THE LARGE-SCALE GEOMETRY OF LOCALLY COMPACT SOLVABLE GROUPS

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Abstract. This short survey deals with the large-scale geometry of solvable groups. Instead of giving a global overview of this wide subject, we chose to focus on three aspects which illustrate the broad diversity of methods employed in this subject. The first one has probabilistic and analytic flavours, the second is related to cohomological properties of unitary representations, while the third one deals with the Dehn function. To keep the exposition concrete, we discuss lots of examples, mostly among solvable linear groups.

Contents

Date: April 15, 2010.
2000 *Mathematics Subject Classification.* 20F65 (primary); 20F69, 20F16, 53A10 (secondary).

1. Introduction

Since Gromov's major input in the eighties and early nineties, the large-scale geometry of finitely generated groups has attracted a lot of attention. In spite of their apparent simplicity, solvable and even nilpotent groups are considerably hard to deal with. For instance, current methods to obtain quasi-isometry rigidity results for solvable groups are so technically involved that they only apply to very specific classes of solvable groups (see for instance [FM98, FM99, FM00, EFW07, EFW12, EFW13]). By contrast, a theory of geometric invariants for amenable groups have been successfully developed since the 80's with the work of Varopoulos, and has more recently gained momentum with new tools coming from ergodic theory introduced by Shalom. In this survey we will focus on geometric invariants of three types.

Some invariants are specific to amenable groups, and in a way, provide tools to *quantify* amenability. The general approach is to start from one of the various characterizations of amenability, and to attach a numerical invariant to it. Here, we will choose Kesten's characterization, and therefore consider the return probability in large time of a symmetric random walk. Its study started with a theorem of Varopoulos computing this invariant for nilpotent groups. Later, it was computed for polycyclic groups, and for many examples of solvable groups, showing a wide range of possible behaviors. Surprisingly, the fact that its asymptotic behavior is invariant under quasi-isometry has been proved only relatively recently [PS01'].

Other invariants are not *a priori* of geometric nature, and can be defined for any finitely generated group, amenable or not, but will be quasi-isometry invariant *only* when the groups are amenable, and sometimes only for certain subclasses. *Measure equivalence* (ME) is an equivalence relation on finitely generated groups introduced by Gromov in [Gr93], as a measure-theoretic analogue of quasi-isometry. Shalom later realized that if two amenable groups are quasi-isometric, then they can be made measure equivalent and quasi-isometric in a "compatible way" [Sh04]. He used this crucial observation to show that certain cohomological or harmonic analytic properties are invariant under quasi-isometry among amenable groups. In this survey, we shall focus on Shalom's property H_{FD}, which is defined in terms of unitary representations. Among other striking applications, he gives an elementary proof of the quasi-isometry rigidity of virtually abelian finitely generated groups.

A third type of invariants are geometric in nature and are interesting for all groups. The Dehn function of a finitely *presented* group gathers combinatorial, geometrical, and topological aspects. Indeed, the Dehn function is related to the complexity of an algorithm solving the word problem. It also has an interpretation as isoperimetric profile, measuring the difficulty of filling big loops in the two-dimensional cell complex associated to a finite presentation of the group. Finally the Dehn function is related to the simple connectedness of the asymptotic cones of the group. For instance, if the Dehn function grows at most quadratically, then the asymptotic cones are simply connected [Pap96]. Conversely, if the cone is simply connected, then the Dehn function grows at most polynomially [Gr93]. A challenge is to give a complete description of the possible behaviors of the Dehn function (and of its higher dimensional versions) for polycyclic groups. Note that even for nilpotent groups, we are far from a complete answer. We shall give a complete answer for a natural class of metabelian groups arising from arithmetic constructions.

Throughout the study of these geometric invariants, it will become clear that one should not focus uniquely on discrete groups, but instead should work as much as possible with general locally compact groups. The interest of studying the large-scale geometry of locally compact solvable groups is for instance supported by the fact that any Lie group (similarly any algebraic group over a local field) admits a closed, cocompact solvable subgroup.

Section 2 is a short introduction to local fields and embeddings of some discrete solvable groups into groups of triangular matrices with coefficients in local fields. The three following sections devoted to the three types of geometric invariants described above, can be read independently.

Notation Let $f, g : \mathbf{R}_+ \to \mathbf{R}_+$ be monotonous functions. We write respectively $f \preceq g$ if there exists $C > 0$ such that $f(t) = O(g(Ct))$, when $t \to \infty$. We write $f \approx g$ if both $f \preceq g$ and $g \preceq f$. The asymptotic behavior of f is its class modulo the equivalence relation $\approx$.

Acknowledgments and apologies. I am indebted to Mark Sapir, Christophe Pittet and particularly to Yves Cornulier for their numerous valuable comments. Due to its short length, this paper should better be considered as a promotion rather than a survey on solvable groups.

2. Solvable Baumslag-Solitar and Lamplighter groups as cocompact lattices

In this short preliminary section we recall basic facts about local fields, refering to [Wei95] for more details. Let p be a prime, and for every $n \in \mathbf{N}$, we denote by $\mathbf{F}_{p^n}$ the field with p^n elements. By *local field* we mean a non-discrete locally compact normed field, which equivalently means a finite extension of either $\mathbf{R}$, $\mathbf{Q}_p$ or of the function fields[1] $\mathbf{F}_p(\!(t)\!)$.

Recall that a *non-archimedean norm* on a field $\mathbf{K}$ is a map $|\cdot| : \mathbf{K} \to [0,\infty)$ satisfying, for all x, $y \in \mathbf{K}$

- $|x| = 0 \Leftrightarrow x = 0$
- $|xy| = |x||y|$
- $|x+y| \leq \max\{\,|x|, |y|\,\}$.

A non-archimedean norm induces a ultrametric invariant distance on $\mathbf{K}$ by $d(x,y) = |x-y|$. The closed unit ball $B(0,1)$ for this distance is clearly a subring, and the unit sphere $S(0,1)$ is the subgroup of invertible elements of this ring. For the sake of concreteness, we shall only consider the cases of $\mathbf{Q}_p$ and $\mathbf{F}_p(\!(t)\!)$ although the discussion readily extends to their finite extensions.

(p-adic numbers) Let p be a prime. A rational number x can be expressed in a unique way as $p^n u/v$, where u and v are integers and $v > 0$ is relatively prime to u and to p. The integer n is called the p-valuation of x, and will be denoted by $v_p(x)$. Now, recall that the field of p-adic numbers $\mathbf{Q}_p$ can be defined as the completion of $\mathbf{Q}$ for the (non-archimedean) norm $|x|_p = p^{-v_p(x)}$. Equivalently, $\mathbf{Q}_p$ can be represented as the field of Laurent series $\sum_{i\geq i_0} a_i p^i$, where $a_i \in \{0,\ldots,p-1\}$ and $i_0 \in \mathbf{Z}$. The unit ball is the ring of p-adic integers $\mathbf{Z}_p$, corresponding to elements of the form $\sum_{i\geq 0} a_i p^i$, and the unit sphere is $\mathbf{Z}_p^\times$. Observe that $\mathbf{Z}_p$ is a compact open subgroup of the additive group $(\mathbf{Q}_p,+)$ which is therefore locally compact.

(function field over $\mathbf{F}_p$) Similarly, the function field $\mathbf{F}_p(\!(t)\!)$ is the field of Laurent series $\sum_{i\geq i_0} a_i t^i$, where $a_i \in \mathbf{F}_p$ and $i_0 \in \mathbf{Z}$. The unit ball is the compact open subring $\mathbf{F}_p[\![t]\!]$ comprising elements such that $i_0 = 0$, and the unit ball is the compact subgroup $\mathbf{F}_p[\![t]\!]^\times$. Equivalently, it can be defined as the completion of $\mathbf{F}_p(t)$ for the non-archimedean norm $|x| = 2^{-\mathrm{val}(x)}$ where $\mathrm{val}(y/z) = \deg(y) - \deg(z)$, with $y, z \in \mathbf{F}_p[t]$, $z \neq 0$.

(Additive and multiplicative groups) It follows from the above that the additive group $(\mathbf{K},+)$ for both $\mathbf{K} = \mathbf{Q}_p$ and $\mathbf{K} = \mathbf{F}_p(\!(t)\!)$ is a locally compact totally disconnected second countable abelian group. As $\mathbf{Z}_p \subset p^{-1}\mathbf{Z}_p \subset \ldots$, (resp. $\mathbf{F}_p[\![t]\!] \subset t^{-1}\mathbf{F}_p[\![t]\!] \subset \ldots$) forms an exhausting increasing sequence of compact subgroups, it turns out that $(\mathbf{K},+)$ is not compactly generated. By contrast, the multiplicative subgroup $\mathbf{Q}_p^\times$ (resp. $\mathbf{F}_p(\!(t)\!)^\times$) identifies to $\mathbf{Z} \times \mathbf{Z}_p^\times$ (resp. to $\mathbf{Z} \times \mathbf{F}_p[\![t]\!]^\times$) via the topological group isomorphim $x \to (\mathrm{val}_p(x), p^{\mathrm{val}_p(x)}x)$ (resp. $x \to (\mathrm{val}(x), 2^{\mathrm{val}(x)}x)$). In particular these multiplicative groups are compactly generated and quasi-isometric to $\mathbf{Z}$.

