+1} \leq |g| \leq 2^{k+2}$, we have

$$\|b'(g)\|_p^p \quad \geq \quad f(2^k) - 1$$

Therefore, the compression ρ' of b' satisfies

$$\rho' \succeq f$$

and we are done. ■

We have the following immediate consequence.

Corollary 2.3.7. For every $1 \leq p < \infty$,

$$B(G,p) = \liminf_{t\to\infty} \frac{\log M\rho_{G,p}(t)}{\log t}.$$

Example 2.3.8. Let F_r be the free group of rank $r \geq 2$ and let $A(F_r)$ be the set of edges of the Cayley graph of F_r associated to the standard set of generators. The standard isometric affine action of F_r on $\ell^p(A(F_r))$, whose linear part is isomorphic to a direct sum $\lambda_{G,p} \oplus_{\ell^p} \ldots \oplus_{\ell^p} \lambda_{G,p}$ of r copies of $\lambda_{G,p}$ has compression $\approx t$. This shows that $M\rho_{\lambda_{F_r,p}}(t) \succeq t^{1/p}$.

2.3.2 Reduction to the regular representation for $p = 2$

In the Hilbert case, we prove that if a group admits a 1-cocycle with large enough compression, then $M\rho_{G,2} = M\rho_{\lambda_{G,2}}$. This result is mainly motivated by Question 2.1.4 since it implies that

$$B(\mathbf{Z} \wr \mathbf{Z}) = B_{\lambda_{G,2}}(\mathbf{Z} \wr \mathbf{Z}).$$

Proposition 2.3.9. Let π be a unitary representation of the group G on a Hilbert space $\mathcal{H}$ and let $b \in Z^1(G, \pi)$ be a cocycle whose compression ρ satisfies

$$\rho(t) \succ t^{1/2}.$$

Then[10],

$$\rho \preceq M\rho_{\lambda_{G,2}}.$$

In particular,

$$M\rho_2 = M\rho_{\lambda_{G,2}}.$$

combining with Proposition 2.3.5, we obtain

Corollary 2.3.10. With the same hypotheses, we have

$$B(G) = B(G, \lambda_{G,2}) = \liminf_{t \to \infty} \frac{\log M\rho_{\lambda_{G,2}}(t)}{\log t}.$$

Proof of Proposition 2.3.9. For every $t > 0$, define

$$\varphi_t(g) = e^{-\|b(g)\|^2/t^2}.$$

By Schoenberg's Theorem [BHV, Appendix C], φ_t is positive definite. It is easy to prove that φ_t is square-summable (see [CTV, Theorem 4.1]). By [Dix, Théorème 13.8.6], it follows that there exists a positive definite, square-summable function ψ_t on G such that $\varphi_t = \psi_t * \psi_t$, where $*$ denotes the convolution product. In other words, $\varphi_t = \langle \lambda(g)\psi_t, \psi_t \rangle$. In particular,

$$\varphi_t(1) = 1 = \|\psi_t\|_2^2$$

and for every $s \in S$,

$$\begin{aligned} \|\psi_t - \lambda(s)\psi_t\|_2^2 &= 2(\|\psi_t\|_2^2 - \langle \lambda(s)\psi_t, \psi_t \rangle) \\ &= 2(1 - \varphi_t(s)) \\ &= 2(1 - e^{-\|b(s)\|^2/t^2}) \\ &\preceq 1/t^2 \end{aligned}$$

[10]Note that the hypotheses of the proposition also imply that G is amenable [CTV, GuKa, Theorem 4.1].

On the other hand, for g such that $\rho(|g|_S) \geq t$, we have

$$\begin{aligned} \|\psi_t - \lambda(g)\psi_t\|_2^2 &= 2(1 - e^{-\|b(g)\|^2/t^2}) \\ &\geq 2(1 - e^{-\rho(|g|_S)^2/t^2}) \\ &\geq 2(1 - 1/e) \end{aligned}$$

So, we have

$$\frac{\|\psi_t - \lambda(g)\psi_t\|_2}{\|\tilde{\nabla}\psi_t\|_2} \geq ct$$

where c is a constant. In other words,

$$\mathrm{Var}_2(\psi_t, \rho^{-1}(t)) \geq ct.$$

It follows from the definitions that $M\rho_{\lambda_{G,2}} \succeq \rho$. ■

2.4 L^p-isoperimetry inside balls

2.4.1 Comparing $\tilde{J^b}_{G,p}$ and $M\rho_{\lambda_{G,p}}$

Let us compare $\tilde{J}^b_{p,G}$ and $M\rho_{\lambda_{G,p}}$ introduced in § 2.3.

Proposition 2.4.1. For every $2 \leq p < \infty$, we have

$$M\rho_{\lambda_{G,p}} \succeq \tilde{J^b}_{G,p}.$$

Proof : Fix some $t > 0$ and choose some $\varphi \in L^p(X)$ whose support lies in $B(1,t)$ such that

$$\frac{\|\varphi\|_p}{\|\tilde{\nabla}\varphi\|_p} \geq \tilde{J^b}_{G,p}(t)/2.$$

Take $g \in G$ satisfying $|g|_S \geq 3t$. Note that $B(1,t) \cap \lambda(g)B(1,t) = \emptyset$. So φ and $\lambda(g)\varphi$ have disjoint supports. In particular,

$$\|\varphi - \lambda(g)\varphi\|_p \geq \|\varphi\|_p$$

and

$$\|\tilde{\nabla}(\varphi - \lambda(g)\varphi)\|_p = 2^{1/p}\|\tilde{\nabla}\varphi\|_p$$

This clearly implies the proposition. ■

Combining with Proposition 2.3.5, we obtain

Corollary 2.4.2. Let $f : \mathbf{R}_+ \to \mathbf{R}_+$ a nondecreasing map be satisfying

$$\int_1^\infty \left(\frac{f(t)}{\tilde{J^b}_{G,p}(t)}\right)^p \frac{dt}{t} < \infty \qquad (CJ_p)$$

for some $1 \leq p < \infty$. Then there exists a 1-cocycle b in $Z^1(G, \lambda_{G,p})$ such that

$$\rho \succeq f.$$

Question 2.4.3. For which groups G do we have $M\rho_{\lambda_{G,p}} \approx \tilde{J^b}_{G,p}$?

We show that the question has positive answer for groups of class $(\mathcal{L})$. On the contrary, note that the group G is nonamenable if and only if $\tilde{J^b}_{G,p}$ is bounded. But we have seen in the previous section that for a free group of rank ≥ 2, $M\rho_{\lambda_{G,p}}(t) \succeq t^{1/p}$. More generally, the answer to Question 2.4.3 is no for every nonamenable group admitting a proper 1-cocycle with values in the regular representation. This question is therefore only interesting for amenable groups.

2.4.2 Sequences of controlled Følner pairs

In this section, we give a method, adapted[11] from [CGP] to estimate $\tilde{J^b}_p$.

Definition 2.4.4. Let G be a compactly generated, locally compact group equipped with a left invariant Haar measure μ. Let $\alpha = (\alpha_n)$ be a nondecreasing sequence of integers. A sequence of α-controlled Følner pairs of G is a family (H_n, H_n') where H_n and H_n' are nonempty compact subsets of G satisfying for some constant $C > 0$ the following conditions :
(1) $S^{\alpha_n} H_n \subset H_n'$
(2) $\mu(H_n') \leq C\mu(H_n)$;
(3) $H_n' \in B(1, Cn)$

If $\alpha_n \approx n$, we call (H_n, H_n') a controlled sequence of Følner pairs.

Proposition 2.4.5. Assume that G admits a sequence of α-controlled Følner pairs. Then

$$\tilde{J^b}_{G,p} \succeq \alpha.$$

Proof : For every $n \in \mathbf{N}$, consider the function $\varphi_n : G \to \mathbf{R}_+$ defined by

$$\varphi_n(g) = \min\{k \in \mathbf{N} : g \in S^k (H_n')^c\}$$

where $A^c = G \smallsetminus A$. Clearly, φ_n is supported in H_n'. It is easy to check that

$$\|\tilde{\nabla}\varphi_n\|_p \leq (\mu(H_n'))^{1/p}$$

and that

$$\|\varphi_n\|_p \geq \alpha_n(\mu(H_n))^{1/p}.$$

Hence by (2),

$$\|\varphi_n\|_p \geq C^{-1/p}\alpha_n\|\tilde{\nabla}\varphi_n\|_p,$$

so we are done. ■

Remark 2.4.6. Note that if H and H' are subsets of G such that $S^k H \subset H'$ and $\mu(H') \leq C\mu(H)$, then there exists by pigeonhole principle an integer $0 \leq j \leq k-1$ such that

$$\mu(\partial S^j H) = \mu(S^{j+1} H \smallsetminus S^j H) \leq \frac{C}{k}\mu(S^j H).$$

[11] In [CGP], the authors are interested in estimating the L^2-isoperimetric profile of a group.

So in particular if (H_n, H'_n) is a α-controlled sequence of Følner pairs, then there exists a Følner sequence (K_n) such that $H_n \subset K_n \subset H'_n$ and

$$\frac{\mu(\partial K_n)}{\mu(K_n)} \leq C/\alpha_n.$$

Moreover, if $\alpha_n \approx n$, then one obtains a controlled Følner sequence in the sense of [CTV, Definition 4.8].

2.5 Isoperimetry in balls for groups of class $(\mathcal{L})$

The purpose of this section is to prove the following theorem.

Theorem 2.5.1. Let G be a group belonging to the class $(\mathcal{L})$. Then, G admits a controlled sequence of Følner pairs. In particular, $\tilde{J^b}_{G,p}(t) \approx t$.

Note that Theorem 1 follows from Theorem 2.5.1 and Corollary 2.4.2.

2.5.1 Wreath products $F \wr \mathbf{Z}$

Let F be a finite group. Consider the wreath product $G = F \wr \mathbf{Z} = \mathbf{Z} \ltimes F^{(\mathbf{Z})}$, the group law being defined as $(n, f)(m, g) = (n + m, \tau_m f + g)$ where $\tau_m f(x) = f(m + x)$. As a set, G is a Cartesian product $\mathbf{Z} \times U$ where U is the direct sum $F^{(\mathbf{Z})} = \bigoplus_{n \in \mathbf{Z}} F_n$ of copies F_n of F. The set $S = S_F \cup S_{\mathbf{Z}} F_n$, where $S_F = F_0$ and $S_{\mathbf{Z}} = \{-1, 0, 1\}$ is clearly a symmetric generating set for G. Define

$$H_n = I_n \times U_n$$

and

$$H'_n = I_{2n} \times U_n$$

where $U_n = F^{[-2n,2n]}$ and $I_n = [-n, n]$.
Let us prove that $(H_n, H'_n)_n$ is a sequence of controlled Følner pairs. We therefore have to show that
(1) $S^n H_n \subset H'_n$
(2) $|H'_n| \leq 2|H_n|$;
(3) there exists $C > 0$ such that $H'_n \subset B(1, Cn)$

Property (2) is trivial. To prove (1) and (3), recall that the length of an element of $g = (k, u)$ of G equals $L(\gamma) + \sum_{h \in \mathbf{Z}} |u(h)|_F$ where $L(\gamma)$ is the length of a shortest path γ from 0 to k in $\mathbf{Z}$ passing through every element of the support of u (see [Par, Theorem 1.2]). In particular,

$$|(u, k)|_S \leq 2L(\gamma)$$

Thus, if $g \in H_n$, then $L(\gamma) \leq 30n$. So (3) follows. On the other hand, if $g = (k, u) \in S^n$, then

$$|k|_{\mathbf{Z}} \leq L(\gamma) \leq n$$

and

$$\mathrm{Supp}(u) \subset I_n.$$

So $H_n g \subset H'_n$. ■

Remark 2.5.2. Note that the proof still works replacing $\mathbf{Z}$ by any group with linear growth. On the other hand, replacing it by a group of polynomial growth of degree d yields to a sequence of $n^{1/d}$-controlled Følner pairs. For instance, as a corollary, we obtain that $B(F \wr \mathbf{Z}^d) \geq 1/d$; but these estimates are not as good as the one that we obtain in Corollary 17.

2.5.2 Semidirect products $\left(\mathbf{R} \oplus \bigoplus_{p \in P} \mathbf{Q}_p\right) \rtimes_{\frac{m}{n}} \mathbf{Z}$.

Note that discrete groups of type (2) of the class $(\mathcal{L})$ are cocompact lattices of a group of the form

$$G = \mathbf{Z} \ltimes_{\frac{m}{n}} \left(\mathbf{R} \oplus \bigoplus_{p \in P} \mathbf{Q}_p\right)$$

with m, n coprime integers and P a finite set of primes (possibly infinite) dividing mn. To simplify notation, we will only consider the case when $P = \{p\}$ is reduced to one single prime, the generalization presenting no difficulty. The case where $p = \infty$ will result from the case of connected Lie groups (see next section) since $\mathbf{Z} \ltimes_{\frac{m}{n}} \mathbf{R}$ embeds as a closed cocompact subgroup of the group of positive affine transformations $\mathbf{R} \ltimes \mathbf{R}$.

So consider the group $G = \mathbf{Z} \ltimes_{1/p} \mathbf{Q}_p$. Define a compact symmetric generating set by $S = S_{\mathbf{Q}_p} \cup S_{\mathbf{Z}}$ where $S_{\mathbf{Q}_p} = \mathbf{Z}_p$ and $S_Z = \{-1, 0, 1\}$. Define (H_k, H_k') by

$$H_k = I_k \times p^{-2k}\mathbf{Z}_p$$

and

$$H_k' = I_{2k} \times p^{-2k}\mathbf{Z}_p,$$

where $I_k = [-k, k]$. Using the same kind of arguments as previously for $F \wr \mathbf{Z}$, one can prove easily that (H_k, H_k') is a controlled sequence of Følner pairs. ■

2.5.3 Amenable connected Lie groups

Let G be a solvable simply connected Lie group. Let S be a compact symmetric generating subset. In [O], it is proved that G admits a maximal normal connected subgroup such that the quotient of G by this subgroup has polynomial growth. This subgroup is called the exponential radical and is denoted $\text{Exp}(G)$. We have $\text{Exp}(G) \subset N$, where N is the maximal nilpotent normal subgroup of G. Let T be a compact symmetric generating subset of $\text{Exp}(G)$. An element $g \in G$ is called strictly exponentially distorted if the S-length of g^n grows as $\log |n|$. The subset of strictly exponentially distorted elements of G coincides with $\text{Exp}(G)$. That is,

$$\text{Exp}(G) = \{g \in G : \ |g^n|_S \approx \log n\}.$$

Moreover, $\text{Exp}(G)$ is strictly exponentially distorted in G in the sense that there exists $\beta \geq 1$ such that for every $h \in \text{Exp}(G)$,

$$\beta^{-1} \log(|h|_T + 1) - \beta \leq |h|_S \leq \beta \log(|h|_T + 1) + \beta \tag{2.5.1}$$

where T is a compact symmetric generating subset of $\mathrm{Exp}(G)$.

We will need the following two lemmas.

Lemma 2.5.3. Let G be a locally compact group. Let H be a closed normal subgroup. Let λ and ν be respectively left Haar measures of H and G/H. Let i be a measurable left-section of the projection $\pi: \ G \to G/H$, i.e. $G = \sqcup_{x\in G/H} i(x)H$. Identify G with the cartesian product $G/H \times H$ via the map $(x,h) \mapsto i(x)h$. Then the product measure $\nu \otimes \lambda$ is a left Haar measure on G.

Proof : We have to prove that $\nu \otimes \lambda$ is left-invariant on G. Fix g in G. Define a measurable map σ_g from G/H to H by

$$\sigma_g(x) = (i(\pi(g)x)^{-1} g i(x).$$

In other words, $\sigma_g(x)$ is the unique element of H such that

$$gi(x) = i(\pi(g)x)\sigma_g(x).$$

Let $\varphi \ : G \to \mathbf{R}$ be a continuous, compactly supported function. We have

$$\int_{G/H\times H} \varphi[gi(x)h] d\nu(x) d\lambda(h) = \int_{G/H\times H} \varphi[i(\pi(g)x)\sigma_g(x)h] d\nu(x) d\lambda(h)$$

As ν and λ are respectively left Haar measures on G/H and H, the Jacobian of the transformation $(x,h) \mapsto (\pi(g)x, \sigma_g(x)h)$ is equal to 1. Hence,

$$\int_{G/H\times H} \varphi[i(\pi(g)x)\sigma_g(x)h] d\nu(x) d\lambda(h) \ = \ \int_{G/H\times H} \varphi[i(x)h] d\nu(x) d\lambda(h).$$

Thus $\nu \otimes \lambda$ is left-invariant. ■

Lemma 2.5.4. Let G be a connected Lie group and H be a normal subgroup. Consider the projection $\pi : G \to G/H$. There exists a compact generating set S of G and a σ-compact cross-section σ of G/H inside G such that $\sigma(\pi(S)^n) \subset S^{n+1}$.

Proof : Since π is a submersion, there exists a compact neighborhood S of 1 in G such that $\pi(S)$ admits a continuous cross-section σ_1 in S. Now, let X be a minimal (discrete) subset of G/H satisfying $G/H = \cup_{x\in X} x\pi(S)$. Since this covering is locally finite and $\pi(S)$ is compact, one can construct by induction a partition $(A_x)_{x\in X}$ of G/H such that every A_x is a constructible, and therefore σ-compact subset of $x\pi(S)$. Let $\sigma_2 : X \to G$ be a cross-section of X satisfying $\sigma_2(X \cap \pi(S)^n) \subset S^n$. Now, for every $z \in A_x$, define

$$\sigma(z) = \sigma_2(x)\sigma_1(x^{-1}z).$$

Clearly, σ satisfies to the hypotheses of the lemma. ■

Equip the group $P = G/\mathrm{Exp}(G)$ with a Haar measure ν and with the symmetric generating subset $\pi(S)$, where π is the projection on P. Assume that S satisfies to the hypotheses of Lemma 2.5.4 and let σ be a σ-compact cross-section of P inside G such that $\sigma(\pi(S)^n) \subset S^{n+1}$. For every $n \in \mathbf{N}$, write $F_n = \sigma(\pi(S)^n)$. Let α be some large enough positive number that we will determine later. Denote by $\lfloor x \rfloor$ the integer part of a real number x. Define, for every $n \in \mathbf{N}$,

$$H_n = S^n T^{\lfloor \exp(\alpha n) \rfloor}$$

and

$$H'_n = S^{2n} T^{\lfloor \exp(\alpha n) \rfloor}.$$

Note that $H'_n = S^n H_n$. On the other hand, since $\mathrm{Exp}(G)$ is strictly exponentially distorted, there exists $a \geq 1$ only depending on α and β such that, for every $n \in \mathbf{N}$,

$$S^n T^{\lfloor \exp(\alpha n) \rfloor} \subset S^{an}.$$

Hence, to prove that (H_n, H'_n) is a sequence of controlled Følner pairs, it suffices to show that $\mu(H'_n) \leq C\mu(H_n)$. Consider another sequence (A_n, A'_n) defined by, for every $n \in \mathbf{N}^*$,

$$A_n = F_{n-1} T^{\lfloor \exp(\alpha n) \rfloor}$$

and

$$A'_n = F_{2n} T^{2\lfloor \exp(\alpha n) \rfloor}.$$

As F_n is σ-compact, A_n and A'_n are measurable. To compute the measures of A_n and A'_n, we choose a normalization of the Haar measure λ on $\mathrm{Exp}(G)$ such that the measure μ disintegrates over λ and the pull-back measure of ν on $\sigma(P)$ as in Lemma 2.5.3. We therefore obtain

$$\mu(A_n) = \nu(\pi(S)^{n-1}) \lambda(T^{\lfloor \exp(\alpha n) \rfloor})$$

and

$$\mu(A'_n) = \nu(\pi(S)^{2n}) \lambda(T^{2\lfloor \exp(\alpha n) \rfloor}).$$

Since P and $\mathrm{Exp}(G)$ have both polynomial growth, there is a constant C such that, for every $n \in \mathbf{N}^*$,

$$\mu(A'_n) \leq C\mu(A_n).$$

So now, it suffices to prove that

$$A_n \subset H_n \subset H'_n \subset A'_n,$$

where the only nontrivial inclusion is $H'_n \subset A'_n$. Let $g \in S^{2n}$; let $f \in F_{2n}$ be such that $\pi(g) = \pi(f)$. Since $F_{2n} \subset S^{2n+2} \subset S^{3n}$,

$$gf^{-1} \in S^{6n} \cap \mathrm{Exp}(G).$$

On the other hand, by (2.5.1),

$$S^{6n} \cap \mathrm{Exp}(G) \subset T^{2\lfloor \exp(6\beta n) \rfloor}.$$

Therefore, for every $n \in \mathbf{N}^*$,

$$H'_n \subset F_{2n} T^{2\lfloor \exp(6\beta n) \rfloor} T^{\lfloor \exp(\alpha n) \rfloor} = F_{2n} T^{2\lfloor \exp(6\beta n) \rfloor + \lfloor \exp(\alpha n) \rfloor}.$$

Hence, choosing $\alpha \geq 6\beta + \log 2$, we have

$$H'_n \subset F_{2n} T^{2\lfloor \exp(\alpha n) \rfloor} = A'_n,$$

and we are done. ■

2.6 On embedding of finite trees into uniformly convex Banach spaces

Definition 2.6.1. A Banach space X is called q-uniformly convex ($q > 0$) if there is a constant $a > 0$ such that for any two points x, y in the unit sphere satisfying $\|x - y\| \geq \varepsilon$, we have

$$\left\| \frac{x+y}{2} \right\| \leq 1 - a\varepsilon^q.$$

Note that by a theorem of Pisier [Pis], every uniformly convex Banach space is isomorphic to some q-uniformly convex Banach space.

In this section, we prove that the compression of a Lipschitz embedding of a finite binary rooted tree into a q-uniformly convex space X always satisfies condition (C_q). Theorem 4 follows from the fact that a L^p-space is $\max\{p, 2\}$-uniformly convex.

Theorem 2.6.2. Let T_J be the binary rooted tree of depth J and let $1 < q < \infty$. Let F be a 1-Lipschitz map from T_J to some q-uniformly convex Banach space X and let ρ be the compression of F. Then there exists $C = C(q) < \infty$ such that

$$\int_1^{2J} \left(\frac{\rho(t)}{t} \right)^q \frac{dt}{t} \leq C. \tag{2.6.1}$$

Corollary 2.6.3. Let F be any uniform embedding of the 3-regular tree T into some q-uniformly convex Banach space. Then the compression ρ of F satisfies Condition (C_q). ■

As a corollary, we also reobtain the theorem of Bourgain.

Corollary 2.6.4. [Bou, Theorem 1] With the notation of Theorem 2.6.2, there exists at least two vertices x and y in T_J such that

$$\frac{\|F(x) - F(y)\|}{d(x,y)} \leq \left(\frac{C}{\log J} \right)^{1/q}.$$

Proof : For every $1 \leq t \leq 2J$, there exist $z, z' \in T_J$, $d(z,z') \geq t$ such that :

$$\frac{\rho(t)}{t} = \frac{\|F(z)-F(z')\|}{t} \geq \frac{\|F(z)-F(z')\|}{d(z,z')}.$$

Therefore

$$\min_{z \neq z' \in T_J} \frac{\|F(z)-F(z')\|}{d(z,z')} \leq \min_{1 \leq u \leq 2J} \frac{\rho(u)}{u}.$$

But by (2.6.1)

$$\left(\min_{1 \leq u \leq 2J} \frac{\rho(u)}{u}\right)^q \int_1^{2J} \frac{1}{t} dt \leq \int_1^{2J} \left(\frac{\rho(t)}{t}\right)^q \frac{dt}{t} \leq C.$$

We then have

$$\min_{z \neq z' \in T_J} \frac{\|F(z)-F(z')\|}{d(z,z')} \leq \left(\frac{C}{\log J}\right)^{1/q}. \blacksquare$$

Proof of Theorem 2.6.2. Since the proof follows closely the proof of [Bou, Theorem 1], we keep the same notation to allow the reader to compare them. For $j = 1, 2 \ldots$, denote $\Omega_j = \{-1,1\}^j$ and $T_j = \bigcup_{j' \leq j} \Omega_{j'}$. Thus T_j is the finite tree with depth j. Denote d the tree-distance on T_j.

Lemma 2.6.5. [Pis, Proposition 2.4] There exists $C = C(q) < \infty$ such that if $(\xi_s)_{s \in \mathbf{N}}$ is an X-valued martingale on some probability space Ω, then

$$\sum_s \|\xi_{s+1} - \xi_s\|_q^q \leq C \sup_s \|\xi_s\|_q^q \tag{2.6.2}$$

where $\| \, \|_q$ stands for the norm in $L^q_X(\Omega)$.

Lemma 2.6 is used to prove

Lemma 2.6.6. If $x_1, \ldots, x_J$, with $J = 2^r$, is a finite system of vectors in X, then

$$\sum_{s=1}^r 2^{-qs} \min_{2^s \leq j \leq J-2^s} \|2x_j - x_{j-2^s} - x_{j+2^s}\|^q \leq C \sup_{1 \leq j \leq J-1} \|x_{j+1} - x_j\|^q. \tag{2.6.3}$$

Denote $\mathcal{D}_0 \subset \mathcal{D}_1 \subset \ldots \subset \mathcal{D}_r$ the algebras of intervals on $[0,1]$ obtained by successive dyadic refinements. Define the X-valued function

$$\xi = \sum_{1 \leq j \leq J-1} 1_{[\frac{j}{J}, \frac{j+1}{J}[} (x_{j+1} - x_j)$$

and consider expectations $\xi_s = \mathbf{E}[\xi|\mathcal{D}_s]$ for $s = 1, \ldots, r$. Since ξ_s form a martingale ranging in X, it satisfies inequality (2.6.2). On the other hand

$$\begin{aligned} \|\xi_{s+1} - \xi_s\|_q^q &= 2^{-r+s} 2^{qs} \sum_{1 \leq t \leq 2^{r-s}}^r 2^{-qs} \|2x_{t2^s} - x_{(t-1)2^s} - x_{(t+1)2^s}\|^q \\ &\leq 2^{-qs} \min_{2^s \leq j \leq J-2^s} \|2x_j - x_{(j-2^s} - x_{j+2^s}\|^q. \end{aligned}$$

So (2.6.3) follows from the fact that

$$\|\xi_s\|_q^q \leq \|\xi_{s+1} - \xi_s\|_\infty^q = \sup_j \|x_{j+1} - x_j\|^q. \ \blacksquare$$

Lemma 2.6.7. If $f_1, \ldots, f_J$, with $J = 2^r$, is a finite system of functions in $L_X^\infty(\Omega)$. Then

$$\sum_{s=1}^{r} 2^{-qs} \min_{2^s \leq j \leq J-2^s} \|2f_j - f_{j-2^s} - f_{j+2^s}\|^q \leq C \sup_{1 \leq j \leq J-1} \|f_{j+1} - f_j\|_\infty^q. \quad (2.6.4)$$

Proof : Replace X by $L_X^q(\Omega)$, for which (2.6.2) remains valid, and use (2.6.3). $\blacksquare$

Lemma 2.6.8. Let $f_1, \ldots, f_J$, with $J = 2^r$, be a sequence of functions on $\{1, -1\}^J$ where f_j only depends on $\varepsilon_1, \ldots, \varepsilon_j$. Then

$$\sum_{s=1}^{r} 2^{-qs} \min_{2^s \leq j \leq J-2^s} \left(\int_{\Omega_j \times \Omega_{2^s} \times \Omega_{2^s}} \|f_{j+2^s}(\varepsilon, \delta) - f_{j+2^s}(\varepsilon, \delta')\|^q d\varepsilon d\delta d\delta' \right)$$
$$\leq 2^q C \sup_{1 \leq j \leq J-1} \|f_{j+1} - f_j\|_\infty^q.$$

Proof : For every $d \leq j \leq J - d$, using the triangle inequality, we obtain

$$\begin{aligned} \|2f_j - f_{j-d} - f_{j+d}\|_q^q &= \int_{\Omega_j \times \Omega_d} \|2f_j - f_{j-d} - f_{j+d}\|^q d\varepsilon d\delta \\ &\geq 2^{-q} \int_{\Omega_j \times \Omega_d \times \Omega_d} \|f_{j+2^s}(\varepsilon, \delta) - f_{j+2^s}(\varepsilon, \delta')\|^q d\varepsilon d\delta d\delta'. \end{aligned}$$

The lemma then follows from (2.6.4). $\blacksquare$

Now, let us prove Theorem 2.6.2. Fix J and consider a 1-Lipschitz map $F : T_J \to X$. Apply Lemma 2.6.8 to the functions $f_1, \ldots, f_J$ defined by

$$\forall \alpha \in \Omega_j, \quad f_j(\alpha) = F(\alpha).$$

By definition of the compression, we have

$$\rho\left(d\left((\varepsilon, \delta), (\varepsilon, \delta)\right)\right) \leq \|f_{j+2^s}(\varepsilon, \delta) - f_{j+2^s}(\varepsilon, \delta')\| \quad (2.6.5)$$

where $\varepsilon \in \Omega_j$ and $\delta, \delta' \in \Omega_{2^s}$.

But, on the other hand, with probability $1/2$, we have

$$d\left((\varepsilon, \delta), (\varepsilon, \delta)\right) = 2.2^s.$$

So combining this with Lemma 2.6.8, (2.6.5) and with the fact that F is 1-Lipschitz, we obtain

$$\sum_{s=1}^{r} 2^{-qs} \rho(2^s)^q \leq 2^{q+1} C$$

But since ρ is decreasing, we have

$$2^{-qs} \rho(2^s)^q \geq 2^{-q-1} \int_{2^{s-1}+1}^{2^s} \frac{1}{t} \left(\frac{\rho(t)}{t}\right)^q dt.$$

So (2.6.1) follows. $\blacksquare$

2.7 Applications and further results

2.7.1 Hilbert compression, volume growth and random walks

Let G be a locally compact group generated by a symmetric compact subset S containing 1. Let us denote $V(n) = \mu(S^n)$ and $S(n) = V(n+1) - V(n) = \mu(S^{n+1} \smallsetminus S^n)$. Extend V as a piecewise linear function on $\mathbf{R}_+$ such that $V'(t) = S(n)$ for $t \in]n, n+1[$.

Proposition 2.7.1. Let G be a compactly generated locally compact group. For any $2 \leq p < \infty$,

$$J_{G,p}(t) \preceq \frac{t}{\log V(t)}.$$

Proof : For every $n \in \mathbf{N}$, define

$$k(n) = \sup\{k,\ V(n-k) \geq V(n)/2\}$$

and

$$j(n) = \sup_{1 \leq j \leq n} k(j).$$

For every positive integer $l \leq n/j(n)$,

$$V(n) \geq 2^l V(n - lj(n))$$

Hence, as $V(0) = 1$,

$$V(n) \geq 2^{n/(j(n)+1)}.$$

Thus, there is a constant $c > 0$ such that

$$j(n) \geq \frac{cn}{\log V(n)}.$$

Let $q_n \leq n$ be such that $j(n) = k(q_n)$. Now define

$$\varphi_n = \sum_{k=1}^{q_n - 1} 1_{B(1,k)}.$$

Note that the subsets $SB(1,k) \vartriangle B(1,k) = B(1,k+1) \smallsetminus B(1,k)$, for $k \in \mathbf{N}$, are piecewise disjoint. Thus, an easy computation shows that

$$\|\tilde{\nabla}\varphi_n\|_p \leq V(q_n)^{1/p}.$$

On the other hand

$$\|\varphi_n\|_p \geq j(n)V(q_n - j(n))^{1/p} \geq \frac{cn}{\log V(n)}(V(q_n)/2)^{1/p}.$$

Since $\tilde{J}^b{}_{G,p}(n) \geq \|\varphi_n\|_p / \|\tilde{\nabla}\varphi_n\|_p$, we conclude that $\tilde{J}^b{}_{G,p}(n) \succeq n/\log V(n)$. ■

Now, consider a symmetric probability measure ν on a finitely generated group G, supported by a finite generating subset S. Given an element φ of $\ell^2(G)$, a simple calculation shows that

$$\frac{1}{2}\int\int |\varphi(sx)-\varphi(x)|^2 d\nu^{(2)}(s)d\mu(x) = \int(\varphi-\nu^{(2)}*\varphi)\varphi d\mu = \|\varphi\|_2^2 - \|\nu*\varphi\|_2^2$$

where μ denotes the counting measure on G. Let us introduce a (left) gradient on G associated to ν. Let φ be a function on G ; define

$$|\tilde{\nabla}\varphi|_2^2(g) = \int |\varphi(sg)-\varphi(g)|^2 d\nu^{(2)}(s).$$

This gradient satisfies

$$\||\tilde{\nabla}\varphi|_2\|_2^2 = 2(\|\varphi\|_2^2 - \|\nu*\varphi\|_2^2).$$

We have

$$\mu(S)^{-1/2}|\tilde{\nabla}\varphi|_2 \leq |\tilde{\nabla}\varphi| \leq |\tilde{\nabla}\varphi|_2.$$

Proposition 2.7.2. Assume that $\nu^{(n)}(1) \succeq e^{-Cn^b}$ for some $b<1$. Then

$$\tilde{J^b}_{G,2}(t) \succeq Ct^{1-b}.$$

Proof : Let us prove that there exists a constant $C'<\infty$ such that for every $n\in\mathbf{N}$, there exists $n\leq k\leq 2n$ such that

$$\frac{\| \, |\tilde{\nabla}\nu^{(2k)}|_2 \, \|_2^2}{\| \, \nu^{(2k)} \, \|_2^2} \leq C'n^{b-1}.$$

Since $\nu^{(2k)}$ is supported in $S^{2k}\subset S^{4n}$, this will prove the proposition. Let C_n be such that for every $n\leq q\leq 2n$

$$\frac{\| \, |\tilde{\nabla}\nu^{(2q)}|_2 \, \|_2^2}{\| \, \nu^{(2q)} \, \|_2^2} \geq C_n n^{b-1}.$$

Since the function defined by $\psi(q)=\| \, \nu^{(2q)} \, \|_2^2$ satisfies

$$\psi(q+1)-\psi(q) = -\frac{1}{2}\| \, |\tilde{\nabla}\nu^{(2q)}|_2 \, \|_2^2,$$

we can extend ψ as a piecewise linear function on $\mathbf{R}_+$ such that

$$\psi'(t) = \frac{1}{2}\| \, |\tilde{\nabla}\nu^{(2q)}|_2 \, \|_2^2$$

for every $t\in[q,q+1[$. Then, for every $n\leq t\leq 2n$ we have

$$-\frac{\psi'(t)}{\psi(t)} \geq C_n n^{b-1}$$

which integrates in

$$-\log\left(\frac{\psi(2n)}{\psi(n)}\right) \geq C_n n^b$$

Since $\psi(n) < 1$, this implies

$$\psi(2n) \leq e^{-C_n n^b}.$$

But on the other hand,

$$\psi(2n) \geq \| \nu^{(4n)} \|_2^2 \geq \nu^{(8n)}(1) \geq e^{-8Cn^b}.$$

So $C_n \leq 8C$. ■

2.7.2 A direct construction to embed trees

Here, we propose to show that the method used in [Bou, GuKa, BrSo] to embed trees in L^p-spaces can also be exploited to obtain optimal estimates (i.e. a converse to Theorem 2.6.2). Moreover, no hypothesis of local finitude is required for this construction.

Theorem 2.7.3. Let T be a simplicial tree. For every increasing function $f : \mathbf{R}_+ \to \mathbf{R}_+$ satisfying, for $1 \leq p < \infty$

$$\int_1^\infty \left(\frac{f(t)}{t}\right)^p \frac{dt}{t} < \infty, \qquad (C_p)$$

there exists a uniform embedding F of T into $\ell^p(T)$ with compression $\rho \succeq f$.

Proof : Let us start with a lemma.

Lemma 2.7.4. For every nonnegative sequence (ξ_n) such that

$$\sum_n |\xi_{n+1} - \xi_n|^p < \infty,$$

there exists a Lipschitz map $F : T \to \ell^p(T)$ whose compression ρ satisfies

$$\forall n \in \mathbf{N}, \quad \rho(n) \geq \left(\sum_{j=0}^n \xi_j^p\right)^{1/p}.$$

Proof : The following construction is a generalization of those carried out in [GuKa] and [BrSo]. Fix a vertex o. For every $y \in T$, denote δ_y the element of $\ell^p(T)$ that takes value 1 on y and 0 elsewhere. Let x be a vertex of T and let $x_0 = x, x_1 \ldots, x_l = o$ be the minimal path joining x to o. Define

$$F(x) = \sum_{i=1}^l \xi_i \delta_{x_i}.$$

To prove that F is Lipschitz, it suffices to prove that $\|F(x)-F(y)\|_p$ is bounded for neighbor vertices in T. So let x and y be neighbor vertices in T such that $d(o,y) = d(x,o)+1 = l+1$. We have

$$\|F(y)-F(x)\|_p^p \leq \xi_0^p + \sum_{j=0}^{l} |\xi_{n+1}-\xi_n|^p.$$

On the other hand, let x and y be two vertices in T. Let z be the last common vertex of the two geodesic paths joining o to x and y. We have

$$d(x,y) = d(x,z) + d(z,y)$$

and

$$\begin{aligned} \|F(x)-F(y)\|_p^p &= \|F(x)-F(z)\|_p^p + \|F(z)-F(y)\|_p^p \\ &\geq \max\{\|F(x)-F(z)\|_p^p\,,\ \|F(z)-F(y)\|_p^p\}. \end{aligned}$$

Let $k = d(z,x)$; we have

$$\|F(x)-F(z)\|_p^p \ \geq\ \sum_{j=0}^{k} \xi_j^p,$$

which proves the lemma. ■

Now, let us prove the proposition. Define (ξ_j) by

$$\xi_0 = \xi_1 = 0;$$

$$\forall j \geq 1, \quad \xi_{j+1} - \xi_j = \frac{1}{j^p}\frac{f(j)}{j}$$

and consider the associated Lipschitz map F from T to $\ell^p(T)$. Clearly, we have

$$\sum |\xi_{n+1}-\xi_n|^p < \infty$$

and

$$\begin{aligned} \sum_{j=0}^{n} \xi_j^p &\geq \sum_{j=[n/2]}^{n} \left(\sum_{k=0}^{j-1} |\xi_{k+1}-\xi_k|\right)^p \\ &\geq n/2 \left(\sum_{k=0}^{[n/2]-1} |\xi_{k+1}-\xi_k|\right)^p \\ &\geq cf([n/2]) \end{aligned}$$

using the fact that f is nondecreasing. So the proposition now follows from the lemma. ■

2.7.3 Cocycles with lacunar compression

Proposition 2.7.5. For any increasing sublinear function $h : \mathbf{R}_+ \to \mathbf{R}_+$ and every $2 \leq p < \infty$, there exists a function f satisfying (C_p), a constant $c > 0$ and a increasing sequence of integers (n_i) such that

$$\forall i \in \mathbf{N}, f(n_i) \geq ch(n_i).$$

Proof : Choose a sequence (n_i) such that

$$\sum_{i \in \mathbf{N}} \left(\frac{h(n_i)}{n_i}\right)^p < \infty$$

Define

$$\forall i \in \mathbf{N}, n_i \leq t < n_{i+1}, \quad f(t) = h(n_i)$$

We have

$$\begin{aligned}
\int_1^\infty \frac{1}{t}\left(\frac{f(t)}{t}\right)^p dt &\leq \sum_i (h(n_i))^p \int_{n_i}^{n_{i+1}} \frac{dt}{t^{p+1}} \\
&\leq (p+1) \sum_i \left(\frac{h(n_i)}{n_i}\right)^p \\
&< \infty
\end{aligned}$$

So we are done. ■

2.7.4 H-metric

Let G be a locally compact, compactly generated group and let S be a compact symmetric generating set. A Hilbert length function is a length function associated to some Hilbert 1-cocycle b, i.e. $L(g) = \|b(g)\|$. Consider the supremum of all Hilbert length functions on G, bounded by 1 on S : it defines a length function on G which in general is no longer a Hilbert length function. This length function has been introduced by Cornulier [Co, § 2.6] who called the corresponding metric "H-metric". Observe that if the group G satisfies $M\rho_{G,2}(t) \approx t$, then the H-metric of G is quasi-isometric to the word length. As a consequence of Theorem 2.5.1 and Proposition 2.4.1, we get

Proposition 2.7.6. For every group in the class (Ł), the H-metric is quasi-isometric to the word length.

Chapitre 3

Etude dynamique des actions par isométries sur un espace de Hilbert I : croissance des cocycles

Résumé

We study growth of 1-cocycles of locally compact groups, with values in unitary representations. Discussing the existence of 1-cocycles with linear growth, we obtain the following alternative for a class of amenable groups G containing polycyclic groups and connected amenable Lie groups : either G has no quasi-isometric embedding into Hilbert space, or G admits a proper cocompact action on some Euclidean space.

On the other hand, noting that almost coboundaries (i.e. 1-cocycles approximable by bounded 1-cocycles) have sublinear growth, we discuss the converse, which turns out to hold for amenable groups with "controlled" Følner sequences ; for general amenable groups we prove the weaker result that 1-cocycles with sufficiently small growth are almost coboundaries. Besides, we show that there exist, on a-T-menable groups, proper cocycles with arbitrary small growth.

Notation. Let G be a locally compact group, and $f, g : G \to \mathbf{R}_+$. We write $f \preceq g$ if there exists $M > 0$ and a compact subset $K \subset G$ such that $f \leq Mg$ outside K. We write $f \sim g$ if $f \preceq g \preceq f$. We write $f \prec g$ if, for every $\varepsilon > 0$, there exists a compact subset $K \subset G$ such that $f \leq \varepsilon g$ outside K.

3.1 Introduction

The study of affine isometric actions on Hilbert spaces has proven to be a fundamental tool in geometric group theory. Let G be a locally compact group, and α an affine isometric action on an affine Hilbert space $\mathcal{H}$ (real or complex). In this paper, we focus on the function $b : G \to \mathcal{H}$ defined by $b(g) = \alpha(g)(0)$; we call such a function a 1-cocycle (see Section 3.2 for details), and we call the function $g \mapsto \|b(g)\|$ a *Hilbert length function* on G.

We focus on the *asymptotic* behaviour of Hilbert length functions on a given group G. A general question is the following : how is it related to the structure of G ?

For instance, if G is σ-compact, G has the celebrated Kazhdan's Property (T) if and only if every Hilbert length function is bounded (see [HV]). This is known to have strong group-theoretic consequences on G : for instance, this implies that G is compactly generated and as compact abelianization (see [BHV, Chap. 2] for a direct proof).

In this paper, we rather deal with groups which are far from having Kazhdan's Property (T) : a locally compact group G is called a-T-menable if it has a proper Hilbert length function. The class of a-T-menable locally compact groups contains (see [CCJJV]) amenable groups, Coxeter groups, isometry groups of locally finite trees, isometry groups of real and complex hyperbolic spaces and all their closed subgroups, such as free and surface groups.

We show in §3.3.4 that, for a-T-menable locally compact groups (for instance, $\mathbf{Z}$), there exist proper Hilbert length functions of arbitrary slow growth.

The study of Hilbert length functions with non-slow growth is more delicate. An easy but useful observation is that, for a given compactly generated, locally compact group, any Hilbert length function L is linearly bounded, i.e. $L(g) \preceq |g|_S$, where $|\cdot|_S$ denotes the word length with respect to some bounded open generating subset.

We discuss, in Section 3.3, Hilbert length functions with sublinear growth. These include those Hilbert length functions whose corresponding 1-cocycle (see Section 3.2) is an almost coboundary, i.e. can be approximated, uniformly on compact subsets, by bounded 1-cocycles. We discuss the converse.

Denote by $(\mathcal{L})$ the class of groups including :

- polycyclic groups and connected amenable Lie groups,
- semidirect products $\mathbf{Z}[\frac{1}{mn}] \rtimes_{\frac{m}{n}} \mathbf{Z}$, with m, n co-prime integers with $|mn| \geq 2$ (if $n = 1$ this is the Baumslag-Solitar group $BS(1, m)$) ; semidirect products $\left(\mathbf{R} \oplus \bigoplus_{p \in S} \mathbf{Q}_p\right) \rtimes_{\frac{m}{n}} \mathbf{Z}$ or $\left(\bigoplus_{p \in S} \mathbf{Q}_p\right) \rtimes_{\frac{m}{n}} \mathbf{Z}$, with m, n co-prime integers, and S a finite set of prime numbers dividing mn.

– wreath products $F \wr \mathbf{Z}$ for F a finite group.

Proposition 3.1.1 (see Propositions 3.3.6, 3.3.4 and 3.3.8)**.**

(1) If G is a compactly generated, locally compact amenable group, then every 1-cocycle with sufficiently slow growth is an almost boundary.

(2) For groups in the class $(\mathcal{L})$, every sublinear 1-cocycle is an almost coboundary.

(3) If Γ is a cocompact lattice in $\mathrm{SO}(n,1)$ or $\mathrm{SU}(n,1)$ for some $n \geq 2$, then Γ admits a 1-cocycle with sublinear growth (actually $\preceq |g|^{1/2}$) which is not an almost coboundary.

In §3.3.5, we show that there exist, on $\mathbf{R}^n$ or $\mathbf{Z}^n$, Hilbert length functions with arbitrary large sublinear growth, showing that, in a certain sense, there is no gap between Hilbert length functions of linear and of sublinear growth.

In Section 3.4, we discuss the existence of a Hilbert length function on G with linear growth. Such a function exists when $G = \mathbf{Z}^n$. We conjecture that the converse is essentially true.

Conjecture 1. Let G be a locally compact, compactly generated group having a Hilbert length function with linear growth. Then G has a proper, cocompact action on a Euclidean space. In particular, if G is discrete, then it must be virtually abelian.

Our first result towards Conjecture 1 is a generalization of a result by Guentner and Kaminker [GuKa, §5] to the non-discrete case.

Theorem 3.1.2 (see Theorem 3.4.1)**.** *Let G be a locally compact, compactly generated group. If G admits a Hilbert length function with growth $\succ |g|^{1/2}$ (in particular, if it admits a Hilbert length function with linear growth), then G is amenable.*

We actually provide a new, simpler proof, while it is not clear how to generalize the proof in [GuKa] to the non-discrete case.

To prove that locally compact groups in the class $(\mathcal{L})$ satisfy Conjecture 1, we use Shalom's Property H_{FD} : a locally compact group has Property H_{FD} if any unitary representation with nontrivial reduced cohomology has a finite-dimensional nonzero subrepresentation. All groups in the class $(\mathcal{L})$ are known to satisfy Property H_{FD}. We prove

Theorem 3.1.3 (see Theorem 3.4.3)**.** *Locally compact, compactly generated groups with Property H_{FD} satisfy Conjecture 1.*

We next consider uniform embeddings into Hilbert spaces. There is a nice trick, for which we are indebted to Gromov, allowing to construct, if the group is amenable, a 1-cocycle with the same growth behaviour as the initial embedding. See Proposition 3.4.4 for a precise statement. In particular, the existence of a quasi-isometric embedding into Hilbert space implies, for an amenable group, the existence of a Hilbert length function with linear growth. Thus we get :

Theorem 3.1.4. *If G is any locally compact, compactly generated group with Property H_{FD} (e.g. in the class $(\mathcal{L})$), then*

- *either G does not admit any quasi-isometric embedding into Hilbert space,*
- *or G acts properly cocompactly on some Euclidean space (i.e. a finite-dimensional real Hilbert space).*

Let us observe that the proof of Theorem 3.1.4 does *not* appeal to asymptotic cones. It contains, as a particular case, the fact that a simply connected nilpotent non-abelian Lie group has no quasi-isometric embedding into Hilbert space, a result due to S. Pauls [Pau]. Morover, Theorem 3.1.4 provides a new proof of the two following results (see §3.4.3 for proofs) :

Corollary 3.1.5 (Quasi-isometric rigidity of $\mathbf{Z}^n$)**.** *If a finitely generated group is quasi-isometric to $\mathbf{Z}^n$, then it has a finite index subgroup isomorphic to $\mathbf{Z}^n$.*

Corollary 3.1.6 (Bourgain [Bou])**.** *For $r \geq 3$, the regular tree of degree r does not embed quasi-isometrically into a Hilbert space.*

Acknowledgments. We are indebted to Misha Gromov for a decisive remark. We also thank Urs Lang for pointing out [Pau] to us, and Pierre de la Harpe for useful corrections.

3.2 Preliminaries

3.2.1 Growth of 1-cocycles

Let G be a locally compact group, and π a unitary or orthogonal representation (always assumed continuous) on a Hilbert space $\mathcal{H} = \mathcal{H}_\pi$. The space $Z^1(G, \pi)$ is defined as the set of continuous functions $b : G \to \mathcal{H}$ satisfying, for all $g, h \in G$, the 1-cocycle condition $b(gh) = \pi(g)b(h) + b(g)$. Observe that, given a continuous function $b : G \to \mathcal{H}$, the condition $b \in Z^1(G, \pi)$ is equivalent to saying that G acts by affine transformations on $\mathcal{H}$ by $\alpha(g)v = \pi(g)v + b(g)$. The space $Z^1(G, \pi)$ is endowed with the topology of uniform convergence on compact subsets.

The subspace of coboundaries $B^1(G, \pi)$ is the subspace (not necessarily closed) of $Z^1(G, \pi)$ consisting of functions of the form $g \mapsto v - \pi(g)v$ for some $v \in \mathcal{H}$. It is well-known [HV, §4.a] that $b \in B^1(G, \pi)$ if and only if b is bounded on G.

The subspace of almost coboundaries $\overline{B^1(G, \pi)}$ is the closure of $B^1(G, \pi)$. A 1-cocycle b is an almost coboundary if and only if the corresponding affine action *almost has fixed points*, i.e. for every compact subset $K \subset G$ and $\varepsilon > 0$, there exists v such that $\sup_{g \in K} \|\alpha(g)v - v\| \leq \varepsilon$ (see [BHV, §3.1]). When G is generated by a symmetric compact subset S, it suffices to check this condition for $K = S$, and a sequence of *almost fixed points* is defined as a sequence (v_n) such that $\sup_{g \in S} \|\alpha(g)v_n - v_n\| \to 0$.

The first cohomology space of π is defined as the quotient space $H^1(G,\pi) = Z^1(G,\pi)/B^1(G,\pi)$, and the first reduced cohomology space of π is defined as $\overline{H^1}(G,\pi) = Z^1(G,\pi)/\overline{B^1(G,\pi)}$.

Now suppose that G is a locally compact, compactly generated group. For $g \in G$, denote by $|g|_S$ the word length of g with respect to an open, relatively compact generating set $S \subset G$.

Let $b \in Z^1(G,\pi)$ be a 1-cocycle with respect to a unitary representation π of G. We study the growth of $\|b(g)\|$ as a function of g.

Definition 3.2.1. The *compression* of the 1-cocycle b is the function

$$\rho : \mathbf{R}_+ \to \mathbf{R}_+ \cup \{\infty\} : \ x \mapsto \rho(x) = \inf\{\|b(g)\| : \ g \in G, \ |g|_S \geq x\}.$$

Remark 3.2.2. A related notion is the *distortion function*, defined in [Far] in the context of an embedding between finitely generated groups. The distortion function of the 1-cocycle b is defined as the function $\mathbf{R}_+ \to \mathbf{R}_+ \cup \{\infty\}$ by $f(x) = \sup\{|g|_S \ : \ \|b(g)\| \leq x\}$. The reader can check that, except in trivial cases[1], the compression ρ and the distortion f are essentially reciprocal to each other.

Recall that a *length function* on a group Γ is a function $L : \Gamma \to \mathbf{R}_+$ satisfying $L(1) = 0$ and, for all $g,h \in \Gamma$, $L(g^{-1}) = L(g)$ and $L(gh) \leq L(g) + L(h)$, so that $d(g,h) = L(g^{-1}h)$ is a left-invariant pseudo-distance (*"écart"*) on Γ.

It is immediate from the 1-cocycle relation that the function $g \mapsto \|b(g)\|$ is a length function on the group G. In particular, if G is locally compact, compactly generated, then it is dominated by the word length. We thus obtain the following obvious bound :

Proposition 3.2.3. *For $b \in Z^1(G,\pi)$, we have $\|b(g)\| \preceq |g|_S$.* ■

Define

$$\operatorname{lin}(G,\pi) = \{b \in Z^1(G,\pi), \ \|b(g)\| \succeq |g|_S\}$$
$$\operatorname{sublin}(G,\pi) = \{b \in Z^1(G,\pi), \|b(g)\| \prec |g|_S\},$$

namely, the set of cocycles with linear (resp. sublinear) growth. Here are immediate observations :

- $\operatorname{sublin}(G,\pi)$ is a linear subspace of $Z^1(G,\pi)$.
- $B^1(G,\pi) \ \subset \ \operatorname{sublin}(G,\pi) \ \subset \ Z^1(G,\pi) \smallsetminus \operatorname{lin}(G,\pi)$.
- If $G = \mathbf{Z}$ or $\mathbf{R}$, then it is easy to check that $Z^1(G,\pi) = \operatorname{lin}(G,\pi) \cup \operatorname{sublin}(G,\pi)$ (this follows either from Proposition 3.3.6 below, or from a direct computation involving von Neumann's ergodic theorem). On the other hand, this does not generalize to arbitrary G. Indeed, take any nontrivial action of $\mathbf{Z}^2$ by translations on $\mathbf{R}$: then the associated cocycle is neither linear nor sublinear.

[1]Trivial cases are : when G is compact, so that ρ is eventually equal to ∞ and f is eventually equal to a finite constant, and when b is not proper, so that ρ is bounded, and f is eventually equal to ∞.

3.2.2 Conditionally negative definite functions and Bernstein functions

A conditionally negative definite function on a group G is a function $\psi : G \to \mathbf{R}_+$ such that $\psi^{1/2}$ is a Hilbert length function. Equivalently [HV, 5.b], $\psi(1) = 0$, $\psi(g) = \psi(g^{-1})$ for all g, and, for all $\lambda_1, \ldots, \lambda_n \in \mathbf{R}$ such that $\sum_{i=1}^n \lambda_i = 0$ and for all $g_1, \ldots, g_n \in G$, we have $\sum_{i,j=1}^n \lambda_i \lambda_j \psi(g_i^{-1} g_j) \leq 0$. Continuous conditionally negative definite functions on a locally compact group G form a convex cone, closed under the topology of uniform convergence on compact subsets.

A continuous function $F : \mathbf{R}_+ \to \mathbf{R}_+$ is a *Bernstein function* if there exists a positive measure μ on Borel subsets of $\mathbf{R}_+^*$ such that $\mu([\varepsilon, \infty[) < \infty$ for all $\varepsilon > 0$, $\int_0^1 x \, d\mu(x) < \infty$, and such that, for some $a \geq 0$,

$$\forall t > 0, \quad F(t) \,=\, at \,+\, \int_0^{+\infty} (1 - e^{-tx}) \, d\mu(x).$$

Note that such a function is real analytic on $\mathbf{R}_+^*$. We note for reference the following well-known result due to of Bochner and Schoenberg [Sch, Theorem 8] :

Lemma 3.2.4. *Let ψ be a conditionally negative definite function on G, and let F be a Bernstein function. Then $F \circ \psi$ is conditionally negative definite on G.* ■

Examples of Bernstein functions are $x \mapsto x^a$ for $0 < a \leq 1$, and $x \mapsto \log(x + 1)$. For more on Bernstein functions, see for instance [BF].

3.3 Cocycles with sublinear growth

3.3.1 Almost coboundaries are sublinear

Proposition 3.3.1. *Let G be a locally compact, compactly generated group. In $Z^1(G, \pi)$, endowed with topology of uniform convergence on compact subsets,*

1) $\mathrm{sublin}(G, \pi)$ *is a closed subspace;*

2) $\mathrm{lin}(G, \pi)$ *is an open subset.*

Proof : Fix a symmetric open, relatively compact generating subset $S \subset G$. Let b be the limit of a net $(b_i)_{i \in I}$ in $Z^1(G, \pi)$. Write $b'_i = b - b_i$, and fix $\varepsilon > 0$. For i large enough (say, $i \geq i_0$), $\sup_{s \in S} \|b'_i(s)\| \leq \varepsilon/2$. Since $g \mapsto \|b'_i(g)\|$ is a length function, this implies that for every $g \in G$ and $i \geq i_0$, $\|b'_i(g)\| \leq \varepsilon |g|_S / 2$, i.e. $\|b'_i(g)\| / |g|_S \leq \varepsilon/2$.

1) Suppose that all b_i's belong to $\mathrm{sublin}(G, \pi)$. Fix $i \geq i_0$. Then $\|b_i(g)\| / |g|_S \leq \varepsilon/2$ for g large enough (say, $g \notin K$ compact). So $\|b(g)\| / |g|_S \leq \varepsilon$ for $g \notin K$. This shows that $b \in \mathrm{sublin}(\Gamma, \pi)$, so we are done.

2) Suppose that $b \in \mathrm{lin}(G,\pi)$. Then, if ε has been chosen sufficiently small, $\|b(g)\|/\|g\| \geq \varepsilon$ for large g (say, $g \notin K$ compact). Hence, $\|b_i(g)\|/|g|_S \geq (\|b(g)\| - \|b'_i(g)\|)/|g|_S \geq \varepsilon/2$ for $i \geq i_0$, $g \notin K$, showing that $b_i \in \mathrm{lin}(G,\pi)$ for $i \geq i_0$. $\square$

Corollary 3.3.2. *If $b \in \overline{B^1}(G,\pi)$, then $\|b(g)\| \prec |g|_S$.*

Proof : $B^1(G,\pi) \subset \mathrm{sublin}(G,\pi)$, so that $\overline{B^1(G,\pi)} \subset \overline{\mathrm{sublin}(G,\pi)} = \mathrm{sublin}(G,\pi)$ by Proposition 3.3.1. $\square$

3.3.2 Groups with controlled Følner sequences

In this section, we prove that the converse to Corollary 3.3.2 is true for unimodular groups in the class $(\mathcal{L})$: that is, a cocycle has sublinear growth if and only if it is an almost coboundary.

Let G be a compactly generated, locally compact group with Haar measure μ, and let S be a compact generating subset. Let (F_n) be a sequence of measurable, bounded subsets of nonzero measure. Set

$$\varepsilon_n = \frac{\sup_{s\in S}\mu(sF_n \triangle F_n)}{\mu(F_n)}.$$

Consider an isometric affine action α of G on a Hilbert space, and let b be the corresponding 1-cocycle. Set

$$v_n = \frac{1}{\mu(F_n)}\int_{F_n} b(g)d\mu(g).$$

This is well-defined.

Lemma 3.3.3. *Suppose that $\sup_{g\in F_n}\|b(g)\| \prec 1/\varepsilon_n$. Then (v_n) is a sequence of almost fixed points for the affine action α associated with b.*

Proof : For $s \in S$, we have

$$\alpha(s)v_n - v_n = \frac{1}{\mu(F_n)}\int_{F_n}(b(sg) - b(g))d\mu(g).$$

Thus

$$\|\alpha(s)v_n - v_n\| \;\leq\; \frac{2}{\mu(F_n)}\int_{sF_n\triangle F_n}\|b(g)\|d\mu(g) \;\leq\; 2\varepsilon_n \sup_{g\in F_n}\|b(g\|. \;\blacksquare$$

Recall (see [BHV, Appendix G]) that "G amenable" exactly means that we can choose (F_n) so that $\varepsilon_n \to 0$, and (F_n) is then called a Følner sequence. In this case, we obtain, as a consequence of Lemma 3.3.3, that a 1-cocycle of sufficiently slow growth (depending on the behaviour of the Følner sequence, i.e. on the asymptotic behaviour of ε_n and the diameter of (F_n)) must be an almost coboundary. We record this as :

Proposition 3.3.4. *Let G be a compactly generated, locally compact amenable group. Then there exists a proper function $u : G \to \mathbf{R}_+$ such that, for every 1-cocycle b of G, $\|b(g)\| \prec u(g)$ implies that b is an almost coboundary.* ■

To obtain more quantitative statements we introduce a more restrictive notion of Følner sets.

Definition 3.3.5. We say that the Følner sequence (F_n) of the amenable, compactly generated locally compact group G is *controlled* if there exists a constant $c > 0$ such that, for all n,

$$F_n \subset B(1, c/\varepsilon_n).$$

In [Tes2], it is proved that a unimodular group in the class $(\mathcal{L})$ admits a controlled Følner sequence.

Proposition 3.3.6. *Let G be a compactly generated, locally compact amenable group admitting a controlled Følner sequence (F_n), and keep the notation as above. Then the following statements are equivalent :*
(1) $b \in \overline{B^1}(\Gamma, \pi)$
(2) $b \in \mathrm{sublin}(\Gamma, \pi)$
(3) The sequence (v_n) is a sequence of almost fixed points for α.

Proof : (3)⇒(1) is immediate , while (1)⇒(2) follows from Corollary 3.3.2. The remaining implication is (2)⇒(3) : suppose that b is sublinear. Write $\sup_{|g|\leq r} \|b(g)\| = f(r)$ where $f(r) \prec r$. Then

$$\sup_{g\in F_n} \|b(g)\| \;\leq\; \sup_{|g|\leq c/\varepsilon_n} \|b(g)\| \;=\; f(c/\varepsilon_n) \;\prec\; 1/\varepsilon_n,$$

so that we can apply Lemma 3.3.3 to obtain that (v_n) is a sequence of almost invariant vectors. ■

We use this to prove a conjecture of Shalom [Sh3, Section 6]. Recall that a representation of a group Γ is said to be *finite* if it factors through a finite group.

Proposition 3.3.7. *Let π be a unitary representation of a finitely generated, virtually nilpotent group Γ and let S be a finite generating subset of Γ. Suppose that π has no finite subrepresentation[2]. For every cocycle $b \in Z^1(\Gamma, \pi)$, define :*

$$v_n = \frac{1}{|S^n|} \sum_{g\in S^n} b(g).$$

Then, there exists a subsequence (v_{n_i}) which is a sequence of almost fixed points for the affine action α associated with b :

$$\|\alpha(s)v_{n_i} - v_{n_i}\| \to 0, \; \forall s \in S.$$

[2] In the conjecture of [Sh3] the assumptions are slightly stronger : S is assumed symmetric, and π is supposed to have no finite-dimensional subrepresentation.

Proof : First recall that there exists $d > 0$ such that $|S^n| \preceq n^d$ for all n. By an elementary argument, there exists an infinite sequence (n_i) such that :

$$\frac{|S^{n_i+1} \setminus S^{n_i}|}{|S^{n_i}|} \preceq \frac{1}{|n_i|}. \tag{3.3.1}$$

It follows that the balls $(B(n_i))_i$ constitute a controlled Følner sequence of Γ.

Since Γ is virtually nilpotent, by Corollary 5.1.3 and Lemma 4.2.2 in [Sh3], it has property H_F, i.e. every representation with non-zero first reduced cohomology has a finite subrepresentation. Here, by our assumption : $\overline{H^1}(\Gamma, \pi) = 0$. So the conclusion follows from Proposition 3.3.6. □

Remark : Shalom proved in the final section of [Sh3] that, if the result of Proposition 3.3.7 was proved under the bare assumption that Γ has polynomial growth, this would give rise to a new, simpler[3] proof of Gromov's celebrated theorem [Gro1] : a finitely generated group of polynomial growth is virtually nilpotent.

3.3.3 A sublinear cocycle with nontrivial reduced 1-cohomology

We show that the converse of Corollary 3.3.2 is not true in general, for finitely generated groups.

Proposition 3.3.8. *Let Γ be a cocompact lattice either in $G = \mathrm{SO}(n, 1)$ $(n \geq 2)$ or $G = \mathrm{SU}(m, 1)$ $(m \geq 1)$. There exists a unitary representation σ of Γ, and $b \in Z^1(\Gamma, \sigma) - \overline{B^1}(\Gamma, \sigma)$, such that*

$$\|b(g)\| \preceq |g|_S^{1/2}.$$

If $n \geq 3$ or $m \geq 2$, the representation σ may be taken to be irreducible.

Proof : A result of Delorme [Del] says that there exists a unitary irreducible representation π with $\overline{H^1}(G, \pi) \neq 0$: so we choose $b \in Z^1(G, \pi) - \overline{B^1}(G, \pi)$. Let K be a maximal compact subgroup of G ; replacing b by a cohomologous 1-cocycle, we may assume that $b|_K \equiv 0$. Then $b : G \to \mathcal{H}_\pi$ factors through a map $F : G/K \to \mathcal{H}_\pi$, which is equivariant with respect to the corresponding affine action on $\mathcal{H}_\pi$. By an unpublished result of Shalom (for a proof, see Corollary 3.3.10 in [BHV]), the map F is harmonic. We may now appeal to Gromov's results ([Gro6], section 3.7.D' ; see also Proposition 3.3.21 in [BHV]) on the growth of harmonic, equivariant maps from a rank 1, Riemannian, symmetric space to a Hilbert space. If $d(x, x_0)$ denotes the Riemannian distance between x and the point x_0 with stabilizer K in G/K, then for some constant $C > 0$:

$$\|F(x)\|^2 \;=\; C\, d(x, x_0) + o(d(x, x_o)).$$

Set $\sigma = \pi|_\Gamma$; since Γ is quasi-isometric to G/K, we get :

$$\|b|_\Gamma(g)\|^2 \;=\; O(|g|_S).$$

[3] The proof would be simpler in that it would not appeal to the solution of Hilbert's 5th problem about the structure of locally compact groups.

Finally, if $n \geq 3$ or $m \geq 2$, then Delorme also showed that π is not in the discrete series of G, so that $\sigma = \pi|_\Gamma$ is irreducible, by a result of Cowling and Steger [CowSte].

It remains to show that $b|_\Gamma$ is not an almost coboundary. But σ, as the restriction to Γ of a non-trivial unitary irreducible representation of G, does not weakly contain the trivial representation of Γ (this follows e.g. from Theorem C in [Bek]). By Guichardet's well-known criterion (see [Gu2], Cor. 2.3 in Chap. III), this implies that $B^1(\Gamma, \sigma) = \overline{B^1(\Gamma, \sigma)}$, in particular every almost coboundary is bounded. Since $b|_\Gamma$ is unbounded, this concludes the proof. $\square$

3.3.4 Cocycles with slow growth

We observe here that, on an a-T-menable group (e.g. $\mathbf{Z}$), there exist cocycles with arbitrarily slow growth.

Proposition 3.3.9. *Assume that G is locally compact, a-T-menable. For every proper function $f : G \to \mathbf{R}_+$ such that $f \geq 1$, there exists a continuous conditionally negative definite, proper function ψ on G such that $\psi \leq f$.*

We need a lemma.

Lemma 3.3.10. *Let u be a proper function on $\mathbf{R}_+$, with $u(t) \geq 1$ for $t \in \mathbf{R}_+$. There exists a proper Bernstein function F such that $F(t) \leq u(t)$ for $t \in \mathbf{R}_+$.*

Proof : We are going to define inductively a sequence $(x_n)_{n\geq 1}$ of positive real numbers such that $0 < x_n < 2^{-n}$, and define

$$F(t) = \sum_{n=1}^{\infty}(1 - e^{-tx_n}).$$

Since $1 - e^{-tx_n} \leq tx_n$, the series defining F will converge uniformly on compact subsets of $\mathbf{R}_+$, so F will be a Bernstein function (in fact associated with $\mu = \sum_{n=1}^{\infty} \delta_{x_n}$ and $a = 0$). Let $F_m(t) = \sum_{n=1}^{m}(1 - e^{-tx_n})$ be the m-th partial sum. For fixed m, we will have $F \geq F_m$, hence

$$\liminf_{t\to\infty} F(t) \;\geq\; \lim_{t\to\infty} F_m(t) \;=\; m;$$

since this holds for every m, we have $\lim_{t\to\infty} F(t) = \infty$, i.e. F is proper.

It remains to manage to construct the x_n's so that $F \leq u$ on $\mathbf{R}_+$. We will construct x_n inductively so that $u > F_n + 2^{-n}$ on $\mathbf{R}_+$. Setting $F_0 \equiv 0$, the construction will also apply to $n = 1$. So assume $0 < x_{n-1} < 2^{-n+1}$ has been constructed so that $u > F_{n-1} + 2^{-n+1}$ on $\mathbf{R}_+$. Since u is proper and F_{n-1} is bounded, we find $K_n > 0$ large enough so that $u(t) > F_{n-1}(t) + 2$ for $t > K_n$, $t \in \mathbf{R}_+$. By taking $x_n > 0$ very small (with $x_n < 2^{-n}$ anyway), we may arrange to have $1 - e^{-tx_n} < 2^{-n}$ for $t < K_n$. Then, for $t < K_n$, $t \in \mathbf{R}_+$:

$$u(t) - F_n(t) = u(t) - F_{n-1}(t) - (1 - e^{-tx_n}) > 2^{-n+1} - 2^{-n} = 2^{-n};$$

while for $t \geq K_n$, $t \in \mathbf{R}_+$:

$$u(t) - F_n(t) = u(t) - F_{n-1}(t) - (1 - e^{-tx_n}) > 2 - 1 = 1 > 2^{-n}.$$

This concludes the induction step. $\square$

Proof of Proposition 3.3.9 : If G is compact, we can take $\psi = 0$; thus suppose G noncompact. As G is a-T-menable, we may choose a proper conditionally negative definite function ψ_0 on G.

Define a proper function $u \geq 1$ on $\mathbf{R}_+$ by

$$u(t) = \inf\{f(g) : \ g \in \psi_0^{-1}(\,[t, \infty[\,)\}.$$

By lemma 3.3.10, we can find a proper Bernstein function F such that $F \leq u$ on $\mathbf{R}_+$. Then, by construction, $F(\psi_0(g)) \leq f(g)$, and by Lemma 3.2.4, $F \circ \psi_0$ is conditionally negative definite. $\square$

3.3.5 Cocycles with arbitrary large sublinear growth

As we observed earlier, a cocycle on $\mathbf{Z}^n$ (or $\mathbf{R}^n$) has either linear or sublinear growth. This raises the question whether there is a gap between the two. We show here that it is not the case.

Lemma 3.3.11. *Let $w : \mathbf{R}_+ \to \mathbf{R}_+$ be any function with sublinear growth. Then there exists a sublinear Bernstein function F such that $F(x) \geq w(x)$ for x large enough.*

Proof : The function $x \mapsto w(x)/x$ tends to zero. It is easy to construct a decreasing function $x \mapsto u(x)$ of class C^1, such that $u(x) \geq w(x)/x$ for x large enough, and such that $u(x) \to 0$ when $x \to \infty$.

Now define the measure

$$d\mu(s) = \frac{-u'(1/s)}{s^3} 1_{[0,1]}(s) ds.$$

An immediate calculation gives, for $0 < \varepsilon \leq 1$, $\int_\varepsilon^1 s d\mu(s) = u(1) - u(1/\varepsilon)$, which is bounded, so that $\int_0^1 s d\mu(s) < \infty$. So we can define the Bernstein function associated to μ : $F(t) = \int_0^\infty (1 - e^{-ts}) d\mu(s)$. Then, for all $t \geq 1$, using the inequality $1 - e^{-ts} \geq (1 - e^{-1})ts$ on $[0, 1/t]$:

$$\begin{aligned} F(t) &\geq \int_0^{1/t} (1 - e^{-ts}) d\mu(s) \\ &\geq (1 - e^{-1}) t \int_0^{1/t} s d\mu(s) \\ &= (1 - e^{-1}) t \int_0^{1/t} \frac{-u'(1/s) ds}{s^2} \\ &= (1 - e^{-1}) t\, u(t) \end{aligned}$$

$$\geq (1-e^{-1})w(t) \qquad \text{for large } t.$$

The Bernstein function $x \mapsto (1-e^{-1})^{-1}F(x)$ satisfies our purposes, as it is easy to see that it is sublinear. ■

An example of application of Lemma 3.3.11 is the following result.

Proposition 3.3.12. *Let G be a locally compact, compactly generated group having a 1-cocycle of linear growth (e.g. $G = \mathbf{Z}^n$ or $\mathbf{R}^n$ for $n \geq 1$). Then, for every function $f : G \to \mathbf{R}_+$ with sublinear growth, there exists on G a sublinear 1-cocycle b such that $\|b(g)\| \succeq f(g)$.*

Proof : Let b' denote a 1-cocycle with linear growth, and write $|g| = \|b'(g)\|$, so that $g \mapsto |g|$ is equivalent to the word length, and its square is conditionally negative definite on G.

By hypothesis, $f(g) \prec |g|$. Define $w : \mathbf{R}_+ \to \mathbf{R}_+$ by

$$w(x) = \sup\{f(h) : |h| \leq x\}.$$

Then w is sublinear on $\mathbf{R}_+$, and so is the function $x \mapsto w(x^{1/2})^2$. By Lemma 3.3.11, we find a sublinear Bernstein function F such that $F(x) \geq w(x^{1/2})^2$ for large x. Using Lemma 3.2.4, the function $g \mapsto F(|g|^2)$ is conditionally negative definite on G ; moreover $F(|g|^2)^{1/2} \prec |g|$, and $F(|g|^2)^{1/2} \geq f(g)$ for $g \in G$ with $|g|$ large enough. ■

3.4 Cocycles with non-slow growth

3.4.1 Amenability

Here is a generalization of a result by Guentner and Kaminker [GuKa, §5] who proved it in the case of discrete groups.

Theorem 3.4.1. *Let G be a locally compact group, and S a symmetric, compact generating subset. Suppose that G admits a 1-cocycle b with compression $\rho(g) \succ |g|^{1/2}$. Then G is amenable.*

Corollary 3.4.2. *If a locally compact, compactly generated group admits a linear 1-cocycle, then it is amenable.* ■

Proof of Theorem 3.4.1 For $t > 0$, define $f_t(g) = e^{-t\|b(g\|^2}$. By Schoenberg's Theorem [BHV, Appendix C], f_t is definite positive. We claim that f_t is square summable. Denote $S_n = \{g \in G :\ |g|_S = n\}$, and fix a left Haar measure μ on G. There exists $a < \infty$ such that $\mu(S_n) \leq e^{an}$ for all n. Since $\rho(g) \succ |g|^{1/2}$, there exists n_0 such that, for all $n \geq n_0$, and all $g \in S_n$, $2t\|b(g)\|^2 \geq (a+1)n$. Then, for all $n \geq n_0$,

$$\int_{S_n} f_t(g)^2 d\mu(g) = \int_{S_n} e^{-2t\|b(g)\|^2} d\mu(g)$$

$$\leq \int_{S_n} e^{-(a+1)n} d\mu(g) \leq \mu(S_n)e^{-(a+1)n} \leq e^{-n}.$$

Therefore, the sequence $(\int_{S_n} f_t(g)^2 d\mu(g))$ is summable, so that f_t is square-summable.

By [Dix, Théorème 13.8.6], it follows that there exists a positive definite, square-summable function φ_t on G such that $f_t = \varphi_t * \varphi_t$, where $*$ denotes convolution. In other words, $f_t = \langle \lambda(g)\varphi_t, \varphi_t \rangle$, where λ denotes the left regular representation of G on $L^2(G)$. Note that f_t converges to 1, uniformly on compact subsets, when $t \to 0$. For $t = 0$, this means that $\|\varphi_t\| \to 1$. We conclude that $(\varphi_t/\|\varphi_t\|)$ provides a sequence of almost invariant vectors for the regular representation of G in $L^2(G)$, so that G is amenable. ■

3.4.2 Cocycles with linear growth

Let us recall a property introduced by Shalom in [Sh3] : a group has Property H_{FD} if every unitary representation such that $\overline{H^1}(\Gamma, \pi) \neq 0$ has a finite-dimensional subrepresentation.

Here are a few useful results about Property H_{FD}.

1) Property H_{FD} is a quasi-isometry invariant among *discrete* amenable groups (Shalom, [Sh3, Theorem 4.3.3]).
2) A finitely generated amenable group with Property H_{FD} has a finite index subgroup with infinite abelianization [Sh3, Theorem 4.3.1].
3) A connected amenable Lie group has Property H_{FD} (F. Martin, [Ma, Theorem 3.3]). A polycyclic group has Property H_{FD} [Sh3, Theorem 5.1.4]. Both results rely on a deep result by Delorme [Del] : connected solvable Lie groups have Property H_{FD}.
4) The semidirect product $\mathbf{Z}[1/mn] \rtimes_{m/n} \mathbf{Z}$, and the wreath product $F \wr \mathbf{Z}$, where F is any finite group, have Property H_{FD} [Sh3, Theorems 5.2.1 and 5.3.1]. Semidirect products $\left(\mathbf{R} \oplus \bigoplus_{p\in S} \mathbf{Q}_p\right) \rtimes_{\frac{m}{n}} \mathbf{Z}$ or $\left(\bigoplus_{p\in S} \mathbf{Q}_p\right) \rtimes_{\frac{m}{n}} \mathbf{Z}$, with m, n co-prime integers, and S a finite set of prime numbers dividing mn also have Property H_{FD} [Sh3, Proof of Theorem 5.3.1].
5) The wreath product $\mathbf{Z} \wr \mathbf{Z}$ does *not* have Property H_{FD} [Sh3, Theorem 5.4.1].

We prove :

Theorem 3.4.3. *Let G be a locally compact, compactly generated group with property H_{FD}. Suppose that G admits a unitary representation π such that* $\mathrm{lin}(G, \pi)$ *is nonempty. Then, G has a compact normal subgroup K such that G/K is isomorphic to some closed subgroup of* $\mathrm{Isom}(\mathbf{R}^n)$. *In particular,*
– *G is quasi-isometric to $\mathbf{R}^m$ for some unique m,*
– *If G is discrete, then G is virtually abelian.*

Proof : Let $(\mathcal{H}, \pi)$ be a unitary representation of G and suppose that there exists $b \in \mathrm{lin}(G, \pi)$. Replacing G by G/K for some compact normal subgroup if necessary, we can suppose by [Com, Theorem 3.7] that G is separable, and thus we can also assume that $\mathcal{H}$ is separable.

As G has Property H_{FD}, $\mathcal{H}$ splits into a direct sum $\mathcal{H} = \mathcal{H}' \oplus (\bigoplus_{n\in\mathbf{N}} \mathcal{H}_n)$ where $\mathcal{H}_n$ are finite dimensional subrepresentations and where $\mathcal{H}'$ is a subrepresentation with trivial reduced cohomology. By Proposition 3.3.1, and since b has linear growth, its orthogonal projection on $\oplus_{n\in\mathbf{N}}\mathcal{H}_n$ still has linear growth, so we can assume that $\mathcal{H} = \oplus_{n\in\mathbf{N}}\mathcal{H}_n$. Now, let b_n be the projection of b on $\oplus_{k\leq n}\mathcal{H}_k$. Then $b_n \to b$ uniformly on compact subsets. So, as $\mathrm{lin}(G,\pi)$ is open and $b \in \mathrm{lin}(G,\pi)$, there exists n such that $b_n \in \mathrm{lin}(G,\pi)$. Hence b_n defines a proper morphism $G \to \mathrm{Isom}(\mathcal{H}_n)$; denote by H its image.

If G is discrete, by Bieberbach's Theorem (see for instance [Bus]), this implies that G has a morphism with finite kernel onto a virtually abelian group, hence is itself virtually abelian.

In general, by Corollary B.0.3, G acts properly and cocompactly on some Euclidean space $\mathbf{R}^n$, hence is quasi-isometric to $\mathbf{R}^n$. $\square$

3.4.3 Uniform embeddings into Hilbert spaces

Let G be a locally compact group, and $|\cdot|_S$ the length function with respect to a compact symmetric generating subset S. For an arbitrary map f of G to a Hilbert space $\mathcal{H}$, define its *dilatation* as :

$$\delta(x) = \sup\{\|f(g) - f(h)\| : \ |g^{-1}h|_S \leq x\} \in \mathbf{R}_+ \cup \{\infty\},$$

and its *compression* as :

$$\rho(x) = \inf\{\|f(g) - f(h)\| : \ |g^{-1}h|_S \geq x\} \in \mathbf{R}_+ \cup \{\infty\},$$

We call f a *uniform map* if $\delta(x) < \infty$ for all $x \in \mathbf{R}_+$ (by an easy standard argument, this implies that δ has at most linear growth). The map f is called a *uniform embedding* if, in addition, $\rho(x) \to \infty$ when $x \to \infty$. It is called a quasi-isometric embedding if, in addition, it has compression with linear growth, i.e. $\rho(g) \succeq |g|$.

The following result, which was pointed out to us by M. Gromov (who provided a proof in the discrete case), is very useful.

Proposition 3.4.4. *Let G be an locally compact, compactly generated, amenable group. Let f be a uniform map of G into a Hilbert space, and ρ its compression, δ its dilatation. Then G admits a 1-cocycle with compression $\geq \rho - a$ and dilatation $\leq \delta + a$, for some constant $a \geq 0$. If G is discrete, we can take $a = 0$.*

Proof : Let m be a mean on G, that is, a continuous, linear map on $L^\infty(G)$ such that $m(1) = 1$, $m(f) \geq 0$ whenever $f \geq 0$ locally almost everywhere. Since G is amenable, we can choose m invariant, i.e. $m(g \cdot f) = m(f)$ for all $g \in G$ and $f \in L^\infty(G)$, where $(g \cdot f)(h)$ is by definition equal to $f(g^{-1}h)$, but we first do *not* assume m invariant.

For $g, h \in G$, set $\Psi(g,h) = \|f(g) - f(h)\|^2$. By assumption,

$$\rho(g^{-1}h) \leq \Psi(g,h)^{1/2} \leq \delta(g^{-1}h), \quad \forall g,h \in G.$$

The upper bound by δ implies that f is a uniform map. By Lemma A.0.1 in the appendix, there exists a uniformly continuous function f' at bounded distance from f (if G is discrete we do not need Lemma A.0.1 since it suffices to take $f' = f$). Write $\Psi'(g,h) = \|f'(g) - f'(h)\|^2$. Then $\Psi^{1/2} - (\Psi')^{1/2}$ is bounded.

Now set $u_{g_1,g_2}(h) = \Psi'(hg_1, hg_2)$ for $g_1, g_2, h \in G$. The upper bound by δ and the uniform continuity of f' imply that the mapping $(g_1, g_2) \mapsto u_{g_1,g_2}$ is a continuous function from $G \times G$ to $L^\infty(G)$, so that the function $\Psi_m(g_1, g_2) = m(u_{g_1,g_2})$ is continuous on $G \times G$.

If m is given by a non-negative function in $L^1(G)$, it is immediate that Ψ_m is conditionally negative definite on G. By continuity of the mapping $m \mapsto \Psi_m(g_1, g_2)$ (when $L^\infty(G)^*$ is endowed with the weak-* topology), it follows that Ψ_m is conditionally negative definite on G for all m. Now assume that m is G-invariant. It follows that Ψ_m is G-invariant, so that we can write $\Psi_m(g_1, g_2) = \psi(g_1^{-1}g_2)$ for some continuous, conditionally negative definite function ψ on G. Let b be the corresponding 1-cocycle. The estimates on ψ, and thus on $\|b\|$ immediately follow from the positivity of m. ■

Corollary 3.4.5. *If a locally compact, compactly generated amenable group G quasi-isometrically embeds into Hilbert space, then it admits a 1-cocycle with linear growth.* ■

From Corollary 3.4.5 and Theorem 3.4.3, we deduce immediately :

Corollary 3.4.6. *Let G be a locally compact, compactly generated amenable group with property H_{FD}. The group G admits a quasi-isometric embedding into a Hilbert space if and only if G acts properly on a Euclidean space. In particular, if G is discrete, this means that it is virtually abelian.*

Combining this corollary with Shalom's results mentioned in §3.4.2, we immediately obtain Theorem 3.1.4 in the introduction.