[1]Finite extensions of $\mathbf{F}_p(\!(t)\!)$ are of form $\mathbf{F}_{p^n}(\!(t)\!)$ for some $n \in \mathbf{N}$,

(Solvable Baumslag-Solitar groups) Note that for every $n \in \mathbf{N}$, the ball of radius p^n of $\mathbf{Q}_p$ is the compact additive subgroup $p^{-n}\mathbf{Z}_p$. Hence contrary to what happens in $\mathbf{R}$, the sequence p^n converges to zero as $n \to \infty$ and has norm going to infinity as $n \to -\infty$. This counter-intuitive behavior has the following advantage. Consider the additive subgroup $\mathbf{Z}[1/p]$ of $\mathbf{Q}$. Clearly it is dense both in $\mathbf{R}$ and in $\mathbf{Q}_p$. However, the diagonal embedding $\mathbf{Z}[1/p] \to \mathbf{R} \oplus \mathbf{Q}_p$ is easily shown to be discrete and cocompact. Indeed, one can check that the relatively compact subset with non-empty interior $X_0 = [0,1) \times \mathbf{Z}_p$ is a section of the projection $\mathbf{R} \oplus \mathbf{Q}_p \to (\mathbf{R} \oplus \mathbf{Q}_p)/\mathbf{Z}[1/p]$.

Recall that for every p prime, Baumslag-Solitar's group

$$\mathrm{BS}(1,p) = \langle t, x \mid txt^{-1} = x^p \rangle$$

is isomorphic to the semi-direct product $\mathbf{Z}[1/p] \rtimes \mathbf{Z}$, where $\mathbf{Z}$ acts on $\mathbf{Z}[1/p]$ by multiplication by p. Now it is easy to see that $\mathrm{BS}(1,p)$ sits as a cocompact lattice inside the group $G = (\mathbf{Q}_p \oplus \mathbf{R}) \rtimes \mathbf{Z}$, where $\mathbf{Z}$ acts by multiplication by p on both factors, which can be also described as

$$G = \left\{ \begin{pmatrix} p^n & 0 & x \\ 0 & p^n & y \\ 0 & 0 & 1 \end{pmatrix} ; x \in \mathbf{R} \times \{0\}, y \in \{0\} \times \mathbf{Q}_p, n \in \mathbf{Z} \right\} \subset \mathrm{GL}(3, \mathbf{R} \times \mathbf{Q}_p).$$

(Lamplighter group) Similarly, the ring $\mathbf{F}_p[t,t^{-1}]$ embeds densely in $\mathbf{F}_p(\!(t)\!)$, but the diagonal embedding $\mathbf{F}_p[t,t^{-1}] \to \mathbf{F}_p(\!(t)\!) \oplus \mathbf{F}_p(\!(t)\!)$ sending t to (t, t^{-1}) is easily seen to be discrete and cocompact. The lamplighter group $L_p = \mathbf{F}_p \wr \mathbf{Z}$ can be described as $\mathbf{F}_p[t,t^{-1}] \rtimes \mathbf{Z}$, where $\mathbf{Z}$ acts by multiplication by t. Therefore L_p embeds as a cocompact lattice in $G = (\mathbf{F}_p(\!(t)\!))^2 \rtimes \mathbf{Z}$, where $\mathbf{Z}$ acts by multiplication by t on the first factor and by t^{-1} on the second factor, i.e.

$$G = \left\{ \begin{pmatrix} t^n & 0 & x \\ 0 & t^{-n} & y \\ 0 & 0 & 1 \end{pmatrix} ; x, y \in \mathbf{F}_p(\!(t)\!), n \in \mathbf{Z} \right\}.$$

3. Probability of return of symmetric random walks

Let G be a locally compact, compactly generated group, and let μ and be a Borel probability measure on G. We shall interpret μ as the probability transition of a random walk on G starting at the neutral element 1_G. Calling this process $(X_n)_{n\geq 0}$, we have $X_0 = 1_G$, and $X_{n+1} = X_n \xi_n$, where $(\xi_n)_{n\geq 0}$ is a i.i.d. sequence of G-valued random variables of distribution μ.

Recall that if ν and ν' are two probability measures on G, the convolution product $\nu * \nu'$ is the push-forward measure under the map

$$\begin{array}{rcl} G \times G & \to & G \\ (g, g') & \mapsto & gg' \end{array}$$

Observe that if G is non-abelian, $\nu * \nu' \neq \nu' * \nu$ in general. One easily checks that the distribution of X_n is the n-fold convolution product $\mu^{(n)} = \mu * \ldots * \mu$.

Let λ be a left Haar measure on G. We will abbreviate $L^2(G,\lambda)$ in $L^2(G)$. Note that μ acts by convolution on $L^2(G)$, and more generally on $L^p(G)$ for every $0 \leq p \leq \infty$. Let us denote this operator P_μ:

$$\begin{array}{rccc} P_\mu: & L^2(G) & \to & L^2(G) \\ & f & \mapsto & P_\mu f := f * \mu. \end{array}$$

3.1. **The invariant ϕ.** As mentioned in the introduction, we shall deduce our invariant from one of the various characterizations of amenability. Here, we adopt Kesten's definition of amenability for locally compact groups. According to this definition, G is non-amenable if and only if there exists a compactly supported, spread-out symmetric probability measure μ on G such that $\mu^{(n)}(U) \preceq e^{-n}$ for all compact neighborhood U of 1_G. See [Ke59] for the historical proof of the equivalence between this definition and Følner's criterion for discrete groups (see also [KV83, Section 5] for a short and elegant proof, and for instance [BHV08, Appendix G4] for general locally compact groups). Let us mention that one of the main interests of ϕ from the point of view of "quantifying amenability" is that it also quantifies the fact that amenability passes to subgroups and quotients [PS01'].

In order to define our invariant, we shall impose more restrictions on G and μ. A crucial hypothesis will be to assume P_μ self-adjoint. One way to achieve this is to assume the measure μ symmetric, and the group G unimodular. Recall that μ is symmetric if for every Borel subset U, $\mu(U) = \mu(U^{-1})$. The standard assumption is to suppose that μ is absolutely continuous with respect to the Haar measure and that its support is compact and generates G. Such assumptions are designed so that eventually our invariant (which is yet to be defined) does not depend on a specific choice of measure among this restricted class[2]. To simplify the exposition, we shall assume further that the density of μ is continuous. Let us denote[3]

$$\phi(n) = \frac{d\mu^{(2n)}}{d\lambda}(1),$$

where λ denotes some Haar measure on G. In this more restricted setting, Kesten's theorem can be reformulated as "G is non-amenable if and only if $\phi(n) \preceq e^{-n}$ ".

[2]Note that Kesten's theorem does not require G to be unimodular, nor that it is compactly generated.

The study of symmetric random walks on non-unimodular groups is also a very interesting topic, but much more involved, mostly because P_μ fails to be self-adjoint [V94].

[3]The 2 in the exponent $2n$ serves to avoid problems arising when the probability is supported on the non-trivial coset of a subgroup of index 2. For instance the simple random walk on $\mathbf{Z}$ associated to $\mu = (\delta_1 + \delta_{-1})/2$ can never come back to zero at even times.

3.2. **A functional analytic invariant.** A simple but crucial observation is that $\phi(n)$ coincides with the square of the norm of the operator $P_\mu^n : L^1(G) \to L^2(G)$, i.e.

$$\phi(n) = \|P_\mu^n\|_{1\to 2}^2. \tag{3.1}$$

In particular, it implies that this operator is bounded (which is not clear *a priori* if G is non-discrete). In order to prove this identity, let us denote $h_n(g) = \frac{d\mu^{(n)}}{d\lambda}(g)$: note that h_n represent the density of the transition probability between 1 and g (or equivalently between k and kg for all $k \in G$) after n steps of the random walk. First, let us check that

$$h_{2n}(g) \leq h_{2n}(1) \tag{3.2}$$

for all $g \in G$. Since μ is symmetric, we have $h(g) = h(g^{-1})$. Using Cauchy-Schwartz' inequality, we obtain

$$\begin{aligned} h_{2n}(g) &= \int h_n(k)h_n(k^{-1}g)d\lambda(k) \\ &\leq \left(\int h_n(k)^2 d\lambda(k)\right)^{1/2}\left(\int h_n(k^{-1}g)^2 d\lambda(k)\right)^{1/2} \\ &= \int h_n(k)^2 d\lambda(k) \\ &= \int h_n(k)h_n(k^{-1})d\lambda(k) \\ &= h_{2n}(1). \end{aligned}$$

Now, let us prove that $\|P_\mu^n\|_{1\to 2}^2 \leq \phi(n)$. Let f be a compactly supported continuous function on G such that $\|f\|_1 = 1$. We observe that

$$\|P_\mu^n f\|_2^2 = \langle P_\mu^{2n} f, f\rangle \leq \langle P_\mu^{2n}|f|, |f|\rangle,$$

so we can assume that $f \geq 0$. We then deduce from (3.2) that

$$P_\mu^{2n} f(g) = \int f(gk)h_{2n}(k)d\lambda(k) \leq h_{2n}(1),$$

from which we obtain that $\phi(n) \geq \|P_\mu^n\|_{1\to 2}^2$. The reverse inequality follows by a straightforward Dirac approximation argument.

3.3. **Link with Sobolev inequalities.** Most results regarding the asymptotic behavior of $\phi(n)$ over the past 25 years have their source in the seminal work of Varopoulos [V83', V85, V85', V85", V86, V87, V91, VSC92] (see also the precursor work of Polya in $\mathbf{Z}^d$ [Po21]). The methods developped by Varopoulos are based on functional analysis, involving Poincaré and Sobolev inequalities. We refer to the nice surveys of Couhon [Cou00], of Saloff-Coste [Sal04] and of Pittet and Saloff-Coste [PS01] for full treatments of these methods. The theory was developed by these authors in two settings: Riemannian manifolds and graphs. Regarding groups, this translates into focussing either on unimodular connected Lie groups, equipped with some left-invariant Riemannian metric, or on finitely generated group equipped with some word metric. A unifying approach for metric measured spaces was given in [T08, T13], which allows in particular to treat all unimodular locally compact, compactly generated groups on the same footing.

Before introducing this more general setting, let us warm up with a more classical setup. Let G be a finitely generated group and S is a finite symmetric generating subset. Then for every function f defined on G, one can define a "gradient" $\nabla f : G \times S \to \mathbf{R}$ by $\nabla f(g,s) = (f(gs) - f(g))/\sqrt{|S|}$. Note that if f is seen as a function on the vertex set of the Cayley graph (G,S), then ∇f coincides (up to the normalization) with the discrete gradient of f, defined as a function of the set of edges of the Cayley graph. One has

$$\|\nabla f\|_2^2 = \frac{1}{|S|} \sum_{g,s \in G \times S} |f(gs) - f(g)|^2.$$

Observe that if μ is the uniform probability measure on S, then the above formula can be written as

$$\|\nabla f\|_2^2 = \sum_{g,s \in G \times S} |f(gs) - f(g)|^2 \mu(s) = \|f\|_2^2 - \langle P_\mu f, f\rangle,$$

which is the starting point of the functional analytic methods relating Sobolev inequalities and upper bounds on ϕ.

Now, let us come back to our more general setting, where G is no longer supposed to be discrete, and μ satisfies our standing assumptions. For every locally integrable function f on G, we define the L^2-norm of the gradient as

$$\|\nabla f\|_2^2 = \int_{g,s \in G \times G} |f(g) - f(gs)|^2 d\lambda(g) d\mu(s).$$

Similarly as above, we deduce the fundamental relation

$$\|\nabla f\|_2^2 = \|f\|_2^2 - \langle P_\mu f, f\rangle.$$

A great achievement of this theory is the following result (a proof is given for finitely generated groups in [Cou00, Theorem 4.2]).

Theorem 3.1. *The upper bound $\phi(n) \preceq n^{-D/2}$ is equivalent to the Sobolev inequality*

$$\|f\|_{\frac{2D}{D-2}} \leq C \|\nabla f\|_2. \tag{3.3}$$

It can be proved without too much difficulty that Sobolev inequalities such as (3.3) do not depend on the choice of μ and that they are stable under quasi-isometry [CS95, T08]. However, in complete generality, it is not known whether the asymptotic behavior of ϕ can be encoded in such functional inequalities. For this reason, the fact that the asymptotic behavior of $\phi(n)$ is a quasi-isometry invariant, and in particular does not depend on the measure, was only proved relatively recently [PS01']. The proof relies on a connexion between the behavior of $\phi(n)$ and the "bottom of the spectrum" of the self-adjoint non-negative operator $1 - P_\mu$ on $L^2(G)$, often refered as a "discrete laplacian" on G (see also [BPS12, T08]).

Let us now discuss the relations between two important invariants: ϕ and the volume growth V, which is defined as follows. We consider a compact symmetric generating subset S of G, and denote $V(r) = \lambda(S^r)$, for all $r \in \mathbf{N}$ (recall that λ is a Haar measure on G which is supposed unimodular).

3.4. **The case of nilpotent groups.** Recall that every finitely generated nilpotent group G has a finite normal subgroup F such that G/F is torsion-free. On the other hand, each torsion-free finitely generated nilpotent group admits a canonical embedding as a cocompact lattice in a simply connected nilpotent Lie group called its real Malcev completion [R72]. Therefore the study of the Dehn function of nilpotent finitely generated groups reduces to that of simply connected nilpotent Lie groups[4]. Recall that by a theorem of Guivarc'h [Guiv73], if G is a nilpotent simply connected Lie group, then $V(n) \approx n^D$, where

$$D = \sum_{i \geq 1} i \dim(C^i(G)/C^{i+1}(G)),$$

with $C^i(G)$ is the lower central series, defined inductively as follows

$$C^1(G) = G, \quad C^{i+1}(G) = [G, C^i(G)].$$

The integer D is called the growth exponent of G. Note that D is a quasi-isometry invariant. It turns out that the probability of return does not provide a more refined invariant, as the following theorem, due to Varopoulos shows (see [V85, VSC92]).

Theorem 3.2. *Let G be a nilpotent connected Lie group of volume growth exponent D. Then*

$$\phi(n) \approx n^{-D/2}.$$

For instance, in the case of the Heisenberg group, defined as

$$H_3(\mathbf{Z}) = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbf{Z} \right\},$$

one has $V(r) \approx r^4$, and therefore $\phi(n) \approx n^{-2}$ (this estimate also holds for Heisenberg group with real coefficients $H_3(\mathbf{R})$).

More elementary proofs have been given since then (which are gathered in the nice survey [PS01]). For instance the fact that $V(r) \succeq r^D$ implies $\phi(n) \preceq n^{-D/2}$

[4]Observe that the converse is false as there exist simply connected nilpotent Lie group without lattices [R72, Remark 2.14].

can be obtained directly without appealing to Gromov's theorem (see [Heb92, HS93]). On the other hand, Barlow, Coulhon and Grigory'an [BCG01] gave a nice and elementary proof of the lower bound $\phi(n) \succeq n^{-D/2}$, which we cannot resist sketching here.

Theorem 3.3. *Assume that there exists $C < \infty$ such that for all r large enough, $V(2r) \leq CV(r)$. Then*

$$\phi(n) \succeq \frac{1}{V(n^{1/2})}.$$

In particular if $V(r) \approx r^D$ then $\phi(n) \succeq n^{-D/2}$.

Proof. To simplify notation we shall write $P_\mu = P$. It is a nice and easy exercise to show that for every non-zero $f \in L^2(G)$ the sequence $\|P^n f\|_2^2/\|P^{n-1}f\|_2^2$ is non-decreasing. It follows that for all $n \geq 1$,

$$\frac{\|P^n f\|_2^2}{\|f\|_2^2} \geq \left(\frac{\|Pf\|_2^2}{\|f\|_2^2}\right)^n$$

Then for every Borel subset $A \subset G$, define

$$\beta(A) = \sup_{\mathrm{Supp}(f) \subset A, f \neq 0} \frac{\|Pf\|_2^2}{\|f\|_2^2}.$$

Hence for every $f \in L^2(G) \cap L^1(G)$ supported in A and satisfying $\|f\|_1 = 1$,

$$\phi(n) = \|P^n\|_{1\to 2}^2 \geq \|f\|_2^2 \left(\frac{\|Pf\|_2^2}{\|f\|_2^2}\right)^n.$$

So by taking the supremum over such f, and using that $\|f\|_2^2 \geq |A|^{-1}$ we deduce

$$\phi(n) \geq \sup_A \frac{\beta(A)^n}{|A|}. \tag{3.4}$$

Note that

$$\beta(A) = 1 - (1 - \beta(A)) = 1 - \inf_{\mathrm{Supp} f \subset A} \frac{\|f\|_2^2 - \|Pf\|_2^2}{\|f\|_2^2}.$$

To obtain a lower bound on $\beta(A)$, it suffices to pick a test function f. This is where we use the doubling property: let $A = B(2r)$ and f be the "hat function" $f(g) = \max\{2r - |g|, 0\}$. Then $\|f\|_2^2 \geq r^2 V(r)$ and, using that μ is compactly supported, we have

$$\|f\|_2^2 - \|Pf\|_2^2 = \frac{1}{2}\sum_{g,h} |f(gh) - f(g)|^2 \mu^{(2)}(h) \preceq V(r),$$

from which we deduce that $1 - \beta(B(r)) \preceq 1/r^2$, or equivalently that $\beta(B(r)) \succeq e^{-1/r^2}$. Together with (3.4), we obtain $\phi(n) \succeq e^{-n/r^2}/V(r)$ which by choosing $r = n^{1/2}$ proves the theorem. □

3.5. **Solvable groups arising as closed linear groups.** By contrast with the polynomial growth case, we will see that solvable groups with exponential growth offer a large diversity of behaviors for ϕ. The only general relation between V and ϕ is given by the following celebrated result.

Theorem 3.4. [HS93]
Let G be any group such that $V(n) \approx \exp(n)$, then

$$\phi(n) \preceq \exp(-n^{1/3}).$$

In this section, we consider the problem of characterizing the class of groups for which this relation is sharp, i.e. such that $\phi(n) \approx \exp(-n^{1/3})$.

The reverse estimate

$$\phi(n) \succeq \exp(-n^{1/3}) \tag{3.5}$$

was first proved for polycyclic groups [Al92, V91], and for unimodular amenable connected Lie groups in [Heb92]. Other examples such as solvable Baumslag-Solitar and Lamplighter groups have been treated in [CGP01]. Pittet and Saloff-Coste proved (3.5) for torsion-free[5] solvable finitely generated groups with finite Prüfer rank [PS03]. Recall that a group G has Prüfer rank at most r if every finitely generated subgroup admits a generating subset of cardinality at most r. For instance polycyclic and solvable Baumslag-Solitar groups have finite Prüfer rank, while Lamplighter groups do not. Denote by $\mathrm{T}(d, \mathbf{K})$ the group of invertible upper triangular matrices of size d with coefficients in $\mathbf{K}$. We believe that the following (classical) fact would deserve to be better known.