Proof of Corollary 3.1.5. We must prove that if a finitely generated group Γ is quasi-isometric to $\mathbf{Z}^n$, then it has a finite index subgroup isomorphic to $\mathbf{Z}^n$. We recall that this result, which was known as a consequence of Gromov's polynomial growth Theorem, has been given a new proof in [Sh3]. As in [Sh3], the first step is the fact that, since Property H_{FD} is a quasi-isometric invariant of amenable groups, Γ has Property H_{FD}. Now, being quasi-isometric to $\mathbf{Z}^n$, Γ quasi-isometrically embeds in a Hilbert space, hence is virtually isomorphic to $\mathbf{Z}^m$ for some m, by Theorem 3.1.4.

Finally, it is well-known that $\mathbf{Z}^m$ and $\mathbf{Z}^n$ being quasi-isometric implies $m = n$. For instance, it suffices to observe that the degree of growth of $\mathbf{Z}^n$ is n. ■

Proof of Corollary 3.1.6. It is enough to show that the regular tree of degree 3 does not embed quasi-isometrically into Hilbert space. But such a tree is quasi-isometric to $\mathbf{Q}_2 \rtimes_2 \mathbf{Z}$, since this group acts cocompactly and properly on the Bass-Serre tree of $\mathrm{SL}_2(\mathbf{Q}_2)$. On the other hand, $\mathbf{Q}_2 \rtimes_2 \mathbf{Z}$ has no compact normal subgroup, so it does not act properly cocompactly on a Euclidean space.Since it has H_{FD}, by Theorem 3.1.4, it does not quasi-isometrically embed into Hilbert space. □

Remark 3.4.7. Proposition 3.4.4 is specific to amenable groups. For instance, if Γ is a free group on $n \geq 2$ generators, then it has uniform embeddings with compression $\geq |g|^a$ for arbitrary $a < 1$ [GuKa, §6], while it has no 1-cocycle with compression $\succ |g|^{1/2}$ since it is non-amenable ([GuKa, §5] or Theorem 3.4.1).

3.4.4 Equivariant Hilbert space compression

The following definition is due to E. Guentner and J. Kaminker [GuKa]. Let G be a compactly generated group, endowed with its word length $|.|_S$.

Definition 3.4.8. The *equivariant Hilbert space compression* of G is defined as :

$$B(G) = \sup\{\alpha \geq 0,\ \exists \pi \text{ unitary representation}, \exists b \in Z^1(G,\pi),\ \|b(g)\| \succeq |g|_S^{\alpha}\}.$$

It is clear that $0 \leq B(G) \leq 1$, and if G admits a linear 1-cocycle, then $B(G) = 1$. The converse in not true : it is shown in [Tes2] that $B(G) = 1$ for groups of the class $(\mathcal{L})$ whereas we have shown above (Theorem 3.1.4) that these groups do not admit linear cocycles unless they act properly on a Euclidean space.

Another immediate observation is that if $B(G) > 0$, then G is a-T-menable. We know nothing about the converse : actually we know no example of a-T-menable group with $B < 1/2$; at the other extreme, we do not know if solvable groups always satisfy $B > 0$.

There is a large class of groups for which a 1-cocycle $b(g) \sim |g|^{1/2}$ can be constructed, so that $B(G) \geq 1/2$ (this must be an equality for non-amenable G by Proposition 3.4.1), including :

- Coxeter groups [BoJS] ;
- groups acting cocompactly on a finite dimensional $CAT(0)$ cubical complex [NR] ;
- groups acting cocompactly on a real or complex hyperbolic space [FH], i.e. $\mathrm{SO}(n,1)$, $\mathrm{SU}(n,1)$ and their cocompact subgroups (e.g. Fuchsian groups) ;
- a large class of "diagram groups", including Thompson's group F of the interval and $\mathbf{Z} \wr \mathbf{Z}$ [AGS].

Also note that by [AGS], $1/2 \leq B(\mathbf{Z} \wr \mathbf{Z}) \leq 3/4$, and $B(\mathbf{Z} \wr \Gamma) \leq 1/2$ for every finitely generated group Γ of non-polynomial growth. It follows that there are solvable (hence amenable) groups with $B \leq 1/2$, as we see by taking Γ solvable of exponential growth such as $\mathbf{Z} \wr \mathbf{Z}$.

It follows from Proposition 3.4.4 that, for *amenable groups* G, the number $B(G)$ is a quasi-isometry invariant.

Chapitre 4

Etude dynamique des actions par isométries sur un espace de Hilbert II : structure des orbites

Résumé

Our main result is that a finitely generated nilpotent group has no isometric action on an infinite-dimensional Hilbert space with dense orbits. In contrast, we construct such an action with a finitely generated metabelian group.

4.1 Introduction

The study of isometric actions of groups on affine Hilbert spaces has, in recent years, found applications ranging from the K-theory of C^*-algebras [HiKa], to rigidity theory [Sh2] and geometric group theory [Sh3, CTV]. This renewed interest motivates the following general problem : *How can a given group act by isometries on an affine Hilbert space ?*

This paper is a sequel to [CTV], but can be read independently. In [CTV], given an isometric action of a finitely generated group G on a Hilbert space $\alpha : G \to \mathrm{Isom}(\mathcal{H})$, we focused on the growth of the function $g \mapsto \alpha(g)(0)$. Here the emphasis is on the structure of orbits.

We will mainly focus on actions of nilpotent groups. Let us begin by a simple example : every isometric action of $\mathbf{Z}$ on a Euclidean space is the direct sum of an action with a fixed point and an action by translations. This actually remains true for general locally compact nilpotent groups. The situation becomes more subtle when we study actions on infinite-dimensional Hilbert spaces. However, something remains from the finite-dimensional case.

We say that a convex subset of a Hilbert space is *locally bounded* if its intersection with any finite-dimensional subspace is bounded. The main result of the paper is the following theorem, proved in §4.4.

Theorem 1. *Let G be a nilpotent group. Let G act isometrically on a Hilbert space $\mathcal{H}$, with linear part π. Let $\mathcal{O}$ be an orbit under this action. Then there exist*

- *a closed subspace T of $\mathcal{H}$ (the "translation part"), contained in the subspace of invariant vectors of π, and*
- *a closed, locally bounded convex subset U of the orthogonal subspace $T^{\perp}$,*

such that $\mathcal{O}$ is contained in $T \times U$.

We owe the following general question to A. Navas : which locally compact groups have an isometric action on an infinite-dimensional separable Hilbert space with dense orbits (i.e. a minimal action) ? Theorem 1 allows us to provide a negative answer in the case of finitely generated nilpotent groups.

Corollary 2. (see Corollary 4.4.6) *A compactly generated, nilpotent-by-compact locally compact group does not admit any affine isometric action with dense orbits on an infinite-dimensional Hilbert space.*

Actually, for compactly generated nilpotent groups, one can describe all affine isometric actions on Euclidean spaces with dense orbits ; see Corollary 4.4.5.

In the course of our proof, we introduce the following new definition : a unitary or orthogonal representation π of a group is *strongly cohomological* if $H^1(G, \rho) \neq 0$ for every nonzero subrepresentation $\rho \leq \pi$. It is easy to observe that the linear part of an affine isometric action with dense orbits is strongly cohomological. The main non-trivial step in the proof of Theorem 1 is the following result.

Proposition 3. (see Proposition 4.3.9) *Let π be an orthogonal or unitary representation of a second countable, nilpotent locally compact group G. Suppose that π is strongly cohomological. Then π is a trivial representation.*

Another case for which we answer negatively Navas' question is the following.

Theorem 4. (see Theorem 4.4.7) *Let G be a connected semisimple Lie group. Then G has no isometric action on a nonzero Hilbert space with dense orbits.*

It is not clear how Theorem 1 and Corollary 2 can be generalized, in view of the following example.

Proposition 5. (see Proposition 4.2.1) *There exists a finitely generated metabelian group admitting an affine isometric action with dense orbits on an infinite-dimensional separable Hilbert space.*

Another construction provides

Proposition 6. (see Proposition 4.2.3) *There exists a countable group admitting an affine isometric action with dense orbits on an infinite-dimensional Hilbert space, in such a way that every finitely generated subgroup has a fixed point.*

Acknowledgements. *We thank A. Navas for useful discussions and encouragement.*

4.2 Existence results

Here is a first positive result regarding Navas' question.

Proposition 4.2.1. *There exists an isometric action of a metabelian 3-generator group on $\ell^2_{\mathbf{R}}(\mathbf{Z})$, all of whose orbits are dense.*

Proof : Observe that $\mathbf{Z}[\sqrt{2}]$ acts on $\mathbf{R}$ by translations, with dense orbits. So the free abelian group of countable rank $\mathbf{Z}[\sqrt{2}]^{(\mathbf{Z})}$ acts by translations, with dense orbits, on $\ell^2_{\mathbf{R}}(\mathbf{Z})$. Observe now that the latter action extends to the wreath product $\mathbf{Z}[\sqrt{2}] \wr \mathbf{Z} = \mathbf{Z}[\sqrt{2}]^{(\mathbf{Z})} \rtimes \mathbf{Z}$, where $\mathbf{Z}$ acts on $\ell^2_{\mathbf{R}}(\mathbf{Z})$ by the shift. That wreath product is metabelian, with 3 generators. ■

Corollary 4.2.2. *There exists an isometric action of a free group of finite rank on a Hilbert space, with dense orbits.* □

Recall that an isometric action $\alpha : G \to \mathrm{Isom}(\mathcal{H})$ *almost has fixed points* if for every $\varepsilon > 0$ and every compact subset $K \subset G$ there exists $v \in \mathcal{H}$ such that $\sup_{g\in K} \|v - \alpha(g)v\| \leq \varepsilon$.

In the example given by Proposition 4.2.1, the given isometric action clearly does not almost have fixed points, i.e. it defines a nonzero element in reduced 1-cohomology. The next result shows that this is not always the case.

Proposition 4.2.3. *There exists a countable group Γ with an affine isometric action α on an infinite-dimensional Hilbert space, such that α has dense orbits, and every finitely generated subgroup of Γ has a fixed point. In particular, the action almost has fixed points.*

Proof : We first construct an uncountable group G and an affine isometric action of G having dense orbits and almost having fixed points.

In $\mathcal{H} = \ell^2_{\mathbf{R}}(\mathbf{N})$, let A_n be the affine subspace defined by the equations

$$x_0 = 1,\ x_1 = 1, ...,\ x_n = 1,$$

and let G_n be the pointwise stabilizer of A_n in the isometry group of $\mathcal{H}$. Let G be the union of the G_n's. View G as a discrete group.

It is clear that G almost has fixed points in $\mathcal{H}$, since any finite subset of G has a fixed point. Let us prove that G has dense orbits.

Claim 1. For all $x, y \in \mathcal{H}$, we have $\lim_{n\to\infty} |d(x, A_n) - d(y, A_n)| = 0$.

By density, it is enough to prove Claim 1 when x, y are finitely supported in $\ell^2_{\mathbf{R}}(\mathbf{N})$. Take $x = (x_0, x_1, ..., x_k, 0, 0, ...)$ and choose $n > k$. Then

$$d(x, A_n)^2 \;=\; \sum_{j=0}^{k}(x_j - 1)^2 + \sum_{j=k+1}^{n} 1^2 \;=\; n + 1 - 2\sum_{j=0}^{k} x_j + \sum_{j=0}^{k} x_j^2,$$

so that $d(x, A_n) = \sqrt{n} + O(\frac{1}{\sqrt{n}})$, which proves Claim 1.

Denote by p_n the projection on the closed convex set A_n, namely $p_n(x_0, x_1, \dots) = (1, 1, \dots, 1, x_{n+1}, x_{n+2}, \dots)$.

Claim 2. For all $x, y \in \mathcal{H}$, we have $\lim_{n\to\infty} \|p_n(x) - p_n(y)\| = 0$.

This is a straightforward computation.

Claim 3. G has dense orbits in $\mathcal{H}$.

Observe that two points $x, y \in \mathcal{H}$ are in the same G_n-orbit if and only if $d(x, A_n) = d(y, A_n)$ and $p_n(x) = p_n(y)$. Fix $x_0, z \in \mathcal{H}$. We want to show that $\lim_{n\to\infty} d(G_n x_0, z) = 0$. So fix $\varepsilon > 0$. By the second claim, for some n_0, $\|p_n(x_0) - p_n(z)\| \le \varepsilon/2$ whenever $n \ge n_0$. Set

$$W = \{x \in \mathcal{H} |\, p_n(x) = p_n(z)\};$$

this is the orthogonal affine subspace of A_n passing through z. Then $y_0 = x_0 + (p_n(z) - p_n(x_0)) \in W$. By the first claim, there exists $n_1 \ge n_0$ such that $|d(y_0, A_n) - d(z, A_n)| \le \varepsilon/2$ for every $n \ge n_1$. Therefore there exists $y \in W$ such that $\|y - z\| \le \varepsilon/2$ and $d(y, A_n) = d(y_0, A_n) = d(x_0, A_n)$. By the previous observation, there exists $g \in G_n$ such that $y = gy_0$. Then

$$d(gx_0, z) \le d(gx_0, gy_0) + d(gy_0, z) \le \varepsilon,$$

so that $d(G_n x_0, z) \le \varepsilon$ for every $n \ge n_1$, proving the last claim.

Using separability of $\mathcal{H}$, it is now easy to construct a countable subgroup Γ of G also having dense orbits on $\mathcal{H}$. ■

Question 1. Does there exist an affine isometric action of a *finitely generated* group on a Hilbert space, having dense orbits and almost having fixed points?

4.3 Cohomology of unitary representations of nilpotent groups

Our non-existence results concerning nilpotent locally compact groups will be based on the following study of their unitary representations.

Definition 4.3.1. If G is a topological group and π a unitary representation, we say that π is *strongly cohomological* if every nonzero subrepresentation of π has nonzero first cohomology.

The following lemma is Proposition 3.1 in Chapitre III of [Gu2].

Lemma 4.3.2. *Let π be a unitary representation of G that does not contain the trivial representation. Let z be a central element of G. Suppose that $1-\pi(z)$ has a bounded inverse (equivalently, 1 does not belong to the spectrum of $\pi(z)$). Then $H^1(G,\pi)=0$.*

Proof : If $g \in G$, expanding the equality $b(gz) = b(zg)$, we obtain that $(1-\pi(z))b(g)$ is bounded by $2\|b(z)\|$, so that b is bounded by $2\|(1-\pi(z))^{-1}\|\|b(z)\|$. ■

Lemma 4.3.3. *Let G be a locally compact, second countable group, and π a strongly cohomological representation. Then π is trivial on the centre $Z(G)$.*

Proof : Fix $z \in Z(G)$. As G is second countable, we may write $\pi = \int_{\hat{G}}^{\oplus} \rho \, d\mu(\rho)$, a disintegration of π as a direct integral of irreducible representations. Let $\chi : \hat{G} \to S^1 : \rho \mapsto \rho(z)$ be the continuous map given by the value of the central character of ρ on z. For $\varepsilon > 0$, set $X_\varepsilon = \{\rho \in \hat{G} : |\chi(\rho) - 1| > \varepsilon\}$ and $\pi_\varepsilon = \int_{X_\varepsilon}^{\oplus} \rho \, d\mu(\rho)$, so that π_ε is a subrepresentation of π. Since $|\rho(z)-1|^{-1} < \varepsilon^{-1}$ for $\rho \in X_\varepsilon$, the operator

$$(\pi_\varepsilon(z) - 1)^{-1} = \int_{X_\varepsilon}^{\oplus} (\rho(z) - 1)^{-1} \, d\mu(\rho)$$

is bounded. We can now apply Lemma 4.3.2 to conclude that $H^1(G, \pi_\varepsilon) = 0$. By definition, this means that π_ε is the zero subrepresentation, meaning that the spectral measure μ is supported in $\hat{G} - X_\varepsilon$. As this holds for every $\varepsilon > 0$, we see that μ is supported in $\{\rho \in \hat{G} | \rho(z) = 1\}$, to the effect that $\pi(z) = 1$. ■

Proposition 4.3.4. *Let G be a topological group, and π a unitary representation of G. Suppose that $\overline{H^1}(G,\pi) \neq 0$. Then π has a nonzero subrepresentation that is strongly cohomological.*

Proof : Suppose the contrary. Then, by a standard application of Zorn's Lemma, π decomposes as a direct sum $\pi = \bigoplus_{i \in I} \pi_i$, where $H^1(G, \pi_i) = 0$ for every $i \in I$, so that $\overline{H^1}(G,\pi) = 0$ by Proposition 2.6 in Chapitre III of [Gu2]. ■

Remark 4.3.5. The converse is false, even for finitely generated groups : indeed, it is easy to check (see [Gu1]) that every nonzero representation of the free group F_2 has non-vanishing H^1, so that every unitary representation of F_2 is strongly cohomological. But it turns out that F_2 has an irreducible representation π such that $\overline{H^1}(F_2, \pi) = 0$ (see Proposition 2.4 in [MaVa]).

Corollary 4.3.6. *Let G be a locally compact, second countable group, and let π be a unitary representation of G without invariant vectors. Write $\pi = \pi_0 \oplus \pi_1$, where π_1 consists of the $Z(G)$-invariant vectors. Then*

(1) π_0 does not contain any nonzero strongly cohomological subrepresentation (in particular, $\overline{H^1}(G, \pi_0) = 0$);

(2) every 1-cocycle of π_1 vanishes on $Z(G)$, so that $H^1(G, \pi_1) \simeq H^1(G/Z(G), \pi_1)$.

Proof : (1) follows by combining Lemma 4.3.3 and Proposition 4.3.4. For (2), we use the idea of proof of Theorem 3.1 in [Sh2] : if $b \in Z^1(G, \pi_1)$, then for every $g \in G$, $z \in Z(G)$,

$$\pi_1(g)b(z) + b(g) \;=\; b(gz) \;=\; b(zg) \;=\; b(g) + b(z)$$

as $\pi_1(z) = 1$. So $\pi_1(g)b(z) = b(z)$; this forces $b(z) = 0$ as π has no G-invariant vector. So b factors through $G/Z(G)$. ■

Observe that Corollary 4.3.6 provides a new proof of Shalom's Corollary 3.7 in [Sh2] : under the same assumptions, every cocycle in $Z^1(G, \pi)$ is almost cohomologous to a cocycle factoring through $G/Z(G)$ and taking values in a subrepresentation factoring through $G/Z(G)$. From Corollary 4.3.6 we also immediately deduce

Corollary 4.3.7. *Let G be a locally compact, second countable, nilpotent group, and let π be a representation of G without invariant vectors. Let (Z_i) be the ascending central series of G ($Z_0 = \{1\}$, and Z_i is the centre modulo Z_{i-1}). Let σ_i denote the subrepresentation of G on the space of Z_i-invariant vectors, and finally let π_i be the orthogonal of σ_{i+1} in σ_i, so that $\pi = \bigoplus \pi_i$.*

Then $H^1(G, \pi_i) \simeq H^1(G/Z_i, \pi_i)$ for all i, and π is not a strongly cohomological subrepresentation. In particular, $\overline{H^1}(G, \pi) = 0$. □

Note that the latter statement is a result of Guichardet [Gu1, Théorème 7], which can be stated as : G has Property H_T (i.e. every unitary representation with non-vanishing reduced 1-cohomology contains the trivial representation).

Definition 4.3.8. We say that a locally compact group G has Property H_{CT} if every strongly cohomological unitary representation of G is trivial.

It is a straightforward verification that this is equivalent to : every strongly cohomological *orthogonal* representation of G is trivial. This will be useful in the next paragraph since we will deal with orthogonal rather than unitary representations. In this section we have proved :

Proposition 4.3.9. *If G is a locally compact, second countable nilpotent group, then G has Property H_{CT}.* □

As a corollary of Proposition 4.3.4, Property H_{CT} implies Property H_T. However the converse is not true, as shown by the following example.

Example 1. Let G be the full affine group of the real line. The dual $\hat{G}$ (i.e. the space of unitary irreducible representations of G with the Fell-Jacobson topology) was described in [Fe62] : it consists of two copies of the real line (corresponding to one-dimensional representations, i.e. characters) plus one point $\{\sigma\}$ which is both open and dense. The only irreducible representation with non-vanishing reduced 1-cohomology is the trivial representation 1_G, so that G has Property H_T ; on the other hand, since σ weakly contains 1_G, one has $H^1(G,\sigma) \neq 0$ by [Gu1, Théorème 1]. So σ is strongly cohomological, meaning that G does not have Property H_{CT}.

4.4 Non-existence results

Definition 4.4.1. 1) We say that a subset Y of a metric space (X,d) is *coarsely dense* if there exists $C \geq 0$ such that, for every $x \in X$,

$$d(x,Y) \leq C.$$

2) We say that a subset Y of a Hilbert space $\mathcal{H}$ is *enveloping* if its closed convex hull is all of $\mathcal{H}$.

Observe that every dense subset of a metric space is coarsely dense. Besides, in a Hilbert space $\mathcal{H}$, every coarsely dense subset Y is enveloping. Indeed, suppose that Y is contained in a closed, convex proper subset X of $\mathcal{H}$. Consider $v \notin X$ and let y denote its projection on X (excluding the trivial case $Y = \emptyset$). Then, for every $\lambda \geq 0$, we have $d(y+\lambda(v-y),Y) \geq d(y+\lambda(v-y),X) = \lambda$, which is unbounded, so that Y is not coarsely dense.

Example 2. In $\ell^2_{\mathbf{R}}(\mathbf{Z})$, let X denote the subset of elements with integer coefficients. Then X is enveloping : indeed, its intersection with the subspace $F_n = \ell^2_{\mathbf{R}}(\{-n,\dots,n\})$ is coarsely dense, hence enveloping in F_n, and the increasing union $\bigcup F_n$ is dense in $\ell^2_{\mathbf{R}}(\mathbf{Z})$. But X is not coarsely dense : indeed, for every $n \geq 0$, the element $\frac{1}{2}\mathbf{1}_{\{1,\dots,4n\}}$ is at distance $\sqrt{n}$ to X.

Note that X is the orbit of 0 for the natural action of the wreath product $\mathbf{Z} \wr \mathbf{Z} = \mathbf{Z}^{(\mathbf{Z})} \rtimes \mathbf{Z}$ on $\ell^2_{\mathbf{R}}(\mathbf{Z})$, where $\mathbf{Z}^{(\mathbf{Z})}$ acts by translations and the factor $\mathbf{Z}$ acts by shifting (compare to the example in the proof of Proposition 4.2.1).

Lemma 4.4.2. *Let G be a topological group and π an orthogonal representation, admitting a 1-cocycle b with enveloping orbits. Then π is strongly cohomological.*

Proof : If σ is a nonzero subrepresentation of π, let b_σ be the orthogonal projection of b on $\mathcal{H}_\sigma$, so that $b_\sigma \in Z^1(G,\sigma)$. Then $b_\sigma(G)$ is enveloping in $\mathcal{H}_\sigma$, in particular b_σ is unbounded. So b_σ defines a nonzero class in $H^1(G,\sigma)$. ■

Theorem 4.4.3. *Let G be a locally compact group with Property H_{CT}. Let G act isometrically on a Hilbert space $\mathcal{H}$, with linear part π. Let $\mathcal{O}$ be an orbit under this action. Then there exist*

– a subspace T of $\mathcal{H}$, contained in $\mathcal{H}^{\pi(G)}$, and

– *a closed, locally bounded convex subset U of $T^\perp$,*
such that $\mathcal{O}$ is contained in $T \times U$.

Proof : We immediately reduce to the case when π has no invariant vectors, so that we must prove that the closed convex hull U of $\mathcal{O}$ is locally bounded.

Observe that a convex subset of a Hilbert space is locally bounded if and only if it contains no affine half-line. Thus denote by $\mathcal{D}$ the set of affine half-lines contained in U, and suppose by contradiction that $\mathcal{D} \neq \emptyset$. Denote by $\mathcal{D}_0$ the corresponding set of linear half-lines (where the linear half-line corresponding to a half-line $x + \mathbf{R}_+ v$ is simply $\mathbf{R}_+ v$). Then $\mathcal{D}_0$ is invariant under the linear action π of G. Let W be the closed subspace of $\mathcal{H}$ generated by all the half-lines in $\mathcal{D}_0$, and denote by σ the corresponding subrepresentation. By assumption, σ is nonzero.

We claim that σ is strongly cohomological, contradicting that π has no invariant vectors along with the H_{CT} assumption. Let ρ be a nonzero subrepresentation of σ. Then by the definition of W, there exists an half-line of U which projects injectively into the subspace of ρ. Thus $H^1(G, \rho) \neq 0$, proving the claim, and ending the proof. ■

Proof of Theorem 1. We can suppose that π has no invariant vector. Suppose that the convex hull of $\alpha(G)(0)$ is not locally bounded. Then it contains a half-line $D = x + \mathbf{R}_+ v$. Let (x_n) be an unbounded sequence in D. Every x_n is a convex combination of elements of the form $\alpha(g)(0)$, where g ranges over a finite subset F_n of G. Besides, since $\pi(G)$ has no invariant vector, there exists $g \in G$ such that $\pi(g)v \neq v$. Let H be the subgroup of G generated by the countable subset $\{g\} \cup \bigcup_n F_n$. Then the convex hull of $\alpha(H)(0)$ contains D. By Proposition 4.3.9, H has Property H_{CT} ; it follows by Theorem 4.4.3 that D is parallel to the invariant vectors of $\pi(H)$, so that v is contained in the $\pi(H)$-invariant vectors, a contradiction. ■

Corollary 4.4.4. *Let G be a locally compact group with Property H_{CT}. Let $\mathcal{H}$ be a Hilbert space on which G acts with enveloping (respectively coarsely dense, resp. dense) image. Then the action is by translations, defined by a continuous morphism : $u : G \to (\mathcal{H}, +)$ with enveloping (resp. coarsely dense, resp. dense) image.* ■

Corollary 4.4.5. *Let G be a locally compact, compactly generated group with Property H_{CT}, and let $\mathcal{H}$ be a (real) Hilbert space. Then*

- *G has an isometric action on $\mathcal{H}$ with coarsely dense (respectively enveloping) orbits if and only $\mathcal{H}$ has finite dimension k, and G has a quotient isomorphic to $\mathbf{R}^n \times \mathbf{Z}^m$, with $n + m \geq k$.*
- *G has an isometric action on $\mathcal{H}$ with dense orbits if and only $\mathcal{H}$ has finite dimension k, and G has a quotient isomorphic to $\mathbf{R}^n \times \mathbf{Z}^m$, with* $\max(n + m - 1, n) \geq k$.

Proof : Let α be an affine isometric action of G with enveloping orbits (this encompasses all possible assumptions). By Corollary 4.4.4, the action is by translations; let u be the morphism $G \to (\mathcal{H}, +)$; its image generates $\mathcal{H}$ as a topological vector space. Let W denote the kernel of u.

Then $A = G/W$ is a locally compact, compactly generated abelian group, which embeds continuously into a Hilbert space. By standard structural results, A has a compact subgroup K such that A/K is a Lie group. Since K embeds into a Hilbert space, it is necessarily trivial, so that A is an abelian Lie group without compact subgroup. Accordingly, A is isomorphic to $\mathbf{R}^n \times \mathbf{Z}^m$ for some integers n, m; the embedding of A into $\mathcal{H}$ extends canonically to a linear mapping of $\mathbf{R}^{n+m}$ into $\mathcal{H}$. In particular $\mathcal{H}$ is finite-dimensional, of dimension $k \leq n + m$.

If the action has dense orbits, then either $m = 0$ and $n \geq k$, or $m \geq 1$ and $m \geq k - n + 1$; this means that $k \leq \max(n + m - 1, n)$. Conversely, if $k \leq n + m - 1$, then, since $\mathbf{Z}$ has a dense embedding into the torus $\mathbf{R}^k/\mathbf{Z}^k$, $\mathbf{Z}^{k+1}$ has a dense embedding into $\mathbf{R}^k$, and this embedding can be extended to $\mathbf{R}^n \times \mathbf{Z}^m$. ■

From Proposition 4.3.9 and Corollary 4.4.5, we deduce

Corollary 4.4.6. *A compactly generated, nilpotent-by-compact group does not admit any isometric action with enveloping (e.g. dense) orbits on an infinite-dimensional Hilbert space.* □

Proposition 4.2.1 on the one hand, and Corollary 4.4.6 on the other, isolate the first test-case for Navas' question :

Question 2. Does there exist a polycyclic group admitting an affine isometric action with dense orbits on an infinite-dimensional Hilbert space?

Let us prove a related result for semisimple groups.

Theorem 4.4.7. *Let G be a connected, semisimple Lie group. Then G cannot act on a Hilbert space $\mathcal{H} \neq 0$ with coarsely dense (e.g. dense) orbits.*

Proof : Suppose by contradiction the existence of such an action α, and let π denote its linear part. Then π is strongly cohomological. By Lemma 4.3.3, π is trivial on the centre of G. Thus the centre acts by translations, generating a finite-dimensional subspace V of $\mathcal{H}$. The action induces a map $p : G \to V \rtimes \mathrm{O}(V)$. Since G is semisimple, the kernel of p contains the sum G_{nc} of all noncompact factors of G, and thus factors though the compact group G/G_{nc}. Thus $H^1(G, V) = 0$, and since π is strongly cohomological, this implies that $V = 0$.

It follows that α is trivial on the centre of G, so that we can suppose that G has trivial centre. Then G is a direct product of simple Lie groups with trivial centre. We can write $G = H \times K$ where K denotes the sum of all simple factors S of G such that $\alpha(S)(0)$ is bounded (in other words, $H^1(S, \pi|_S) = 0$). Then the restriction of α to H also has coarsely dense orbits. Moreover, every simple factor of H acts in an unbounded way, so that, by a result of Shalom [Sh1, Theorem 3.4][1], the action of H is proper. That is, the map $i : H \to \mathcal{H}$ given by

[1] Shalom only states the result for a simple group, but the proof generalizes immediately. See for instance [CLTV] for another proof, based on the Howe-Moore Property.

$i(h) = \alpha(h)(0)$ is metrically proper and its image is coarsely dense. By metric properness, the subset $X = i(H) \subset \mathcal{H}$ satisfies : X is coarsely dense, and every ball in X (for the metric induced by $\mathcal{H}$) is compact.

Suppose that $\mathcal{H}$ is infinite-dimensional and let us deduce a contradiction. For some $d > 0$, we have $d(x, X) \leq d$ for every $x \in \mathcal{H}$. If $\mathcal{H}$ is infinite-dimensional, there exists, in a fixed ball of radius $7d$, infinitely many pairwise disjoint balls $B(x_n, 3d)$ of radius $3d$. Taking a point in $X \cap B(x_n, 2d)$ for every n, we obtain a closed, infinite and bounded discrete subset of X, a contradiction.

Thus $\mathcal{H}$ is finite-dimensional ; since every simple factor of H is non-compact, it has no non-trivial finite-dimensional orthogonal representation, so that the action is by translations, and hence is trivial, so that finally $\mathcal{H} = \{0\}$. ■

Remark 4.4.8. 1) The same argument shows that a semisimple, linear algebraic group over any local field, cannot act with coarsely dense orbits on a Hilbert space.

2) The argument fails to work with enveloping orbits : indeed, in $\ell^2_{\mathbf{R}}(\mathbf{N})$, let X denote the set sequences (x_n) such that $x_n \in 2^n\mathbf{Z}$ for every $n \in \mathbf{N}$. Then X is coarsely dense in $\ell^2_{\mathbf{R}}(\mathbf{N})$, but, for the metric induced by $\mathcal{H}$, every ball in X is finite, hence compact. We do not know if a semisimple Lie group (e.g. $\mathrm{SL}_2(\mathbf{R})$) can act isometrically on a nonzero Hilbert space with enveloping orbits.

Chapitre 5

Annulation de la cohomologie réduite en degré 1 à valeur dans une représentation sur un espace L^p

Résumé

We prove that the first reduced cohomology with values in a mixing L^p-representation vanishes for a class of amenable groups including amenable Lie groups. In particular this solves for a large class of amenable groups a conjecture of Gromov saying that every finitely generated amenable group has no first reduced ℓ^p-cohomology. Another consequence is to prove a Pansu's conjecture about vanishing of the first reduced L^p-cohomology on a homogeneous, closed at infinity, Riemannian manifold.

5.1 Introduction

5.1.1 Main results

Let G be a locally compact group acting by measure-preserving bijections on a measure space (X, m). We say that the action is mixing if for every measurable subset of finite measure $A \subset X$, $m(gA \cap A) \to 0$ when g leaves every compact in G. Let π be the corresponding continuous representation of G in $L^p(X, m)$, where $1 < p < \infty$. We say that π is C_0 (or mixing) if its coefficients vanish at infinity, or equivalently, if the G-action on (X, m) is mixing. In this paper, we will call such a representation a mixing L^p-representation of G.

We consider a class of amenable groups, called class $(\mathcal{L})$ (see also [CTV, **?**] where this class is introduced), including

(1) polycyclic groups and connected amenable Lie groups,

(2) semidirect products $\mathbf{Z}[\frac{1}{mn}] \rtimes_{\frac{m}{n}} \mathbf{Z}$, with m, n co-prime integers with $|mn| \geq 2$ (if $n = 1$ this is the Baumslag-Solitar group $BS(1, m)$) ; semidirect products $\left(\bigoplus_{i \in I} \mathbf{Q}_{p_i}\right) \rtimes_{\frac{m}{n}} \mathbf{Z}$ with m, n co-prime integers, and $(p_i)_{i \in I}$ a finite family of primes (including $\infty : \mathbf{Q}_\infty = \mathbf{R}$)) dividing mn.

(3) wreath products $F \wr \mathbf{Z}$ for F a finite group.

Our main result is the following theorem.

Theorem 18. Let G be a group of class $(\mathcal{L})$ and let π be a mixing L^p-representation of G. Then the first reduced cohomology of G with values in π vanishes, i.e. $\overline{H^1}(G, \pi) = 0$.

It is well known [Pu] that for finitely generated groups G, the first reduced l^p-cohomology with values in the left regular representation is isomorphic to the space $HD_p(G)$ of p-harmonic functions with gradient in ℓ^p modulo the constants. We therefore obtain the following corollary.

Corollary 19. Let G be a discrete group of class $(\mathcal{L})$. Then every p-harmonic function on G with gradient in l^p is constant.

Using Von Neumann algebra techniques, Cheeger and Gromov [CG] proved that every finitely generated amenable group G has no nonconstant harmonic function with gradient in ℓ^2, the generalization to every $1 < p < \infty$ being conjectured by Gromov.

With some work, we can also deduce the following result, conjectured by Pansu in [Pa2]. Recall that a manifold M of dimension d is called closed at infinity if there exists an exhausting sequence of compact subsets of M with smooth boundaries (A_n) satisfying $\mu_{d-1}(\partial A_n)/\mu_d(A_n) \to 0$, where ∂A_n is the boundary of A_n and where μ_k denotes the Riemannian measure on submanifolds of dimension k.

Corollary 20. Let M be a homogeneous Riemannian manifold. If it is closed at infinity, then for every $p > 1$, every p-harmonic function on M with gradient in $L^p(TM)$ is constant. In other words, $HD_p(M) = 0$.

This is proved in § 5.3. Together with Pansu's results [Pa2, Théorème H], we obtain the following dichotomy.

Theorem 21. Let M be a homogeneous Riemannian manifold. Then :
- either M is quasi-isometric to a homogeneous Riemannian manifold with strictly negative curvature, and then there exists $p_0 \geq 1$ such that $HD_p(M) \neq 0$ if and only if $p > p_0$;
- or $HD_p(M) = 0$ for every $p > 1$.

5.1.2 Ideas of the proof

The proof of Theorem 18 splits into two steps. First (see Theorem 5.2.1), we prove that for any locally compact compactly generated group G and any mixing L^p-representation π of G, every 1-cocycle $b \in Z^1(G, \pi)$ is *sublinear*, which means that for every compact symmetric generating subset S of G, we have

$$\|b(g)\| = o(|g|_S)$$

when $|g|_S \to \infty$, $|g|_S$ being the word length of g with respect to S. Then, we adapt to this context a result of [CTV] saying that for a group of class $(\mathcal{L})$, a 1-cocycle belongs to $\overline{B}^1(G, \pi)$ if and only if it is sublinear. The part "only if" is an easy exercise. To prove the other implication, we consider the affine action σ of G associated to the 1-cocycle b and we use isoperimetric properties of groups of class $(\mathcal{L})$ that we established in [Tes2, Theorem 11] to construct a sequence of almost fixed points [1] for σ.

5.2 Sublinearity of cocycles

Theorem 5.2.1. Let G be a locally compact compactly generated group and let S be a compact symmetric generating subset. Let π be a mixing L^p-representation of G. Then, every 1-cocycle $b \in Z^1(G, \pi)$ is sublinear, i.e.

$$\|b(g)\| = o(|g|_S)$$

when $|g|_S \to \infty$, $|g|_S$ being the word length of g with respect to S.

We will need the following lemma.

Lemma 5.2.2. Let us keep the assumptions of the theorem. For any fixed $j \in \mathbf{N}$,

$$\|\pi(g_1)v_1 + \ldots \pi(g_j)v_j\|_p^p \to \|v_1\|_p^p + \ldots + \|v_j\|_p^p$$

when $d_S(g_k, g_l) \to \infty$ whenever $k \neq l$, uniformly with respect to $(v_1, \ldots, v_j)$ on compact subsets of $(L^p(X))^j$.

[1] Note that this is an analogue of [CTV, Proposition 3.6].

Proof of Lemma 6.3.6. Clearly, it suffices to prove the lemma for $v_1, \ldots, v_j$ belonging to a subset D of $L^p(X,m)$ such that the vector space generated by D is dense in $L^p(X,m)$. Thus, assume that for every $1 \leq k \leq j$, v_k is an indicator function of a subset of finite measure A_k. For $u, v \in L^2(G,m)$, write $\langle u,v\rangle = \int_X u(x)v(x)dm(x)$. For every $1 \leq i \leq j$,

$$\begin{aligned} m\left((\cup_{l\neq i} g_l \cdot A_l) \cap g_k \cdot A_k\right) &= \langle \sum_{l\neq i} \pi(g_l)v_l, \pi(g_k)v_k\rangle \\ &= \sum_{l\neq i} \langle \pi(g_l)v_l, \pi(g_k)v_k\rangle \\ &= \sum_{l\neq i} \langle \pi(g_k^{-1}g_l)v_l, v_k\rangle \to 0 \end{aligned}$$

when $d_S(g_k, g_l) \to \infty$. This clearly implies the lemma. ■

Proof of Theorem 5.2.1. Fix some $\varepsilon > 0$. Let $g = s_1 \ldots s_n$ be a minimal decomposition of g into a product of elements of S. Let $m \leq n$, q and $r < m$ be positive integers such that $n = qm + r$. To simplify notations, we assume $r = 1$. For $1 \leq i < j \leq n$, denote by g_j the prefix $s_1 \ldots s_j$ of g and by $g_{i,j}$ the subword $s_{i+1} \ldots s_j$ of g. Developing $b(g)$ with respect to the cocycle relation, we obtain

$$b(g) = b(s_1) + \pi(g_1)b(s_2) + \ldots + \pi(g_{n-1})b(s_n).$$

Let us group the terms in the following way

$$\begin{aligned} b(g) = &\left[b(s_1) + \pi(g_m)b(s_{m+1}) + \ldots + \pi(g_{(q-1)m})b(s_{(q-1)m+1})\right] \\ &+ \left[\pi(g_1)b(s_2) + \pi(g_{m+1})b(s_{m+2}) + \ldots + \pi(g_{(q-1)m+1})b(s_{(q-1)m+2})\right] \\ &+ \ldots + \left[\pi(g_{m-1})b(s_m) + \pi(g_{2m-1})b(s_{2m}) + \ldots + \pi(g_{qm})b(s_{qm+1})\right] \end{aligned}$$

In the above decomposition of $b(g)$, consider each term between $[\cdot]$, e.g. of the form

$$\pi(g_k)b(s_{k+1}) + \ldots + \pi(g_{(q-1)m+k})b(s_{(q-1)m+k+1}) \tag{5.2.1}$$

for $0 \leq k \leq m-1$ (we decide that $s_0 = 1$). Note that since S is compact and π is continuous, there exists a compact subset K of E containing $b(s)$ for every $s \in S$. Clearly since $g = s_1 \ldots s_n$ is a minimal decomposition of g, the length of $g_{i,j}$ with respect to S is equal to $j - i - 1$. For $0 \leq i < j \leq q-1$ we have

$$d_S(g_{im+k}, g_{jm+k}) = |g_{im+k,jm+k}|_S = (j-i)m \geq m.$$

So by Lemma 6.3.6, for $m = m(q)$ large enough, the p-power of the norm of (8.3.3) is less than

$$\|b(s_{k+1})\|_p^p + \|b(s_{m+k+1})\|_p^p + \ldots \|b(s_{(q-1)m+k+1})\|_p^p + 1.$$

Up to change the constant C, the above term is therefore less than Cq. So again up to change C, we obtain

$$\|b(g)\|_p \leq Cmq^{1/p}.$$

So for $q \geq q_0 = (C\varepsilon)^{p/(p-1)}$, we have

$$\|b(g)\|_p/n \leq Cq^{1-1/p} \leq \varepsilon.$$

Now, let n be larger than $m(q_0)q_0$. We have $\|b(g)\|_p/|g| \leq \varepsilon$. ■

Proof of Theorem 18. Theorem 18 results from Theorem 5.2.1 and the following result, which is an adaptation of [CTV, Proposition 3.6].

Proposition 5.2.3. Keeping the notations of Theorem 18, a 1-cocycle b belongs to $\overline{B(G,\pi)}$ if and only if b is sublinear.