Proposition 3.5. *Any torsion-free finitely generated solvable group with finite Prüfer rank virtually sits as a discrete subgroup of* $\mathrm{T}(d, \mathbf{C}) \times \mathrm{T}(d, \mathbf{K}_1) \times \ldots \times \mathrm{T}(d, \mathbf{K}_m)$*, where $\mathbf{K}_i$'s are local fields of zero characteristic.*

Proof. By [Weh73, pp. 25-26], torsion-free solvable groups with finite Prüfer rank are $\mathbf{Q}$-linear, i.e. subgroups of $\mathrm{GL}(d, \mathbf{Q})$ for some $d \in \mathbf{N}$. Since G is finitely generated, its entries lie in $\mathbf{Z}[1/n]$ for some integer $n = p_1^{\alpha_1} \ldots p_m^{\alpha_m}$. Therefore G embeds discretely in $\mathrm{GL}(d, \mathbf{R}) \times \mathrm{GL}(d, \mathbf{Q}_{p_1}) \times \ldots \times \mathrm{GL}(d, \mathbf{Q}_{p_m})$. Then by Lie Kolchin's theorem, up to passing to a finite-index subgroup, G embeds in $\mathrm{T}(d, \mathbf{C}) \times \mathrm{T}(d, \mathbf{K}_1) \times \ldots \times \mathrm{T}(d, \mathbf{K}_m)$, where $\mathbf{K}_i$ is a finite extension of $\mathbf{Q}_{p_i}$. $\square$

It turns out that all known examples of groups satisfying (3.5) belong to the following class of amenable locally compact groups. The class GES of *geometrically elementary solvable groups* is the smallest class of compactly generated locally compact groups

- comprising all unimodular closed compactly generated subgroups of the group $\mathrm{T}(d, \mathbf{K})$, for any integer d and any local field $\mathbf{K}$;

[5]The mention "torsion-free" is missing from the statements of [PS03], although their proof only applies under this assumption as they use some linear embedding of the nilpotent radical.

- stable under taking finite direct products, unimodular closed compactly generated subgroups, and unimodular quotient;
- stable under quasi-isometry.

Theorem 3.6. [T13, Theorem 7] *GES groups*[6] *satisfy* $\phi(n) \succeq \exp(-n^{1/3})$.

The class GES contains non-virtually solvable groups such as $F \wr \mathbf{Z}$ for any non-trivial finite group F. It also contains finitely generated groups which are not residually finite (in particular not linear) as shown by the following example due to Hall [Hal61]. Fix a prime q and consider the group of upper triangular 3 by 3 matrices:

$$G = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & q^n & y \\ 0 & 0 & 1 \end{pmatrix}; x, y, z \in \mathbf{Z}[1/q]; n \in \mathbf{Z} \right\}.$$

Taking the quotient by the central infinite cyclic subgroup of unipotent matrices $I + mE_{1,3}$ where $m \in \mathbf{Z}$, we obtain an elementary solvable group which is non-residually finite since its center is isomorphic to $\mathbf{Z}[1/q]/\mathbf{Z}$.

In [T13, Question 1.4], it is asked whether the converse of Theorem 3.6 holds: are all unimodular locally compact compactly generated groups G satisfying $\phi(n) \succeq \exp(-n^{1/3})$ in the class GES? It should be pointed though that the class GES is somewhat not very natural as it is defined in both algebraic and geometric terms. Here is a more evasive though more natural question.

Question 3.7. Can (3.5) be characterized algebraically?

On a philosophical level, groups satisfying (3.5) are the smallest possible among those with exponential growth, while for a linear group, being embeddable as a *discrete* (or more generally *closed*) subgroup of a finite product of groups of matrices over local fields is also a "size" constraint (implying for instance finite asymptotic dimension).

As we shall see in the next section, the "smallest" finitely generated solvable groups with exponential growth failing to satisfy (3.5) are $\mathbf{Z} \wr \mathbf{Z}$ nor $\mathbf{F}_p \wr \mathbf{Z}^2$ for some prime p. On the other hand the decay of ϕ gets slower when passing to a subgroup or a quotient [PS03]. These remarks motivate the following conjecture.

Question 3.8. Does a solvable finitely generated group G satisfy $\phi(n) \succeq \exp(-n^{1/3})$ if and only if it does not have a subquotient isomorphic to $\mathbf{Z} \wr \mathbf{Z}$ or $\mathbf{F}_p \wr \mathbf{Z}^2$ for any prime p? Although this is not directly related to this problem, it seems relevant to mention a result of Kropholler [Kr84] who characterizes solvable groups not having $\mathbf{C}_p \wr \mathbf{Z}$ as a subquotient.

This should be compared with the following conjecture of Pittet and Saloff-Coste.

Question 3.9. [Sal04, Conjecture 2] Does a finitely generated torsion-free solvable group G satisfy $\phi(n) \succeq \exp(-n^{1/3})$ if and only if it has finite Prüfer rank?

[6] There is an issue in [T13]: Theorem 7 uses a definition of GES groups which is *a priori* larger the one given here. However its proof is only valid for the latter.

3.6. **Estimates for wreath products.** Recall that $G \wr H = \bigoplus_H G \rtimes H$, where H acts by permuting the factors. Building on an idea of Varopoulos [V83, V83'], Pittet and Saloff-Coste [PS99, PS02] compute the long range estimates of ϕ for various interesting solvable groups, providing the first examples of solvable groups with decay faster than $\exp(-n^{1/3})$. For instance regarding the discussion of last section, they prove that for $\mathbf{Z} \wr \mathbf{Z}$,

$$\phi(n) \approx \exp(-n^{1/3}(\log n)^{2/3}),$$

and for $\mathbf{F}_p \wr \mathbf{Z}^2$,

$$\phi(n) \approx \exp(-n^{1/2}).$$

So these two groups violate (3.5).

The more general estimates for wreath products were obtained by Erschler [E03, E06] who introduced new ideas involving some notion of isoperimetric profile. Here is a sample of the estimates obtained by these authors (see [Sal04, Section 4.3] for a more comprehensive list).

Example 3.10. [E03, PS02, SZ]

- For $A \wr \mathbf{Z}^d$ with A finite (non-trivial):

$$\phi(n) \approx \exp(-n^{d/(d+2)}).$$

- For $A \wr \mathbf{Z}^d$ with A infinite with polynomial growth and $d \neq 0$:

$$\phi(n) \approx \exp(-n^{d/(d+2)}(\log n)^{2/(d+2)}).$$

- For $A \wr \mathbf{Z}^d$ with A polycyclic with exponential growth:

$$\phi(n) \approx \exp(-n^{(d+1)/(d+3)}).$$

- For $A \wr B$ with B polycyclic with exponential growth, and A non-trivial polycyclic (not necessarily infinite):

$$\phi(n) \approx \exp(-n/(\log n)^2).$$

- Denote $A \wr_k B$ the iterated wreath product defined inductively by $A \wr_0 B = A$ and $A \wr_{k+1} B = (A \wr_k B) \wr B$. Then for $\mathbf{Z} \wr_k \mathbf{Z}$,

$$\phi(n) \approx \exp(-n(\log_k n/\log_{k-1} n)^2),$$

where $\log_k$ is defined inductively by $\log_0 n = n$, and $\log_{k+1} n = \log(1 + \log_k n)$.
- For the free d-solvable group of rank r, one has, for $d \geq 2$

$$\phi(n) \approx \exp(-n(\log_{d-1} n/\log_{d-2} n)^{2/r}),$$

and for $d = 1$ (the free metabelian case),

$$\phi(n) \approx \exp(-n^{r/(r+2)}(\log n)^{2/(r+2)}).$$

As an interesting application of some of these computations, Erschler [E06] deduced new fine estimates on the multiplicity of self-intersections of the simple random walk on $\mathbf{Z}^d$.

We end this section with the following open question, essentially motivated by the lack of counterexamples combined with the fact that there are only countably finitely generated many metabelian groups

Question 3.11. [Sal04] Is-it true that if Γ is metabelian of exponential growth then there exist two rational numbers α, β, such that $\phi(n) \approx \exp(-n^{\alpha}(\log n)^{\beta})$?

4. Cohomological properties as geometric invariants for amenable groups

For this section we refer almost exclusively to Shalom's seminal paper [Sh04]. Instead of giving a complete list of results obtained by Shalom, which would take too long, we shall outline a few key notions introduced in that article. Using harmonic analysis tools, Shalom introduces new quasi-isometry invariants for amenable groups, from which he deduces interesting rigidity properties for certain classes of solvable groups. As a first striking application he finds an elementary proof of the quasi-isometry rigidity of the class of groups which are virtually isomorphic to $\mathbf{Z}^d$, for every d. Indeed, the proof uses relatively light spectral theory, so in particular avoids Montgomery-Zippin's book on Hilbert's fifth problem [MZ74] involved in Gromov's characterization of groups with polynomial groups [Gr81].

Let G be a locally compact second countable group, and π a continuous unitary representation on a Hilbert space $\mathcal{H}$. For any integer $n \geq 0$, define

$$C^n(G,\pi) = \{\omega : G^n \to \mathcal{H} : \omega(gg_0,\dots,gg_{n-1}) = \pi(g)\omega(g_0,\dots,g_{n-1})\}.$$

Let $d_n : C^n(G,\pi) \to C^{n+1}(G,\pi)$ be the standard differential:

$$(d_n\omega)(g_0,\dots,g_n) = \sum_{i=1}^{n}(-1)^i\omega(g_0,\dots,\hat{g}_i,\dots,g_n).$$

Denote the spaces of n-cocycles, n-coboundaries, n-cohomology and reduced n-cohomology respectively by $Z^n(G,\pi)$, $B^n(G,\pi)$, $H^n(G,\pi) = Z^n(G,\pi)/B^n(G,\pi)$, and $\overline{H}^n(G,\pi) = Z^n(G,\pi)/\overline{B^n(G,\pi)}$, where we take the closure for the topology of uniform convergence on compact sets.

4.1. Property H_{FD} and variants.

Let us recall the following properties introduced by Shalom.

Definition 4.1. The group G has Property H_{FD} or H_T, if for every unitary representation π with $\overline{H^1}(G,\pi) \neq 0$, there exists a non-zero subrepresentation $\sigma \subset \pi$ which is finite dimensional, or respectively trivial.

Clearly

$$(H_T) \Longrightarrow (H_{FD}).$$

A crucial reason for working with the reduced cohomology is that contrary to the usual one, the reduced cohomology desintegrates over irreducible representations.

Theorem 4.2. [Guic80, p 190, Proposition 2.6] *If $\pi = \int^{\oplus} \pi_x d\mu(x)$ is a direct integral decomposition of the unitary representation π of G, and for μ-a.e x one has $\overline{H^n}(G, \pi_x) = 0$, then $\overline{H^n}(G, \pi) = 0$.*

Hence Property H_T can be reformulated as "any irreducible unitary representation of G with non-trivial first reduced cohomology is trivial".

Shalom proves that nilpotent groups have Property H_T (see Section 4.2 for a proof in the case of $\mathbf{Z}$).

A result of Delorme [De77] states that any connected solvable Lie group has Property H_{FD}. Using basic spectral theory, Shalom proves that $(\mathbf{Q}_p \oplus \mathbf{R}) \rtimes_{(p,p)} \mathbf{Z}$, and $\mathbf{F}_n(\!(t)\!)^2 \rtimes_{(t,t^{-1})} \mathbf{Z}$ have property H_T. These two examples are interesting as they contain respectively Baumslag-Solitar's group $\mathrm{BS}(1,p)$ and the lamplighter group $\mathbf{F}_p \wr \mathbf{Z}$ as cocompact lattices (see Section 2). It is very likely that this property holds more generally for groups of the form $G = U \rtimes A$, where $A \simeq \mathbf{Z}^d$ acts semi-simply on U, unipotent group over a finite product of local fields (e.g. finite extensions of $\mathbf{R}$, $\mathbf{Q}_p$, or $\mathbf{F}_p(\!(X)\!)$).

Using induction of cocycles (see section 4.4), Shalom proves that Property H_{FD} is stable under quasi-isometry, and is inherited by cocompact lattices. He deduces that polycyclic groups, solvable Baumslag-Solitar groups, and lamplighter groups have Property H_{FD}. On the other hand he shows that $\mathbf{Z} \wr \mathbf{Z}$ does not have it. It is therefore tempting to ask whether countable discrete groups with H_{FD} necessarily arise from similar arithmetic constructions (compare this to the discussion of Section 3.5, and more specifically to Conjecture 3.8).

The following result makes a crucial use of the quasi-invariance of Property H_{FD}.

Theorem 4.3. *Any infinite group Γ which is quasi-isometric to a polycyclic group has a finite index subgroup with infinite abelianization.*

One might ask if this can be strengthened to the statement that a group which is quasi-isometric to a polycyclic group has greater or equal first virtual Betti number. However, this turns out to be false [Sh04, Theorem 5.5.1] (even assuming that both groups are polycyclic).

These results rely on a general process of *induction* of cocycles we shall describe in Sections 4.3 and 4.4. But, before explaining these techniques, we would like to give an elementary proof that $\mathbf{Z}$ has property H_T, which has the advantage of showing an interesting interpretation of the first reduced cohomology.

4.2. **A proof that Z has property H_T.** Let us recall a convenient characterization of first cohomology in terms of isometric affine actions. Here we let G be a locally compact σ-compact group. Let $c(g_0, g_1)$ be a 1-cochain. Remember that $c(gg_0, gg_1) = c(g_0, g_1)$, justifying the "non-homogeneous" notation $b(g) = c(1, g)$ for cochains. Then b is a 1-cocycle if and only it satisfies the relation

$$b(gh) = \pi(g)b(h) + b(g),$$

and b is coboundary if and only if there exists $v \in \mathcal{H}$ such that $b(g) = v - \pi(g)v$. Now, given a cocycle b, one can define an affine isometric action σ of G on $\mathcal{H}$ by the formula

$$\sigma(g)w = \pi(g)w + b(g),$$

for all $w \in \mathcal{H}$. Notice that σ fixes $v \in \mathcal{H}$ if and only if b is a coboundary, more precisely $b(g) = v - \pi(g)v$. Moreover, one checks that b is in the closure of coboundaries if and only if σ "asymptotically" fixes a point, i.e. if there is a sequence $v_n \in \mathcal{H}$ such that for all $g \in G$,

$$\|\sigma(g)v_n - v_n\| \to 0,$$

the convergence being uniform on compact sets.

Proposition 4.4. *The group* **Z** *has Property H_T.*

Proof. Let $(\pi, \mathcal{H})$ be a unitary representation of **Z** without invariant vectors. We must prove that for every cocycle $b \in Z^1(G, \pi)$, the affine representation σ almost has fixed points. We claim that the sequence

$$v_n = \frac{1}{n}\sum_{j=0}^{n-1} b(j) = \frac{1}{n}\sum_{j=0}^{n-1} \sigma(j)0$$

is asymptotically fixed by σ. Indeed, one has

$$\sigma(1)v_n - v_n = \frac{1}{n}\sum_{j=0}^{n-1}(\sigma(j+1)0 - \sigma(j)0) = \frac{1}{n}\sum_{j=0}^{n-1}(b(j+1) - b(j)) = \frac{b(n)}{n}.$$

Moreover, the cocycle relation implies

$$b(n) = \sum_{i=0}^{n-1} \pi(i)b(1) = \sum_{i=0}^{n-1} \pi(1)^i b(1).$$

But since $\pi(1)$ does not have invariant vectors, Von Neumann's ergodic theorem implies that

$$\frac{b(n)}{n} = \frac{1}{n}\sum_{i=0}^{n-1} \pi(1)^i b(1) \to 0.$$

So it follows that $b \in \overline{B^1(G, \pi)}$, hence that $\overline{H}^1(G, \pi) = 0$. □

4.3. **Topological and measurable couplings.** Underpinning Shalom's strategy are two notions due to Gromov: topological and measurable couplings.

A *topological* coupling between two countable discrete groups Λ and Γ is a locally compact space X equipped with two commuting actions (by homeomorphisms) of Λ and Γ on X such that both actions are proper and cocompact. The main observation of Gromov is that two groups are quasi-isometric if and only if they admit a topological coupling.

A *measured* coupling between Λ and Γ is a Borel measured space (X, μ) equipped with two commuting, measure preserving actions of Λ and Γ such that each of these actions admits a fundamental domain of finite measure. Two groups are called *measure equivalent* (ME) if they admit a measured coupling (also called ME coupling).

A measured coupling (X, μ) is called *uniform* if there exist fundamental domains X_Λ and X_Γ for respectively Λ and Γ such that any Λ-translate of X_Γ intersects only finitely many Γ-translates of X_Γ (and similarly, interchanging the roles of Λ and Γ). The groups are then called uniformly measure equivalent (UME).

Observe that two lattices in a same locally compact group G are ME. Here the space is of course the ambient group G, equipped with a Haar measure, the groups Λ and Γ acting respectively by left and right translations. Moreover if the two lattices are cocompact, then G defines a topological coupling. Hence in the latter case, the same coupling is both measurable and topological. By taking relatively compact fundamental domains, one easily sees that such a coupling is uniform.

Shalom's main observation is that a topological coupling between two amenable groups can be equipped with a bi-invariant Borel measure, and therefore gives rise to a UME coupling. Namely, since Γ is amenable, there exists a Γ-invariant probability measure on $\Lambda\backslash X$, which can then be lifted to a Borel bi-invariant measure on X.

4.4. **Induction of representations via quasi-isometries.** Let us recall the construction of a topological coupling between quasi-isometric groups. Given a quasi-isometry $f : \Lambda \to \Gamma$, one can define the space X to be the closure for the pointwise topology in Γ^Λ of the set of functions $\lambda \to \gamma_0 f(\lambda_0^{-1}\lambda)$, where λ_0 and γ_0 vary respectively in Λ and Γ. One easily checks that X is indeed a topological coupling with respect to the obvious Λ-action at the source and Γ-action at the target. We shall denote the Λ-action on X as right multiplication $x\lambda^{-1}$, while the Γ-action will be denoted as left multiplication γx. Up to replacing Γ by $\Gamma \times F$ for some finite group F we can always assume that f is injective, and for the sake of simplicity, we will even assume in the sequel that f is a bijection. Then the actions of both Λ and Γ share a common compact open fundamental domain:

$$X_\Lambda = X_\Gamma = X_0 = \{h \in X : \ h(1) = 1\}.$$

We now define the **cocycles** $\alpha : \Gamma \times X_0 \to \Lambda$ and $\beta : X_0 \times \Lambda \to \Gamma$ by the rule

$$\alpha(\gamma, x) = \lambda \Longleftrightarrow \beta(x, \lambda) = \gamma \Longleftrightarrow \gamma^{-1}x\lambda \in X_0,$$

Observe X_0 naturally identifies with X/Λ and via this identification, the Γ-action (by homeomorphisms) on X/Λ becomes, for all $x \in X_0$,

$$\gamma \cdot x = \gamma x \alpha(\gamma^{-1}, x).$$

Let us suppose from now on that Γ is *amenable.* Then there exists a Γ-invariant probability measure μ on X_0.

Now, given a unitary Λ-representation $(\pi, \mathcal{H})$, we define the **induced representation** $\mathrm{Ind}_\Lambda^\Gamma \pi$ where the representation space is

$$L^2(X_0, \mathcal{H}) = \left\{ \psi : X_0 \to \mathcal{H} : \int_{X_0} \|\psi(x)\|_2^2 d\mu(x) < \infty \right\},$$

and the (unitary) Γ-action is defined as

$$(\gamma\psi)(x) = \pi(\alpha(\gamma, x))\psi(\gamma^{-1} \cdot x).$$

The next result is one of the two main step in the proof of the invariance of H_{FD} under quasi-isometry.