Proof : Assume that b is sublinear. We will need the following result.

Theorem 5.2.4. [Tes2] Let G be a group of class $(\mathcal{L})$ and let S be some compact generating subset of G. Then G admits a sequence of compact subsets $(F_n)_{n\in\mathbf{N}}$ satisfying the two following conditions
(i) there is a constant $c > 0$ such that

$$\mu(sF_n \vartriangle F_n) \leq c\mu(F_n)/n \quad \forall s \in S, \forall n \in \mathbf{N};$$

(ii) for every $n \in \mathbf{N}$, F_n is contained [2] in S^n. ■

Let (F_n) be such a Følner sequence in G. Define a sequence $(v_n) \in E^{\mathbf{N}}$ by

$$v_n = \frac{1}{\mu(F_n)} \sum_{g\in F_n} b(g).$$

We claim that (v_n) defines a sequence of almost fixed points for the affine action σ defined by $\sigma(g)v = \pi(g)v + b(g)$. Indeed, we have

$$\begin{aligned}
\|\sigma(s)v_n - v_n\| &= \left\| \frac{1}{\mu(F_n)} \sum_{g\in F_n} \sigma(s)b(g) - \frac{1}{\mu(F_n)} \sum_{g\in F_n} b(g) \right\| \\
&= \left\| \frac{1}{\mu(F_n)} \sum_{g\in F_n} b(sg) - \frac{1}{\mu(F_n)} \sum_{g\in F_n} b(g) \right\| \\
&= \left\| \frac{1}{\mu(F_n)} \sum_{g\in s^{-1}F_n} b(g) - \frac{1}{\mu(F_n)} \sum_{g\in F_n} b(g) \right\| \\
&\leq \frac{1}{\mu(F_n)} \sum_{g\in s^{-1}F_n \vartriangle F_n} \|b(g)\|.
\end{aligned}$$

Since $F_n \subset S^n$, we obtain that

$$\|\sigma(s)v_n - v_n\| \leq \frac{C}{n} \sup_{|g|_S \leq n+1} \|b(g)\|$$

which converges to 0 by Theorem 5.2.1. This proves the non-trivial implication of Proposition 5.2.3. ■

[2] Actually, they also satisfy $S^{[cn]} \subset F_n$ for a constant $c > 0$.

5.3 Liouville D_p-Properties.

5.3.1 Generalities

Let M be a homogenous Riemannian manifold, equipped with its Riemannian measure m. Fix $p > 1$. Denote by D_p the vector space of differentiable functions whose gradient is in $L^p(TM)$. A function $f \in D_p(M)$ is called p-harmonic if it is a weak solution of

$$div(|\nabla f|^{p-2}\nabla f) = 0,$$

that is,

$$\int_M \langle |\nabla f|^{p-2}\nabla f, \nabla \varphi\rangle dm = 0,$$

for every $\varphi \in C_0^\infty(M)$. Equivalently, p-harmonic functions are the minimizers of the variational integral

$$\int_M |\nabla f|^p dm.$$

We say that M satisfies a Liouville D_p-Property if every p-harmonic with gradient in L^p function on M is constant. Equip $D_p(M)$ with a pseudo-norm $\|f\|_{D_p} = \|\nabla f\|_p$, which induces a norm on $D_p(M)$ modulo the constants. Denote by $\mathbf{D}_p(M)$ the completion of this normed vector space. Denote by $W^{1,p}(M) = L^p(M) \cap D_p(M)$. $W^{1,p}(M)$ canonically embeds in $\mathbf{D}_p(M)$ as a subspace. The first reduced L^p-cohomology of M is the quotient space $\overline{H_p}^1(M) = D_p(M)/\overline{W^{1,p}(M)}$ where $\overline{W^{1,p}(M)}$ is the closure of $W^{1,p}(M)$ in the Banach space $\mathbf{D}_p(M)$. As $\mathbf{D}_p(M)$ is a strictly convex, reflexive Banach space, every $f \in \mathbf{D}_p(M)$ admits a unique projection $\tilde{f}$ on the closed subspace $\overline{W^{1,p}(M)}$ such that $d(f, \tilde{f}) = d(f, \overline{W^{1,p}(M)})$. One can easily check that $\tilde{f}$ is p-harmonic. In conclusion, the reduced cohomology class of $f \in \mathbf{D}_p$ admits a unique p-harmonic element, modulo the constants. Hence, M has Liouville D_p-Property if and only if $\overline{H_p}^1(M) = 0$.

5.3.2 Proof of Corollary 20

Let M be an closed at infinity homogeneous manifold. Let G be its group of isometries. As the stabilizer of a point of M is compact, M is quasi-isometric to G, which is a Lie group with a finite number of connected components. Since M is closed at infinity, it is well known that G is amenable and unimodular. In [Ho], it is proved that Liouville D_p-Property is invariant under quasi-isometries between manifolds with bounded geometry, a condition that is automatically satisfied on a homogeneous manifold. Hence, it suffices to prove Liouville D_p-Property for G, equipped with a left-invariant Riemannian metric. It is not difficult to prove in the context of Lie groups that if G is unimodular, then the first reduced cohomology with values in the regular L^p-representation $\lambda_{G,p}$ is isomorphic to the first reduced L^p-cohomology $\overline{H_p}^1(G)$. Then one can deduce directly Corollary 20 from Theorem 18. However, we propose a different approach here. Instead of using Theorem 18, we reformulate the proof, only using Theorem 5.2.1. The interest is to provide an explicit approximation of an element of $\mathbf{D}_p(G)$ by a sequence of functions in $W^{1,p}(G)$ using a convolution-type argument. Since Liouville D_p-Property is equivalent to the vanishing of $\overline{H_p}^1(G)$, we have to show that for every p-Dirichlet function on G, there exists a sequence of functions (f_n) in $W^{1,p}(G)$ such that the sequence $(\|\nabla(f-f_n)\|_p)$ converges to zero. Let (F_n) be a Følner sequence as in Theorem 5.2.4. By a standard regularization argument, we can construct for every n, a smooth 1-Lipschitz function φ_n such that

– $0 \leq \varphi_n \leq 1$;
– for every $x \in F_n$, $\varphi_n(x) = 1$;
– for every y at distance larger than 2 from F_n, $\varphi_n(y) = 0$.

Denote by $F'_n = \{x \in G : \ d(x, F_n) \leq 2\}$. By Theorem 5.2.4, there exists a constant $C < \infty$ such that

$$\mu(F'_n \smallsetminus F_n) \leq C\mu(F'_n)/n$$

and

$$F'_n \subset B(1, Cn).$$

Define

$$p_n = \frac{\varphi_n}{\int_G \varphi_n d\mu}.$$

Note that p_n is a probability density satisfying for every $x \in X$,

$$|\nabla p_n(x)| \leq \frac{1}{\mu(F_n)}.$$

For every $f \in D_p(G)$, write $P_n f(x) = \int_X f(y)p_n(y^{-1}x)d\mu(y)$. As G is unimodular,

$$P_n f(x) = \int_X f(yx^{-1})p_n(y^{-1})d\mu(y).$$

We claim that $P_n f - f$ is in $W^{1,p}$. Note that by triangular inequality, for every $g \in G$ and every $f \in D_p$, we have

$$\|f - \rho(g)f\|_p \leq |g|\|\nabla f\|_p$$

where $|g| = d(1, g)$ and where ρ is defined by $\rho(g)f(x) = f(xg)$. Recall that the support of p_n is included in F'_n which itself is included in $B(1, Cn)$. Thus, integrating the above inequality, we get

$$\|f - P_n f\|_p \leq Cn\|\nabla f\|_p,$$

so $f - P_n f \in L^p(G)$.

It remains to show that the sequence $(\|\nabla P_n f\|_p)$ converges to zero. We have

$$\nabla P_n f(x) = \int_G f(y)\nabla p_n(y^{-1}x)d\mu(y)$$

Since $\int_G \nabla p d\mu = 0$, we get

$$\begin{aligned}\nabla P_n f(x) &= \int_G (f(y) - f(x^{-1}))\nabla p_n(y^{-1}x)d\mu(y) \\ &= \int_G (f(yx^{-1}) - f(x^{-1}))\nabla p_n(y^{-1})d\mu(y).\end{aligned}$$

Hence,

$$\begin{aligned}\|\nabla P_n f\|_p &\leq \int_G \|\lambda(y)f - f\|_p |\nabla p_n(y^{-1})| d\mu(y) \\ &\leq \frac{1}{\mu(F_n)} \int_{F_n' \smallsetminus F_n} \|\lambda(y)f - f\|_p d\mu(y) \\ &\leq \frac{\mu(F_n' \smallsetminus F_n)}{F_n} \sup_{|g| \leq Cn} \|b(g)\|_p \\ &\leq \frac{C}{n} \sup_{|g| \leq Cn} \|b(g)\|_p\end{aligned}$$

where $b(g) = \lambda(g)f - f$. Note that $b \in Z^1(G, \lambda_{G,p})$. Thus, by Theorem 5.2.1,

$$\|\nabla P_n f\|_p \to 0.$$

This completes the proof of Corollary 20. ■

Deuxième partie

Isopérimétrie dans les espaces métriques mesurés et les groupes localement compacts

Chapitre 6

Inegalités de Sobolev à grande échelle sur les espaces métriques mesurés

Résumé

We introduce different notions of "large-scale" gradient of a Lipschitz function defined on a metric measure space. We then prove the invariance under large-scale equivalence (maps that generalize the quasi-isometries) of Sobolev inequalities. Moreover, we provide a criterion on the space allowing to obtain such an inequality at a given scale when it holds at large scale. We extend to this very general setting the well-known relation between the large time on-diagonal behavior of random walks and Sobolev inequalities. Our main application of this new approach is a very general characterization of the existence of a spectral gap on a quasi-transitive metric measure space X, providing a natural point of view to understand this phenomenon. As another application concerning locally compact groups, we prove that the L^p-isoperimetric profile is asymptotically smaller on a closed unimodular subgroup than on the group itself.

6.1 Introduction

We introduce a notion of "gradient at a certain scale" of a function defined on a general metric measure space. We then give a meaning to the notion of "large-scale" Sobolev inequalities for metric measure spaces and we show their invariance under large-scale equivalence. Moreover, we show that under some controlled connectivity assumption, the large scale Sobolev inequalities are equivalent to Sobolev and inequalities at positive any scale. We also study the relations between our notion of gradient at given scale and the well-known infinitesimal notion of generalized upper-gradient. We generalize some stability results [CouSa1] for Sobolev inequalities from the contexts of Riemannian manifolds and of weighted graphs to our general context. The improvement of our point of view is to get rid of any condition at small scale since it is rubbed out by the definition of the large-scale gradient, and to work in possibly non-geodesic spaces. This latter generality may be really useful, for instance when one has to deal with subspaces, which are not quasi-geodesic in general. This level of generality is also necessary for the study of non-compactly generated locally compact groups. This functional analysis approach generalizes the purely geometric notion of large-scale isoperimetry that we introduced in [Tes1].

As an application, we extend to this setting the well-known relation between the large time on-diagonal behavior of random walks and Sobolev inequalities. This enables us to prove that a reversible random walk on a quasi-transitive metric measure space has spectral radius equal to 1 if and only if the group acting is amenable and unimodular. This provides a general explanation for particular cases[1] treated in [Kest, Bro, Salv, SoW, Pit2, SW].

We prove a general statement that generalizes the monotonicity [Er] of the isoperimetric profile on finitely generated groups when passing to a subgroup. In particular, our statement applies to unimodular closed subgroups of locally compact groups.

6.1.1 Functional analysis at a given scale

Modulus of gradient at scale h.

Let (X, d) be a metric space. The purpose of this section is to define a notion of (modulus of) gradient at a scale h, where h is some positive number. The first naive idea to do so is to define

$$|\nabla f|_h(x) = \sup_{y \in B(x,h)} |f(y) - f(x)|$$

for any Lipschitz function $f : X \to \mathbf{R}$, $B(x, h)$ denoting the closed ball of center x and radius h. Note that this can be written in the following form :

$$|\nabla f|_h(x) = \|f - f(x)\|_{\infty, B(x,h)}$$

[1]Note that some of the results of these articles are more precise than ours and in a sense, more general when they manage to deal with non-reversible random walk.

which emphasizes the fact that we actually consider a "local" L^∞-norm. So we naturally generalize this and define a local L^p-norm for every $1 \leq p \leq \infty$. For this, we obviously need some measure. What we could do is start from a measure on X and define a local L^p-norm as the L^p norm restricted to balls with respect to this measure. However, when we consider a random process on X, the notion of local L^2-norm that naturally emerges is the L^2-norm with respect to the probability transition. This motivates the following definition.

Let (X, d, μ) be a metric measure space. Consider a family $P = (P_x)_{x\in X}$ of probability measures on X. Then for every $p \in [1, \infty]$, we define an operator $|\nabla|_{P,p}$ on Lip(X) by

$$\forall f \in \mathrm{Lip}(X), \quad |\nabla f|_{P,p}(x) = \|f - f(x)\|_{P_x,p} = \left(\int |f(y) - f(x)|^p dP_x(y)\right)^{1/p},$$

if $p < \infty$; and for $p = \infty$, we decide that

$$|\nabla f|_{P,\infty}(x) = \|f - f(x)\|_{P_x,\infty} = \sup\{|f(y) - f(x)|,\ y \in Supp(P_x)\}.$$

Definition 6.1.1. A family of probabilities $P = (P_x)_{x\in X}$ on X is called a viewpoint at scale $h > 0$ on X if there exist a large constant $1 \leq A < \infty$ and a small constant $c > 0$ such that for (μ-almost) every $x \in X$:
- $P_x \ll \mu$;
- $p_x = dP_x/d\mu$ is supported in $B(x, Ah)$;
- p_x is larger than $c/V(x, h)$ on $B(x, h)$.

Remark 6.1.2. Note that if X is doubling at any scale, then a viewpoint at scale h is also a viewpoint at scale h' for any $h' < h$.

Example 6.1.3. A basic example of viewpoint at scale h is given by

$$P_x = \frac{1}{V(x,h)} 1_{B(x,h)}, \quad \forall x \in X.$$

We denote the associated L^p-gradient by $|\nabla|_{h,p}$. Note that with the notation of the beginning of the §,

$$|\nabla|_h = |\nabla|_{h,\infty}.$$

Remark 6.1.4. A viewpoint at scale h has at least two interesting interpretations : one as an operator transition of a random walk on X ; the other as a Markov operator acting on $L^p(X)$ for every $p \geq 1$. This operator is defined by

$$Pf(x) = \int_X f(y) dP_x(y).$$

Consequently, there is a natural semi-group structure on the set of viewpoints at scale h on space X. Indeed, it is straightforward to check[2] that if P is a viewpoint at scale h and Q is a viewpoint at scale h', then $P \circ Q$ is a viewpoint at any scale $h" < h + h'$.

[2]One has to suppose that the space is doubling at any scale : see Definition 6.1.18.

Remark 6.1.5. Let us indicate another way of describing the objects that we introduced. Instead of directly defining a *local norm of the gradient at scale* h, we could first define a true *gradient at scale* h on a fiber space over X and then take a local norm of the gradient on the fibers. Here the fiber space would be $Y_h = \{(x,y) \in X^2, d(x,y) \leq h\}$ with projection $\pi : Y \to X$ on the first factor, so that $\pi^{-1}(x) = B(x,h)$. The gradient at scale h of f is then $\nabla_h f(x,y) = f(x) - f(y)$, where $(x,y) \in Y_h$. A viewpoint at scale h on X is now a probability measure on every fiber of some Y_{Ah} for A large enough ; and the L^p-gradient of f associated to such a viewpoint corresponds to the L^p-norm of f in every fiber with respect to this measure[3].

Remark 6.1.6. We can also define a Laplacian w.r.t. a viewpoint $P = (P_x)_{x\in X}$ by

$$\Delta_P f(x) = (P - id)f(x),$$

and more generally a p-Laplacian for any $p > 1$ by

$$\Delta_{P,p} f(x) = \int |f - f(x)|^{p-2}(f - f(x))dP_x.$$

If P is self-adjoint with respect to the scalar product associated to μ, then we have the usual relations

$$\langle \Delta_{P,p} f, g\rangle = \int \left(\int |f(y) - f(x)|^{p-2}(f(y) - f(x))(g - g(x))dP_x(y) \right) d\mu(x),$$

$$\langle \Delta_{P,p} f, g\rangle = \int |\nabla f|^p_{P,p} d\mu = \int \int |f(y) - f(x)|^p p_x(y)d\mu(y)d\mu(x),$$

and in particular, for $p = 2$,

$$\langle \Delta_P f, g\rangle = \int \int (f(y) - f(x))(g(y) - g(x))p_x(y)d\mu(x)d\mu(y).$$

Sobolev inequalities at scale h.

Let $\varphi : \mathbf{R}_+ \to \mathbf{R}_+$ be an increasing function and let $p \in [1, \infty]$. The following formulation of Sobolev inequality was first introduced in [Cou2].

Definition 6.1.7. One says that X satisfies a Sobolev inequality (S^p_φ) at scale (at least) $h > 0$ if there exists some finite positive constants C, C' depending only on h, p and φ such that

$$\|f\|_p \leq C\varphi(C'|\Omega|)\||\nabla f|_h\|_p$$

where Ω ranges over all compact subsets of X, $|\Omega|$ denotes the measure $\mu(\Omega)$, and $f \in \mathrm{Lip}(\Omega)$, $\mathrm{Lip}(\Omega)$ being the set of Lipschitz functions in X with support in Ω.

[3] Note that we can also define the gradient of f without referring to the scale : $\nabla f : X \times X \to \mathbf{R}$, $\nabla f(x,y) = f(x) - f(y)$, looking at $X \times X$ as a fiber space over the first factor. Then the scale appears when choosing a norm on every fiber

Definition 6.1.8. We say that X satisfies a large-scale Sobolev inequality (S^p_φ) if it satisfies (S^p_φ) at some scale h (equivalently, for h large enough).

Crucial remark 6.1.9. Note that we defined those inequalities with $|\nabla|_h$ whereas we could have defined them with $|\nabla|_{P,q}$ for any viewpoint $(P_x)_{x\in X}$ at scale h and any $q \geq 1$. A crucial fact that we prove in §6.2 is that satisfying a *large-scale* Sobolev inequality **does not** depend on this choice. Trivial implications are given by Proposition 6.2.1 and the other ones by Proposition 6.2.2.

Remark 6.1.10. Note that for large scale Sobolev inequalities, only Ω with large volume are involved. In fact, we will only be interested in the asymptotic behavior of φ. This motivates the following notations. Let $\varphi, \psi : \mathbf{R}_+ \to \mathbf{R}_+^*$ be nondecreasing strictly positive functions. We write respectively $\varphi \preceq \psi$, $\varphi \prec \psi$ if there exists $C > 0$ such that $\varphi(t) = O(\psi(Ct))$, resp. $\varphi(t) = o(\psi(Ct))$ when $t \to \infty$. We write $\varphi \approx \psi$ if both $\varphi \preceq \psi$ and $\psi \preceq \varphi$. The asymptotic behavior of φ is its class modulo the equivalence relation $\approx$.

Remark 6.1.11. It is easy to prove that (S^p_φ) implies (S^q_φ) whenever $p \leq q < \infty$ for any choice of gradient (see [Cou4] for a proof in the Riemannian setting). It is proved in [CL] that the converse is false for general Riemannian manifolds. This is still an open question for groups, although it is likely to be true in this case.

Link with Sobolev inequalities for infinitesimal gradients

Other notions of “modulus of gradient” have been introduced and studied for general metric spaces. In particular the notion of upper gradient plays a crucial role in the study of doubling metric spaces equipped with the Hausdorff measure (see for instance [Hei]). Such spaces naturally occur as boundaries of Gromov-hyperbolic spaces and are often studied up to quasi-conformal maps. Such a point of view is quite different from ours since it focuses on the local properties of the space, which is often supposed compact. However, it is natural to ask when a Sobolev inequality at large scale is equivalent to the same Sobolev inequality w.r.t. some upper gradient. In particular, given a Riemannian manifold, is it true that it satisfies a Sobolev inequality at large scale if and only if it satisfies it for its usual gradient ? The answer is no if for instance the Riemannian manifold contains a sequence of open submanifolds isometric to open half-spheres of radius going to zero. A sufficient condition for obtaining a positive answer is to ask for a local Poincaré inequality (see Proposition 6.6.5).

Note that different strategies have been used to ignore the local geometric properties of a manifold. In [ChFel] for instance, they avoid the local behavior of the isoperimetric profile on a manifold by restricting it to subsets containing a geodesic ball of fixed radius. In [Cou1], they consider Nash inequalities restricted to functions convoluted by the heat kernel at time 1 and obtain in this way the invariance under quasi-isometries of certain upper bounds of the on-diagonal behaviour of the heat kernel : this idea is quite closed to ours (see Remark 6.6.9). This issues are discussed in Sections 6.6.3 and 6.6.2. Among other things, we prove under a very weak property of bounded geometry that a manifold satisfies a Sobolev inequality at large scale if and only if it satisfies it for the usual gradient in restriction to functions of the form $g = Pf$, where P is the Markov operator associated to any viewpoint at some scale $h > 0$.

6.1.2 Sobolev inequalities (S^p_φ) at scale h for $p = 1, 2, \infty$

Now let us give characterizations of (S^p_φ) at given scales for some important values of $p = 1, 2, \infty$ (see [Cou4] for the classical setting of manifolds).

Geometric interpretations of (S^p_φ) at scale h for $p = 1, \infty$

In [Cou1] (see also [Cou4, proposition 22]), it is proved that (S^∞_φ) can only hold if φ is unbounded and then is equivalent to the volume lower bound

$$V(x, r) \geq \varphi^{-1}(r)$$

where $\varphi^{-1}(r) = \{v, \varphi(v) \geq r\}$, for every $x \in X$ and every $r > 0$. The original proof works formally in our setting.

Proposition 6.1.12. Let $(X, d\mu)$ be a metric measure space. The Sobolev inequality (S^∞_φ) at scale h can only hold if φ is unbounded and then is equivalent to the volume lower bound

$$V(x, r) \geq \varphi^{-1}(r)$$

for $r \geq h$. ■

The inequality (S^1_φ) at scale h is equivalent to the isoperimetric inequality (at scale h)

$$\frac{|\partial_h \Omega|}{|\Omega|} \geq \frac{1}{C\varphi(C'|\Omega|)}$$

where the boundary of A is defined by

$$\partial_h A = [A]_h \cap [A^c]_h$$

with the usual notation $[A]_h = \{x \in X,\ d(x, A) \leq h\}$. The usual proof of this equivalence (see [Cou4]) works formally in our context, using the following version of the co-area formula :

$$\frac{1}{2}\int_{\mathbf{R}_+} \mu\left(\partial_h\{f \geq t\}\right) dt \leq \int_X |\nabla f|_h(x) d\mu(x) \leq \int_{\mathbf{R}_+} \mu\left(\partial_h\{f \geq t\}\right) dt \qquad (6.1.1)$$

where f is a non-negative measurable function defined on X. Indeed, for every measurable subset $A \subset X$, we have

$$\mu(\partial_h A) = \int_X |\nabla 1_A|_h(x) d\mu(x).$$

Thus, (6.1.1) follows by integrating over X the following local inequalities

$$\frac{1}{2}\int_{\mathbf{R}_+} |\nabla 1_{\{f\geq t\}}|_h(x)dt \leq |\nabla f|_h(x) \leq \int_{\mathbf{R}_+} |\nabla 1_{\{f\geq t\}}|_h(x)dt, \tag{6.1.2}$$

for every $x \in X$. The right-hand inequality results from the fact that $f = \int_{\mathbf{R}_+} 1_{\{f\geq t\}}dt$ and from the sub-additivity of $|\nabla|_h$. To prove the left-hand, note that $|\nabla 1_{\{f\geq t\}}(x)|_h = 1$ if and only if

$$\inf_{B(x,h)} f < t \leq \sup_{B(x,h)} f$$

or

$$\inf_{B(x,h)} f \leq t < \sup_{B(x,h)} f;$$

Hence,

$$\int_{\mathbf{R}_+} |\nabla 1_{\{f\geq t\}}|_h(x)dt \leq \sup_{B(x,h)} f - \inf_{B(x,h)} f \leq 2|\nabla f|_h(x),$$

which proves (6.1.2).

Probabilistic interpretation of (S^2_φ) at scale h

The case $p = 2$ is of particular interest since it contains some probabilistic information on the space X. It is proved in [CG] that for manifolds with bounded geometry, upper bounds of the large-time on-diagonal behavior of the heat kernel are equivalent to some Sobolev inequality (S^2_φ). In [Cou3], a similar statement is proved for the standard random walk on a weighted graph. In § 6.5, we give a discrete-time version of this theorem in our general setting. The point is that the original proof of [Cou3, Theorem 7.2] is formal enough to be adapted to our setting. The proof of Theorem 6.1.16 below emphasizes the fact that the notion of symmetric viewpoint at scale h that we introduce below is likely to be the most natural way of capturing the link between large-scale geometry and the long-time behavior of random walks on X.

Definition 6.1.13. Let (X, d, μ) be a metric measure space and consider some $h > 0$. A view-point $P = (P_x)_{x\in X}$ at scale h on X is called symmetric if one of the following equivalent statement holds.

- The random walk whose probability of transition is P is reversible with respect to the measure μ.
- The associated operator on $L^2(X, \mu)$ defined by

$$Pf(x) = \int_X f(y)dP_x(y)$$

  is self-adjoint.
- For every a.e. $x, y \in X$, $p_x(y) = p_y(x)$.

Definition 6.1.14. We call a reversible random walk at scale h a random walk whose probability transition is a symmetric view-point at scale h.

Example 6.1.15. Let (X, d, μ) be a metric measure space. Consider the standard viewpoint at scale h of density $p_x = 1_{B(x,h)}/V(x, h)$ with respect to μ. In general, this is not a symmetric viewpoint, i.e. the random walk of probability transition $dP_x(y) = p_x(y)d\mu(y)$ is not reversible with respect to μ. However, it is reversible with respect to the measure μ' defined by

$$d\mu'(x) = V(x, h)d\mu(x).$$

It is easy to check that if (X, d, μ) is doubling at any scale, then so is (X, d, μ'). Moreover, if $x \mapsto V(x, h)$ is bounded from above and from below, then P defines a symmetric viewpoint on (X, d, μ').

Theorem 6.1.16. Let $X = (X, d, \mu)$ be a metric measure space and let $P = (P_x)_{x\in X}$ be a symmetric view-point at scale h on X. Let φ be some increasing positive function. Define γ by

$$t = \int_0^{1/\gamma(t)} (\varphi(v))^2 \frac{dv}{v}.$$

(i) Assume that X satisfies a Sobolev inequality (S^2_φ) w.r.t. $|\nabla|_{P^2,2}$. Then

$$p_x^{2n}(x) \leq \gamma(cn) \quad \forall n \in \mathbf{N}, a.e \forall x \in X,$$

for some constant $c > 0$.

(ii) If γ satisfies a numerical condition (δ) (see [Cou3, p 18]) and if

$$p_x^{2n}(x) \leq \gamma(n) \quad \forall n \in \mathbf{N}, a.e \forall x \in X,$$

then X satisfies (S^2_φ) w.r.t. $|\nabla|_{P,2}$.

6.1.3 Sobolev and isoperimetry at scale h

Isoperimetric profile at scale h

Generalizing the case $p = 1$, Sobolev inequalities (S^p_φ) can be also understood as L^p-isoperimetric inequalities. Let A be a measurable subset of X. For every $p > 0$, define

$$J_p(A) = \sup_f \frac{\|f\|_p}{\||\nabla f|_h\|_p}$$

where the supremum is taken over functions $f \in \mathrm{Lip}(A)$. Now, taking the supremum over subsets A with measure less than $m > 0$, we get an increasing function $j_{X,p}$ sometimes called the L^p-isoperimetric profile. Note that the terminology "isoperimetric profile" is somewhat ambiguous since there exist various nonequivalent definitions (see in particular [CouSa1, Chapter 1]). One of them is

$$j_X(m) = \sup_{|A|\leq m} \frac{|A|}{|\partial_h A|}$$

which satisfies

$$j_X \approx j_{X,1},$$

taking the same h in the definition of the gradient and in the definition of the boundary. Clearly, the space X always satisfies the Sobolev inequality (S^p_φ) with $\varphi = j_{X,p}$. Conversely, if X satisfies (S^p_φ) for a function φ, then

$$j_{X,p} \succeq \varphi.$$

Consequently, we have

$$j_{X,p} \preceq j_{X,q}$$

whenever $p \leq q < \infty$ (see Remark 6.1.11 about Sobolev inequalities).

Isoperimetric profile inside balls

Definition 6.1.17. Let us fix a gradient at scale h on X. L^p-isoperimetric profile inside balls is the nondecreasing function $J^b_{G,p}$ defined by

$$J^b_{X,p}(t) = \sup_{x\in X} J_p(B(x,t)).$$

Note that $J^b_{X,p}(t)$ is the supremum of $J_p(A)$ over subsets A of diameter[4] less than t. The L^p-isoperimetric profile inside balls plays a crucial role in the study of uniform embeddings of amenable groups into L^p-spaces (see [Tes2]). It is also central in the proof [Tes3] that a closed at infinity, homogenous manifold does not carry any non-constant p-harmonic function with gradient in L^p.

Connection with the large-scale isoperimetry introduced in [Tes1]

One can also define another kind of isoperimetric profile at scale h :

$$I(t) = \inf_{\mu(A)\geq t} \mu(\partial_h A)$$

which can be specialized on a family of (measurable) subsets of finite volume $\mathcal{A}$: we call lower (resp. upper) profile at scale h restricted to $\mathcal{A}$ the nondecreasing function $I^{\downarrow}_{\mathcal{A}}$ defined by

$$I^{\downarrow}_{\mathcal{A}}(t) = \inf_{\mu(A)\geq t, A\in\mathcal{A}} \mu(\partial_h A)$$

(resp. $I^{\uparrow}_{\mathcal{A}}(t) = \sup_{\mu(A)\leq t, A\in\mathcal{A}} \mu(\partial_h A)$). We can then study the large scale isoperimetric properties of a family $\mathcal{A}$ considering the asymptotic behavior of these two increasing functions [Tes1]. In [Tes1], we used this variant to investigate the question : are balls always asymptotically isoperimetric in a metric measure space with doubling property ? For that purpose, we introduced a general setting adapted to the study of asymptotic isoperimetry on metric measure spaces. An

[4]This profile is associated to another kind of Sobolev inequalities, where the function φ of the volume is replaced by a function Φ of the diameter.

important consequence of the geometric interpretation of Sobolev inequalities in L^1 (see § 6.1.2) is that every geometric notion that we introduced in [Tes1, §3] appears as a particular case of the functional point of view adopted in the present paper. In particular, [Tes1, Theorem 3.10] that implied the invariance under large-scale equivalence of isoperimetric properties is now covered by the lemmas of § 6.3.3. Moreover, we choose here to treat separately the large-scale setting, where no connectivity hypotheses are required on the spaces, and the control on the scale that really *depends* on a connectivity assumption (see § 6.6).

6.1.4 Large-scale equivalence between metric measure spaces

In this section, we define the *objects* and the *isomorphisms* of the category of *metric measure spaces at large scale* that we will consider in this paper. The objects are metric measure spaces with a very weak property of bounded geometry.

Definition 6.1.18. We say[5] that a space X is doubling at scale $r > 0$ if there exists a constant C_r such that

$$\forall x \in X, \quad V(x, 2r) \leq C_r V(x, r)$$

where $V(x, r) = \mu(B(x, r))$.

Crucial remark 6.1.19. Since the constant C_r depends on r, the doubling property at any scale has *absolutely no* influence on the volume growth. In particular, one should be careful to distinguish it from the well-known *doubling property* stating that there exists a constant $C < \infty$ (not depending on the radius) such that $V(x, 2r) \leq CV(x, r)$ for all $x \in X$ and $r > 0$. Contrary to the doubling property at any scale, the doubling property implies polynomial growth, i.e. that there exists a constant $D < \infty$ such that $V(x, r) \leq r^D V(x, 1)$ for every $x \in X$ and $r \geq 1$.

For most of the results proved in this paper[6], we only use the doubling property at scale $r \geq h/2$, if the gradient considered is at scale h. However, to simplify the exposition, we will always assume that the space is doubling at any positive scale.

Clearly, doubling at scale r for every $r > 0$ is a very weak property of controlled geometry : for instance, every graph with bounded degree, equipped with the counting measure is doubling at any scale. Other examples are Riemannian manifolds with Ricci curvature bounded from below. Assume that the volume of balls of fixed radius is bounded from above and from below by constants depending on r. Then one can check easily that X is doubling at any scale. It is important to note that the doubling property at any scale is strictly weaker than this property. One can easily construct weighted graphs or Riemannian

[5] In [CouSa1] and in [Tes1], the doubling property at any scale r is denoted $(DV)_{loc}$ property.

[6] In fact all the results except the few ones where the infinitesimal structure of the space is clearly involved.

manifolds which are doubling at any scale but with unbounded volume for balls of radius 1.

Example 6.1.20. Let X be a connected graph with degree bounded by d, equipped with the counting measure. The volume of balls of radius r satisfies

$$\forall x \in X, \quad 1 \leq V(x,r) \leq d^r.$$

In particular, X is doubling at any scale.

The isomorphisms are maps that we call *large-scale equivalences*.

Definition 6.1.21. Let (X, d, μ) and (X', d', μ) two spaces satisfying the doubling property at any scale. Let us say that X and X' are large-scale equivalent if there is a function F from X to X' with the following properties

(a) for every sequence of pairs $(x_n, y_n) \in (X^2)^{\mathbb{N}}$

$$(d(F(x_n), F(y_n)) \to \infty) \Leftrightarrow (d(x_n, y_n) \to \infty)\,.$$

(b) F is almost onto, i.e. there exists a constant C such that $[F(X)]_C = X'$.

(c) For $r > 0$ large enough, there is a constant $C_r > 0$ such that for all $x \in X$

$$C_r^{-1} V(x,r) \leq V(F(x), r) \leq C_r V(x,r).$$

Crucial remark 6.1.22. Note that being large-scale equivalent is an equivalence relation between metric measure spaces with doubling property at any scale.

Remark 6.1.23. If X and X' are quasi-geodesic, then (a) and (b) imply that F is roughly bi-Lipschitz : there exists $C \geq 1$ such that

$$C^{-1} d(x,y) - C \leq d(F(x), F(y)) \leq C d(x,y) + C.$$

This is very easy and left to the reader. In this case, (a) and (b) correspond to the classical definition of a *quasi-isometry*.

Example 6.1.24. Consider the subclass of metric measure spaces including graphs with bounded degree, equipped with the countable measure ; Riemannian manifolds with Ricci curvature bounded from below and sectional curvature bounded from above, equipped with the Riemannian measure. In this class, quasi-isometries are always large-scale equivalences.

6.1.5 Examples

Discretization

Recall that a weighted graph is a connected graph X equipped with a structure of metric measure space on the set of its vertices, the distance being the usual geodesic one. Similarly, a weighted manifold is a Riemannian manifold equipped with a measure $d\mu$ absolutely continuous with respect to the Riemannian measure. A discretization [Gro2, Kan] of a weighted Riemannian manifold

X can be defined as a weighted graph large-scale equivalent to X. More generally, a discretization of a metric measure space is a weighted graph large-scale equivalent to X.

Consider some $b > 0$ and define a roughly geodesic distance on X by setting

$$d_b(x, y) = \inf_{\gamma} l(\gamma)$$

where γ runs over every chains $x = x_0 \dots x_m = y$ and where $l(\gamma) = \sum_{i=1}^{m} d(x_i, x_{i-1})$ is the length of γ.

Definition 6.1.25. We say that X is metrically proper (resp. quasi-geodesic) if there exists $b > 0$ such that the identity map $(X, d_b) \to (X, d)$ is a uniform embedding (resp. a quasi-isometry onto its image).

Definition 6.1.26. [Tes1] Let $X = (X, d)$ be a metric space and fix some $b > 0$. We call a b-chain between two points $x, y \in X$ a chain $x = x_1 \dots x_m = y$ such that for every $1 \leq i < m$, $d(x_i, x_{i+1}) \leq b$. Let us say that X is uniformly b-connected if every $x, y \in X$ can be connected by a b-chain whose length m only depends on $d(x, y)$. We say that it is large-scale uniformly connected if it there exists $b > 0$ such that it is uniformly b-connected.

Clearly, being metrically proper or large-scale uniformly connected are preserved by large-scale equivalence. Note that a quasi-geodesic metric space is both metrically proper and large-scale uniformly connected ; so are graphs and Riemannian manifolds.

Proposition 6.1.27. A metric measure space with Doubling Property at any scale admits a discretization if and only if it is metrically proper and large-scale uniformly connected. Moreover X is quasi-isometric to a graph if and only if it is quasi-geodesic.

Proof. Assume that $X = (X, d, \mu)$ is metrically proper, large-scale uniformly connected and doubling at any scale. Consider a minimal covering of X with balls of radius h. We construct a weighted graph $G(X)$ as follows ; the vertices of $G(X)$ are the centers of the balls ; we put an edge between two vertices if the balls intersect. By large-scale uniform connectedness, $G(X)$ is connected as soon as h is large enough. Moreover, large-scale uniform properness and doubling Property at any scale clearly imply that the injection map $G(X) \hookrightarrow X$ is a large-scale equivalence. The converse is obvious. ■

Locally compact groups

Let G be a group. Recall that a length function on G is function $L : G \to \mathbf{R}_+$ such that $L(1) = 0$ and

$$\forall g, h \in G, \quad L(gh) \leq L(g) + L(h).$$

If L is a length function, then $d(g, h) = L(g^{-1}h)$ defines a left-invariant pseudo-metric on G. Conversely, if d is a left-invariant pseudo-metric on G, then $L(g) = d(1, g)$ defines a length function on G.

Definition 6.1.28. Let G be a locally compact group. A metric d on G is called uniform if for any of sequence $(g_n, h_n) \in (G \times G)^{\mathbf{N}}$, $d(g_n, h_n) \to \infty$ if and only if $g_n^{-1}h_n$ leaves every compact eventually.

It is well-known that G admits uniform left-invariant metrics if and only if G is second countable. The following proposition is straightforward and left to the reader.

Proposition 6.1.29. Let d and d' be two uniform metrics on G. The spaces (G, d) and (G, d') are doubling at any (large enough) scale and the identity map $(G, d) \to (G, d')$ is a large scale equivalence. ■

Definition 6.1.30. Let G be a second countable locally compact group. The asymptotic class of a metric d is the set of metrics d' on G such that the identity map $(G, d) \to (G, d')$ is a quasi-isometry.

Remark 6.1.31. Note that the set of uniform quasi-geodesic metrics on a locally compact group forms a (possibly empty) asymptotic class.

Proposition 6.1.32. Let G be a locally compact group. The following statements are equivalent.

(i) G admits a uniform, large-scale uniformly connected metric ;
(ii) G admits a uniform quasi-geodesic metric ;
(iii) G admits a left-invariant quasi-geodesic metric ;
(iv) G is quasi-isometric to a graph with bounded degree ;
(v) G is compactly generated.

Proof : Clearly, $(iii) \Rightarrow (ii) \Rightarrow (i)$ are obvious, $(iii) \Leftrightarrow (iv)$ results from Proposition 6.1.27. Let us prove that $(v) \Rightarrow (iv)$. Assume that G is compactly generated and let S be a compact symmetric subset S. One can equip G with a uniform quasi-geodesic length function setting

$$\forall g \in G, \quad |g|_S = \inf\{n \in \mathbf{N}, g \in S^n\}.$$

Now, let us prove that $(i) \Rightarrow (v)$. Suppose that G has a uniform, large-scale uniformly connected metric d with constant C. Since d is uniform, there exists $R < \infty$ such that for all $g \in G$, the closed ball $B(g, C)$ is compact and contained in $g \cdot B(1, R)$.

We claim that G is generated by $B(1, R)$. Fix $g \in G$. Indeed, let $g_1 = 1, \ldots, g_n = g$ be a chain such that $d(g_i, g_{i+1}) \leq C$ for every $1 \leq i \leq n-1$. We have $g_{i+1} \in B(g_i, C) \subset g_i \cdot B(1, R)$. Hence, an immediate induction shows that $g \in B(1, R)^n$ and we are done. ■

6.2 Equivalence of Sobolev inequalities with respect to different gradients

Here, we show that *large-scale* Sobolev inequalities do not really depend on the kind of gradient that we use to write them. In spite of its easy and short proof, this result is crucial for our purpose since it shows that our definitions are natural.

The following proposition results immediately from the definitions.

Proposition 6.2.1. If $h' \geq h > 0$, then

$$\| |\nabla f|_{h'} \|_p \geq \| |\nabla f|_h \|_p.$$

Moreover, if P is a viewpoint at scale h with constants c and A (see the definition below) and if $q \leq q' \leq \infty$, then

$$c|\nabla f|_{h,q} \leq |\nabla f|_{P,q} \leq C|\nabla f|_{P,q'} \leq Ch|\nabla f|_{Ah} \quad \forall f \in \mathrm{Lip}(X)$$

where C is a constant[7] depending on h' and h. ■

The non-trivial comparisons between different gradient are summarized in the following proposition.

Proposition 6.2.2. Let X be some metric measure space satisfying a Sobolev inequality (S^p_φ) at scale h. Then, for any viewpoint $P = (P_x)_{x \in X}$ at scale $2h$, X satisfies (S^p_φ) w.r.t. $|\nabla|_{P,q}$ for any $q \geq 1$.