Proposition 4.5. [Sh04, Theorem 3.1.2] *If $\mathrm{Ind}_\Lambda^\Gamma \pi$ contains a finite dimensional Γ-subrepresentation, then π contains a finite dimensional Λ-subrepresentation.*

This (classically) results from the following (more symmetrical) statement.

Proposition 4.6. *Let σ be a unitary Γ-representation. Then the trivial Γ-representation is contained in $\mathrm{Ind}_\Lambda^\Gamma \pi \oplus \sigma$ if and only if the trivial Λ-representation is contained in $\mathrm{Ind}_\Gamma^\Lambda \sigma \oplus \pi$.*

4.5. **Induction of cohomology via uniform measurable coupling.** We keep the notation of the last section. Shalom defines two maps at the level of cochains

$$\begin{array}{rcl} I : C^{n+1}(\Lambda, \pi) & \to & C^{n+1}(\Gamma, \mathrm{Ind}_\Lambda^\Gamma \pi) \\ I\omega(\gamma_0, \dots \gamma_n)(x) & = & \omega(\alpha(\gamma_0, x), \dots, \alpha(\gamma_n, x)), \end{array}$$

and

$$\begin{array}{rcl} T : C^{n+1}(\Gamma, \mathrm{Ind}_\Lambda^\Gamma \pi) & \to & C^{n+1}(\Lambda, \pi) \\ T\sigma(\lambda_0, \dots \lambda_n) & = & \displaystyle\int_{X_0} \sigma(\beta(x, \lambda_0), \dots, \beta(x, \lambda_n)) d\mu(x). \end{array}$$

Proposition 4.7. *Both the "induction" morphism I and the "transfer" morphism T commute with the differential, inducing morphisms at the level of the (reduced) cohomogy such that TI is the identity on $H^n(\Lambda, \pi)$ (and on $\overline{H}^n(\Lambda, \pi)$). In particular, the induction morphism induces an injection in (reduced) cohomology.*

Before turning to the proof of the invariance under quasi-isometry of Property H_{FD}, let us mention an interesting by-product of the above construction. The following statement is a corollary of [Sh04, Theorem 1.5]. Recall that the cohomological dimension of a discrete group Γ over a ring R is the supremum over all integers n such that there exists an $R\Gamma$-module V with $H^n(\Gamma, V) \neq 0$.

Theorem 4.8. *The cohomological dimension over* $\mathbf{Q}$ *is stable under quasi-isometry among the class of all amenable groups.*

Actually, to be precise, the proof of this theorem uses a variant of the previous induction techniques, where the cohomology with values in a unitary representation is replaced by ordinary cohomology with values in $\mathbf{Q}$, and where the probability μ is replaced by an invariant mean $\mu_{\mathbf{Q}}$ with values in $\mathbf{Q}$, defined on the boolean algebra of clopen subsets of X_0 (which is therefore adapted to this purely algebraic setting). To find such a $\mu_{\mathbf{Q}}$, one can compose μ with $\mathbf{Q}$-linear projection from $\mathbf{R}$ to $\mathbf{Q}$ (which exists by the axiom of choice!).

To put Theorem 4.8 into perspective, recall that Gromov [Gr93, 1.H] asked whether cohomological dimension is a quasi-isometry invariant in all generality. Gersten [Ge93'] proved that two quasi-isometric groups with finite classifying spaces have same cohomological dimension. Let us finally mention that [Sh04, Theorem 1.5] has also implications for non-amenable groups (see also [Sh04, 6.1]).

4.6. **Proof of the invariance of Property H_{FD} under quasi-isometry.** All the ingredients of that proof are squattered in the previous sections, so let us put them together. Suppose Λ and Γ are quasi-isometric and Γ has Property H_{FD}. Let π be a unitary Λ-representation with non-trivial first reduced cohomology. By Proposition 4.7, $\mathrm{Ind}_{\Lambda}^{\Gamma}\pi$ has non-trivial first reduced cohomology, hence by Property H_{FD} of Γ it contains a non-zero finite dimensional subrepresentation. Therefore Proposition 4.5 implies that π also contains a non-zero finite dimensional subrepresentation, which finishes the proof.

4.7. **A stronger result for nilpotent groups.** Shalom proves that nilpotent groups satisfy an even stronger property than property H_T, namely that for *every* $n \geq 0$, any π with $\overline{H^n}(G, \pi) \neq 0$ contains the trivial representation. Observe the connection with betti numbers: $b_n(G) = \dim_{\mathbf{C}} H^n(G, 1) = \dim_{\mathbf{C}} \overline{H^n}(G, 1)$. Using the induction techniques described in the previous sections, he shows

Theorem 4.9. *Betti numbers are invariant under quasi-isometry among nilpotent groups.*

This was subsequently improved by Sauer:

Theorem 4.10. [Sau06, Theorem 1.5] *If* Λ *and* Γ *are quasi-isometric nilpotent groups, then the real cohomology rings* $H^*(\Lambda, \mathbf{R})$ *and* $H^*(\Gamma, \mathbf{R})$ *are isomorphic as graded rings.*

In the same paper, Sauer also proves that the Hirsch rank is stable under quasi-isometry among solvable groups (this statement had been proved earlier for polycyclic groups [Ro03]). Recall that the Hirsch rank of a solvable group is

$$\mathrm{hr}(G) = \sum_{i\geq 1} \dim(\mathbf{Q}\otimes A_i),$$

where $A_i = D^i(G)/D^{i+1}(G))$, $(D^i(G))_i$ being the derived series of G, defined inductively as $D^0(G) = G$, and $D^{i+1}(G) = [D^i(G), D^i(G)]$.

4.8. **Open problems.** Once again we refer to Shalom's article for further results and also lots of interesting open questions (see [Sh04, Section 6]). A natural open problem is to characterize among finitely generated solvable groups, those with Property H_{FD}. More specifically does the group $\mathbf{F}_p \wr \mathbf{Z}^2$ satisfy H_{FD}? Also does Delorme's result generalize to solvable algebraic p-adic groups?

Shalom asks whether Property H_{FD} can be characterized by the non-existence of an irreducible infinite-dimensional unitary representation with non-trivial first reduced cohomology, suggesting as a possible counterexample the group $\mathbf{Z}\wr\mathbf{Z}$.

Finally let us mention that higher degree versions of Property H_{FD} are also quasi-isometry invariants between amenable groups, and are yet to be studied more systematically. In particular, can Property H_{FD} and its higher dimensional versions be characterized in purely geometric terms?

5. Dehn functions of solvable groups

The notion of compact presentation provides a unified setting to define the Dehn function in the context of arbitrary locally compact groups. It was introduced by Kneser [Kn64] who applied it to the p-completion of p-arithmetic groups.

Definition 5.1. [Ab87, Section 1.1] A locally compact group G is *compactly presented* if for some/any compact generating symmetric set S and some k (depending on S), there exists a presentation of the abstract group G with S as set of generators, and relators of length $\leq k$.

Kneser proves that if an algebraic group G is defined over $\mathbf{Z}$, then $G(\mathbf{Z}[1/n])$ is finitely generated/presented if and only if $G(\mathbf{Q}_p)$ is compactly generated/presented for all primes p dividing n. In his seminal work, Abels [Ab87] characterizes p-adic algebraic groups which are compactly presented (which reduces to the solvable case), and uses Kneser's theorem to deduce an algebraic characterization of finitely presented p-arithmetic groups.

Let G be any locally compact compactly generated group and let S be a compact generating subset. If $k\in\mathbf{N}$ is as in Definition 5.1, a relation means a word w in the letters of S which represents the trivial element of G; its *area* is the least m such that w is a product, in the free group, of $\leq m$ conjugates of relations of length $\leq k$.

Definition 5.2. The *Dehn function* of G is defined as

$$\delta(n) = \sup\{\text{area}(w) : w \text{ relation of length } \leq n\}.$$

When G is compactly presented, one can check that $\delta(n) < \infty$ for all $n \in \mathbf{N}$, provided that k is large enough. The precise value of $\delta(n)$ depends on (S, k), but not the $\simeq$-asymptotic behavior, where $u(n) \simeq v(n)$ if for suitable positive constants $a_1, \dots, b_4$ independent of n

$$a_1 u(a_2 n) - a_3 n - a_4 \leq v(n) \leq b_1 u(b_2 n) + b_3 n + b_4, \quad \forall n \geq 0.$$

In the combinatorial point of view, the Dehn function of a finitely presented group is a measure of the complexity of its word problem. In the geometric point of view, compact presentability means simple connectedness at large scale [Gr93, 1.C$_1$], and the Dehn function appears as a quantified version. See also the nice introduction by Gersten [Ge93]. Lower bounds on the Dehn function are often referred as an "isoperimetric inequalities", while upper bounds are sometimes called "filling inequalities". Let us only mention here that for a simply connected Lie group equipped with a left-invariant riemannian metric, the usual 2-dimensional isoperimetric function has the same asymptotic behavior as the Dehn function defined in terms of presentation (which makes sense as connected Lie groups are compactly presented). We refer the reader to Bridson's survey [Bri02] for a detailed discussion (which extends readily to our more general setting).

5.1. **Nilpotent groups.** The study of filling invariants of nilpotent groups is a whole area in itself, involving difficult analytic, combinatorial and homological techniques. We refer to Sapir's survey [Sap11, Section 3.2] for a more complete account. Recall that the study of large-scale invariants of nilpotent groups reduces to that of simply connected nilpotent Lie groups (see Section 3.4).

The "(c+1)-upper bound". The "(c+1)-upper bound", formulated by Gromov [Gr93, 5A'$_5$] states that a simply connected nilpotent group of class c satisfies a n^{c+1}-filling inequality. This upper bound on the Dehn function is proved in complete generality in [GHR03] by combinatorial methods, while in [Gr93, Gr96] Gromov outlines an analytic proof based on infinitesimally invertible operators.