Proof : By Proposition 6.2.1, it suffices to prove that X satisfies (S^p_φ) w.r.t. $|\nabla|_{2h,1}$. Write

$$P_x = \frac{1}{V(x,h)} 1_{B(x,h)} \quad \forall x \in X.$$

For every $f \in \mathrm{Lip}(X)$ we write

$$Pf(x) = \int f dP_x, \quad \forall x \in X.$$

Lemma 6.2.3. There exists $C < \infty$ such that

$$|\nabla Pf|_h(x) \leq C|\nabla f|_{h,1}(x) \quad \forall f \in \mathrm{Lip}(X), \forall x \in X.$$

Proof : Consider some $y \in B(x,h)$.

$$|Pf(x) - Pf(y)| \leq |Pf(x) - f(x)| + |Pf(y) - f(x)| \leq C|\nabla|_{2h,1} f(x).$$

with $C < \infty$ depending only on the doubling constant at scale h. ■

Now apply the Sobolev inequality (S^p_φ) at scale h to Pf,

$$\| |\nabla Pf|_h \|_p \geq \varphi^{-1}(\Omega) \|Pf\|_p \geq \varphi^{-1}(\Omega) \|f\|_p - \varphi^{-1}(\Omega) | \|f\|_p - \|Pf\|_p |.$$

Now, if $\| |\nabla f|_{h,1} \|_p \geq \|f\|_p/2$, there is nothing to prove. Hence, assuming the contrary, and since $| \|f\|_p - \|Pf\|_p | \leq \| |\nabla f|_{h,1} \|_p$, we obtain

$$\| |\nabla Pf|_h \|_p \geq \varphi^{-1}(\Omega) \|f\|_p / 2,$$

which yields

$$\| |\nabla f|_{h,1} \|_p \leq C^{-1} \varphi^{-1}(\Omega) \|f\|_p / 2$$

thanks to the lemma. ■

[7] It comes from the doubling property at any scale.

6.3 Invariance of Sobolev inequalities under large-scale equivalence

The aim of this section is to prove the following theorem.

Theorem 6.3.1. Let $F : X \to X'$ be a large-scale equivalence between two spaces X and X' satisfying the doubling property at any scale. Assume that for $h > 0$ fixed, the space X satisfies a Sobolev inequality (S^p_φ) at scale h, then there exists h', only depending on h and on the constants of F such that X' satisfies (S^p_φ) at scale h'. In particular, large-scale Sobolev inequalities are invariant under large scale equivalence.

For that purpose, we will first prove some preliminary results.

6.3.1 Thick subsets

Definition 6.3.2. A subset A of a metric space is called h-thick if it is a reunion of closed balls of radius h.

Denote $\mathrm{Lip}_0(X)$ the set of Lipschitz functions in X with compact support. Roughly speaking, the following proposition says that we can focus on functions with thick support.

Proposition 6.3.3. Let $X = (X, d, \mu)$ be a metric measure space. Fix some $h > 0$ and some $p \in [1, \infty]$. There exists a constant $C > 0$ such that for any $f \in \mathrm{Lip}_0(X)$, there is a function $\tilde{f} \in \mathrm{Lip}_0(X)$ whose support is included in a $h/2$-thick subset Ω such that

$$\mu(\Omega) \leq \mu(Supp(f)) + C$$

and for every $p \in [1, \infty]$,

$$\frac{\||\nabla \tilde{f}|_{h/2}\|_p}{\|\tilde{f}\|_p} \leq C \frac{\||\nabla f|_h\|_p}{\|f\|_p}.$$

Proof : Let us prove the proposition for $p < \infty$. Let $f \in \mathrm{Lip}_0(X)$ be such that $\|f\|_p = 1$. Assume that f satisfies

$$\||\nabla|_h f\|_p \geq \frac{1}{2}.$$

Then, for $\tilde{f}$, consider for instance the indicator function of a ball $B(x, a)$ of volume 1 (so that $\|\tilde{f}\|_p = 1$). We have

$$\||\nabla \tilde{f}|_{h/2}\|_p^p \leq \mu(B(1 + h/2)) \leq C\mu(B(x, a)) = C.$$

Thus, let us assume that

$$\||\nabla f|_h\|_p \leq \frac{1}{2}.$$

Let Ω be the subset of $Supp(f)$ defined by

$$\Omega = \{x \in X, d(x, Supp(f)^c) \geq h/2\}$$

and set

$$\tilde{f} = f \cdot 1_\Omega.$$

Note that for every $x \in Supp(f) \setminus \Omega$, there exists some $y \in B(x,h)$ such that $f(y) = 0$. Therefore, we have $|f(x)| \leq |\nabla f|_h(x)$. Hence,

$$\int_X |\tilde{f}|^p d\mu \geq \int_X |f|^p d\mu - \int_X (|\nabla f|_h)^p d\mu \geq \frac{1}{2}.$$

On the other hand, let $x \in \Omega$. If $d(x, Supp(f)) \geq h$, then

$$|\nabla \tilde{f}|_{h/2} = |\nabla|_{h/2} f \leq |\nabla|_h f.$$

Otherwise,

$$|\nabla \tilde{f}|_{h/2} \leq \max \left\{ |f(x)|, \sup_{y \in B(x,h/2)} |f(x) - f(y)| \right\}$$

and

$$|\nabla f|_h = \sup_{y \in B(x,h)} |f(x) - f(y)| = \max \left\{ |f(x)|, \sup_{y \in B(x,h)} |f(x) - f(y)| \right\}.$$

Thus

$$|\nabla \tilde{f}|_{h/2} \leq |\nabla f|_h;$$

so we are done. ■

On the other hand, the doubling property at any scale "extends" to thick subsets in the following sense.

Proposition 6.3.4. Let X be a metric measure space satisfying the doubling property at any scale. Fix two positive numbers u and v. There exists a constant $C = C(u,v) < \infty$ such that for any u-thick subset $A \subset X$, we have

$$\mu([A]_v) \leq C\mu(A).$$

Proof : The proof follows from standard covering arguments. ■

6.3.2 Rough volume preserving property

Let us prove a useful *rough* volume preserving property of large scale equivalences.

Proposition 6.3.5. Let $X = (X, d, \mu)$ and $X' = (X', d', \mu')$ be two spaces satisfying the doubling property at any scale and let $F : X \to X'$ be a large-scale equivalence. Let $u > 0$, then there exists a constant $C = C(u, F)$ such that
(1) If $A \subset X$ and $A' \subset X'$ are such that $[F^{-1}(A')]_u \subset A$, then $\mu'(A') \leq C\mu(A)$.
(2) If $A \subset X$ and $A' \subset X'$ are such that $[F(A)]_u \subset A'$, then $\mu(A) \leq C\mu'(A')$.

Proof : Let us prove (1). Let Z be a maximal set of $2u$-separated points of $F^{-1}(A')$. Clearly, the balls $(B(z,u))_{z\in Z}$ are disjoint and included in A. On the other hand, maximality of Z implies that the family $(B(z,2u))_{z\in Z}$ forms a covering of A. So we have

$$\sum_{z\in Z} \mu(B(z,u)) \leq \mu(A) \leq \sum_{z\in Z} \mu(B(z,2u)) \tag{6.3.1}$$

By property (a) of a large-scale equivalence, there exists v such that for every $x \in X$, $F(B(x,2u)) \subset B(F(x),v)$. In particular, the family $((B(F(z),v))_{z\in Z}$ forms a covering of $F(A)$. Using Property (c) of a large-scale equivalence and Doubling Property at any scale of X together with (8.3.3), we get

$$\begin{aligned} \mu(A') \leq \mu'(F(A)) \leq \sum_{z\in Z} \mu'(B(F(z),v)) &\leq C' \sum_{z\in Z} \mu(B(z,v)) \\ &\leq C \sum_{z\in Z} \mu(B(z,u)) \leq C\mu(A) \end{aligned}$$

which proves the proposition. ■

6.3.3 Proof of the invariance under large-scale equivalence

Let $F : X \to X'$ be a large-scale equivalence between two spaces X and X' satisfying the doubling property at any scale. Assume that $f \in \mathrm{Lip}(X')$. For every $h > 0$, define a function on X

$$\forall x \in X, \quad \psi_h(x) = \sup_{y\in B(x,h)} |f \circ F(y)|.$$

Lemma 6.3.6. For h large enough, there exists a constant $c = c(h, f) > 0$ such that

$$\mu(\{\psi_h^p \geq t\}) \geq c\mu'(\{|f|^p \geq t\}).$$

In particular, for every $p > 0$,

$$\|\psi_h\|_p \geq c\|f\|_p.$$

Proof : We can obviously assume that $p = 1$ and that $f \geq 0$. Thanks to Proposition 6.3.5, we only have to check that

$$[F^{-1}(\{f \geq t\})]_h \subset \{\psi_h \geq t\}.$$

Indeed, let $x \in F^{-1}(\{f \geq t\})$. Then $f \circ F(x) \geq t$. So for all $y \in B(x,h)$, we have $\psi_h(y) \geq t$. ■

Lemma 6.3.7. For h' large enough, there exists a constant $C < \infty$ such that

$$\mu(\{(|\nabla\psi_h|_h)^q > t\}) \leq C\mu'(\{|(\nabla f|_{h'})^q > t/2\}).$$

In particular, for every $q > 0$,

$$\||\nabla\psi_h|_h\|_q \leq C\||\nabla f|_{h'}\|_q.$$

Proof : We can of course assume that $q = 1$. Thanks to Proposition 6.3.5, it suffices to prove that for h' large enough,

$$[F(\{|\nabla\psi_h|_h > t\})]_{h'/2} \subset \{|\nabla f|_{h'} > t/2\}.$$

Indeed, let $x \in X$ be such that $|\nabla\psi_h|_h(x) > t$. This means that there exists $y \in B(x, h)$ such that $|f \circ F(x) - f \circ F(y)| > t$. On the other hand, by property (a) of a large-scale equivalence, one can choose h' such that $d(F(x), F(y)) \leq h'/2$. Hence,

$$\forall z \in B(F(x), h'/2), \quad |\nabla f|_{h'}(z) \geq \max\{|f(x) - z|, |f(y) - z|\} \geq t/2.$$

So $z \in \{|\nabla f|_{h'} > t/2\}$. ■

Lemma 6.3.8. For u large enough, there exists a constant $C < \infty$ such that

$$\mu'\left(Supp(\psi_h)\right) \leq C\mu\left([Supp(f)]_u\right).$$

Proof : This follows trivially from Proposition 6.3.5.

Proof of Theorem 6.3.1 Let Ω be a compact subset of X' of measure m. We want to prove that every $f \in \mathrm{Lip}(\Omega)$ satisfies

$$\|f\|_p \leq C\varphi(Cm)\||\nabla f|_h\|_p$$

with h' and C depending only on F, h and X. Thanks to Proposition 6.3.3 and up to choose a larger h', we can assume that Ω is v-thick for any $v > 0$. Then, thanks to Lemma 6.3.8 and to Proposition 6.3.4, we have

$$Supp(\psi_h) \leq C'm$$

for some constant C'. So apply (S^p_φ) to ψ_h and then conclude thanks to Lemmas 6.3.6 and 8.3.2. ■

6.4 Large-scale foliation of a metric measure space

6.4.1 Monotonicity of the isoperimetric profile

Definition 6.4.1. Let $X = (X, d_X, \mu)$ and $Y = (Y, d_Y, \lambda)$ be two metric measure spaces satisfying the doubling property at any scale. We say that X is large-scale foliated by Y if it admits a measurable partition $X = \sqcup_{z\in Z} Y_z$ satisfying the following conditions.

(i) There exists a measure ν on Z and a measure λ_z on ν-almost every Y_z such that for every continuous compactly supported function f on X,

$$\int_X f(x)d\mu(x) = \int_Z \left(\int_{Y_z} f(t)d\lambda_z(t)\right) d\nu(z).$$

(ii) For ν-almost every z in Z, $Y_z = (Y_z, d_X, \mu)$ is large-scale equivalent to (Y, d_Y, λ) uniformly with respect to $z \in Z$.

(iii) We finally impose a normalization condition on the measures λ_z : for every (equivalently for one) radius $r > 0$, there exists a constant $1 \leq C < \infty$ such that for every $z \in Z$ and every $x \in Y_z$,

$$C^{-1}V_X(x,1) \leq V_{Y_z}(x,1) \leq CV_X(x,1).$$

Recall that the compression of a map F between two metric space X and Y is the function ρ defined by

$$\forall t > 0, \quad \rho(t) = \inf_{d_X(x,x')\geq t} d_Y(F(x), F(x')).$$

Definition 6.4.2. We call the compression of a large-scale foliation of X by Y the function

$$\rho(t) = \inf_{z\in Z} \rho_z(t)$$

where ρ_z is the compression function of the large-scale equivalence $Y \to Y_z$. Since these large-scale equivalences are uniform with respect to z, we actually have $\rho \approx \rho_z$ uniformly with respect to z.

A crucial example that we will consider in some details in the next § is the case when $Y = H$ is a closed subgroup of a locally compact group $G = X$ such that G/H carries a G-invariant measure. In [Er, Lemma 4], it is proved that if H is finitely generated subgroup of a finitely generated group G, then $j_H \preceq j_G$. Here is a generalization of this easy result.

Proposition 6.4.3. Let $X = (X, d, \mu)$ and $Y = (Y, \delta, \lambda)$ be two metric measure spaces satisfying the doubling property at any scale. Assume that X is large-scale foliated by Y. Then if Y satisfies a Sobolev inequality (S^p_φ) at scale h, then X satisfies (S^p_φ) at scale h' for h' large enough. In other words, if $j_{X,p}$ and $j_{Y,p}$ denote respectively the L^p-isoperimetric profiles of X and Y at scale h and h', then

$$j_{Y,p} \succeq j_{X,p}.$$

Moreover, if ρ is the compression of the large-scale equivalence, then

$$J^b_{Y,p} \succeq J^b_{X,p} \circ \rho.$$

Let us start with a lemma.

Lemma 6.4.4. For every $z \in Z$, let $[Y_z]_1$ be the 1-neighborhood of Y_z in X. The inclusion map $Y_z \to [Y_z]_1$ is a large-scale equivalence, uniformly w.r.t. z.

Proof of the lemma. The two metric conditions (a) and (b) for being a large-scale equivalence (see Definition 6.1.21) are trivially satisfied here, the uniformity w.r.t. z resulting from the one of $Y \to Y_z$. It remains to compare the volume of balls of fixed radius. But this is done by Condition (iii) of Definition 6.4.1. ■

Proof of Proposition 6.4.3. All along the proof, C will denote a positive constant.

Assume that Y satisfies the Sobolev inequality (S_φ^p). Let Ω be a compact subset of X and $f \in \mathrm{Lip}(\Omega)$. We want to prove that f satisfies (S_φ^p) at some scale h'. By Proposition 6.3.3, we can assume that Ω is 1-thick. For every $z \in Z$, denote by f_z the restriction of f to Y_z and $\Omega_z = \Omega \cap Y_z$.

Claim 6.4.5. There exists a constant $C < \infty$ such that for every $z \in Z$ $\lambda_z(\Omega_z) \leq C\mu(\Omega)$.

Proof : As Ω is 1-thick, the claim follows from the previous lemma and Proposition 6.3.5. ■

By Theorem 6.3.1, there exists $h' > 0$ such that Y_z satisfies (S_φ^p) at scale h', uniformly with respect to $z \in Z$. So for every $z \in Z$,

$$\|f_z\|_p \leq C\varphi(C\lambda_z(\Omega_z))\||\nabla f_z|_{h'}\|_p.$$

Since $\lambda_z(\Omega_z) \leq C\mu(\Omega)$ and φ is nondecreasing, we have

$$\|f_z\|_p \leq C\varphi(C\mu(\Omega))\||\nabla f_z|_{h'}\|_p.$$

Moreover, we have

$$\|f\|_p^p = \int_Z \|f_z\|_p^p d\nu(z)$$

and

$$\||\nabla f|_{h'}\|_p^p = \int_Z \||\nabla f_z|_{h'}\|_p^p d\nu(z).$$

Clearly, since Y_z is equipped with the induced distance, for every $z \in Z$ and every $x \in Y_z$,

$$|\nabla f|_{h'}(x) \geq |\nabla f_z|_{h'}(x).$$

Therefore,

$$\||\nabla f|_{h'}\|_p^p \geq \int_Z \||\nabla f_z|_{h'}\|_p^p.$$

We then have

$$\|f\|_p \leq C\varphi(C\mu(\Omega))\||\nabla f|_{h'}\|_p,$$

and we are done. ■

6.5 Sobolev inequality (S^2_φ) and on-diagonal upper bounds for random walks

In this section, we revisit the relations [Cou4] between Sobolev inequalities (S^2_φ) and on-diagonal upper bounds for random walks in our general context. The main purpose is to prove the following result which is a generalization of [Cou4, Theorem 7.2] to our context.

Theorem 6.5.1. Let $X = (X, d, \mu)$ be a metric measure space and let $P = (P_x)_{x \in X}$ be a symmetric view-point at scale h on X. Let φ be some increasing positive function. Define γ by

$$t = \int_0^{1/\gamma(t)} (\varphi(v))^2 \, \frac{dv}{v}.$$

(i) Assume that X satisfies a Sobolev inequality (S^2_φ) w.r.t. $|\nabla f|_{P^2,2}$. Then

$$p_x^{2n}(x) \leq \gamma(cn) \quad \forall n \in \mathbf{N},$$

for some constant $c > 0$.

(ii) If γ satisfies a numerical condition (δ) (see [Cou3, p 18]) and if

$$p_x^{2n}(x) \leq \gamma(n) \quad \forall n \in \mathbf{N},$$

then X satisfies (S^2_φ) w.r.t. $|\nabla|_{P,2}$.

Proof : In [Cou3, Theorem 7.2], the same result is proved for a weighted graph (X, μ) using the usual notion of gradient on graphs and where P is the standard random walk on (X, μ). Nevertheless, their proof only relies on the following formal link between P and the gradient.

$$c(\|f\|_2^2 - \|Pf\|_2^2) \leq \|\nabla f\|_2^2 \leq C(\|f\|_2^2 - \|Pf\|_2^2).$$

Here, this relation is satisfied when considering the gradient $|\nabla|_{P^2,2}$ and we even have

Lemma 6.5.2. For every $f \in L^2(X)$, we have

$$\||\nabla f|_{P^2,2}\|_2^2 = 2(\|f\|_2^2 - \|Pf\|_2^2).$$

Proof : Write

$$\begin{aligned}
\||\nabla f|_{P^2,2}\|_2^2 &= \int\int_{X^2} |f(x) - f(y)| dP_x(y) d\mu(y) \\
&= \int_X f^2(x) d\mu(x) + \int_X f^2(y) dP_x(y) d\mu(y) - 2\int_X f(x) P_x(f) d\mu(x) \\
&= \|f\|_2^2 + \langle Pf^2, 1\rangle - 2\langle f, Pf\rangle
\end{aligned}$$

As P is self-adjoint, $\langle Pf^2, 1\rangle = \langle f^2, P1\rangle = \|f\|_2^2$ and we are done. ■

So, the proof of [Cou3, Theorem 7.2] can be used formally in our context. However, for the sake of completeness, we give a sketch of this proof. First, using that P^n is symmetric, one checks easily that

$$\sup_{x,y\in X} p_x^{2n}(x) = \|P^{2n}\|_{1\to\infty}$$

where $\|\cdot\|_{p\to q}$ denotes the operator norm form $L^p(X,\mu)$ to $L^q(X,\mu)$.
Proof of (i). Assume that (S_φ^2) holds. Let us start with an important lemma.

Lemma 6.5.3. The Sobolev inequality (S_φ^2) for the L^2-gradient w.r.t. the view-point P is equivalent to the so-called Nash inequality

$$\|f\|_2^2 \le C\phi\left(C\frac{\|f\|_1^2}{\|f\|_2^2}\right)\||\nabla|_{P,2}\|_2^2.$$

Proof : Assume that a function f satisfies Nash inequality. Using Schwarz inequality and the fact that φ is nondecreasing, we obtain

$$\|f\|_2 \le \varphi^2\left(\frac{\|f\|_2^2}{\|f\|_1^2}\right)\||\nabla f|_{P^2,2}\|_2^2 \le \varphi^2(|\Omega|)\||\nabla f|_{P^2,2}\|_2^2.$$

The proof of the other implication relies on an argument of Grigor'yan in [Gri]. Assume that (S_φ^2) holds. Let $f\in \mathrm{Lip}_0(X)$. For every $\lambda>0$, since $f<2(f-\lambda)$ on $\{f>2\lambda\}$, we may write

$$\begin{aligned}\int f^2 &= \int_{f>2\lambda} f^2 + \int_{f\le 2\lambda} f^2\\ &\le 4\int_{f>2\lambda}(f-\lambda)^2 + 2\lambda\int_{f\le 2\lambda} f\\ &\le 4\int_{f>2\lambda}(f-\lambda)^2 + 2\lambda\|f\|_1\end{aligned}$$

Now applying (S_φ^2) to $(f-\lambda)_+$ gives

$$\int (f-\lambda)_+^2 \le \varphi(\mu(\{f>2\lambda\}))\||\nabla f|_{P^2,2}\|_2^2,$$

that is, since

$$\mu(\{f>\lambda\}) \le \frac{\|f\|_1}{\lambda}$$

and φ is non-decreasing,

$$\int (f-\lambda)_+^2 \le \varphi\left(\frac{\|f\|_1}{\lambda}\right)\||\nabla f|_{P^2,2}\|_2^2.$$

Therefore

$$\int f^2 \le 4\varphi\left(\frac{\|f\|_1}{\lambda}\right)\||\nabla f|_{P^2,2}\|_2^2 + 2\lambda\|f\|_1.$$

Letting $\varepsilon > 0$ and taking $\lambda = \varepsilon\|f\|_2^2/\|f\|_1$ in this equation yields

$$\|f\|_2^2 \le 4\phi\left(\frac{\|f\|_2^2}{\varepsilon\|f\|_1^2}\right)\||\nabla f|_{P^2,2}\|_2^2 + 2\varepsilon\|f\|_2^2$$

or equivalently,

$$\|f\|_2^2 \le \frac{4}{1-2\varepsilon}\phi\left(\frac{\|f\|_2^2}{\varepsilon\|f\|_1^2}\right)\||\nabla f|_{P^2,2}\|_2^2$$

Taking $\varepsilon = 1/4$, for example yields

$$\|f\|_2^2 \le 8\phi\left(4\frac{\|f\|_2^2}{\|f\|_1^2}\right)\||\nabla f|_{P^2,2}\|_2^2$$

which is the expected Nash inequality. ■

Now, consider $f \in L^1(X,\mu)$, non-negative, with $\|f\| = 1$ and define a sequence $u_n = \|P^n f\|_2^2$. The above inequality applied to the function $P^n f$ thus reads as

$$u_n \le \varphi^2(1/u_n)(u_n - u_{n+1})$$

since $\|P^n f\|_1 = \|f\|_1 = 1$ by Markov property of P. Let $t \to u_t$ be the increasing, piecewise linear function extending u_n on $\mathbf{R}_+$. If we put $v_t = 1/u_t$, then the above inequality becomes

$$dt \le \varphi^2(v_t)\frac{dv_t}{v_t},$$

hence, by integrating between 0 and t, we obtain

$$t \le \int_{v_0}^{1/v_t} \varphi^2(s)\frac{ds}{s};$$

and since by definition

$$t = \int_0^{1/\gamma(t)} (\varphi(v))^2\,\frac{dv}{v},$$

this means that $\gamma(t) \le v_t$, i.e.

$$\|P^n f\|_2^2 \le \gamma(n)$$

from which we deduce

$$\|P^n\|_{1\to 2} \le \sqrt{\gamma(n)}.$$

Now, using the fact that P^n is symmetric,

$$\|P^n\|_{2\to\infty} = \|P^n\|_{1\to 2} \le \sqrt{\gamma(n)}.$$

Hence

$$\|P^{2n}\|_{1\to\infty} \le \|P^n\|_{2\to\infty}\|P^n\|_{1\to 2} \le \gamma(n).$$

So (i) follows.

Proof of (ii). Assume that the decay $\|P^{2n}\|_{1\to\infty} \le \gamma(n)$ holds. Observe that $\|P^{2n}\|_{1\to\infty} = \|P^n\|_{1\to 2}$, then take f with $\|f\|_1 = 1$ and define as above $u_n = \|P^n f\|_2^2$. Since P is self-adjoint,

$$\|P^n f\|_2^2 = \langle P^n f, P^n f\rangle = \langle P^{n-1} f, P^{n+1} f\rangle \le \|P^{n-1} f\|_2 \|P^{n+1} f\|_2.$$

In other words, $u_n^2 \le u_{n-1}u_{n+1}$ and u_{n+1}/u_n is nondecreasing in n. It follows that

$$\left(\frac{u_1}{u_0}\right)^n \le \frac{u_1}{u_0}\frac{u_2}{u_1}\dots\frac{u_n}{u_{n-1}} = \frac{u_n}{u_0}.$$

Now, since by assumption $u_n \le \gamma(n)$,

$$\log\frac{\|f\|_2^2}{\gamma(n)} \le \log\frac{u_0}{u_n} \le n\log\frac{u_0}{u_n} \le n\left(\frac{u_0}{u_1} - 1\right),$$

hence

$$\|Pf\|_2^2 \le \left(\frac{n}{\log\frac{\|f\|_2^2}{\gamma(n)}}\right)(\|f\|_2^2 - \|Pf\|_2^2),\ \forall n \in \mathbf{N}.$$

Finally, for all f such that $\|f\|_1 = 1$,

$$\|f\|_2^2 \le \left(\frac{n}{\log\frac{\|f\|_2^2}{\gamma(n)}} + 1\right)(\|f\|_2^2 - \|Pf\|_2^2),\ \forall n \in \mathbf{N}.$$

An optimization[8] in n yields the Nash inequality that is equivalent to (S_φ^2) by Lemma 6.5.3. ■

6.6 Controlling the scale

6.6.1 Going down the scale

In this section, we address the following question. Let $X = (X, d, \mu)$ be a metric measure space X satisfying a Sobolev inequality at scale h ; we know that it automatically satisfies the same Sobolev inequalities at any larger scale ; but under what assumptions does it satisfy this inequality at some smaller scale h' ? This can be compared to a similar discussion in [Tes1] where we considered the isoperimetric properties of a metric measure space[9].

For example, consider $X = \mathbb{Z}^d$ $(d \ge 2)$ equipped with the distance $d(x, y) = \sum_{i=1}^d |y_i - x_i|$ and with the countable measure. It is well known that X satisfies a Sobolev inequality $S(d/(d-1), 1)$ at any scale ≥ 1. But no Sobolev inequality is available at a scale $s < 1$ since for every $f \in \mathrm{Lip}(X)$, $|\nabla f|_s = 0$. Clearly, the problem comes from the lack of connectivity at scale < 1.

The following proposition shows that Property of uniform b-connectedness (see Definition 6.1.26) together with Property of doubling at any scale are sufficient to control the minimal scale at which Sobolev inequalities may be valid.

[8]This is where condition (δ) is needed.

[9]This is a particular case of the present discussion corresponding to $p = 1$.

Proposition 6.6.1. Assume that X is a b-uniformly connected space satisfying the doubling property at any scale $r \geq b$. Then X satisfies a large-scale Sobolev inequality if and only if it satisfies the same Sobolev inequality at scale $2b$ (but with different constants).

Proof : Let $f : X \to \mathbf{R}$ be some Lipschitz function. Let us prove that for all $h \geq 2b$, there is a constant $C = C(h) < \infty$ such that for every $t > 0$

$$\mu(\{|\nabla f|_h > t\}) \leq C\mu(\{|\nabla f|_{2b} > t/C\}). \tag{6.6.1}$$

Consider a point $x \in \{|\nabla f|_h > t\}$: there is $y \in B(x,h)$ such that $|f(x) - \varphi(y)| > t$. Now, let $x = x_1 \dots x_m = y$ be a b-connecting chain between x and y (with m only depending on h). Clearly, there exists $1 \leq i < m$ such that $|\varphi(x_i) - \varphi(x_{i+1})| > t/m$. So in particular, for all $z \in B(x_i, b)$, $|\nabla f|_{2b}(z) > t/(2m)$. Let Z be a maximal $2E$-separated subset of $\{|\nabla f|_h > t\}$. The balls $(B(z,2E))_{z\in Z}$ form a covering of $\{|\nabla f|_h > t\}$. On the other hand, by the previous discussion, in each ball $B(z,E)$, one can find a ball $B(x_z, b)$ included in $\{|\nabla f|_{2b} > t/(2m)\}$. Since the balls $(B(x_z,b))_{z\in Z}$ are disjoint, (8.3.4) follows from doubling property at any scale $r \geq b$. ■

6.6.2 From finite scale to infinitesimal scale

Definition 6.6.2. (see for instance [Sem, Definition 1.18]) Let (X,d) be a metric space, and let u and g be two Borel measurable functions defined on X, with u real-valued and g taking values in $[0,\infty]$. We say that g is an *generalized gradient* of u if

$$|u(\gamma(a)) - u(\gamma(b))| \leq \int_a^b g(\gamma(t))dt$$

whenever $a, b \in \mathbf{R}$ and $\gamma : [a,b] \to X$ is 1-Lipschitz (so that $d(\gamma(s),\gamma(t)) \leq |s-t|$ for all $s,t \in [a,b]$).

Example 6.6.3. [Sem, Lemma 1.20] The function g defined by

$$g(x) = \liminf_{r\to 0} r^{-1} \sup_{y\in B(x,r)} |u(y) - u(x)|$$

is a generalized gradient of u. Let us call g the standard upper gradient of u and we denote it by $|\overline{\nabla} u|$.

The following proposition is obvious by passing to the limit.

Proposition 6.6.4. Fix $p \in [1,\infty]$. Assume that for every $h > 0$, (X,d,μ) satisfies a Sobolev inequality (S^p_φ) w.r.t. the gradient $|\nabla|_h$. Suppose that the constants appearing in these inequalities are uniform with respect to h, then X satisfies (S^p_φ) w.r.t. the standard upper gradient.

The following fact had already been noticed in the case of a discretization of a manifold [CouSa1]. Its proof, here, is straightforward from the definition of $|\nabla|_{P,p}$.

Proposition 6.6.5. Fix some $h > 0$ and $p \in [1, \infty]$. Let (X, d, μ) be a metric measure space with doubling property at radius $\geq h$, and let P be a viewpoint at scale h on X. Suppose that a function $u \in \mathrm{Lip}(X)$ satisfies (S^p_φ) w.r.t. $|\nabla|_{P,p}$. Let g be an generalized gradient of u. We assume that u satisfies the following local Poincaré inequality $(P(1,p))_{loc}$

$$\int_{B(x,h)} |h(y) - h(x)|^p dP_x(y) \leq C \int_{B(x,h')} g^p(y) d\mu(y)$$

for some constants $C, h < \infty$. Then u satisfies (S^p_φ) w.r.t. g.

Example 6.6.6. Let M be a Riemannian manifold. Then the local norm of its usual gradient trivially coincides with the standard upper gradient on M. Now, assume that M satisfying a local Poincaré inequality (as in the Proposition) and let X be a discretization of M. According to Theorem 6.3.1, if X satisfies (S^p_φ), then M also satisfies (S^p_φ) w.r.t. its usual gradient.

6.6.3 From infinitesimal scale to finite scale

In this last section, we will prove that if a metric measure space satisfies a Sobolev inequality w.r.t. the standard upper gradient (see Exemple 6.6.3), then it satisfies this Sobolev inequality at any scale.

Theorem 6.6.7. Fix $p \in [1, \infty]$. Let (X, d, μ) be a metric measure space satisfying the doubling property at any scale. Assume that (X, d, μ) satisfies a Sobolev inequality (S^p_φ) w.r.t. the standard upper gradient $|\overline{\nabla}|$. Then X satisfies (S^p_φ) w.r.t. $|\nabla|_h$ for every $h > 0$.

Proof : Assume that X satisfies (S^p_φ) w.r.t. the standard upper gradient. Using the same tools as in the proof of Proposition 6.2.2, one can see that it suffices to show that for every $h > 0$ and every function f, there exists a viewpoint P at scale $h/2$ such that

$$\|Pf\|_p \leq C\varphi(\mu(\Omega))\||\nabla Pf|_h\|_p \tag{6.6.2}$$

where Ω is a measurable subset containing the support of f. According to Proposition 6.3.3, we can assume that $Supp(f)$ is thick. Thus, thanks to Proposition 6.3.4, we can replace Ω by $[\Omega]_{Ah}$ that[10] contains $Supp(Pf)$. Finally, it suffices to prove that (S^p_φ) w.r.t. $|\nabla|_h$ is satisfied for functions of the form Pf, with $f \in \mathrm{Lip}(X)$.

Define a 1-Lipschitz map $\theta : X \times X \to \mathbf{R}_+$ by $\theta(x, y) = d(y, B(x, h)^c)$. Write

$$p_x(y) = \frac{\theta(x, y)}{K(x)},$$

where $K(x) = \int_{B(x,h)} \theta(x, z) d\mu(z)$. Since X is doubling at any scale, one can easily check that $p_x(y)$ is the density of a viewpoint P at scale h. Moreover,

[10] A is the large constant appearing in the definition of a viewpoint at scale h.

$D^{-1}V(x,h) \leq K(x') \leq DV(x,h)$ where $D \geq 1$ only depends on the doubling constant at scale h.

Let x' be a point distinct from x. We have

$$
\begin{aligned}
Pf(x') - Pf(x) &= \int_X (p_{x'}(y) - p_x(y)) f(y) d\mu(y) \\
&= \int_X (p_{x'}(y) - p_x(y))(f(y) - f(x)) d\mu(y) \\
&= \int_X \frac{\theta(x', y)K(x) - \theta(x,y)K(x')}{K(x)K(x')} (f(y) - f(x)) d\mu(y) \\
&= \int_X \frac{(\theta(x', y) - \theta(x,y))K(x) - \theta(x,y)\left(K(x') - K(x)\right)}{K(x)K(x')} (f(y) - f(x)) d\mu(y)
\end{aligned}
$$

Since X is doubling at any scale, it is not difficult to see that for x' closed to x, $C^{-1}K(x) \leq K(x') \leq CK(x)$ where $C \geq 1$ only depends on the doubling constant at scale h. Hence,

$$
|\overline{\nabla} Pf|(x) \leq C \int_X \frac{|\overline{\nabla}_x \theta|(x,y)K(x) + \theta(x,y)|\overline{\nabla} K|(x)}{K(x)^2} |f(y) - f(x)| d\mu(y)
$$

On the other hand, note that

$$
|\overline{\nabla} K|(x) \leq \int_X |\overline{\nabla}_x \theta|(x,z) d\mu(z) \leq V(x,h).
$$

Up to change the constant C, we conclude that

$$
\begin{aligned}
|\overline{\nabla} Pf|(x) &\leq C \frac{1}{V(x,h)} \int_{B(x,h)} |f(y) - f(x)| d\mu(y) \\
&\leq C|\nabla f|_h(x).
\end{aligned}
$$

Now, to conclude, it remains to apply (S^p_φ) w.r.t. the standard upper gradient to Pf. Together with the above inequality, we obtain (6.6.2). ■

Corollary 6.6.8. If a Riemannian manifold M with doubling property at any scale satisfies (S^p_φ) for the usual gradient, then it satisfies it at any scale. If X is a discretization of M, then it also satisfies (S^p_φ).

Remark 6.6.9. Assume that X is uniformly connected (e.g. X is a Riemannian manifold), so that Proposition 6.6.1 applies. Note that in the proof of Theorem 6.6.7, we actually show that a Sobolev inequality at large scale is equivalent to the Sobolev inequality for the standard upper gradient restricted to functions of the form $g = Pf$, where P is a viewpoint at some positive scale.

6.7 Applications to quasi-transitive spaces

6.7.1 Existence of a spectral gap on a quasi-transitive space

The main result

The following theorem generalizes results from [Kest, Bro, Salv, SoW, Pit2, SW]. We say that a locally compact metric measure space (X, d, μ) is quasi-transitive if there exists a locally compact group G acting properly and co-compactly by measure-preserving isometries on X. The quasi-transitivity of the action easily implies that X is doubling at any scale. Note that this implies that G is second countable.

Theorem 6.7.1. Let (X, d, μ) be a quasi-G-transitive metric measure space. Then G is unimodular and amenable if and only if for h large enough and every reversible viewpoint P at scale h on (X, d, μ), the spectral radius $\rho(P) = 1$, or in other words, if the discrete Laplacian $\Delta = I - P$ has no spectral gap around zero.

Proof : The proof splits in three parts. First, by Theorem 6.1.16, one checks easily that $\rho(P) = 1$ if and only if the large scale profile $j_{X,2}(t) \to \infty$ when $t \to \infty$. Indeed, $j_{X,2}(t) \leq C$ means that X satisfies a large-scale Sobolev inequality (S^2_φ) with $\varphi(t) = C$. Thus by Theorem 6.1.16, this happens if and only if $p^{2n}_x(x)$ has exponential decay, i.e. if and only if $\rho(P) < 1$.

Second, take a uniform left-invariant metric on G. The co-compactness and properness of the G-action on X imply that G and X are large-scale equivalent (this is straightforward). Hence, by Theorem 6.3.1, it is enough to prove Theorem 6.7.1 for $X = G$. This third step will be achieved by Corollary 6.7.10.

Remark 6.7.2. Note that if we assume G compactly generated, then it is classical and not difficult to see that a quasi-G-transitive metric measure space is quasi-isometric to G, equipped with the word metric d_S corresponding to a compact generating subset S of G.

Corollary 6.7.3. Assume that M is a Riemannian manifold, then G is unimodular and amenable if and only if the spectral radius of the heat kernel equals 1, or in other words, if the usual Laplacian on M has no spectral gap around zero.

Proof : The Laplacian has a spectral gap if and only if M satisfies a Sobolev inequality $\|\nabla f\|_2 \geq c\|f\|_2$ for the usual gradient. As M is quasi-transitive, it is easy to check that it satisfies a local Poincaré inequality as in Proposition 6.6.5. Indeed, one has to prove that such a local Poincaré inequality $(P(1,1))_{loc}$ holds, for any $q \geq 1$ on a compact subset K such that $X = \cup_{g \in G} gK$. But this results from the fact that such inequality holds in $\mathbf{R}^d$. Now, applying Proposition 6.6.5 and Theorem 6.6.7, we see that the spectral gap is equivalent to a large-scale Sobolev inequality. We conclude thanks to Theorem 6.7.1. ■

Second countable locally compact groups.

All the locally compact groups considered here are second countable.

Recall that a locally compact group can be endowed with a "large-scale" structure of metric measure space. Let us consider the following natural question : is amenability a geometric property among compactly generated locally compact groups ? Recall that a locally compact group is called amenable if it admits a left invariant mean [Pi]. By *geometric property*, we mean a property characterized in terms of metric measure space. Moreover, we expect such a property to be invariant under large-scale equivalence. Følner's characterization of amenability implies that the answer is positive when the group is *finitely generated*. On the opposite, note that any connected Lie group admits a co-compact amenable subgroup (take for instance a maximal solvable subgroup) and therefore is always quasi-isometric to a compactly generated locally compact amenable group. So the answer is negative in general. Actually, we will see that the answer is yes if and only if the group is unimodular.

Let G be a locally compact group equipped with some proper left-invariant metric d and with its Haar measure μ. Fix some $h > 0$. We define the boundary of a subset A of G by

$$\partial_h A \quad = \quad AB(e,h) \cap A^c B(e,h).$$

It is important to note that the multiplication by elements of $B(e,h)$ is on the *right*, so that $AB(e,h)$ has the following metric interpretation :

$$AB(e,h) = \cup_{x\in A} B(x,h) = [A]_h$$

where $[A]_h = \{x \in G,\ d(x,A) \leq h\}$. In particular, this definition of boundary coincides with the one we gave in introduction for a general metric space.

For any sequence of compact subsets with positive measure (F_n) of G and for every $g \in G$, we define $\phi_n(g) = \mu(gF_n \vartriangle F_n)/\mu(F_n)$. Note that here, the multiplication by g is on the left.

Recall [Pi] that the group G is amenable if and only one of the following equivalent statements holds :

(1) There exists a sequence (F_n) such that $\phi_n(g)$ is pointwise converging to zero.

(2) There exists a sequence (F_n) such that $\phi_n(g)$ converges to zero uniformly on compact sets.

(3) There exists a sequence (F_n) such that $\mu(QF_n \cap QF_n^c)/\mu(F_n) \to 0$ for every compact subset Q.

If a sequence (F_n) satisfies (1), or equivalently, (2), then it is called a Følner sequence.

Remark 6.7.4. Generally, in the definition of Følner sequence, (F_n) is also asked to be an increasing exhaustion of G (this also characterizes amenability).

Here, the multiplication by Q is on the *left*, so that amenability is *not a priori* characterized in terms of isoperimetry, or in other words, in terms of metric measured space properties. This is not surprising since amenability is not invariant under quasi-isometry (for instance, every connected Lie group admits a cocompact solvable closed subgroup). Let us define a *geometric* version of amenability.

Definition 6.7.5. The group G is called geometrically amenable if it admits a sequence of compact subsets (F_n) such that one of the following equivalent statements holds :
(1) $\mu(F_n \triangle F_n g)/\mu(F_n) \to 0$ for every $g \in G$.
(2) For every compact subset Q of G,

$$\mu(F_n Q \cap F_n^c Q)/\mu(F_n) \to 0.$$

The following proposition justifies the term "geometric".

Proposition 6.7.6. A locally compact group G is geometrically amenable if and only if for h large enough, the isoperimetric profile $j_{G,1}$ (resp. $j_{G,p}$ for any $p \geq 1$) at scale h is unbounded.

Proof : Clearly, (2) of the definition of geometrically amenable implies that $j_{G,1}$ is unbounded at any scale. Conversely, the negation of (2) together with the σ-compacity of G yields the existence of a compact subset K of G such that for every measurable subset A with finite measure,

$$\mu(A) \leq C\mu(AK \triangle A)$$

for some constant $C < \infty$. Let h be such that $K \subset B(e, h)$. It follows that

$$\mu(A) \leq C\mu(\partial_h A),$$

which means that the profile $j_{X,1}$ at scale h is bounded. ■

If G is unimodular, up to replacing F_n with F_n^{-1}, it is equivalent for G to have left or right Følner sequences. In particular, if a group is unimodular, then it is geometrically amenable if and only if it is amenable. Actually, we have better : geometric amenability is equivalent to amenability plus unimodularity.

Lemma 6.7.7. If the group G is non-unimodular, then it satisfies the following isoperimetric inequality for h large enough

$$\mu(\partial_h A) \geq c\mu(A) \quad \forall A \subset G$$

where c is some positive constant.

Proof : Let δ be the modular function of G. Since G is non-unimodular, there exists $g \in G$ such that $\delta(g) > 1$. So, choosing h large enough, we can assume that $g \in B(e, h)$. Then for any compact subset $A \subset G$, we have

$$\mu(\partial_h A) \geq \mu(Ag \triangle A) \geq \mu(Ag) - \mu(A) = (\delta(g) - 1)\mu(A). \blacksquare$$

Proposition 6.7.8. Let G be a locally compact group equipped with a left Haar measure. Then G is amenable and unimodular if and only if it admits a geometric Følner sequence. In particular if G is compactly generated, then G is amenable and unimodular if and only if it is geometrically amenable.

Proof : This is a direct consequence of Lemma 6.7.7 and of the above discussion. ■

Recall that quasi-isometries between homogeneous metric measure spaces are large-scale equivalences. We have the following corollaries to Theorem 6.3.1.

Corollary 6.7.9. Geometric amenability is invariant under large-scale equivalence between second countable locally compact groups.

Corollary 6.7.10. Geometric amenability is invariant under quasi-isometry between compactly generated locally compact groups.

Corollary 6.7.11. Being amenable and unimodular is invariant under large-scale equivalence between second countable locally compact groups. ■

6.7.2 Isoperimetric profile of subgroups

Proposition 6.7.12. Let H be a closed subgroup of a locally compact, countable at infinity group G. Assume that the quotient G/H carries a G-invariant Borel measure. Then, G is large-scale foliated by H.

Proof : Equip G and H with left Haar measures μ and λ and with left invariant uniform metrics d_G and d_H. Let ν be a G-invariant σ-finite measure on the quotient $Z = G/H$. Since ν is G-invariant, up to normalize it, one can assume that for every continuous compactly supported function f on G,

$$\int_G f(g)d\mu(x) = \int_Z \left(\int_H f(gh)d\lambda(h)\right) d\nu(gH).$$

We claim that the partition $G = \sqcup_{gH \in Z} gH$ satisfies Conditions (i) to (iii) of Definition 6.1.21. Clearly, (i) follows from the above decomposition of μ. For every $g \in G$, the left-translation by g is an isometry on G. On the other hand, since H is a closed subgroup, the inclusion map $H \to G$ is a uniform embedding, i.e. satisfies condition (i) of Definition 6.1.21. This proves (ii). Finally, (iii) follows immediately from the left-invariance of μ. ■

Corollary 6.7.13. Let H be a closed subgroup of a locally compact group G such that G/H carries a G-invariant measure. Then,
(1) $j_{H,p} \succeq j_{G,p}$;
(2) If ρ is the compression of the inclusion map $H \hookrightarrow G$, then $J^b_{H,p} \succeq J^b_{G,p} \circ \rho$. ■

Remark 6.7.14. Corollary 6.7.13 holds for exemple if G and H are both unimodular. Actually this is the only interesting situation since, by Lemma 6.7.7, a non-unimodular group always satisfies the "best" Sobolev inequality at large scale : $\||\nabla f|_h\|_p \geq c_p\|f\|_p$ for every $p \geq 1$. On the other hand, if H is non-unimodular and if G is unimodular and amenable, then, by Proposition 6.7.8 the conclusion of Corollary 6.7.13 is never true[11].

[11] For exemple, consider the non-unimodular group H of positive affine transformations of $\mathbf{R}$: this group, equipped with its left-invariant Riemannian metric is isometric to the Hyperbolic plane. In particular, it has a bounded isoperimetric profile. On the other hand, it is a closed subgroup of the solvable unimodular Lie group Sol, whose isoperimetric profile $j_{G,p}$ is asymptotically equivalent to $\log t$.

Chapitre 7

Isopérimétrie asymptotique des boules dans un espace à croissance polynomiale

Résumé

In this paper, we study the asymptotic behavior of the volume of spheres in metric measure spaces. We first introduce a general setting adapted to the study of asymptotic isoperimetry in a general class of metric measure spaces. Let $\mathcal{A}$ be a family of subsets of a metric measure space (X, d, μ), with finite, unbounded volume. For $t > 0$, we define

$$I^{\downarrow}_{\mathcal{A}}(t) = \inf_{A \in \mathcal{A}, \mu(A) \geq t} \mu(\partial A).$$

We say that $\mathcal{A}$ is asymptotically isoperimetric if $\forall t > 0$

$$I^{\downarrow}_{\mathcal{A}}(t) \leq CI(Ct),$$

where I is the profile of X. We show that there exist graphs with uniform polynomial growth whose balls are not asymptotically isoperimetric and we discuss the stability of related properties under quasi-isometries. Finally, we study the asymptotically isoperimetric properties of connected subsets in a metric measure space. In particular, we construct graphs with uniform polynomial growth whose connected subsets are not asymptotically isoperimetric.

7.1 Introduction

The study of large scale isoperimetry on metric measure spaces has proven to be a fundamental tool in various fields ranging from geometric group theory [Gro3, PS] to analysis and probabilities on graphs and manifolds [CouSa1, CouSa2]. One of the targets of this paper is to find a simple setting adapted to the large scale study of isoperimetric properties. This includes some general assumptions on metric measure spaces, a convenient notion of "large scale" boundary of a subset, and a family of maps preserving the large scale isoperimetric properties. There are two kinds of questions concerning isoperimetry [Ros] : what is the isoperimetric profile ? What are the subsets that optimize the isoperimetric profile ? Here, we will formulate similar questions in a large scale setting : we will not be interested in the exact values of the isoperimetric profile but in its asymptotic behavior and we will consider sequences of subsets that optimize "asymptotically" the isoperimetric profile. Dealing with general metric measure spaces, the family of balls seems to be a natural candidate for optimizing asymptotically the isoperimetric profile. Nevertheless, we will see that even under apparently strong assumptions on the space X, this is not always the case. Let us be more precise.

7.1.1 Boundary of a subset and isoperimetric profile

Let (X, d, μ) be a metric measure space. Let us denote $B(x, r)$ the closed ball of center x and radius r. We suppose that the measure μ is Borel, supported on X and σ-finite. For any measurable subset A of X, any $h > 0$, write

$$A_h = \{x \in X, d(x, A) \leq h\},$$

and

$$\partial_h A = A_h \cap (A^c)_h.$$

Let us call $\partial_h A$ the h-boundary of A, and $\partial_h B(x, r)$ the h-sphere of center x and radius r.

Definition 7.1.1. Let us call the h-profile the nondecreasing function defined on $\mathbf{R}_+$ by

$$I_h(t) = \inf_{\mu(A) \geq t} \mu(\partial_h A),$$

where A ranges over all μ-measurable subsets of X with finite measure.

This definition of large-scale boundary has the following advantage : under some weak properties on the metric measure space X, we will see in Section 7.3.1 that in some sense, the boundary of a subset $A \subset X$ has a thickness "uniformly comparable to h". This will be play a crucial role in the proof of the invariance of "asymptotic isoperimetric properties" under large-scale equivalence (see § 7.1.3).

7.1.2 Lower/upper profile restricted to a family of subsets

Let (X, d, μ) be a metric measure space. In order to study isoperimetric properties of a family of (measurable) subsets of X with finite, unbounded volumes, it is useful to introduce the following notions

Definition 7.1.2. Let $\mathcal{A}$ be a family of subsets of X with finite, unbounded volumes. We call lower (resp. upper) h-profile restricted to $\mathcal{A}$ the nondecreasing function $I^{\downarrow}_{h,\mathcal{A}}$ defined by

$$I^{\downarrow}_{h,\mathcal{A}}(t) = \inf_{\mu(A)\geq t, A\in\mathcal{A}} \mu(\partial_h A)$$

(resp. $I^{\uparrow}_{h,\mathcal{A}}(t) = \sup_{\mu(A)\leq t, A\in\mathcal{A}} \mu(\partial_h A)$).

Definition 7.1.3. Consider two monotone functions f and $g : \mathbf{R}_+ \to \mathbf{R}_+$. Say that $f \approx g$ if there exist some constants C_i such that $C_1 f(C_2 t) \leq g(t) \leq C_3 f(C_4 t)$ for all $t \in \mathbf{R}_+$.

The asymptotic behavior of a monotone function $\mathbf{R}_+ \to \mathbf{R}_+$ may be defined as its equivalence class modulo $\approx$.

We get a natural order relation on the set of equivalence classes modulo $\approx$ of monotone functions defined on $\mathbf{R}_+$ by setting

$$(f \preceq g) \Leftrightarrow (\exists C_1, C_2 > 0, \forall t > 0, \quad f(t) \leq C_1 g(C_2 t)).$$

We say that the family $\mathcal{A}$ is **asymptotically isoperimetric** (resp. **strongly asymptotically isoperimetric**) if for all $A \in \mathcal{A}$

$$I^{\downarrow}_{h,\mathcal{A}} \preceq I_h$$

(resp. $I^{\uparrow}_{h,\mathcal{A}} \preceq I_h$).

Remark 7.1.4. Note that asymptotically isoperimetric means that for any t we can always choose an optimal set among those of $\mathcal{A}$ whose measure is larger than t whereas strongly asymptotically isoperimetric means that every set of $\mathcal{A}$ is optimal (but the family $(\mu(A))_{A\in\mathcal{A}}$ may be lacunar). In almost all cases we will consider, the family $(\mu(A))_{A\in\mathcal{A}}$ will not be lacunar, and strong asymptotic isoperimetry will imply asymptotic isoperimetry.

7.1.3 Large scale study

Let us recall the definition of a quasi-isometry (which is also sometimes called rough isometry).

Definition 7.1.5. Let (X, d) and (X', d') two metric spaces. One says that X and X' are quasi-isometric if there is a function f from X to X' with the following properties.

(a) there exists $C_1 > 0$ such that $[f(X)]_{C_1} = X'$.

(b) there exists $C_2 \geq 1$ such that, for all $x, y \in X$,

$$C_2^{-1} d(x,y) - C_2 \leq d'(f(x), f(y)) \leq C_2 d(x,y) + C_2.$$

Example 7.1.6. Let G be a finitely generated group and let S_1 and S_2 two finite symmetric generating sets of G. Then it is very simple to see that the identity map $G \to G$ induces a quasi-isometry between the Cayley graphs (G, S_1) and (G, S_2). At the beginning of the 80's, M. Gromov (see [Gro3]) initiated the study of finitely generated groups up to quasi-isometry.

Example 7.1.7. The universal cover of a compact Riemannian manifold is quasi-isometric to every Cayley graph of the covering group (see [Gro3] and [S]).

Note that the notion of quasi-isometry is purely metric. So, when we look for quasi-isometry invariant properties of a metric measure space like, for instance, volume growth, we are led to assume some uniformity properties on the volume of balls. This is the reason why, for instance, this notion is well adapted to geometric group theory. But since we want to deal with more general spaces, we will define a more restrictive class of maps. Those maps will be asked to preserve locally the volume of balls. On the other hand, we want local properties to be stable under bilipschitz fluctuations of the metric. Precisely, let (X, d, μ) be a metric measure space and let d' be another metric on X such that d/d' and d'/d are bounded. The following definition (see [CouSa1]) prevents wild changes of the volume of balls with bounded radii under the identity map between (X, d, μ) and (X, d', μ).

Definition 7.1.8. Let us say that (X, d, μ) is **doubling at fixed radius**, or has property $(DV)_{loc}$ if for all $r > 0$, there exists $C_r > 0$ such that, for all $x \in X$

$$\mu(B(x, 2r)) \leq C_r \mu(B(x, r)).$$

Remark 7.1.9. Note that Property $(DV)_{loc}$ is local in r but uniform in x.

Example 7.1.10. Bounded degree graphs or Riemanniann manifolds with Ricci curvature bounded from below satisfy $(DV)_{loc}$.

The following notion was introduced by Kanai [Kan] (see also [CouSa1]).

Definition 7.1.11. Let (X, d, μ) and (X', d', μ') two metric measure spaces with property $(DV)_{loc}$. Let us say that X and X' are large scale equivalent (we can easily check that it is an equivalence relation) if there is a function f from X to X' with the following properties : there exist some constants $C_1 > 0$, $C_2 \geq 1$, $C_3 \geq 1$ such that

(a) f is a quasi-isometry of constants C_1 and C_2 ;

(b) for all $x \in X$

$$C_3^{-1} \mu(B(x,1)) \leq \mu'(B(f(x),1)) \leq C_3 \mu(B(x,1)).$$

Focusing our attention on balls of radius 1 may not seem very natural. Nevertheless, this is not a serious issue since property $(DV)_{loc}$ allows to make no distinction between balls of radius 1 and balls of radius C for any constant $C > 0$.

Remark 7.1.12. Note that for graphs with bounded degree (equipped with the counting measure), or Riemannian manifolds with bounded Ricci curvature (equipped with the Riemannian measure), quasi-isometries are automatically large-scale equivalences.

7.1.4 Volume of balls and growth function

Let (X, d, μ) be a metric measure space. The equivalence class modulo $\approx$ of $\mu(B(x,r))$ is independent from x. We call it the volume growth of X and we write it $V(r)$. We have the following easy fact (see [CouSa1]).

Proposition 7.1.13. The volume growth is invariant under large-scale equivalence (among $(DV)_{loc}$ spaces).

Definition 7.1.14. Let X be a metric measure space. We say that X is doubling if there exists a constant $C > 0$ such that, $\forall x \in X$ and $\forall r \geq 0$

$$\mu(B(x, 2r)) \leq C\mu(B(x,r)). \tag{7.1.1}$$

We will call this property (DV).

Remark 7.1.15. It is easy to see that (DV) is invariant under large scale equivalence between $(DV)_{loc}$ spaces. To be more general, we could define an asymptotic doubling condition $(DV)_\infty$, restricting (7.1.1) to balls of radius more than a constant (depending on the space). Property $(DV)_\infty$ is also stable under large-scale equivalence between $(DV)_{loc}$ spaces and has the advantage to focus on large scale properties only. Actually, in every situation met in this paper, the assumption (DV) can be replaced by $(DV)_\infty + (DV)_{loc}$ (note that they are equivalent for graphs). Nevertheless, for the sake of simplicity, we will leave this generalization aside.

Example 7.1.16. A crucial class of doubling spaces is the class of spaces with polynomial growth : we say that a metric measure space has (strict) polynomial growth of degree d if there exists a constant $C \geq 1$ such that, $\forall x \in X$ and $\forall r \geq 1$

$$C^{-1}r^d \leq \mu(B(x,r)) \leq Cr^d.$$

Gromov proved [Gro1] that if a finitely generated group G satisfies

$$\mu(B(1,r)) \leq Cr^d$$

for some constant $C > 0$, then it has polynomial growth with integer degree. Another very interesting class of examples are fractals as for instance, the (unbounded) Sierpinski gasket or more generally, polygaskets (see [Fal, Str]).

7.2 Organization of the paper

In the next section, we present a setting adapted to the study of asymptotic isoperimetry in general metric measure spaces. The main interest of this setting is that the "asymptotic isoperimetric properties" are invariant under large-scale equivalence. In particular, it will imply that if X is a $(DV)_{loc}$ and uniformly connected space (see next section), then the class modulo $\approx$ of I_h will not depend any more on h provided h is large enough. For that reason, we will simply denote I instead of I_h. Then, we introduce a notion of weak geodesicity which is invariant under Hausdorff equivalence (see § 7.3.2) but not under quasi-isometry. We call it property (M) since it can be formulated in terms of existence of some "monotone" geodesic chains between any pair of points. This property plays a crucial role when we want to obtain upper bounds for the volume of spheres (see [Tes0]). It will also appear as a natural condition for some properties discussed in this paper.

Here are the two main problems concerning isoperimetry in metric measure spaces : first, determining the asymptotic behavior of the profile ; second, finding families of subsets that optimize the profile. The asymptotic behavior of I is more or less related to volume growth (see [CouSa2] and [Pit2] for the case of finitely generated groups). In the setting of groups, the two problems have been solved for Lie groups (and for polycyclic groups) in [PS] and [CouSa2] and for a wide class of groups constructed by wreath products in [Er]. It seems very difficult (and probably desperate) to get general statements for graphs with bounded degree without any regularity assumption (like doubling property or homogeneity). On the other hand, let us emphasize the fact that doubling condition appears as a crucial assumption in many fields of analysis. So in this article, we will deal essentially with doubling metric measure spaces. Without any specific assumption on the space, balls seem to be natural candidates for being isoperimetric subsets, especially when the space is doubling (see Corollary 7.4.4).

One could naively think that thanks to Theorem 8.2.4, properties like asymptotic isoperimetry of balls are stable under large-scale equivalence. Unfortunately, it is not the case. Indeed, Theorem 8.2.4 roughly says that if $f : X \to X'$ is a large scale equivalence between two metric measure spaces, then for $a > 0$ large enough and for every measurable subset A of X, the measure of the boundary of $[f(A)]_a$ is smaller than the measure of the boundary of A up to a multiplicative constant. So, in order to apply Theorem 8.2.4 to asymptotic isoperimetry of balls, we need the existence of some $C > 0$ such that

$$B(f(x), r - C) \subset [f(B(x,r))]_C \subset B(f(x), r + C) \quad \forall x \in X, \forall r > 0. \tag{7.2.1}$$

This condition is satisfied if f is a Hausdorff equivalence (see § 7.3.2 for the definition). But if f is a quasi-isometry, we only have

$$B(f(x), C^{-1}r - C) \subset [f(B(x,r))]_C \subset B(f(x), Cr + C) \quad \forall x \in X, \forall r > 0. \tag{7.2.2}$$

(Note that (7.2.1) and (7.2.2) are purely metric conditions).

Let us introduce some terminology. First, let us write $\mathcal{B}$ for the family of all closed balls of X.

Definition 7.2.1. Let X be a metric measure space.

– We say that X is **(IB)** if balls are asymptotically isoperimetric, i.e. if

$$I_{\mathcal{B}}^{\downarrow} \preceq I.$$

Otherwise, we will say that X is **(NIB)**.

– We say that X is **strongly-(IB)** if balls are strongly asymptotically isoperimetric, i.e. if

$$I_{\mathcal{B}}^{\uparrow} \preceq I.$$

– Finally, we say that a metric measure space is **stably-(IB)** (resp. stably-(NIB)) if every (M)-space (see Definition 8.2.1) large scale equivalent to X is (IB) (resp. (NIB)). If necessary, we will restrict our study to a certain class of metric measure spaces.

Definition 7.2.2. We say that a space (X, d, μ) satisfies a strong (isoperimetric) inequality —or that X has a strong profile— if $I \succeq id/\phi$ where ϕ is the equivalence class modulo $\approx$ of the function

$$t \to \inf\{r, \mu(B(x,r)) \geq t\}.$$

We will show that every doubling space satisfying a strong isoperimetric inequality satisfies (IB). This actually implies that such a space satisfies stably-(IB). In particular, any compactly generated, locally compact group of polynomial growth satisfies (IB). In contrast, apart from the Abelian case [Tes0], it is still unknown whether such a group G satisfies strongly-(IB) or not, or, in other words, if we have $\mu(K^{n+1} \smallsetminus K^n) \approx n^{d-1}$ where K is a compact generating set of G and μ is a Haar measure on G.

Conversely, we will show that every strongly-(IB) doubling space satisfies a strong isoperimetric inequality. On the other hand, we will see that the strong isoperimetric inequality does not imply strongly-(IB), even if the volume growth is linear ($V(r) \approx r$).

To see that strongly-(IB) is not stable under large scale equivalence, even among graphs with polynomial growth, we shall construct a graph quasi-isometric to $\mathbf{Z}^2$ whose volume of spheres is not dominated by $r^{\log 3/\log 2}$ (where r is the radius). Note that this can be compared with the following result (see [Tes0], theorem 1)

Theorem 7.2.3. [Tes0] Let X be a metric measure space with properties (M) and (DV) (for instance, a graph or a complete Riemannian manifold with the doubling property). There exists $\delta > 0$ and a constant $C > 0$ such that, $\forall x \in X$ and $\forall r > 0$

$$\mu\left(B(x, r+1) \smallsetminus B(x,r)\right) \leq Cr^{-\delta}\mu(B(x,r)).$$

In particular, the ratio $\mu(\partial B_{x,r}(x))/\mu(B(x,r))$ tends to 0 uniformly in x when r goes to infinity.

When the profile is not strong, we will see that many situations can happen. All the counterexamples built in the corresponding section will be graphs of polynomial growth.

The case of a bounded profile is quite specific.[1] Indeed, in that case, and under some hypothesis on X (including graphs and manifolds with bounded geometry), we will prove that if $(P_n)_{n\in\mathbf{N}}$ is an asymptotically isoperimetric sequence of connected subsets of X, one can find a constant $C \geq 1$ and $\forall n \in \mathbf{N}$, some $x_n \in X, r_n > 0$ such that

$$B(x_n, r_n) \subset P_n \subset B(x_n, Cr_n).$$

Note that here, we don't ask X to be doubling.

Nevertheless, we will see that there exist graphs with polynomial growth (with unbounded profile) such that no asymptotically isoperimetric family has this property. In particular, those graphs are stably-(NIB).

To be complete, we also build graphs with polynomial growth, bounded profile and satisfying stably-(NIB).

Concerning the stability under large-scale equivalence, we will see that even among graphs with polynomial growth, with bounded or unbounded profile, property (IB) is not stable under large-scale equivalence (in the case of graphs equipped with the counting measure, a large-scale equivalence is simply a quasi-isometry).

Finally, we shall examine isoperimetric properties of connected subsets.

Definition 7.2.4. Let us say that a subset A is (metrically) connected if for any partition $A = A_1 \sqcup A_2$ such that $\partial A_1 \cap \partial A_2 = \emptyset$, either A_1 or A_2 is empty.

Clearly, since balls of a (M)-space are connected, the strong isoperimetric inequality implies that connected sets are asymptotically isoperimetric (see also Theorem 7.6.1).

On the other hand, we will show that there exist graphs with polynomial growth whose connected subsets are not asymptotically isoperimetric : namely there exists an increasing sequence of integers (N_n) such that to optimize (asymptotically) the isoperimetric profile at these values, one has to take a sequence of subsets with an number of connected components that tends to infinity and such that the distance between these connected components also tends to infinity.

Remark 7.2.5. Note that all our conterexamples are far from being homogeneous. So many of the properties discussed in this paper should also be discussed in a more restrictive class of spaces such as spaces with fractal properties.

7.3 Isoperimetry at infinity : a general setting

7.3.1 Isoperimetry at a given scale

The purpose of this section is to find some minimal conditions under which "isoperimetric properties at infinity" are invariant under large-scale equivalence.

[1] Note that there exist unbounded fractals like the unbounded Serpinsky gasket [Str], with polynomial growth and with bounded asymptotic isoperimetric profile (this is a trivial fact).

In the introduction, namely in Section 7.1.1, we justified our definition of the boundary by the fact that we want it to have a uniform thickness. Nevertheless, it is not suffisant to our purpose : it is also important for the space X to look connected at scale h. Indeed, let X be a graph ; if $h = 1/2$, then every subset of X has a trivial boundary, so that all the isoperimetric properties of X are trivial.

Definition 7.3.1. *Let X be a metric space and fix $b > 0$. Let us call a b-chain of length n from x to y, a finite sequence $x_0 = x, \dots, x_n = y$ such that $d(x_i, x_{i+1}) \leq b$.*

The following definition can be used to study the isoperimetry at a given scale, although we will only use it "large-scale version" in this paper.

Definition 7.3.2.
Scaled version : Let $b > 0$ and $E_1 \gg b$. Let us say that X is uniformly b-connected at scale $\leq E_1$ if there exists a constant $E_2 \geq E_1$ such that for every couple $x, y \in X$ such that $d(x, y) \leq E_1$, there exists a b-chain from x to y totally included in $B(x, E_2)$.
Large-scale version : If, for all $E_1 \gg b$, X is uniformly b-connected at scale $\leq E_1$, then we say that X is uniformly b-connected (or merely uniformly connected).

Remark 7.3.3. Note that in the scaled version, the space X is allowed to have a proper nonempty subset A such that $d(A, A^c) > E_1$: in this case X is not b-connected at all.

Invariance under quasi-isometry : Note that if X is uniformly b-connected at scale $\leq E_1$ and if $f : X \to X'$ is a quasi-isometry of constants C_1 and C_2, then X' is uniformly $C_2 b + C_1$-connected at scale $\leq E_1/C_2 - C_1$. In particular, if X is uniformly b-connected, then X' is uniformly $(C_2 b + C_1)$-connected.

Remark 7.3.4. Let us write $d_b(x, y)$ for the b-distance from x to y, that is, the minimal length of a b-chain between x and y (note that if every couple of points of X can be joined by a b-chain, then d_b is a pseudo-metric on X).

If there exists $C > 0$ such that, for all $x, y \in X$, one has $d_b(x, y) \leq C d(x, y) + C$, then in particular, X is uniformly b-connected (we can call this property quasi-geodesic property).

Example 7.3.5. A graph and a Riemannian manifold are respectively uniformly 1-connected and uniformly b-connected for all $b > 0$.

Proposition 7.3.6. *Let X be a uniformly b-connected space at scale $\leq E_1$. Let h be such that $h \geq 2b$.*

(i) For every subset A of X and every $x \in A^c$ such that $d(x, A) < E_1$ (resp. $x \in A$ such that $d(x, A^c) < E_1$), there exists a point $z \in \partial_h A$ at distance $\leq E_2$ of x such that

$$B(z, b) \subset \partial_h A.$$

(ii) If, moreover, X is $(DV)_{loc}$ and $h \ll E_1$, then there exists a constant $C' \geq 1$ such that, for every subset A, there exists a family $(B(y_i, b))_i$ included in $\partial_h A$, such that, for all $i \neq j$, $d(y_i, y_j) \geq E_2$ and such that

$$\sum_i \mu(B(y_i, b)) \leq \mu(\partial_h A) \leq C' \sum_i \mu(B(y_i, b)).$$

(iii) The h-boundary measure of a subset of a $(DV)_{loc}$, uniformly b-connected space does not depend on h up to a multiplicative constant, provided $E_1 \gg h \geq 2b$.

Proof : Let $x \in A^c$ such that $d(x, A) < E_1$ and let $y \in A$ be such that $d(x, y) \leq E_1$. We know from the hypothesis that there exists a finite chain $x_0 = x, x_1, \ldots, x_n = y$ satisfying

– $x_n \in A$,

– $d(x, x_i) \leq E_2$ for all i,

– for all $1 \leq i \leq n$, $d(x_{i-1}, x_i) \leq b$.

Since $x \in A^c$ and $y \in A$, there exists $j \leq n$ such that $x_{j-1} \in A^c$ and $x_j \in A$. Clearly, $x_j \in A_b \cap [A^c]_b = \partial_b A$. But since $[\partial_b A]_b \subset \partial_{2b} A \subset \partial_h A$, the ball $B(x_j, b)$ is included in $\partial_h A$, which proves the first assertion.

Let us show the second assertion. Consider a maximal family of disjoint balls $(B(x_i, 2E_2))_{i \in I}$ with centers $x_i \in \partial_h A$. Then $(B(x_i, 5E_2))_{i \in I}$ forms a covering of $\partial_h A$.

Using the first assertion and the fact that $h \ll E_1$, one sees that each $B(x_i, 2E_2)$ contains a ball $B(y_i, b)$ included in $\partial_h A$. It is clear that the balls $B(y_i, 10E_2)$ form a covering of $\partial_h A$ and that the balls $(B(y_i, b)$ are disjoint. But, by property $(DV)_{loc}$, there exists $C' \geq 1$, depending on b and E_2, such that, for all $i \in I$

$$\mu(B(y_i, 10E_2)) \leq C' \mu(B(y_i, b)).$$

We deduce

$$\sum_i \mu(B(y_i, b)) \leq \mu(\partial_h A) \leq C' \sum_i \mu(B(y_i, b))$$

which proves (ii). The assertion (iii) now follows from (ii). ■

Remark 7.3.7. This proposition gives conditions to study isoperimetry at scale between b and E_1, i.e. choosing h far from those two bounds. Thus, we will always assume that this condition holds and we will simply write ∂A instead of $\partial_h A$. Otherwise, problems may happen. We talked about what can occur if $h < b$ at the beginning of this section. Now, let us give an idea of what can happen if $h > E_1$. Consider a metric measure space X such that $X = \cup_{i \in I} X_i$ where the X_i are subsets such that $d(X_i, X_j) \geq E_1$ whenever $i \neq j$ and such that $\mu(X_i)$ is finite for every $i \in I$ but not bounded. Note that for $h < E_1$ the boundary of every X_i is empty so that the family $(X_i)_{i \in I}$ is trivially asymptotically isoperimetric. But this can change dramatically if $h > E_1$ because the boundary of X_i can meet many X_j's for $j \neq i$.

Remark 7.3.8. If we replace uniformly b-connected at scale $\leq E_1$ by uniformly b-connected, then the proposition gives a setting adapted to the study of large scale isoperimetry. Namely, it says that for a uniformly b-connected, $(DV)_{loc}$ space, the choice of h does not matter, provided $h \geq 2b$.

Corollary 7.3.9. Let X be a $(DV)_{loc}$, uniformly b-connected space. If $h, h' \geq 2b$, we have

$$I_h \approx I_{h'}.$$

So, from now on, we will simply call "profile" (instead of h-profile) the equivalence class modulo $\approx$ of I_h. Note that the same holds for restricted profiles $I^{\downarrow}_{h,\mathcal{A}}$, and $I^{\uparrow}_{h,\mathcal{A}}$ that we will simply denote $I^{\downarrow}_{\mathcal{A}}$ and $I^{\uparrow}_{\mathcal{A}}$ (where $\mathcal{A}$ is a family of subsets of X).

The following theorem shows that a large-scale equivalence f with controlled constants essentially preserves all isoperimetric properties.

Theorem 7.3.10. Let $f\ (X, d, \mu) \to (X', d', \mu')$ be a large-scale equivalence (with constants C_1, C_2 and C_3) where X (resp. X') is $(DV)_{loc}$ and uniformly b-connected at scale $\leq E_1$ (resp. uniformly b'-connected at scale $\leq E_1'$). We suppose also that E_1 and E_1' are far larger than C_1, C_2, $C_2 b$ and $C_2(b' + C_1)$. Then, there exists a constant $K \geq 1$ such that, for any subset A of finite measure

$$\mu'(\partial[f(A)]_{C_1}) \leq K\mu(\partial A).$$

Proof : Let us start with a lemma.

Lemma 7.3.11. Let X be a $(DV)_{loc}$ space and fix some $\alpha > 0$. Then there exists a constant $c > 0$ such that, for all family $(B(x_i, \alpha))_{i \in I}$ of disjoint balls of X, there is a subset J of I such that $\forall j \in J$, the balls $B(x_j, 2\alpha)$ are still disjoint, and such that

$$\sum_{j \in J} \mu(B(x_j, 2\alpha)) \geq c \sum_{i \in I} \mu(B(x_i, \alpha)).$$

Proof : Let us consider a maximal subset J of I such that $(B(x_j, 2\alpha))_{j \in J}$ forms a family of disjoint balls. Then, by maximality, we get

$$\bigcup_{i \in I} B(x_i, \alpha) \subset \bigcup_{j \in J} B(x_j, 4\alpha).$$

We conclude thanks to property $(DV)_{loc}$. ■

To fix ideas, take $h = 2b$ and $h' = 2b'$. Assertion (ii) of Proposition 7.3.6 implies that there exists a family of balls $(B(y_i, b'))_i$ included in $\partial[f(A)]_{C_1}$ such that, for all $i \neq j$, $d(y_i, y_j) \geq E_2'$ and such that

$$\sum_i \mu(B(y_i, b')) \leq \mu(\partial_h[f(A)]_{C_1}) \leq C' \sum_i \mu(B(y_i, b')).$$

By the lemma, and up to changing the constant C', one can even suppose that $d(y_i, y_j) \gg C_2E_2$ for $i \neq j$.

For all i, let x_i be a element of X such that $d(f(x_i), y_i) \leq C_1$. The points x_i are then at distance $\gg E_2$ to one another. Moreover, since y_i is both at distance $\leq 2b + C_1$ of $f(A)$ and of $f(A^c)$, x_i is both at distance $\ll E_1$ of A and of A^c. So, by the assertion (i) of the proposition, there exists a ball $B(z_i, b)$ included in $\partial A \cap B(x_i, E_2)$. Since balls $B(x_i, E_2)$ are disjoint, so are the $B(z_i, b)$. The theorem then follows from property $(DV)_{loc}$ and from property of "almost-conservation" of the volume (property (b)) of large-scale equivalence. ■

Remark 7.3.12. Note that in the case of graphs, the condition $h \geq 2$ can be relaxed to $h \geq 1$ (the proposition and the theorem stay true and their proofs are unchanged).

Corollary 7.3.13. Under the hypotheses of the theorem, we have

(i) if the family $(A_i)_{i \in I}$ is asymptotically isoperimetric, then so is $(f(A_i)_b)_{i \in I}$;

(ii) if I and I' are the profiles of X and X' respectively, we get $I \approx I'$.

The corollary results immediately from the theorem and the following proposition. ■

Proposition 7.3.14. Let f be a large-scale equivalence between two $(DV)_{loc}$ spaces X and X'. Then for every subset A of X, there exists $C \geq 1$ such that

$$\mu(A) \leq C\mu'([f(A)]_{C_1}).$$

Proof : Consider a maximal family of disjoint balls $(B(y_i, C_1))_{i \in I}$ whose centers belong to $f(A)$. These balls are clearly included in $[f(A)]_{C_1}$. By property $(DV)_{loc}$, the total volume of these balls, and therefore $\mu'([f(A)]_{C_1})$, are comparable to the sum of the volumes of balls $B(x_i, 3C_1)_{i \in I}$ that form a covering of $[f(A)]_{C_1}$. The preimages of these balls thus cover A. But, for each i, $f^{-1}(B(y_i, 3C_1))$ is contained in a ball of radius $3C_1C_2 + C_2$ and of center x_i where $x_i \in f^{-1}(\{y_i\})$. By property $(DV)_{loc}$ and property of almost-conservation of the measure of small balls (property (b)) of f, the measure of this ball is comparable to that of $B(y_i, 3C_1)$. So we are done. ■

Finally, let us mention that if we suppose that X and X' are uniformly connected and satisfy the $(DV)_{loc}$ condition, then Theorem 8.2.4 and its corollary hold for any large-scale equivalence f.

7.3.2 Property (M) : monotone geodesicity

Let us introduce a natural (but quite strong) property of geodesicity.

Definition 7.3.15. Let us say that (X, d) has property (M) if there exists $C \geq 1$ such that, $\forall x \in X$, $\forall r > 0$ and $\forall y \in B(x, r + 1)$, we have $d(y, B(x, r)) \leq C$.

Remark 7.3.16. Let (X, d) be a (M) metric space. Then X has "monotone geodesics" (this is why we call this property (M)) : i.e. there exists $C \geq 1$ such that, for all $x, y \in X$, there exists a finite chain $x_0 = x, x_1, \ldots, x_n = y$ such that $\forall 0 \leq i < n$,

$$d(x_i, x_{i+1}) \leq C;$$

and

$$d(x_i, x) \leq d(x_{i+1}, x) - 1.$$

Consequently, $\forall r, k > 0$, $\forall y \in B(x, r+k)$, we have

$$d(y, B(x, r)) \leq Ck.$$

These two properties are actually trivially equivalent to property (M).

Recall (see [Gro4], p 2) that two metric spaces X and Y are said Hausdorff equivalent

$$X \sim_{Hau} Y$$

if there exists a (larger) metric space Z such that X and Y are contained in Z and such that

$$\sup_{x \in X} d(x, Y) < \infty$$

and

$$\sup_{y \in Y} d(y, X) < \infty.$$

Remark 7.3.17. It is easy to see that property (M) is invariant under Hausdorff equivalence. But on the other hand, property (M) is unstable under quasi-isometry. To construct a counterexample, one can quasi-isometrically embed $\mathbf{R}_+$ into $\mathbf{R}^2$ such that the image, equipped with the induced metric does not have property (M) : consider a curve starting from 0 and containing for every $k \in \mathbf{N}$ a half-circle of radius 2^k. So it is strictly stronger than quasi-geodesic property ([Gro4], p 7), which is invariant under quasi-isometry : X is quasi-geodesic if there exist two constants $d > 0$ and $\lambda > 0$ such that for all $(x, y) \in X^2$ there exists a finite chain of points of X

$$x = x_0, \ldots, x_n = y,$$

such that

$$d(x_{i-1}, x_i) \leq d, \quad i = 1 \ldots n,$$

and

$$\sum_{i=1}^{n} d(x_{i-1}, x_i) \leq \lambda d(x, y).$$

Example 7.3.18. A geodesic space has property (M), in particular graphs and complete Riemannian manifolds have property (M). A discretisation (i.e. a discrete net) of a Riemannian manifold X has property (M) for the induced distance.

Remark 7.3.19. Note that in general, if X is a metric measure space, we have

$$\partial_{1/2}B(x, r+1/2) \subset B(r+1) \setminus B(x,r).$$

Moreover, if X has property (M), then, we have

$$B(x, r+1) \setminus B(x,r) \subset \partial_C B(x, r+1).$$

Note that this is not true in general, even for quasi-geodesic spaces.

7.4 Link between isoperimetry of balls and strong isoperimetric inequality

7.4.1 Strong isoperimetric inequality implies (IB)

The spaces we will consider from now on will be $(DV)_{loc}$ and uniformly 1-connected. Let us write $\partial A = \partial_2 A$ for any subset A of a metric space X (note that these conventions are motivated by Proposition 7.3.6).

Let X be a metric measure space. Let V be a nondecreasing function belonging to the volume growth class (for instance $V(r) = \mu(B(x,r))$ for a $x \in X$). Write $\phi(t) = \inf\{r, V(r) \geq t\}$ for the "right inverse" function of V. Remark that if f and g are nondecreasing functions $\mathbf{R}_+ \to \mathbf{R}_+$, then $f \approx g$ if and only if their right inverses are equivalent. In particular, the equivalence class of ϕ is invariant under large-scale equivalence.

Definition 7.4.1. Let us call a strong isoperimetric inequality the following kind of isoperimetric inequality

$$\forall A \subset X, \quad |\partial A| \geq C^{-1}|A|/\phi(C|A|).$$

Remark that this is equivalent to

$$I \succeq id/\phi,$$

Therefore, if X satisfies a strong isoperimetric inequality, we will say that it has a strong profile.

Example 7.4.2. If X has polynomial growth of degree d, we have $\phi(t) \approx t^{1/d}$. So X has a strong profile if and only if

$$I \succeq (id)^{\frac{d-1}{d}}.$$

Write, for all $x \in X$ and for all $0 < r < r'$

$$C_{r,r'}(x) = B(x, r') \setminus B(x,r).$$

Proposition 7.4.3. Let X be a doubling space (here, no other hypothesis is required). There exists a constant $C \geq 1$ such that

$$\forall x \in X, \forall r \geq 1, \quad \inf_{r \leq r' \leq 2r} \mu(C_{r'-1,r'}) \leq C\mu(B(x,r))/r.$$

Proof : Clearly, it suffices to prove the proposition when $r = n$ is a positive integer. First, note that

$$\cup_{k=n}^{2n}(B(x,k) \smallsetminus B(x,k-1)) \subset B(x,2n).$$

So, we have

$$\mu(B(x,2n)) \geq n \inf_{n \leq k \leq 2n} \mu(B(x,k) \setminus B(x,k-1)).$$

We conclude by Doubling property. ■

Corollary 7.4.4. Let X be a uniformly connected doubling space. Then we have

$$I_{\mathcal{B}}^{\downarrow} \preceq id/\phi.$$

Namely, there exists a constant $C \geq 1$ such that

$$\forall x \in X, \forall r > 0, \quad \inf_{r' \geq r} \mu\left(\partial B(x,r')\right) \leq C\mu(B(x,r))/r.$$

Proof. This follows from Remark 7.3.19. ■

Corollary 7.4.5. Let X be a uniformly connected doubling space satisfying a strong isoperimetric inequality. Then, X is stably-(IB).

Proof : It follows from Corollary 7.4.4 and from Corollary 7.3.13. ■

Remark 7.4.6. Varopoulos [V] showed that the strong isoperimetric inequality is satisfied by any group of polynomial growth. Coulhon and Saloff-Coste [CouSa2] then proved it for any unimodular compactly generated locally compact group with a simple and elegant demonstration. We have the following corollary.

Corollary 7.4.7. A Cayley graph of a group of polynomial growth is stably-(IB).

7.4.2 The strong isoperimetric inequality does not imply strongly-(IB)

Note that this will result from the example shown in section 7.4.3. Let us present here a counterexample with linear growth.

For every integer n, we consider the following finite rooted tree G_n : first take the standard binary tree of depth n. Then stretch it as follows : replace each edge connecting a $k-1$'th generation vertex to a k'th generation vertex by a (graph) interval of length $2^{2^{n-k}}$. Then consider the graph G'_n obtained by taking two copies of G_n and identifying the vertices of last generation of the first copy with those of the second copy. Write r_n and r'_n for the two vertices of G'_n corresponding to the respective roots of the two copies of G_n. Finally, glue "linearly" the G'_n together identifying r'_n with r_{n+1}, for all n : it defines a graph X.

Let us show that X has linear growth (i.e. polynomial growth of degree 1). Thus $I \approx 1$, and since the boundary volume of balls is clearly not bounded, we do not have $I_B^{\uparrow} \preceq I$. In particular, X is not strongly-(IB).