It is worth mentioning that this filling inequality has a nice simple proof for connected nilpotent homogeneous Lie groups (outlined in [Gr93, 5A'$_5$] and proved in details in [Pi95]). Let us briefly summarize the argument. Recall that a connected nilpotent Lie group is homogeneous if there exists a one-parameter family of continuous automorphisms δ_t, $t > 0$ such that δ_t induces the multiplication by t^j on the real vector space $C^j(G)/C^{j+1}(G)$ (see Section 3.4). The first example is of course $\mathbf{R}^n$, for which δ_t is simply the multiplication by t. Heisenberg's group,

$$H_3(\mathbf{R}) = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbf{R} \right\},$$

admits a dilation δ_t which acts by multiplying x and y by t, and z by t^2.

For a suitable (Carnot-Carathéodory) left-invariant length metric on G, δ_t is a similitude, i.e. satisfies $d(\delta_t(g), \delta_t(g')) = td(g, g')$ (see for instance [Go76, Pan83, Bre]). Let $B(R)$ denote ball of radius R for such a metric. The key observation is that the homotopy $F : [0, 1] \times B(R) \to B(R)$ defined by $F(t, g) = \delta_t(g)$ for $t > 0$, and $F(0, g) = 1$ is $O(R^c)$-Lipschitz with respect to t and 1-Lipschitz with respect to g. It easily follows that a loop γ can be filled with a disc of area controlled by diameter$(\gamma)^c \times$length(γ), hence by length$(\gamma)^{c+1}$.

In the case of the Heisenberg group $H_3(\mathbf{R})$, this implies that the Dehn function is bounded by n^3.

The n^{c+1}-upper bound on the Dehn function is not sharp in general. For instance, higher dimensional Heisenberg's groups

$$H_{2m+1}(\mathbf{R}) = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & I_m & y \\ 0 & 0 & 1 \end{pmatrix} : x = (x_1, \dots, x_m), y = (y_1, \dots, y_m)^T, z \in \mathbf{R} \right\},$$

have quadratic Dehn function as soon as $m \geq 2$. This fact was claimed by Thurston [ECHLPT92] and proved in [All98, Gr96] (let us also mention a particularly nice combinatorial proof given in [OS99]). More recently Young [Y] constructed examples of simply connected nilpotent Lie groups of class c for every $c \geq 1$ with quadratic Dehn function.

Lower bounds and central extensions. The n^{c+1}-upper bound turns out to be sharp for free nilpotent groups of class c [BMS93, Ge93]. In particular $H_3(\mathbf{R})$ has cubical Dehn function. Let us sketch the combinatorial proof of this fact given in [BMS93, Theorem 7].

One way to obtain lower bounds for the Dehn function is to exhibit distorted central extensions, as shown by the following proposition (for simplicity we state it only for discrete groups).

Proposition 5.3. [BMS93] *Let G be a finitely presented group and let S be a finite generating subset. Let g be an infinite order central element in some finitely presented group G. Suppose moreover that g is "ϕ-distorted", i.e. for every $m \in \mathbf{N}$,*

$$|g^m|_S \leq \phi(m),$$

for some increasing function $\phi : \mathbf{N} \to \mathbf{N}$. Then if $G/\langle g\rangle$ is finitely presented its Dehn function satisfies $D(n) \gtrsim \phi^{-1}(n)$.

Proof. Let π denote the projection $G \to G/\langle g\rangle$. Let $\langle \pi(S); R\rangle$ be a finite presentation of $G/\langle g\rangle$. Let $m \in \mathbf{N}$, and let w be a word in S of length $n \leq \phi(m)$ such that $g^m = w$. Observe that w is a relation relative to the presentation of $G/\langle g\rangle$. Suppose its area is less than $k \in \mathbf{N}$, i.e. one has the following equality in the free group $\langle S\rangle$: $w = r_1^{a_1} \dots r_k^{a_k}$, where $r_i \in R$. Pushing this equality in G eliminates the a_i's (as the r_i's are central). Moreover, in G, each $r_i = g^{m_i}$ for some bounded m_i in G. Therefore we get $k \gtrsim m_1 + \dots + m_k = m$. □

Let $N(c,r)$ be the free nilpotent group of class c and rank $r \geq 2$ (i.e. whose abelianization is isomorphic to $\mathbf{Z}^r$). Note that the kernel of the natural projection $N(c+1,r) \to N(c,r)$ is simply the last non-trivial term of the descending central sequence of $N(c+1,r)$ whose elements are easily seen to be n^{c+1}-distorted. Hence Proposition 5.3 implies that the Dehn function of $N(c,r)$ is at least n^{c+1}.

"Exotic" examples. The precise behavior of the Dehn function of (even 2-step) nilpotent groups is far from being understood. Wenger [Wen11] gave a stunning example of a 2-step nilpotent group with Dehn function satisfying $D(n)/n^2 \to \infty$ and $D(n) \lesssim n^2 \log n$ (the upper bound being due to Ol'shanskii and Sapir [OS99, Remark page 927], see also [Y', Corollary 5]).

Let us close this section with an open question.

Question 5.4. Let G be a simply connected nilpotent Lie group. Does the limit $\alpha_G = \lim \log D(n)/\log n$ always exist? Is α necessarily an integer?

5.2. **Connected solvable Lie groups and non-cocompact lattices.** A motivation for studying filling invariants of solvable connected Lie groups comes from questions about non-cocompact irreducible lattices in semisimple Lie groups. Recall that symmetric spaces being non-positively curved, their filling invariants are easy to compute (and in the non-hyperbolic case, behave as in the euclidean space of the same dimension). This is why we focus on non-cocompact lattices. Gromov [Gr93, $5A_7$] (see [Leu04] for a detailed proof) proved an exponential filling inequality for all lattices in semi-simple Lie groups. Recall that a non-cocompact lattice acts co-compactly on a subset of the symmetric space obtained by removing horoballs whose boundary can be modeled over some connected solvable Lie groups. Let us call these solvable groups horospherical groups. It is generally much simpler to estimate the Dehn function of the horospherical group than of the lattice. Therefore the approach used by most authors is to find ways to transfer isoperimetric/filling inequalities from the horospherical group to the lattice.

The rank 1 case. In rank one, horospherical groups are 2-step nilpotent, and lattices are relatively hyperbolic with respect to them. Gromov [Gr93, $5A_6$] deduces that these lattices satisfy a cubical filling inequality, which turns out to be sharp for non-cocompact lattices in $\mathrm{SU}(2,1)$ [Gr93], and $\mathrm{Sp}(n,1)$ [Pi97] for $n \geq 2$. The precise behaviour of the Dehn function is known for all lattices in rank 1 [Gr93, Pi97], except for the exceptional case $\mathrm{F}_{4(-20)}$, which remains open. Let us insist that the problem here is to understand the Dehn function of the (2-step nilpotent) maximal unipotent subgroup of $\mathrm{F}_{4(-20)}$, which by [Wen11] is known to be non-quadratic[7].

The rank 2 case. The idea for deducing isoperimetric inequalities from the horospherical group to the lattice is to project filling discs onto the horosphere (using convexity of horoballs in symmetric spaces). Difficulties may arise from

[7] The proof of the cubical lower bound in [Pi97] has a mistake, pointed out in [LP04].

the fact that horoballs are not necessarily disjoint (except when the lattice has **Q**-rank one). This approach is initiated by Thurston [ECHLPT92] who proves that $\mathrm{SOL}_3(\mathbf{R})$, and therefore $\mathrm{SL}(3, \mathbf{Z})$ have exponential Dehn function. Using the same approach, Leuzinger and Pittet [LP96] extend this result to all irreducible non-uniform lattices in any semisimple Lie group of real rank 2.

Lattices in higher ($\geq$ 3) rank. Generalizing Gromov [Gr93, 5.A_9], Drutu [Dr04] shows that if the simple Lie group has **R**-rank ≥ 3, then the horospherical groups satisfy a quadratic filling inequality. Using these estimates, she proves that if the lattice has **Q**-rank 1, then its Dehn function is bounded by $n^{2+\varepsilon}$ for any $\varepsilon > 0$.

Unfortunately this approach fails when the **Q**-rank is at least 2, due to the fact that horoballs intersect in an intricate way. However, estimates on horospherical groups are still useful. Young [Y13] uses quadratic filling inequalities proved by Leuzinger and Pittet [LP04] for certain solvable Lie groups in his recent groundbreaking demonstration that $\mathrm{SL}(n, \mathbf{Z})$ has quadratic Dehn function for $n \geq 5$. This fact has been conjectured by Thurston [ECHLPT92] for $n \geq 4$. Observe that the case $n = 4$ remains open.

5.3. **A class of metabelian groups.** Consider a semidirect product

$$G = V \rtimes A,$$

where $A \simeq \mathbf{Z}^d$ is a finitely generated abelian group of rank d, such that $a = \mathrm{diag}(a_i) \in A$ acts diagonally on $V = \bigoplus_{i=1}^m \mathbf{K}_i$, where $\mathbf{K}_i$ is a local field, equipped with a norm $|\cdot|_i$. Denote by

$$\begin{aligned} \alpha_i : \quad A \to & \quad \mathbf{R} \\ a \mapsto & \quad \log |a_i|_i. \end{aligned}$$

Note that $\alpha_i \in \mathrm{Hom}(A, \mathbf{R}) \simeq \mathbf{R}^d$. We will assume that none of the α_i's are trivial. The first part of the following theorem is the main result of [CT10].