Since X is infinite, it is enough to show that there exists a constant $C > 0$ such that

$$|B(x,r)| \leq Cr \tag{7.4.1}$$

for every vertex x of X. But it is clear that among the balls of radius r, those which are centered in points of n'th generation of a G_n for n large enough are of maximal volume. Let us take such an x. Remark that for $\sum_{j=0}^{k} 2^{2^j} \leq r \leq \sum_{j=0}^{k+1} 2^{2^j}$, we have

$$|B(x,r)| \leq 2 \mid B(x, \sum_{j=0}^{k} 2^{2^j}) \mid +2 \left(r - \sum_{j=0}^{k} 2^{2^j} \right)$$

So it is enough to show (7.4.1) for $r = \sum_{j=0}^{k} 2^{2^j}$. We have

$$\mu(B(x, \sum_{j=0}^{k} 2^{2^j})) = \sum_{j=0}^{k} 2.2^j.2^{2^{k-j}} \leq 4.2^{2^k}.$$

Which proves (7.4.1) with $C = 8$. ■

Remark 7.4.8. This example and that of section 7.4.3 show in particular that the strong isoperimetric inequality does not imply (even in linear growth case) strongly-(IB).

7.4.3 Instability of strongly-(IB) under quasi-isometry

Theorem 7.4.9. We can find a graph, quasi-isometric to $\mathbf{Z}^2$ (resp. a Riemannian manifold M bi-Lipschitz equivalent to $\mathbf{R}^2$) whose volume of spheres is not dominated by $r^{\log 3/\log 2}$ (where r is the radius).

Remark 7.4.10. The restriction to dimension 2 is not essential, but was made to simplify the exposition (actually, we merely need the dimension to be greater or equal to 2).

Proof : The general idea of the construction is to get a sequence of spheres which look like finitely iterated Von Koch curves. First, we will build a graph with weighted edges. Actually, this graph will be simply the standard Cayley

graph of $\mathbf{Z}^2$, and the edges will have lengths equal to 1 except for some selected edges which will have length equal to a small, but fixed positive number.
First step of the construction : Let us define a sequence (A_k) of disjoint subtrees of $\mathbf{Z}^2$ (which is identified to its usual Cayley graph). Let (e_1, e_2) be the canonical basis of $\mathbf{Z}^2$ and denote $S = \{\pm e_1, \pm e_2\}$. For every $k \geq 1$, let $a_k = (2^{2k}, 0)$ be the root of the tree A_k and define A_k by

$$x \in A_k \Leftrightarrow x = a_k + 2^k \varepsilon_0(x) + 2^{k-1}\varepsilon_1(x) + \ldots + 2^{k-i(x)}\varepsilon_{i(x)}(x) + r(x)\varepsilon_{i(x)+1}(x) \tag{7.4.2}$$

where
— $0 \leq i(x) \leq k-1$,
— $\varepsilon_j(x)$ belongs to S for every $0 \leq j \leq i(x)+1$ and is such that $\varepsilon_{j+1}(x) \neq -\varepsilon_j(x)$ (for $j \leq i(x)$),
— $r(x) \leq 2^{k-i(x)-1} - 1$.

It is easy to see that A_k is a subtree of $\mathbf{Z}^2$ and that the above decomposition of x is unique. In particular, we can consider its intrinsic graph metric d_{A_k} : let S_k be the sphere of center a_k and of radius $2^{k+1}-1$ for this metric. Clearly, $|S_k| \geq 3^{k-1}$.
Second step of the construction : We define a graph Y with weighted edges as follows : Y is the usual Cayley graph of $\mathbf{Z}^2$; all edges of Y have length 1 but those belonging to $A = \cup_k A_k$ which have length equal to 1/100. The measure on Y is the countable measure and the distance between two vertices v and w is the minimal length of a chain joining v to w, the length of a chain being the sum of the weights of its edges. Clearly, as a metric measure space, Y is large-scale equivalent to $\mathbf{Z}^2$.

For every $k \geq 2$, consider the sphere $S(a_k, r_k) = B(a_k, r_k+1) \smallsetminus B(a_k, r_k)$ of Y, where $r_k = (2^{k+1}-1)/100$.

Claim 7.4.11. We have $S_k \subset S(a_k, r_k)$, so that

$$\mu(S(a_k, r_k)) \geq 3^{k-1} \geq r_k^{\log 3/\log 2}.$$

Proof : Note that the claim looks almost obvious on a drawing. Nevertheless, for the sake of completeness, we give a combinatorial proof. Let us show that a geodesic chain in the tree A_k is also a minimizing geodesic chain in Y. Applying this to a geodesic chain between a_k and any element of S_k (which is of length r_k in Y), we have that $S_k \subset S(a_k, r_k)$, so we are done.

So let x be a vertex of A_k. By (7.4.2), we have

$$x = a_k + 2^k \varepsilon_0(x) + 2^{k-1}\varepsilon_1(x) + \ldots + 2^{k-i(x)}\varepsilon_{i(x)}(x) + r(x)\varepsilon_{i(x)+1}(x)$$

Let us show by recurrence on $d_Y(a_k, x)$ (which takes discrete values) that

$$d_Y(a_k, x) = d_{A_k}(a_k, x)/100 = (2^k + \ldots + 2^{k-i(x)} + r(x))/100 = \frac{2^{k+1}(1 - 2^{-i(x)-1} + r(x))}{100}$$

If $x = a_k$, there is nothing to prove. Consider $c = (c(0) = x, c(2), \ldots, c(m) = a_k)$ a minimal geodesic chain in Y between a_k and x. Clearly, it suffices to prove that $c \subset A_k$. Suppose the contrary. Let t be the largest positive integer such that $c(t)$ belongs to A_k and $c(t+1)$ does not. Let l be the smallest positive integer such that $c(t+l) \in A_k$, so that $(c(t+1), \ldots, c(t+l-1))$ is entirely outside of A_k. By recurrence, the chain $(c(t+l), \ldots, c(m))$ is in A_k. Thus we have

$$d_Y(x, a_k) = d_{A_k}(x, c(t))/10 + |c(t) - c(t+l)|_{\mathbf{Z}^2} + d_{A_k}(c(t+l), a_k)/100.$$

Since c is a minimal chain, we also have

$$d_Y(c(t), a_k) = |c(t) - c(t+l)|_{\mathbf{Z}^2} + d_{A_k}(c(t+l), a_k)/100.$$

The following lemma applied to $u = c(t)$ and $v = c(t+l)$ implies that $t = t+l$ which is absurd since it means that c is included in A_k. ■

Lemma 7.4.12. let u and v be in A_k. We have

$$|u - v|_{\mathbf{Z}^2} \geq (d_{A_k}(u, a_k) - d_{A_k}(v, a_k))/50.$$

Proof : We can of course assume that $d_{A_k}(u, a_k) \geq d_{A_k}(v, a_k)$. Let $u = u_1 + u_2$ and $v = v_1 + v_2$ with

$$u_1 = 2^k \varepsilon_0(u) + \ldots 2^{k-i(v)} \varepsilon_{i(v)}(u)$$

and

$$v_1 = 2^k \varepsilon_0(v) + \ldots 2^{k-i(v)} \varepsilon_{i(v)}(v).$$

Note that by construction,

$$d_{A_k}(u_1, a_k) = d_{A_k}(v_1, a_k)$$

and since A_k is a tree,

$$d_{A_k}(u, a_k) - d_{A_k}(v, a_k) = d_{A_k}(u_2, a_k) - d_{A_k}(v_2, a_k) \leq 2^{k-i(v)+2}. \qquad (7.4.3)$$

On the other hand, we have

$$|u - v|_{\mathbf{Z}^2} \geq ||u_1 - v|_{\mathbf{Z}^2} - |u_2 - v|_{\mathbf{Z}^2}|$$

First, assume that $u_1 \neq v_1$. Then, by (7.4.2), the projection of $u_1 - v_1$ along e_1 or e_2 is not zero and belongs to $2^{k-i(v)}\mathbf{N}$. Moreover, using the fact that $\varepsilon_{j+1}(u) \neq -\varepsilon_j(u))$ for every j, the same projection of $u_2 - v_2$ is (in $\mathbf{Z}^2$-norm) less than

$$2.(2^{k-i(v)-2} + 2^{k-i(v)-4} + \ldots = 2^{k-i(v)-1}(1 + 1/4 + 1/4^2 + \ldots) \leq 2/3.2^{k-i(v)}$$

Thus,

$$|u - v|_{\mathbf{Z}^2} \geq 2^{k-i(v)}/3.$$

So we are done.

Now, assume that $u_1 = v_1$. If $i(u) = i(v)$ or if $i(u) \leq i(v)+1$ and $\varepsilon_{i(v)+1}(u) = \pm\varepsilon_{i(v)+1}(v)$, then we have trivially

$$|u - v|_{\mathbf{Z}^2} = (d_{A_k}(u, a_k) - d_{A_k}(v, a_k)).$$

Otherwise, we have

$$u - v = u_2 - v_2 = (2^{k-i(v)-1} - r(v))\varepsilon_{i(v)+1}(u) + 2^{k-i(v)-2}\varepsilon_{i(v)+2}(u) + \ldots + r(u)\varepsilon_{i(u)+1}.$$

So, projecting this in the direction of $\varepsilon_{i(v)+2}(u)$, and since $\varepsilon_{i(v)+3}(u) \neq -\varepsilon_{i(v)+2}(u)$, we obtain

$$|u-v|_{\mathbf{Z}^2} = |u_2-v_2|_{\mathbf{Z}^2} \geq 2^{k-i(v)-2} - (2^{k-i(v)-4} + \ldots + 2^{k-i(u)} + r(u)) \geq 2^{k-i(v)-2} - 2^{k-i(v)-3} = 2^{k-i(v)-3}.$$

Together with 7.4.3, we get

$$|u - v|_{\mathbf{Z}^2} \geq 32(d_{A_k}(u, a_k) - d_{A_k}(v, a_k))$$

which proves the lemma. ■

Clearly, Y is quasi-isometric to $\mathbf{Z}^2$. It is not difficult (and left to the reader) to see that we can adapt the construction to obtain a graph.

Now, let us explain briefly how we can adapt the construction to obtain a Riemannian manifold bi-Lipschitz equivalent to $\mathbf{R}^2$. First, we embed $\mathbf{Z}^2$ into $\mathbf{R}^2$ in the standard way, so that A_k is now a subtree of $\mathbf{R}^2$. Let $\tilde{A}$ be the 1/100-neighborhood of A in $\mathbf{R}^2$. Let f be a nonnegative function defined on $\mathbf{R}^2$ such that $1 - f$ is supported by $\tilde{A}$, $f \geq a$ and $f(x) = a$ for all $x \in A$. Finally, define a new metric on $\mathbf{R}^2$ multiplying the Euclidean one by f. ■

7.4.4 Strongly-(IB) implies the strong isoperimetric inequality

The converse to Proposition 7.4.5 is clearly false (see the examples of the next section). However, one has

Proposition 7.4.13. Let X be a doubling (M)-space. Suppose moreover that there exists $x \in X$ such that the family of balls of center x is strongly asymptotically isoperimetric. Then we have

$$I^{\downarrow}_{\mathcal{B}} \succeq id/\phi.$$

In particular, X satisfies a strong isoperimetric inequality.

Proof. Since $(B(x, r))_r$ forms an asymptotically isoperimetric family, it is enough to show that there exists $c > 0$ such that

$$\mu(\partial B(x, r)) \geq c\frac{\mu(B(x, r))}{r}.$$

But, let us recall that property (M) implies that there exists $C > 0$ such that, for all $r > 0$

$$\mu(C_{r,r+1}(x)) \leq C\mu(\partial_1 B(x, r)).$$

Since $(B(x,r))_r$ forms an asymptotically isoperimetric family, there exists $C' \geq 1$, such that, for all $r' < r$

$$\mu(\partial B(x,r')) \leq C'\mu(\partial B(x,r)).$$

Using these two remarks, we get

$$\mu(B(x,r)) \leq CC'r\mu(\partial B(x,r)).$$

So we are done. ■

7.5 What can happen if the profile is not strong

All the metric measure spaces built in this section will be graphs with polynomial growth. For simplicity, we write $|A|$ for the cardinal of a finite subset A of a graph.

7.5.1 Bounded profile : connected isoperimetric sets are "controlled" by balls

We will say that a subset A of a metric space is metrically connected (we will merely say "connected" from now on) if there does not exist any nontrivial partition of $A = A_1 \sqcup A_2$ with $d(A_1, A_2) \geq 10$.

Let X be a uniformly 1/2-connected space, with bounded profile, and such that the measures of balls of radius 1/2 is more than a constant $a > 0$. Actually, we can ignore nonconnected sets. Indeed if (A_n) is an isoperimetric family, then the A_n have a bounded number of connected components : otherwise, by Proposition 7.1.13, the boundary of A_n would not be bounded (because the distinct connected components have disjoint 1-boundaries each one containing a ball of radius 1/2). It suffices to replace A_n by its connected component of maximal volume.

Claim 7.5.1. Let (X, d, μ) be a $(DV)_{loc}$, uniformly 1/2-connected space such that the measures of balls of radius 1/2 is more than $a > 0$ and whose profile I is bounded. Then, if (A_n) is an isoperimetric sequence of connected subsets of X, there exist a constant $C > 0$, some $x_n \in X$ and some $r_n > 0$ such that

$$\forall n, \quad B(x_n, r_n) \subset A_n \subset B(x_n, Cr_n).$$

Proof : To fix ideas, let us assume that $\partial A = \partial_1 A$ (for all $A \subset X$). Let y_n be a point of A_n and write $d_n = \sup_{y \in \partial A_n} d(y_n, y)$. Let $r \leq d_n$ be such that $C_{r,r+1}(y_n)$ intersects nontrivially ∂A_n (recall that $C_{r,r'}(x) = B(x,r') \setminus B(x,r)$). Then, by Proposition 7.3.6, there exists a constant $C \geq 1$ such that $C_{r-C,r+C}(y_n) \cap \partial A_n$ contains a ball of radius 1/2 and therefore has measure $\geq a$. Consequently, if $\delta_n = \sup\{r' - r; C_{r,r'}(y_n) \cap \partial A_n = \emptyset\}$, then

$$\mu(\partial A_n) \geq \frac{d_n}{2C\delta_n} a. \tag{7.5.1}$$

Since the boundary of A_n has bounded measure, there exists a constant $c > 0$ and, for all n, two positive reals r'_n and $r"_n$ such that $r"_n - r'_n \geq cd_n$ and $C_{r'_n, r"_n} \cap \partial A = \emptyset$.

Write $s_n = (r'_n + r"_n)/2$. Since A_n is connected, $C_{s_n - 10, s_n + 10}(x) \cap A_n$ is nonempty. But then, if $x_n \in C_{s_n-10,s_n+10}(x) \cap A_n$, we get

$$B\left(x_n, \frac{r"_n - r'_n}{2} - 10\right) \subset A_n.$$

On the other hand

$$A_n \subset B(x_n, 2d_n). \tag{7.5.2}$$

Write $r_n = cd_n/2 - 10$. The proposition follows from (7.5.1) and from (7.5.2). ■

7.5.2 Stably-(NIB) graphs with unbounded profile and where isoperimetric families can never be "controlled" by families of balls

Theorem 7.5.2. For every integer $d \geq 2$, there exists a graph X of polynomial growth of degree d, with unbounded profile, satisfying stably-(NIB) and such that, for all isoperimetric sequences (A_n), it is impossible to find sequences of balls $B_n = B(x_n, r_n)$ and $B'_n = B(x'_n, r'_n)$ of comparable radii (i.e. such that r'_n/r_n is bounded) such that

$$B_n \subset A_n \subset B'_n, \quad \forall n.$$

Consider the graph X obtained from $\mathbf{Z}^d$ deleting some edges. Consider, in the axis $\mathbf{Z}.e_1$, the intervals (S_n) of length $[\sqrt{n}]$ and at distance 2^n from one another. Consider the sequence (A_n) of full parallelepiped defined by the equations $x_1 \in I_n$ and $|x_i| \leq n/2$ for $i \geq 2$.

Then consider a partition of the boundary (in $\mathbf{Z}^d$) of A_n in $(d-1)$-dimensional cubes a_n^k whose edges have length approximatively $\sqrt{n}$. Remove all the edges that connect A_n to its complement but those connected to the "center" of a_n^k (here, the center of a_n^k is a point of $\mathbf{Z}^d$ we choose at distance ≤ 2 from the "true center" in $\mathbf{R}^n$ of the convex hull of a_n^k). We thus obtain a connected graph X. Note that the A_n are such that

$$|A_n| \approx n^{d-1}\sqrt{n}$$

and

$$|\partial_X A_n| \approx \frac{|\partial_{\mathbf{Z}^d} A_n|}{|a_n^0|} \approx n^{d-1}/(\sqrt{n})^{d-1} = (\sqrt{n})^{d-1}.$$

Write A for the union of A_i and A^c for its complement in X.

Claim 7.5.3. The growth in X is polynomial of degree d.

Proof : It will follow from the strong profile of balls. ■

Claim 7.5.4. The profile of X is not strong.

Proof : Let us consider the A_n. If the profile was strong, the sequence $u_n = \frac{|A_n|}{|\partial A_n|^{\frac{d}{d-1}}}$ would be bounded. But there exists a constant $c > 0$ such that

$$u_n \geq cn^{d-1}\sqrt{n}/(\sqrt{n})^d = cn^{\frac{d-1}{2}} \to \infty.$$

■

Claim 7.5.5. Let R be a unbounded subset of $\mathbf{R}_+$ and let $(P_r)_{r\in R}$ be a family of subsets such that there exist two constants $C \geq 1$ and $a > 0$ such that

$$\forall r > 0, \exists x_r \in X, \quad B(x_r, r/C) \subset [P_r]_a \subset B(x_r, Cr).$$

Then there exists a constant c' such that

$$\forall r > 0, \quad \mu(\partial P_r) \geq c'\mu(P_r)^{\frac{d-1}{d}}.$$

The following lemma and its proof will be useful in all examples that we will expose in the following sections. Write A^c for the complement of A (in X or, which is actually the same in $\mathbf{Z}^d$).

Lemma 7.5.6. The profile of A^c (or of A'^c) is strong. That means $I(t) \approx t^{\frac{d-1}{d}}$.

Proof of the lemma.

First of all, it is enough to consider only connected subsets P of A^c. Indeed, if P has many connected components $P_1 \ldots P_k$, then, by subadditivity of the function $\phi : t \to t^{\frac{d-1}{d}}$, if the P_i verify $|\partial P_i| \geq c\phi(|P_i|)$, then so do P.

Note that A^c embeds into X and into $\mathbf{Z}^d$. The idea consists in comparing the profile of A^c to that of $\mathbf{Z}^d$. First of all, let us assume that a connected subset P of A^c—seen in X—intersects the boundary of many A_n. Then, as $|A_n|$ is negligible compared to the distance between the A_n when n goes to infinity, the set of points of $\partial_{\mathbf{Z}^d} P$ at distance 1 of A has negligible volume compared to $|\partial P|$. Thus, if $|P|$ et n are large enough, we get

$$|\partial_{A^c} P| \geq \frac{1}{2}|\partial_{\mathbf{Z}^d} P|.$$

So it is enough to consider subsets meeting only one A_n. But the complement of a convex polyhedron of $\mathbf{Z}^d$ has trivially the same profile (up to a constant) as $\mathbf{Z}^d$. So we are done. ■

Proof of the claim 7.5.5. Let (P_r) be a family of subsets of X satisfying the condition of the proposition. We have to show that $\forall r$, $|\partial P_r| \geq c'|P_r|^{\frac{d-1}{d}}$. If $P \subset A^c$, the claim is a direct consequence of the lemma.

Suppose that P meets some A_n and that $r \geq 100C\sqrt{n}$. Then we have already seen (in the proof of Lemma 7.5.2) that if many A_n intersect P_r, the cardinal of the intersection of this P_r with A are negligible compared to its boundary provided n and $|P_r|$ are large enough. We can thus suppose that P_r meets only one A_n. Furthermore, since $r \geq 100\sqrt(n)$, there is some x' in $B(x_r, r/C)$ such that

$$B(x', r/10C) \in B(x_r, r/C) \cap A^c.$$

Then, observe that since $B(x', r/10C) \subset [P_r]_a$, there is a $B(x', r/10C) \subset [P_r]_a$, there is a constant $c > 0$ such that

$$|P_r \cap B(x', r/C)| \geq c|B(x', r/C)|. \tag{7.5.3}$$

It follows that the intersection of P_r with A^c has volume $\geq c'|P_r|$ where $\bar{c}$ is a constant depending only on C and a. So by Lemma 7.5.2, we have

$$|\partial_X P_r| \geq |\partial_{A^c}(P_r \cap A^c)| \geq c|P_r|^{\frac{d-1}{d}}.$$

We then have to study the case $r \leq 100C\sqrt{n}$. We can assume that $x_r \in A_n$ (otherwise, we conclude with Lemma 7.5.2). Let π be the orthogonal projection on the hyperplane $x_2 = 0$. Then for n large enough, Cr is smaller than $n/2$. Consequently, since $P_r \in B(x_r, Cr)$, every point of $\pi(P_r)$ has at least one antecedent in the boundary of P_r. So, we have

$$|\partial_X P_r| \geq |\pi(P_r)|.$$

Moreover, note that $\pi(B(x_r, r/C)) = B(\pi(x_r), r/C)$ (note that this ball lies in $\mathbf{Z}^{d-1}$). On the other hand, since the projection is 1-Lipschitz, we get

$$\pi([P_r]_a) \subset [\pi(P_r)]_a,$$

so

$$B(\pi(x_r), r/C) \subset [\pi(P_r)]_a.$$

Similarly to (7.5.3), we have

$$|\pi(P_r) \cap B(\pi(x_r), r/C| \geq c|B(\pi(x_r), r/C)|$$

So, finally, we have

$$|\partial_X P_r| \geq c' r^{d-1}$$

so we are done. ■

Corollary 7.5.7. In every space isometric at infinity to X, the volume of spheres $\approx r^{d-1}$. In particular, they are not asymptotically isoperimetric.

Proof of the corollary. Let $f : X' \to X$ a large-scale equivalence between two metric measure spaces X' and X and take $y \in X'$. It comes

$$B\left(f(y), \frac{r}{C_2} - C_1\right) \subset B([f(B(y,r))]_{C_1}) \subset B(f(y), C_2 r + C_1).$$

The corollary follows from Claim 7.5.5 and from Theorem 8.2.4. ■

7.5.3 Graphs stably-(NIB) with bounded profile

Theorem 7.5.8. For any integer $d \geq 2$, one can find a graph of polynomial growth of degree d, with bounded profile, and which is stably-(NIB).

The construction follows the same lines as in the previous section. Consider in $\mathbf{Z}^d$, a sequence (C_n) of subsets defined by

$$C_n = B(x_n, n) \cup B(x'_n, n)$$

where $x_n = (2^{n+1}, n - \log n, 0, \dots, 0)$ and $x'_n = (2^{n+1}, \log n - n, 0, \dots, 0)$.

We disconnect C_n from the rest everywhere but in the axis $\mathbf{Z}.e_1$. Let Y be the corresponding graph. C_n looks like a ball (of $\mathbf{Z}^d$) "constricted" at the equator. Indeed, every point of C_n belonging to the hyperplane $\{x_2 = 0\}$ is at distance at most $\log n$ from the boundary (in Y) of C_n. This is the property that will prevent C_n from being "deformed" into a ball. Write $C = \cup_n C_n$.

Lemma 7.5.9. The graph C^c has a strong profile.

Proof : The demonstration is essentially the same as for Lemma 7.5.2. ■

Claim 7.5.10. The growth in the graph X is polynomial of degree d.

Proof : We have to show that there exists a constant $c > 0$ such that, $\forall x, r$, $|B(x,r)| \geq cr^d$ (the converse inequality following from the fact that X embeds in $\mathbf{Z}^d$). Thanks to Lemma 7.5.9, we can suppose that B is included in a C_{n_0} so that its radius is $\leq n_0$.

The conclusion follows then from the next trivial fact : in $\mathbf{Z}^d$, if $r \leq n_0$, the volume of the intersection of a ball of radius n_0 with a ball of radius $r \leq n_0$ and of center belonging to the first ball is $\geq 2^{-d}|B(x,r)| \geq 2^{-10d}r^d$. Indeed, the worst case is when x is in a "corner" of the ball. So we are done. ■

Claim 7.5.11. If Y' is a (M)-space which is isometric at the infinity to Y, then its balls are not asymptotically isoperimetric.

Proof : The demonstration results from the following lemma and Proposition 7.1.13.

Lemma 7.5.12. Let $\mathbf{P}$ be an asymptotically isoperimetric family of connected subsets of X. Then there exists a constant $C \geq 1$ such that, for all $P \in \mathbf{P}$ of measure $> C$, there exists n such that $|P \bigtriangleup C_n| \leq C$.

Proof : Since the profile of C^c is strong, it is clear that for $|P|$ large enough, $P \cap C^c$ must be bounded. We then have to show that if (P_n) is a sequence of subsets such that for all n, $P_n \subset C_n$ and such that $|P_n|$ and $|C_n \setminus P_n|$ tends to infinity, then $|\partial P_n|$ also tends to infinity. Suppose, for instance that $|P_n| \leq |C_n \setminus P_n|$. But Theorem 8.2.4 makes clear that this problem in $\mathbf{Z}^d$ is equivalent to the similar problem in $\mathbf{R}^d$: that is, replacing C_n with its convex hull $\tilde{C}_n$ in $\mathbf{R}^d$. Since the $\tilde{C}_n$ are homothetic copies of $\tilde{C}_1$, by homogeneity, we only have to show that the profile $I(t)$ of $\tilde{C}_1$ is $\geq ct^{\frac{d-1}{d}}$ for $0 < t < |\tilde{C}_1|/2$, which is a known fact (see [Ros]). ■

Let us finish the demonstration of Claim 7.5.11. We now have to show that the sets C_n cannot be—up to a set of bounded measure—inverse images of balls by some large-scale equivalence. So let (X', d, μ) be a (M)-space and let $f : X \to X'$ be a large-scale equivalence.

Let us consider two points of C_n of respectively maximum and minimum x_2. The distance of each of these points to C^c is $\geq n/2$ and yet, every 1-chain joining them must pass through $C_n \cap \{x_2 = 0\}$ whose points are at distance $\leq 2 \log n$ from C^c. But this is impossible for a ball in a (M)-space. Indeed, in a ball $B = B(o, R)$ with $R \geq N$, if a point x is at distance cN from the boundary, then the points belonging to a ball centered in x and of radius $cN/2$ are at distance at least $cN/2$ from the boundary of B. But this ball intersects the ball centered in o and of radius $R - cN/2$. Moreover, by property (M), there exists a 1-chain joining x to o and staying in $B(o, R - cN/2)$, so at a distance of the order of N from boundary of B. ■

7.5.4 The instability of (IB) under quasi-isometry between graphs of polynomial growth

Theorem 7.5.13. Let d be an integer ≥ 2. There exists two graphs X and X' quasi-isometric, of polynomial growth of degree d and with bounded or unbounded profile, such that X satisfies (IB) but not X'.

Like in the examples of the two previous sections, we will build a graph X removing some edges from $\mathbf{Z}^d$: for $n \in \mathbf{N}$, let A_n be the ball of radius n whose center belongs to the axis $\mathbf{Z}.e_1$ in such a chain that A_{n+1} is at distance 2^n from A_n. We then remove all the edges of the boundary of A_n but those belonging to the line $\mathbf{Z}.e_1$. We write A for the union of A_n. The graph X' is obtained from X by taking its image by the linear map fixing the first coordinate and acting on the orthogonal as an homothetic transformation of ratio 4 (it is clear that it is a quasi-isometry). More precisely, we replace each edge of X parallel to the first axis, by a chain of length 2 also parallel to the first axis. Write A' for the image of A.

Remark 7.5.14. In the previous example, the profile is bounded. Nevertheless, one can slightly modify the construction in order to get an unbounded profile : for instance, removing only edges of the boundary of A_n at distance $\geq \log n$ from the axis $\mathbf{Z}.e_1$ (instead of those which are outside of this axis).

Claim 7.5.15. The graphs X and X' have polynomial growth of degree d.

As these graphs are subgraphs of $\mathbf{Z}^d$, their volume growths are less than the one of $\mathbf{Z}^d$. The converse inequality will follow from the fact that in X', the profile restricted to balls is strong and from the fact that X and X' are quasi-isometric. ■

Claim 7.5.16. In X, the balls are asymptotically isoperimetric.

Proof : It is clear by construction that the A_n are balls and that their boundaries have bounded volume. ■

Claim 7.5.17. In X', the profile restricted to balls is strong $I^{\downarrow}_{\mathcal{B}}(t) \approx t^{\frac{d-1}{d}}$. In particular, X' is not (IB).

Proof : Remark that Lemma 7.5.2 stays true in this context. Let $B = B(x,r)$ be a ball of the graph X'. We have to show that there exists a constant $c > 0$ such that

$$|\partial B| \geq c|B|^{\frac{d-1}{d}}.$$

According to Lemma 7.5.2, we can assume that $B \subset A$. Thus, there exists n_0 such that $B \subset A_{n_0}$.

Let us embed $\mathbf{Z}^d$ into $\mathbf{R}^d$. Let us replace the discrete polyhedron A_n and B by their convex hulls $\tilde{A_n}$ and $\tilde{B}$ in $\mathbf{R}^d$. Let $\tilde{X}$ be the space obtained removing from $\mathbf{R}^d$ (Euclidean) the points of the Euclidean boundary of $\tilde{A_n}$ (for all n) but the two ones belonging to the axis $\mathbf{R}.e_1$ (resp. those at distance $\leq \log n$ of the axe) for the case of bounded profile (resp. for the case of unbounded profile). Let us equip $\tilde{X}$ –seen as a subset of $\mathbf{R}^d$– with Lebesgue measure and with the geodesic metric $d(x,y) = \inf_\gamma l(\gamma)$ with γ taking values in the set of arcs joining x to y in $\tilde{X}$, $l(\gamma)$ being the Euclidean length of γ.

The embedding j of X into $\tilde{X}$ we obtain like this is clearly a large-scale equivalence.

For simplicity, we will write $|A|$ for the (Lebesgue) measure of a subset A of $\tilde{X}$. On the other hand, note that $\partial_{10}\tilde{B}$ contains $[j(B(x,r))]_1 \setminus [j(B(x,r-2))]_1$, which by Proposition 7.3.6 has same measure (up to multiplicative constant) as ∂B. The same holds for $\tilde{B}$ and B. Moreover, since $\tilde{B}$ and A_{n_0} are convex polyhedra, it is clear that the 10-boundary of $\tilde{B}$ has same measure (up to multiplicative constants) as its Euclidean boundary (whose measure is the limit when $h \to 0$ of $|\partial_h \tilde{B}|/h$). Write

$$|\partial_{eucl}\tilde{B}| = \lim_{h\to 0} |\partial_h \tilde{B}|/h$$

Consequently, it is enough to show that there exists $c > 0$ such that

$$|\partial_{eucl}\tilde{B}| \geq c|\tilde{B}|^{\frac{d-1}{d}}$$

Note that by homogeneity, the quantity

$$Q = \frac{1}{r^{d-1}}|\partial_{eucl}\tilde{B}|$$

only depends on the ratio n/r. Fix $n = n_0$. For r small enough (let us say $\leq r_c$ for some $r_c > 0$), $\tilde{B}$ never meets two parallel faces : Q stays larger than a constant

> 0 (i.e. profile of a $1/2^{d-1}$'th of space of $\mathbf{R}^d$). By compactness, it follows that Q reaches its minimum when x and r vary under the conditions $r_c \leq r \leq n_0/2$. On the other hand, as $\tilde{B}$ is strictly included in $\tilde{A_{n_0}}$, this minimum has to be > 0. The ratio Q is therefore larger than a constant $c' > 0$. finally, there is a constant $c > 0$ such that

$$|\partial_{eucl}\tilde{B}| \geq c' r^{d-1} \geq c|B|^{\frac{d-1}{d}}.$$

So we are done. ■

7.6 Asymptotic isoperimetry of connected subsets

Let X be a metric measure space. Set $\partial A = \partial_1 A$ and assume that X is uniformly 1/2-connected (see section 7.3.1). Recall that we say that a subset A of X is connected if there does not exist a nontrivial partition $A = A_1 \sqcup A_2$ with

$$\partial A_1 \cap \partial A_2 = \emptyset.$$

Write $\mathcal{C}$ for the set of connected subsets of finite measure of X.

Theorem 7.6.1.

(i) Let X be such that the measures of balls of radius 1/2 are bounded below by $a > 0$. Suppose that $I(t) = o(t)$. Then there exists a positive and increasing sequence (t_i) tending to infinity such that $I_{\mathcal{C}}^{\downarrow}(t_i) = I(t_i)$.

(ii) Assume that X is a doubling (M)-space and has a strong profile. Then $I_{\mathcal{C}}^{\downarrow} \approx I$.

(iii) Let d be an integer ≥ 2. There exists a graph X of polynomial growth of degree d and a increasing sequence of integers (N_n) such that $I(N_n) = o(I_{\mathcal{C}}^{\downarrow}(N_n))$.

Proof :

Note that (ii) follows from Corollary 7.4.5 and from the fact that property (M) implies that balls are connected.

Let us show the first assertion of the theorem. Suppose that there exists $T > 0$ such that $\forall t \geq T$, $I(t) < I_{\mathcal{C}}^{\downarrow}(t)$. We will show that it implies that

$$I(t) \geq a\frac{t}{T}. \tag{7.6.1}$$

Write t_m for the upper bound of the set of t such that $\forall s \leq t$, one has $I(s) \geq a\frac{s}{T}$. Since I is nondecreasing, if t_m is finite, then it is a maximum.

Remark that $t_m \geq T$ since the boundary of every nonempty subset of X contains a ball of radius 1/2 (see Proposition 7.3.6) and therefore has measure $\geq a$.

Suppose by contradiction that t_m is finite. By definition of t_m, for all $s > t_m$ there exists a subset A such that

$$\mu(A) \geq s$$

and

$$\mu(\partial A) < as/T.$$

Moreover, since $t_m \geq T$, we can suppose that

$$\mu(\partial A) < I_{\mathcal{C}}^{\downarrow}(s)$$

(in particular, A is not connected).

It follows that there exists a smallest positive integer k such that there exist $t_m \leq s \leq t_m + T/2$ and a subset A of measure $\geq s$, with k connected components and whose boundary has measure $< \min\{I_{\mathcal{C}}^{\downarrow}(s), sa/T\}$. Let A be such a subset. Note that $k \geq 2$. Thus, we have

$$A = A_1 \sqcup A_2$$

with $d(A_1, A_2) \geq 10$.

Since k is minimal, one has, for $i = 1, 2$

$$\mu(A_i) < t_m.$$

Indeed, if for instance, one had $\mu(A_1) \geq t_m$, then since the boundary of A_2 has measure $\geq a$, one would have

$$\mu(\partial A_1) \leq (t_m + T/2)\frac{a}{T} - a = \frac{t_m a}{T} - a/2 < \frac{t_m a}{T}.$$

Therefore, as $I_{\mathcal{C}}^{\downarrow}(t_m) \geq I(t_m) \geq t_m a/T$, one would also have

$$\mu(\partial A_1) < I_{\mathcal{C}}^{\downarrow}(t_m).$$

But then, by minimality of k, A_1 should have at least k connected components, which is absurd since it has strictly less components than A.

But, by definition of t_m, this implies that

$$\begin{aligned} \mu(\partial A) &= \mu(\partial A_1) + \mu(\partial A_2) \\ &\geq \frac{\mu(A_1)a}{T} + \frac{\mu(A_2)a}{T} \\ &= \frac{\mu(A)a}{T} \end{aligned}$$

which is absurd. ■

In order to show the second assertion of the theorem, we proceed as in the previous sections : we start from the graph $\mathbf{Z}^d$, and then we remove some edges. Let us consider the following family of cubes $(C_n^m)_{0 \leq m \leq n-1, n \in \mathbf{N}^*}$ of $\mathbf{Z}^d$: the C_n^m are Euclidean cubes of edges' length 2^{2^n} whose centers are disposed along the axis $\mathbf{Z}.e_1$ as follows : C_n^{m+1} is the image of C_n^m by the translation of vector $n2^{2^n}.e_1$ and C_n^{n-1} and C_{n+1}^1 are at distance $(n+1)2^{2^{(n+1)}}$ to one another. To build the graph X, we remove all the edges joining C_n^m to the rest of the graph but those which have a vertex belonging to the Euclidean cube c_n^m of dimension $d-1$ of the boundary of C_n^m, of volume 2^{n^2} and centered in one of the two intersection points of C_n^m with the axis $\mathbf{Z}.e_1$. Write C for the union of cubes C_n^m.

Claim 7.6.2. The growth in X is polynomial of degree d.

Proof : Let $B = B(x,r)$ be a ball. Let us prove that $|B| \geq 2^{-100d} r^d$. If the center of B doesn't belong to any C_n^m, it is clear. Suppose therefore that $x \in C_{n_0}^{m_0}$ for integers n_0 and $m_0 < n_0$. Write D_{n_0} for the diameter of $C_{n_0}^{m_0}$. If $r \geq 3D_{n_0}$, then B contains $B(y, r/2)$ with y belonging to no C_n^m. So we are brought back to the previous case. In the other case, the conclusion follows from the following trivial fact : in $\mathbf{Z}^d$, if $r \leq n$, the volume of the intersection of a cube C of edges' length equal to n with a ball of radius $r \leq n$ and of center $x \in C$ is $\geq 2^{-d}|B(x,r)| \geq 2^{-10d} r^d$. Indeed, the worst case is when x is a corner of the cube. ∎

Claim 7.6.3. Take $N_n = n2^{2^n}$. Then $I(N_n) = o(I_C^{\downarrow}(N_n))$.

Proof : Let us consider the set $C_n = \cup_m C_n^m$. Its volume is equal to N_n and its boundary has volume equal $n2^{n^2}$. On the other hand, let n_1 be an integer and let P be a connected subset of volume $\geq N_{n_1}$. We want to show that $|\partial P| \geq c2^{(n_1+1)^2}$, for a constant $c > 0$, which is clearly enough to conclude.

Thanks to the following lemma, the only remaining case to consider is when P meets a cube C_n^m. But, because of the large distance between two such cubes, we can assume that P meets only one of these cubes, say $C_{n_0}^{m_0}$.

Lemma 7.6.4. The profile of the graph C^c is strong (i.e. $\approx t^{\frac{d-1}{d}}$).

(same demonstration as for Lemma 7.5.2)

If $|P \cap C^c| \geq |P|/2$, then the lemma applied to $P \cap C^c$ allows to conclude. Suppose therefore that $|P \cap C| \geq |P|/2$. This implies in particular that $n_0 \geq n_1+1$. We then remark that $|\partial(P \cap C_{n_0}^{m_0})| \leq |\partial P|$. Indeed, let π be the orthogonal projection onto the hyperplane containing $c_{n_0}^{m_0}$, then every point of $c_{n_0}^{m_0} \cap P$ admits un antecedent by π belonging to the boundary of P. So we can assume that $P \subset C_{n_0}^{m_0}$. If $|P| \leq 3/2|C_{n_0}^{m_0}|$, then there exists $c > 0$ such that

$$|\partial P| \geq c|P|^{\frac{d-1}{d}} \tag{7.6.2}$$

(isoperimetry in the full Euclidean cube : see [Ros]). Otherwise, assume that $|P| \geq 3/2|C_{n_0}^{m_0}|$ and write $Q = C_{n_0}^{m_0} \setminus P$.

- If the volume of Q is $\geq D_{n_0}/2$ where D_{n_0} is the diameter of $C_{n_0}^{m_0}$, then 7.6.2 applied to Q implies that

$$|\partial Q| \geq c2^{(d-1)2^{n_0}/d} \geq c2^{2^{n_0-1}} \geq c2^{n_0^2} = c2^{(n_1+1)^2}.$$

But, the boundary of Q is—up to points belonging to $c_{n_0}^{m_0}$ (whose cardinal is negligible compared to $c2^{2^{n_0-1}}$)—equal to the boundary volume of P. So we are done.

– If $|Q| \leq D_{n_0}/2$, then every point of $c_{n_0}^{m_0}$ has preimages in ∂P by the projector π. But $|c_{n_0}^{m_0}| = 2^{n_0^2} = 2^{(n_1+1)^2}$, which ends the demonstration. ■

Chapitre 8

Mesure des sphères dans un espace doublant et application à la théorie ergodique

Résumé

Let G be a compactly generated locally compact group and let U be a compact generating set. We prove that if G has polynomial growth, then $(U^n)_{n\in\mathbf{N}}$ is a Følner sequence and we give a polynomial estimate of the rate of decay of

$$\frac{\mu(U^{n+1} \smallsetminus U^n)}{\mu(U^n)}.$$

Our proof is based on doubling property. As a matter of fact, the result remains true in a wide class of doubling metric measured spaces including manifolds and graphs. As an application, we obtain a balls averages L^p-pointwise ergodic theorem for probability G-spaces, with G of polynomial growth and for all $1 \leq p < \infty$.

8.1 Introduction

Let G be a compactly generated, locally compact (cglc) group endowed with a left Haar measure μ. Recall that a sequence $(A_n)_{n\in\mathbf{N}}$ of subsets of a locally compact group G is said to be Følner if for any compact set K,

$$\mu(K.A_n \vartriangle A_n) = o(\mu(A_n)).$$

Let U be a compact generating set of G (we mean by this that $\cup_{n\in\mathbf{N}} U^n = G$), non necessarily symmetric. If $\mu(U^n)$ grows exponentially, it is easy to see that the sequence $(U^n)_{n\in\mathbf{N}}$ *cannot* be Følner. On the other hand, if $\mu(U^n)$ grows subexponentially, then there exists trivially a sequence $(n_i)_{i\in\mathbf{N}}$ of integers such that $(U^{n_i})_{i\in\mathbf{N}}$ is Følner. But it is not clear whether the whole sequence $(U^n)_{n\in\mathbf{N}}$ is Følner. This was first conjectured for amenable groups by Greenleaf in 1969 ([Gre], p 69), who also proved it with Emerson [EmGr] in the Abelian case, correcting a former proof of Kawada [Kaw] (see also Proposition 8.5.2). The conjecture is actually not true for all finitely generated amenable groups since there exist amenable groups with exponential growth (for instance, all solvable groups which are not virtually nilpotent). Nevertheless, the conjecture is still open for groups with subexponential growth. In 1983, Pansu [Pa1] proved it for nilpotent finitely generated groups[1]. In fact, he proved that $\mu(U^n) \sim Cn^d$, for a constant $C = C(U) > 0$, which clearly implies that $(U^n)_{n\in\mathbf{N}}$ is Følner. In this article, we prove the conjecture for all compactly generated groups with polynomial growth. More precisely, we prove the following theorem : there exist $\delta > 0$ and a constant $C > 0$, such that

$$\mu\left(U^{n+1} \smallsetminus U^n\right) \leq Cn^{-\delta}\mu(U^n).$$

Interestingly, our proof works in a much more general setting. Recall that a metric measure space (X, d, μ) satisfies the doubling condition (or "is doubling") if there exists a constant $C \geq 1$ such that

$$\forall r > 0, \forall x \in X, \quad \mu(B(x, 2r)) \leq C\mu(B(x, r))$$

where $B(x, r) = \{y \in X, d(x, y) < r\}$. Let $S(x, r)$ denote the "1-sphere" of center x and radius r, i.e. $S(x, r) = B(x, r + 1) \smallsetminus B(x, r)$. Actually, we prove a similar result for doubling metric measured spaces satisfying a weak geodesic property we will call property (M) (see § 8.3.2). In this setting, the result becomes : there exist $\delta > 0$ and a constant $C > 0$, such that

$$\forall x \in X, \forall r > 0, \quad \mu(S(x, r)) \leq Cr^{-\delta}\mu(B(x, r)).$$

In particular, the conclusion of this theorem holds for metric graphs and Riemannian manifolds satisfying the doubling condition.

[1] In [Bre], Breuillard recently generalized the theorem of Pansu, which now holds for general cglc groups of polynomial growth.

In the case of metric measured spaces, our result is somewhat optimal, since in [Tes0], we build a graph X, quasi-isometric to $\mathbb{Z}^2$, such that there exist $0 < a < 1$, an increasing sequence of integers $(n_i)_{i\in\mathbf{N}}$ and $x \in X$ such that

$$|B(x, n_i + 1) \smallsetminus B(x, n_i)| \geq c|B(x, n_i)|/n_i^a \quad \forall i \in \mathbf{N}.$$

Moreover, we will see that our assumptions on X, that is, doubling condition and property (M) (see Definition 8.2.1 below) are also optimal in some sense.

An interesting and historical motivation (see for instance [Gre]) for finding Følner sequences in groups comes from ergodic theory. As a consequence of our result, we obtain a L^p-pointwise ergodic theorem ($1 \leq p < \infty$) for the balls averages, which holds for any cglc group G of polynomial growth (see theorem 8.4.3). We refer to a recent survey of A. Nevo [N] for more details and complete proofs.

8.2 Følner sets in metric measured spaces : statement of the results

8.2.1 Metric measured spaces

Let $X = (X, d, \mu)$ be a metric measured space, with μ a σ-compact Borel measure on (X, d). Recall that X is said to be doubling if there exists $C \geq 1$ such that

$$\mu(B(x, 2r)) \leq C\mu(B(x, r)) \quad \forall x \in X, \forall r > 0.$$

We will ask X to have a slightly weaker property than being a length space :

Definition 8.2.1. Let us say that (X, d) has property (M) if there exists $C \geq 1$ such that, $\forall x \in X$, $\forall r > 0$ and $\forall y \in B(x, r + 1)$, we have $d(y, B(x, r)) \leq C$.

Remark 8.2.2. Let (X, d) be a (M) metric space. Then X has got "monotone geodesics" (that is why we call this property (M)) : i.e. there exists $C \geq 1$ such that, for all $x, y \in X$, there exists a finite chain $x_0 = x, x_1, \ldots, x_m = y$ such that for $0 \leq i < m$,

$$d(x_i, x_{i+1}) \leq C;$$

and

$$d(x_i, x) \leq d(x_{i+1}, x) - 1.$$

Consequently, $\forall r, s \geq 1$, $\forall y \in B(x, r + s)$, we have

$$d(y, B(x, r)) \leq Cs.$$

These two properties are actually trivially equivalent to property (M).

Recall (see [Gro4], p. 2) that two metric spaces X and Y are said Hausdorff equivalent

$$X \sim_{Hau} Y$$

if there exists a (larger) metric space Z such that X and Y that X and Y are contained in Z and such that

$$\sup_{x\in X} d(x,Y) < \infty$$

and

$$\sup_{y\in Y} d(y,X) < \infty.$$

It is easy to see that property (M) is invariant under Hausdorff equivalence. But on the other hand, property (M) is unstable under quasi-isometry : one can easily find a quasi-isometric embedding of $\mathbb{R}$ into $\mathbb{R}^2$ such that the image, equipped with the induced metric does not have property (M). So (M) is strictly stronger than the quasi-geodesic property ([Gro4], p. 7), which is invariant under quasi-isometry : X is quasi-geodesic if there exist two constants $d > 0$ and $\lambda > 0$ such that for all $(x,y) \in X^2$ there exists a finite chain of points of X

$$x = x_0, \dots, x_m = y,$$

such that

$$d(x_{i-1}, x_i) \le d, \quad i = 1 \dots m,$$

and

$$\sum_{i=1}^{n} d(x_{i-1}, x_i) \le \lambda d(x,y).$$

Example 8.2.3. A length space has property (M), so do graphs and Riemannian manifolds. A discretisation (i.e. a discrete net) of a Riemannian manifold M has property (M) for the induced distance.

Our main result says that in a doubling (M) space, balls are Følner sets.

Theorem 8.2.4. Let $X = (X, \mu, d)$ be a doubling, (M) metric measured space. Then, there exists $\delta > 0$ and a constant $C > 0$ such that, $\forall x \in X$ and $\forall n \in \mathbf{N}$

$$\mu\left(B(x, n+1) \smallsetminus B(x,n)\right) \le C n^{-\delta} \mu(B(x,n)).$$

In particular, the ratio $\mu(B(x,n+1) \smallsetminus B(x,n))/\mu(B(x,n))$ tends to 0 uniformly in x when n goes to infinity.

Let us conclude this section by some remarks about the optimality of the assumptions.

Remark 8.2.5. First, note that the doubling assumption cannot be replaced by polynomial growth. Indeed, for every integer n, consider the following finite rooted tree G_n of root o_n : first, take the standard trivalent tree of depth n. Then stretch it as follows : replace each edge connecting a $k-1$'th generation vertex to a k'th generation vertex by a (graph) interval of length 2^{n-k}. Now, to make just one connected infinite graph G, glue G_n and G_{n+1} together for every

$n \in \mathbf{N}$, identifying o_{n+1} with some vertex of the sphere $S(o_n, 2^{n+1})$ of G_n. It is very easy to embed G into $\mathbb{Z}^2$. Thus, the growth of G is bounded by the one of $\mathbb{Z}^2$, so is polynomial. On the other hand,

$$|S(o_n, 2^{n+1} - 1)| = 3^{n-1}$$

and

$$|B(o_n, 2^{n+1})| = \sum_{j=0}^{n} 2^k 3^{n-k} \leq 2.3^{n+1}.$$

So balls are not Følner in G.

Remark 8.2.6. Another interesting point is the fact that property (M) cannot be replaced by any quasi-isometry invariant property like quasi-geodesic property. Indeed, one can very easily build a counterexample, embedding quasi-isometrically $\mathbb{R} \times [0, 1]$ into $\mathbb{R}^2$. In particular, balls being Følner sets is not invariant under quasi-isometry.

8.2.2 An interesting particular case : locally compact groups with polynomial growth

Let (G, μ) be a cglc group endowed with a Haar measure μ. Let U be a compact generating set of G. Define a left invariant distance d on G by :

$$\forall x, y \in G, \quad d(x, y) = \inf\{n \in \mathbf{N}, yx^{-1} \in U^n\}.$$

Note that unless U is symmetric (i.e. $U^{-1} = U$), d is not really a distance since we do not have : $d(x, y) = d(y, x)$. Nevertheless, d is "weakly" symmetric, i.e. there exists a constant $C > 0$ such that

$$\forall x, y \in X, \quad d(x, y) \leq C d(y, x)$$

In fact, we could prove Theorem 8.2.4 only supposing that d is weakly symmetric. But for simplicity, we only wrote the proof in the true metric setting.

Let us start with some generalities. First, note that up to replacing U by U^m, for some fixed $m > 0$, we can assume that $1 \in U$, so that the sequence $(U^n)_{n \in \mathbf{N}}$ is nondecreasing. More generally, we have the following simple fact.

Proposition 8.2.7. Let G be a cglc group and let U and V be two compact sets such that U generates G. Then there exists $m \in \mathbf{N}^*$ such that, for all $n \geq m$, $V \subset U^n$.

Proof : First, note that by a simple Baire argument, U^n contains a nonempty open set for n big enough. Let Ω be a nonempty open subset of U^n. Then, for n big enough, U^n contains the inverse of a given element of Ω. Thus, U^{n+1} contains an open neighborhood of 1. Let $\Omega.x_i$ be a finite covering of V. For n big enough, we can suppose that $x_i \in U^n$, so actually, $V \subset U^{2n+2}$. ∎

Let us say that G has polynomial growth if there exist a generating set U, $D > 0$ and a constant $C \geq 1$ such that

$$\mu(U^n) \leq Cn^D.$$

Theorem 8.2.8. Let G be a cglc group of polynomial growth, then it has strictly polynomial growth. Precisely, there exist a nonnegative integer d, independent of the generating compact set U, and a constant $C = C(U) \geq 1$ such that

$$C^{-1}n^d \leq \mu(U^n) \leq Cn^d. \tag{8.2.1}$$

For the sake of completeness (and also because we have not been able to find it in the litterature), we give a proof of this difficult theorem in appendix. This result is due to Guivarc'h [Gui], Jenkins [Jen], Gromov [Gro1], Losert [Lo] and makes a crucial use of a structure theorem due to Wang [Wa] and Mostow [M].

In the group setting, we obtain a slightly improved version of Theorem 8.2.4.

Theorem 8.2.9. Let G be a cglc group of polynomial growth. Consider a sequence $(U_n)_{n\in\mathbf{N}}$ of measurable subsets such that there exists two generating compact sets K, K' such that, for all $n \in \mathbf{N}$,

$$K \subset U_n \subset K'.$$

Write

$$N_n = U_n.U_{n-1}\ldots U_0 \quad \forall n \in \mathbf{N}$$

Then, there exist $\delta > 0$ and a constant $C \geq 1$ such that

$$\mu\left(N_{n+1} \smallsetminus N_n\right) \leq Cn^{-\delta}\mu(N_n) \quad \forall n \in \mathbf{N}^*.$$

In particular, the sequence $(N_n)_{n\in\mathbf{N}}$ is Følner.

The following corollary is a also a corollary of Theorem 8.2.4.

Corollary 8.2.10. Let G be a cglc group of polynomial growth, and U be a compact generating set of G. Then, there exist $\delta > 0$ and a constant $C \geq 1$ such that

$$\mu\left(U^{n+1} \smallsetminus U^n\right) \leq Cn^{-\delta}\mu(U^n) \quad \forall n \in \mathbf{N}^*.$$

In particular, the sequence $(U^n)_{n\in N}$ is Følner.

In fact, we will not use the full contents of Theorem 8.2.8. All we really need is Doubling Property : the existence of a constant $C = C(U) \geq 1$ such that

$$\mu(U^{2n}) \leq C\mu(U^n) \quad \forall n \in \mathbf{N}.$$

It clearly results from Strict Polynomial Growth. On the other hand, Doubling Property implies trivially Polynomial Growth. Unfortunately, the converses have to this day found no elementary proofs and require to prove Theorem 8.2.8.

8.3 Proofs

We will start proving Theorem 8.2.4 which is our "more general result". Nevertheless, Theorem 8.2.9 is not an immediate consequence of the group version of Theorem 8.2.4, that is, Corollary 8.2.10. So for the convenience of the reader, we will give a proof of Corollary 8.2.10 using notations adapted to the group setting, and then give the additional argument which is needed to obtain Theorem 8.2.9.

8.3.1 A preliminary observation

The following observation is one of the main ingredients of the proofs.

Lemma 8.3.1. Let $X = (X, \mu)$ be a measured space. Let us consider an increasing sequence $(A_n)_{n\in\mathbf{N}}$ of measurable subsets of X. Define $C_{n,n+k} = A_{n+k} \smallsetminus A_n$. We suppose that $\mu(A_n)$ is finite and unbounded with respect to $n \in \mathbf{N}$. Let us suppose that there exists a constant $\alpha > 0$ such that, for all integers $k \leq n$,

$$\mu(C_{n-k,n}) \geq \alpha.\mu(C_{n,n+k}). \tag{8.3.1}$$

Then, there exist $\delta > 0$ and a constant $C \geq 1$ such that $\forall n \geq 1$

$$\frac{\mu(C_{n-1,n})}{\mu(A_n)} \leq Cn^{-\delta}.$$

Proof : Write $i_n = [\log_2 n]$. For $i \leq i_n$, define $b_i = \mu(C_{n-2^i,n})$. Note that

$$C_{n-2^i,n} = C_{n-2^i,n-2^{i-1}} \cup \ldots \cup C_{n-1,n} \quad \forall i \leq i_n$$

and that the reunion is piecewise disjoint. So we have

$$b_i = \mu(C_{n-2^i,n-2^{i-1}}) + \ldots + \mu(C_{n-1,n}).$$

On the other hand, by 8.3.1

$$\begin{array}{rcl} \mu(C_{n-2^i,n-2^{i-1}}) & = & \mu(C_{n-2^{i-1}-2^{i-1},n-2^{i-1}}) \\ & \geq & \alpha.\mu(C_{n-2^{i-1},n-2^{i-1}+2^{i-1}}) \\ & = & \alpha.b_{i-1}. \end{array}$$

But note that

$$b_i = b_{i-1} + \mu(C_{n-2^i,n-2^{i-1}})$$

So

$$b_i \geq (1+\alpha)b_{i-1}.$$

Therefore

$$b_i \geq (1+\alpha)^i \mu(C_{n-1,n}).$$

Thus, it comes

$$\begin{aligned}
\mu(A_n) &\geq b_{i_n} \\
&\geq (1+\alpha)^{i_n}\mu(C_{n-1,n}) \\
&\geq \frac{1}{2}(1+\alpha)^{\log_2 n}\mu(C_{n-1,n}) \\
&\geq \frac{1}{2}n^{\log_2(1+\alpha)}\mu(C_{n-1,n}).
\end{aligned}$$

So we are done. ■

8.3.2 The case of metric measured spaces : proof of Theorem 8.2.4

For all $x \in X$ and $r' > r > 0$, write

$$C_{r,r'}(x) = B(x,r') \smallsetminus B(x,r),$$

and

$$c_{r,r'}(x) = \mu(C_{r,r'}(x)).$$

Thanks to lemma 8.3.1, we only need to prove that shells are doubling, i.e. that there exists a constant $\alpha > 0$ such that, for all $x \in X$, and for any integers $n > k > 10C$ (where C is the constant that appears in the definition of property (M))

$$c_{n-k,n}(x) \geq \alpha c_{n,n+k}(x).$$

So it is enough to prove the following lemma :

Lemma 8.3.2. Let (X,d,μ) a doubling, (M) space. Then, $\forall x \in X$ and for all couples of integers $10C < k \leq r$

$$c_{n-k,n}(x) \geq \alpha . c_{n,n+k}(x).$$

Proof : Let y be in $C_{n,n+k}(x)$. Consider a finite chain $x_0 = y, x_1, \ldots, x_m = x$ such that for $0 \leq i < m$

$$d(x_i, x_{i+1}) \leq C;$$

and

$$d(x_{i+1}, x) \leq d(x_i, x) - 1.$$

Clearly, $x_{2k} \in B(x, n-k/2)$, so that

$$d(y, B(x, r-k/2)) \leq d(y, x_{2k}) \leq 2Ck.$$

Let k_0 be the smallest integer such that $x_{k_0} \in B(x, r-k/2)$. Since $y \in C_{r,r+k}(x)$, k_0 exists and is less than $2k$. Moreover, minimality of k_0 implies that $x_{k_0} \in C_{n-k/2-C,n-k/2}(x)$. So we have

$$d(y, C_{n-k/2-C,n-k/2}(x)) \leq 2Ck. \tag{8.3.2}$$

Let $(z_i)_i$ be a maximal family of k-separated points in $C_{n-k/2-C,n-k/2}(x)$. Clearly, $C_{n-k/2-C,n-k/2}(x)$ is covered by the balls $B(z_i, 2k)$. Consequently, (8.3.2) implies that the balls $B(z_i, (2+2C)k)$ cover $C_{n,n+k}(x)$. On the other hand, for k large enough, the balls $B(z_i, k/2)$ are included in $C_{n-k,n}(x)$. Moreover, they are disjoint. So we conclude by doubling property. ■

8.3.3 The case of groups : proof of Theorem 8.2.9

First, let us reformulate the proof of Corollary 8.2.10 using some notations adapted to the group setting (it may look slightly more complicated than the proof of Theorem 8.2.4, but this is merely due to the fact that U is not assumed to be symmetric : see the beginning of Section 8.2.2).

Let (G, μ) be a cglc group of polynomial growth endowed with a Haar measure μ. Let U be a compact generating set (containing 1). Here, we will denote

$$C_{n,n+k} = U^{n+k} \setminus U^n, \quad \forall n, k \in \mathbf{N}$$

and

$$c_{n,n+k} = \mu(C_{n,n+k}).$$

Recall that we want to find a constant α such that $c_{n-k,n} \geq \alpha.c_{n,n+k}$. To simplify notations, let us assume that k is a multiple of 4.

First, note that

$$C_{n,n+k} \subset U^{2k} C_{n-k/2,n-k/2+1}. \tag{8.3.3}$$

Indeed, let y be in $C_{n,n+k}$, and let $(y_1, \ldots, y_{n+j})$ be a minimal sequence of elements of U such that $y = y_{n+j} \ldots y_1$. By definition of $C_{n,n+k}$ and by minimality, we have $1 \leq j \leq k$. Moreover, it is easy to see that minimality also implies

$$y_{n-k/2+1} \ldots y_1 \in C_{n-k/2,n-k/2+1}.$$

So

$$y \in y_{n+j} \ldots y_{n-k/2+2} C_{n-k/2,n-k/2+1} \subset U^{2k} C_{n-k/2,n-k/2+1}$$

and we are done.

On the other hand, we have

$$U^{k/4} C_{n-k/2,n-k/2+1} \subset C_{n-k,n}.$$

So, let (x_i) be a maximal family of points of $C_{n-k/2,n-k/2+1}$ such that $U^{k/4}x_i \cap U^{k/4}x_j = \emptyset$ for $i \neq j$. By maximality of (x_i), we have

$$C_{n-k/2,n-k/2+1} \subset \cup_i U^{-k/4} U^{k/4} x_i.$$

So by (8.3.3), we get

$$C_{n,n+k} \subset \cup_i U^{2k} U^{-k/4} U^{k/4} x_i \tag{8.3.4}$$

Let S be a symmetric compact set containing U^3. Then, since $U^{2k}U^{-k/4}U^{k/4}$ is included in S^k, proposition 8.2.7 and (8.2.1) imply that there exists a constant $C > 0$ such that

$$\mu\left(U^{2k}U^{-k/4}U^{k/4}x_i\right) \leq C\mu\left(U^{k/4}x_i\right) \tag{8.3.5}$$

for a constant $C > 0$. Thus, since the $U^{k/4}x_i$ are disjoint and included in $C_{n-k,n}$, we get

$$c_{n-k,n} \geq \sum_i \mu\left(U^{k/4}x_i\right). \tag{8.3.6}$$

Finally, using (8.3.4), (8.3.5) and (8.3.6), we deduce

$$c_{n-k,n} \geq C^{-1}c_{n,n+k}. \blacksquare$$

Proof of Theorem 8.2.9

The only significant modification we have to do in order to prove Theorem 8.2.10 concerns the inclusion (8.3.3). Actually, we have to show a kind of property (M) adapted to this context. For simplicity, assume that $1 \in K$, so that the sequence $(U_nU_{n-1}\ldots U_0)_{n\in\mathbf{N}}$ is increasing.

Lemma 8.3.3. There exists $j_0 \in \mathbf{N}$, such that for every $n \in \mathbf{N}$ and every $x \in U_{n+k}\ldots U_0 \setminus U_n \ldots U_0$, we have

$$x \in K'^k\left(U_n \ldots U_0 \setminus U_{n-kj_0}\ldots U_0\right).$$

Proof : Since K' contains U_i for every $i \in \mathbf{N}$, we have

$$x \in K'^k U_n \ldots U_0.$$

On the other hand, let q be an integer such that

$$x \in K'^k U_{n-q}\ldots U_0.$$

Then, let j_0 be such that $K' \subset K^{j_0}$ (see proposition 8.2.7). Since $K \subset U_i$ for every i, it comes

$$x \in U_{n-q+kj_0}\ldots U_0.$$

But this implies $q < kj_0$, so we are done. $\blacksquare$

Write $C_{n,n+k} = U_{n+k}\ldots U_0 \setminus U_n \ldots U_0$. According to the lemma, we have

$$C_{n,n+k} \subset K' C_{n-kj_0,n}$$

for every $k < n/j_0$.

Using the same arguments as in the proof of Theorem 8.2.10, we get $c_{n,n-j_0k} \geq \alpha.c_{n,n+k}$ and we conclude thanks to Theorem 8.3.1. $\blacksquare$

8.4 Consequences in ergodic theory

Let G be a locally compact second countable (lcsc) group, X a standard Borel space on which G acts measurably by Borel automorphisms. Let m be a G-invariant probability measure on X ((X,m) is called a Borel probability G-space). The G-action on X gives rise to a strongly continuous representation π of G as a group of isometries of the Banach space $L^p(X)$ for $1 \leq p < \infty$, given by $\pi(g)f(x) = f(g^{-1}x)$. For any Borel probability measure β on G, and given some $p \geq 1$, we can consider the averaging operator given by

$$\pi(\beta)f(x) = \int_G f(g^{-1}x)d\beta(g), \quad \forall f \in L^p(X).$$

Let (β_n) be a sequence of probability measures on G. We say that (β_n) satisfies a pointwise ergodic theorem in $L^p(X)$ if

$$\lim_{n\to\infty} \pi(\beta_n)f(x) = \int_X f dm$$

for almost every $x \in X$, and in the L^p-norm, for all $f \in L^p(X)$, where $1 \leq p < \infty$. Let μ be a Haar measure on G. We will be interested in the case when β is the normalized average on a set of finite measure N of G.

Definition 8.4.1 (Regular sequences)**.** A sequence of sets of finite measure N_k in G is called regular if

$$\mu(N_k^{-1}.N_k) \leq C\mu(N_k).$$

Let us recall the following general result (also proved in the recent survey of Amos Nevo [N])

Theorem 8.4.2. [Tem][Chat][Bew][Em] Assume G is an amenable lcsc group, and $(N_n)_{n\in\mathbf{N}}$ is an increasing left Følner regular sequence, with $\cup_{n\in\mathbf{N}}N_n = G$. Then, the sequence $(\beta_n)_{n\in\mathbf{N}}$ (associated to (N_k)) satisfies the pointwise ergodic theorem in $L^p(X)$, for every Borel probability G-space (X,m) and every $1 \leq p < \infty$.

Now, let us focus on the case when G is a locally compact, compactly generated group of polynomial growth. Consider a sequence $(U_n)_{n\in\mathbf{N}}$ satisfying the hypothesis of Theorem 8.2.9. According to Theorem 8.2.9 and Proposition 8.2.7, the sequence $N_n = U_0.U_1\dots U_n$ clearly satisfies the hypothesis of Theorem 8.4.2. So we get the following corollary.

Theorem 8.4.3. Let G be a cglc group of polynomial growth. Consider a sequence $(U_n)_{n\in\mathbf{N}}$ of measurable subsets such that there exist two generating compact subsets K, K' such that, for all $n \in \mathbf{N}$

$$K \subset U_n \subset K'.$$

Write $N_n = U_0.U_1\dots U_n$. Then, the sequence $(\beta_n)_{n\in\mathbf{N}}$ (associated to $(N_n)_{n\in\mathbf{N}}$) satisfies the pointwise ergodic theorem in $L^p(X)$, for every Borel probability G-space (X,m) and every $1 \leq p < \infty$.

8.5 Remarks and questions

In this section, we address a (non-extensive) list of remarks and problems related to the subject of this paper.

The general Greenleaf localisation conjecture. The following question is still open : is the Greenleaf conjecture true for all subexponential groups ?

Groups with exponential growth. Let G be a finitely generated group with exponential growth and let U be a finite generating subset. Does there exist a constant $c > 0$ such that[2]

$$\mu(U^{n+1} \smallsetminus U^n) \geq c\mu(U^n)?$$

Asymptotic isoperimetry. Let G be a locally compact, compactly generated group and let U be a compact generating neighborhood of 1. If A is a subset of G, we call boundary of A and denote by ∂A the subset $UA \smallsetminus A$. Let μ be a Haar measure on G. Recall the definition of the monotone isoperimetric profile of G (see [PS])

$$I^{\uparrow}(t) = \inf_{\mu(A) \geq t} \mu(\partial A)/\mu(A)$$

where A runs over measurable subsets of finite measure of G. We can also define a (monotone) profile relatively to a family $\mathbf{A}$ of subsets of G

$$I^{\uparrow}_{\mathbf{A}}(t) = \inf_{\mu(A) \geq t, A \in \mathbf{A}} \mu(\partial A)/\mu(A).$$

By a theorem of Varopoulos ([V] [CouSa2]), G has polynomial growth of degree d if and only if $I^{\uparrow}(t) \approx t^{(d-1)/d}$. An interesting question is for which groups do we have $I^{\uparrow}_{(U^n)_{n\in\mathbf{N}}} \preceq I^{\uparrow}$? It is an easy fact for groups of polynomial growth, and more generally in doubling metric measure spaces.

Proposition 8.5.1. Let X be a doubling metric measure space. There exists a sequence $(r_i)_{i\in\mathbf{N}}$ such that $2^i \leq r_i \leq 2^{i+1}$ and such that

$$\forall i \in \mathbf{N}, \forall x \in X, \quad \mu(S(x, r_i)) \leq C\mu(B(x, r_i)/r_i.$$

In particular, if G has polynomial growth of degree d, and if U is a compact generating set of G, then there exists a subsequence n_i such that $2^i \leq n_{i+1} \leq 2^{i+1}$ and such that

$$\mu(U^{n_i+1} \smallsetminus U^{n_i}) \leq Cn_i^{(d-1)/d}. \tag{8.5.1}$$

To prove Proposition 8.5.1, one just has to remark that

$$\forall n < m \in \mathbf{N}, \quad S(x, n) \cap S(x, m) = \emptyset$$

[2] An erroneous proof of this fact is written in [Pit2].

and that

$$\cup_{k=1}^{2^i} S(x, 2^i + k) \subset B(x, 2^{i+1}),$$

so that

$$2^i \inf_{1 \le k \le 2^i} \mu(S(x, 2^i + k)) \le \mu(B(x, 2^{i+1}))$$

and finally, one can conclude thanks to doubling property.

Conversely, does $I^{\uparrow}_{(\mathbf{U^k})} \preceq I^{\uparrow}$ imply that G has polynomial growth ? Subexponential growth ?

One can also wonder if (8.5.1) holds for any integer n (when G has polynomial growth of degree d), or equivalently, if there is a constant $C > 0$ such that :

$$\forall n \in \mathbf{N}, \quad \mu(U^{n+1} \smallsetminus U^n) \le C \frac{\mu(U^n)}{n}. \tag{8.5.2}$$

Proposition 8.5.2. Let G be a cglc Abelian group and let U be a compact generating set of G. Then, (8.5.2) holds.

Sketch of the proof. First, note that it is an easy fact when $G = \mathbb{R}^d$ (the adaptation to $\mathbb{Z}^d$ is left to the reader) : if K is convex, it is trivial (since $K + K = 2.K$) ; then show that $\hat{K}^n \subset K^{n+k}$ where $\hat{K}$ denotes the convex hull of K, and where k is a positive integer smaller than $d+1$ times the diameter of K. On the other hand, a cglc Abelian group G is isomorphic to a direct product $K \times \mathbb{R}^a \times \mathbb{Z}^b$, with $a, b \in \mathbf{N}$, and K being a compact group. ■

Annexe A

Maps into Hilbert space

Let G be a locally compact group and $\mathcal{H}$ a Hilbert space. Let f be a map : $G \to \mathcal{H}$ (not necessarily continuous). We call f a *uniform map* if, for every compact subset $K \subset G$, we have $\sup_{g \in G, k \in K} \|f(kg) - f(g)\|\} < \infty$. If G is compactly generated, this coincides with the definition given in §3.4.3.

Lemma A.0.1. *Let $f : G \to \mathcal{H}$ be a uniform map. Then there exists $\tilde{f} : G \to \mathcal{H}$ such that :*
- *$\tilde{f}$ is at bounded distance from f, and*
- *$\tilde{f}$ is uniformly continuous on G.*

Proof : Fix an open, relatively compact, symmetric neighbourhood V of 1 in G. Consider a closed, discrete subset $X \subset G$ such that

(1) $\bigcup_{x \in X} xV = G$, and

(2) for all $x, y \in X$, if $x^{-1}y \in V$, then $x = y$.

The existence of such a subset X is immediate from Zorn's Lemma.

Fix a function $\phi : G \to \mathbf{R}_+$, continuous with compact support, such that $\phi \leq 1$, and (3) : $\phi \equiv 1$ on V. Fix a symmetric, compact subset W containing the support of ϕ.

Set $\Phi(g) = \sum_{x \in X} \phi(x^{-1}g)$ and observe that $\Phi(g) \geq 1$ as a consequence of (1) and (3). Define

$$\tilde{f}(g) = \frac{1}{\Phi(g)} \sum_{x \in X} \phi(x^{-1}g) f(x).$$

Let us first check that $\tilde{f}$ is at bounded distance from f. For all $g \in G$,

$$\tilde{f}(g) - f(g) = \sum_{x \in X} \frac{\phi(x^{-1}g)}{\Phi(g)} (f(x) - f(g)).$$

Since f is a uniform map, there exists $M < \infty$ such that for all $g, h \in G$, $h^{-1}g \in W$ implies $\|f(h) - f(g)\| \leq M$. It follows that, for all $g \in G$, we have $\|\tilde{f}(g) - f(g)\| \leq M$.

Now let us show that f is uniformly continuous. Consider a neighbourhood V_0 of 1 such that $V_0^2 \subset V$. It immediately follows that, for every $g \in G$, the set $X \cap gV_0$ contains at most one element. On the other hand, by compactness, there exist $g_1, \ldots, g_n$ such that $W \subset \bigcup_{i=1}^n g_i V_0$. It follows that, for all $g \in G$, the set $gW \cap X$ has cardinality at most n.

Write $u_\phi(g) = \sup_{h \in G} |\phi(h) - \phi(hg)|$. Since ϕ is uniformly continuous, $u_\phi(g) \to 0$ when $g \to 1$.

Then $|\phi(x^{-1}g) - \phi(x^{-1}h)| \leq u_\phi(g^{-1}h)$ and, for all $g, h \in G$, $\Phi(g) - \Phi(h) = \sum_{x \in X} (\phi(x^{-1}g) - \phi(x^{-1}h)) \leq 2n\, u_\phi(g^{-1}h)$, since the only nonzero terms are those for $x \in (gW \cap X) \cup (hW \cap X)$. Accordingly, Φ is uniformly continuous. Since $\Phi \geq 1$, it follows that $1/\Phi$ is also uniformly continuous; let us define $u_{1/\Phi}$ as we have defined u_ϕ.

For $g, h \in G$,

$$\begin{aligned}
\left| \frac{\phi(x^{-1}g)}{\Phi(g)} - \frac{\phi(x^{-1}h)}{\Phi(h)} \right| &\leq \left| \frac{\phi(x^{-1}g)}{\Phi(g)} - \frac{\phi(x^{-1}h)}{\Phi(g)} \right| + \left| \frac{\phi(x^{-1}h)}{\Phi(g)} - \frac{\phi(x^{-1}h)}{\Phi(h)} \right| \\
&\leq \frac{|\phi(x^{-1}g) - \phi(x^{-1}h)|}{\Phi(g)} + |\phi(x^{-1}h)| \left| \frac{1}{\Phi(g)} - \frac{1}{\Phi(h)} \right| \\
&\leq u_\phi(g^{-1}h) + u_{1/\Phi}(g^{-1}h).
\end{aligned}$$

Therefore, fixing some $x_0 \in X$,

$$\begin{aligned}
\|\tilde{f}(g) - \tilde{f}(h)\| &= \sum_{x \in X} \left(\frac{\phi(x^{-1}g)}{\Phi(g)} - \frac{\phi(x^{-1}h)}{\Phi(h)} \right) (f(x) - f(x_0)) \\
&\leq \sum_{x \in X} \left| \frac{\phi(x^{-1}g)}{\Phi(g)} - \frac{\phi(x^{-1}h)}{\Phi(h)} \right| \|f(x) - f(x_0)\| \\
&\leq \sum_{x \in (gW \cap X) \cup (hW \cap X)} \left| \frac{\phi(x^{-1}g)}{\Phi(g)} - \frac{\phi(x^{-1}h)}{\Phi(h)} \right| \|f(x) - f(x_0)\| \\
&\leq (u_\phi(g^{-1}h) + u_{1/\Phi}(g^{-1}h)) \sum_{x \in (gW \cap X) \cup (hW \cap X)} \|f(x) - f(x_0)\|.
\end{aligned}$$

Since f is a uniform map, there exists $M' < \infty$ such that $h^{-1}g \in V^2W$ implies $\|f(h) - f(g)\| \leq M'$ for all $g, h \in G$. Now fix x_0 so that $g \in x_0 V$, and suppose $g^{-1}h \in V$. If $x \in gW \cup hW$, then it follows that $\|f(x) - f(x_0)\| \leq M'$. Accordingly, whenever $g^{-1}h \in V$,

$$\|\tilde{f}(g) - \tilde{f}(h)\| \leq 2n(u_\phi(g^{-1}h) + u_{1/\Phi}(g^{-1}h))M',$$

so that $\tilde{f}$ is uniformly continuous. ■

Annexe B

Actions on Euclidean spaces

Proposition B.0.2. *Let G be a closed subgroup of* $\mathrm{E}_n(\mathbf{R}) = \mathrm{Isom}(\mathbf{R}^n)$*. The following are equivalent :*

(i) G is cocompact in E_n.

(ii) G acts cocompactly on $\mathbf{R}^n$.

(iii) G does not preserve any proper affine subspace of $\mathbf{R}^n$.

Proof : (i)$\Leftrightarrow$(ii)$\Rightarrow$(iii) are trivial.

Let us show (iii)$\Rightarrow$(ii). We use some results of Guivarc'h on the structure of closed (not necessarily connected) subgroups of amenable connected Lie groups. By [Gui, Théorème IV.3 and Lemma IV.1], G has a characteristic closed cocompact solvable subgroup R. Then R has a characteristic subgroup of finite index N which maps to a torus of $\mathrm{O}_n(\mathbf{R})$ through the natural projection $\mathrm{E}_n \to \mathrm{O}_n(\mathbf{R})$.

First case : G does not contain any nontrivial translation. Then N is abelian. If $g \in N$, let d_g denote its displacement length : $d_g = \inf\{\|g.v - v\| : v \in \mathbf{R}^n\}$. The subset $A_g = \{v \in \mathbf{R}^n : \|gv - v\| = d_g\}$ is a (nonempty) affine subspace of $\mathbf{R}^n$, and is N-stable since N is abelian. Also note that if W is any g-stable affine subspace, then[1] $W \cap A_g \neq \emptyset$. It easily follows that finite intersections of subspaces of the form A_g, for $g \in N$, are nonempty, and a dimension argument immediately yields that $A = \bigcap_{g \in N} A_g \neq \emptyset$. This is a G-invariant affine subspace, hence is, by assumption, all of $\mathbf{R}^n$. Therefore, every element in N is a translation, so that $N = \{1\}$ and thus G is compact. This implies that G has a fixed point, so that the assumption implies $n = 0$ (i.e. leads to a contradiction if $n \geq 1$).

General case. Argue by induction on the dimension n. Suppose that $n \geq 1$. Let T_G be the subgroup of translations in G. Let W be the linear subspace generated by T_G. Since T_G is closed, it acts cocompactly on W. Moreover, by the first case, W has positive dimension. Note that the linear action of G clearly preserves W.

[1] Let $v \in A_g$ such that $d(v, W) = d(A_g, W)$. Let p denote the projection on W ; since W is g-stable, p commutes with g. Since $d(v, pv) = d(gv, gpv) \leq d(x, y)$ for all $x \in [v, gv]$, $y \in [pv, gpv]$, we easily obtain that v, gv, pv, gpv form a rectangle, so that $d(pv, gpv) = d(v, gv)$ and thus $pv \in A_g$ by definition of A_g.

Now look at the action of G on the affine space $\mathbf{R}^n/W$. It does not preserve any proper affine subspace, hence is cocompact by the induction hypothesis. Since the action of T_G on W is also cocompact, it follows that the action of G on $\mathbf{R}^n$ is also cocompact.

Corollary B.0.3. *Let G be a locally compact group. Suppose that G has a proper isometric action on a Euclidean space. Then it has a proper cocompact isometric action on a Euclidean space.*

Proof : Let G act on a Euclidean space by isometries. Let V be a G-invariant affine subspace of minimal dimension. Then the action of G on V is clearly proper, and is cocompact by Proposition B.0.2. ■

Annexe C

Groups of polynomial growth

The aim of this section is to sketch a complete proof of Theorem 8.2.8 (due to Guivarc'h/Jenkins, Gromov, Lozert, Mostow and Wang) whose arguments are scattered in the literature.

Theorem C.0.3. Let G be a cglc group of polynomial growth, then it has strictly polynomial growth. Precisely, there exist a nonnegative integer d, independent of the generating compact set U, and a constant $C = C(U) \geq 1$ such that

$$C^{-1}n^d \leq \mu(U^n) \leq Cn^d. \tag{C.0.1}$$

The main step of the proof is the following theorem.

Theorem C.0.4. Let G be a cglc group of polynomial growth. Then G has a normal compact subgroup K such that F/K has a faithful finite dimensional representation π such that $\pi(G)$ is a closed uniform subgroup of a closed linear group with finitely many components.

Then, one can conclude thanks to the following theorem of Guivarc'h.

Theorem C.0.5 (Guivarc'h, Jenkins)**.** Inequalities (C.0.1) hold for a connected solvable Lie group of polynomial growth.

To obtain Theorem C.0.3 from the two previous theorems, one needs the following basic facts proved by Guivarc'h in [Gui].

Proposition C.0.6. Let G be a cglc group and let H be a closed compactly generated subgroup.
(i) If G has polynomial growth, then H does.
(ii) If G has polynomial growth and H is normal, then G/H has polynomial growth.
(iii) If H is compact and normal, then G has polynomial growth if and only if G/H does and if one of both has strict polynomial growth then the other does with the same degree.
(iv) If H is cocompact, then the same holds for G and H.

So now, let us prove Theorem C.0.4. We will need the following results.

Theorem C.0.7 (Gromov)**.** A finitely generated group of polynomial growth is virtually nilpotent.

Theorem C.0.8 (Losert)**.** Let G be a cglc group of polynomial growth. Then G has a normal compact subgroup K such that G/K is a Lie group. Moreover, G has a closed cocompact subgroup F, containing K such that F/K is a solvable Lie group.

Using Losert and Gromov theorems, we can assume that G is an extension of a connected Lie group by a nilpotent finitely generated group.

We say that a locally compact group G is Noetherian if every closed subgroup of G is compactly generated. One can prove easily [Gui] that a solvable group G is Noetherian if and only if each $i \in \mathbf{N}$, $D^iG/D^{i+1}G$ is compactly generated. So Noetherian solvable groups are the smallest class of cglc groups stable under extension and containing all compactly generated Abelian groups. Clearly, an extension of a connected Lie group by a nilpotent finitely generated group is a Noetherian solvable group. Now, we are able to conclude thanks to the following beautiful result of Mostow (and Wang).

Theorem C.0.9. Let G be a Noetherian solvable group. Then G has a normal compact subgroup K such that F/K has a faithful finite dimensional representation π such that $\pi(G)$ is a closed uniform subgroup of a closed linear group with finitely many components.

Remark C.0.10*.* About 15 years before Mostow's proof [M], Wang showed [Wa] this result for a group G such that there is an exact sequence

$$1 \to N \to G \to \mathbb{Z}^k$$

with N being a torsion free nilpotent Lie group such that N° is simply connected and N/N° is finitely generated. Note that it is not so difficult (see [M]) to deduce Theorem C.0.9 from this case (which concentrates the main difficulties).

Remark C.0.11*.* As we said in the introduction, Breuillard has recently proved [Bre] the following improvement of Theorem 8.2.8, generalizing a former result of Pansu [Pa1] for finitely generated nilpotent groups :

Theorem C.0.12. Let G be a cglc group of polynomial growth of degree d and let U be a compact generating set. Then there exists a constant $C = C(U)$ such that :

$$\mu(U^n) \sim Cn^d.$$

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