Theorem 5.5. *Suppose that $\alpha_i \neq 0$ for all i. If for every pair $i \neq j$, the open interval (α_i, α_j) does not contain 0, then G is compactly presented with quadratic Dehn function if $d \geq 2$, and linear Dehn function if $d = 1$. Otherwise either G is not compactly presented or has exponential Dehn function.*

One can be more specific about the second part of the theorem. First observe that if there exists a pair i, j such that $0 \in (\alpha_i, \alpha_j)$, then it is easy to see by a retraction argument that the Dehn function of G is bounded below by the Dehn function of a subgroup $G_{i,j} = (\mathbf{K}_i \oplus \mathbf{K}_j) \rtimes \mathbf{Z}$ where the action of **Z** dilates one factor while contracting the other one. If both fields are non-archimedean, then $G_{i,j}$, and therefore G are not compactly presented (see [Ab87]). If for every such pair, at least one of the two fields is archimedean, then G is compactly presented but has exponential Dehn function (this follows for instance from Gromov's fibration inequality [Gr93, $5B_1$ and $5B_3$]).

Below are a few examples to which Theorem 5.5 applies.

Solvable Baumslag-Solitar. Recall that for every p prime, Baumslag-Solitar's group $\mathrm{BS}(1,p) = \langle t, x \mid txt^{-1} = x^p \rangle$ sits as a cocompact lattice inside the group $(\mathbf{Q}_p \oplus \mathbf{R}) \rtimes \mathbf{Z}$, where $\mathbf{Z}$ acts by multiplication by p on both factors (see Section 2). Hence we deduce from the second part of Theorem 5.5 that $\mathrm{BS}(1,p)$ has exponential Dehn function. This was first proved in [ECHLPT92] (see also [Ge92, Theorem B]). In [GH01], an exponential lower bound is more generally proved for any strictly ascending HNN-extension of a finitely generated nilpotent group.

Lamplighter. Another example to which the second part of Theorem 5.5 applies is the lamplighter group $L_p = \mathbf{F}_p \wr \mathbf{Z}$, which can be described as $\mathbf{F}_p[t, t^{-1}] \rtimes \mathbf{Z}$, where $\mathbf{Z}$ acts by multiplication by t. It is well known that L_p is not finitely presented, but here it can be seen as an immediate corollary of Theorem 5.5 using that L_p embeds as a cocompact lattice in $(\mathbf{F}_p(\!(t)\!))^2 \rtimes \mathbf{Z}$, where $\mathbf{Z}$ acts by multiplication by t and t^{-1} on the first, resp. the second factor (see Section 2).

SOL. Let $\mathrm{SOL}_{2d-1}(\mathbf{K})$ be the semidirect product of $\mathbf{K}^d$ by the set of diagonal matrices with determinant of norm one. It has a cocompact subgroup of the form $\mathbf{K}^d \rtimes \mathbf{Z}^{d-1}$, obtained by reducing to matrices whose diagonal entries are all powers of a given element of $\mathbf{K}$ of norm $\neq 1$. When $d \geq 3$, Theorem 5.5 implies that $\mathrm{SOL}_{2d-1}(\mathbf{K})$ has quadratic Dehn function, which was proved by Gromov for $\mathbf{K} = \mathbf{R}$ [Gr93, 5.A_9]. On the other hand if $d = 2$, then $\mathrm{SOL}_3(\mathbf{R})$ is well-known to have exponential Dehn function [ECHLPT92] and $\mathrm{SOL}_3(\mathbf{K})$ to be not compactly presented if $\mathbf{K}$ is non-archimedean [Ab87]: these statements result from the discussion following Theorem 5.5.

5.4. **Application to the complexity of the word problem.** It is well-known and easy to prove that the word problem of a finitely presented group G is solvable if and only the Dehn function is bounded by a recursive function. Supporting the idea that solvable groups can be wild, Kharlampovich [Kh81] exhibited a solvable finitely presented group with unsolvable word problem.

More recently, Kharlampovich, Myasnikov and Sapir proved the two following quantitative results. Recall that for a finitely presented residually finite group there is an algorithm due to McKinsey solving the word problem [Mc73].

Theorem 5.6. [KMS] *For every recursive function f, there is a residually finite finitely presented* 3*-solvable group G with Dehn function greater than f and the word problem decidable in polynomial time.*

Theorem 5.7. [KMS] *There exists a finitely presented residually finite* 3*-solvable group with NP-complete word problem.*

In [BRS02] it is proved that the word problem of a finitely generated group G is in NP (solvable in polynomial time by a non-deterministic Turing machine) if and only if G can be quasi-isometrically embedded into a finitely presented group with polynomial Dehn function. On the other hand the word problem is known to be in P for all linear finitely generated groups [LZ77, Wa91], therefore including solvable Baumslag-Solitar groups, and more generally all metabelian groups. It follows that these groups can be embedded in larger groups with polynomial Dehn function, and an interesting question is to find explicit bounds on the polynomial exponent. Answering a question of [BRS02], Arzhantseva and Osin [AO02] proved that Baumslag-Solitar's group can be embedded in a metabelian group with at most cubical Dehn function. Using Theorem 5.5 we shall see that both $\mathrm{BS}(1,n)$ and L_n (quasi-isometrically) embed into finitely presented metabelian groups with quadratic Dehn function.

Consider the two commuting matrices $A = \begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix}$, $B = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$, and define the group

$$\Gamma_p = \mathbf{Z}[1/p]^2 \rtimes_{(A,B)} \mathbf{Z}^2.$$

Note that Γ_p contains an obvious copy of $\mathrm{BS}(1,p)$, namely $(\mathbf{Z}[1/p] \times \{0\}) \rtimes (\mathbf{Z} \times \{0\})$. In [CT10], it is proved that Γ_p has quadratic Dehn function. To see why, observe that Γ_p sits as a cocompact lattice in the group

$$(\mathbf{Q}_p^2 \oplus \mathbf{R}^2) \rtimes_{(A,B)} \mathbf{Z}^2,$$

where A and B are viewed as matrices over the ring $\mathbf{Q}_p \times \mathbf{R}$, and apply Theorem 5.5.

Let $p > 0$ be a prime and consider the finitely presented group

$$\Lambda_p = \langle a, s, t \mid \; a^p, \; [s,t], \; [a^t, a], \; a^s = a^t a \rangle$$

which was introduced by Baumslag [Ba72] as a finitely presented metabelian group containing a copy of the lamplighter group $\mathbf{F}_p \wr \mathbf{Z}$. Recall that the group Λ_p embeds as a cocompact lattice into the group $\mathrm{SOL}_5(\mathbf{F}_p(\!(u)\!))$ (see [CT10, Proposition 3.4] for this basic fact). As a consequence we deduce that Λ_p has quadratic Dehn function.

5.5. **Asymptotic cone.** Let us now recall the definition of the asymptotic cone, a notion which was introduced by Gromov [Gr93] (see also the survey of Drutu [Dr02]). Let (X,d) be a metric space and ω a nonprincipal ultrafilter on $\mathbf{N}$. Define

$$\mathrm{Precone}(X) = \{x = (x_n) : x_n \in X, \limsup d(x_n, x_0)/n < \infty\}.$$

Define the pseudo-distance

$$d_\omega(x,y) = \lim_\omega \frac{1}{n} d(x_n, y_n).$$

Definition 5.8. The asymptotic cone $\mathrm{Cone}_\omega(X)$ is defined as the Hausdorffication of $\mathrm{Precone}(X)$, endowed with the resulting distance d_ω.

If $X = G$ is a group and d is left-invariant, then $\mathrm{Precone}(G)$ is obviously a group, d_ω is a left-invariant pseudodistance, and $\mathrm{Cone}_\omega(G) = \mathrm{Precone}(G)/H$, where H is the subgroup of elements at pseudodistance 0 from the identity. It is not normal in general. To summarize, the asymptotic cone of a finitely generated group equipped with a word distance is a homogeneous geodesic metric space, but not necessarily a group (see [Cor11, Proposition 3.1] for a characterization of when it is a group).

5.6. **Fundamental group of the asymptotic cone.** In Section 5.3 we gave an algebraic criterion characterizing when certain metabelian groups have quadratic/exponential Dehn function or are not compactly presented. The same theorem characterizes when the asymptotic cone is simply connected, as by Papazoglou's theorem, quadratic Dehn function implies that the asymptotic cone is simply connected, while in the other cases it is not (see [Gr93, 5.F] or [Dr02, Theorem 4.1]).

When non-trivial, the fundamental group of the asymptotic cone tends be a wild object. In [EO05], it is for instance shown that any countable group can be embedded (even as free factor [DS05, Theorem 7.33]) into a fundamental group of an asymptotic cone of a finitely generated group. Note also that this group is already large for SOL:

Theorem 5.9. [Bu99] *For every nonprincipal ultrafilter ω, the fundamental group of the asymptotic cone of* $\mathrm{SOL}(\mathbf{R})$ *and of solvable Baumslag-Solitar groups is uncountable, contains non-abelian free groups, but is not free.*

By contrast we will now describe examples of solvable groups (some being polycyclic) for which this group is abelian.

The examples we are now going to discuss are based on the following group, introduced by Abels [Ab79]. Let R be a commutative ring with unit.

$$A_4(R) = \left\{ \begin{pmatrix} 1 & x_{12} & x_{13} & x_{14} \\ 0 & t_{22} & x_{23} & x_{24} \\ 0 & 0 & t_{33} & x_{34} \\ 0 & 0 & 0 & 1 \end{pmatrix} : \ x_{ij} \in R; \ t_{ii} \in R^{\times}. \right\} \tag{5.1}$$

Observe that if we denote by $Z(R)$ the subgroup generated by unipotent matrices whose only nonzero off-diagonal entry is x_{14}, then $Z(R)$ is central in $A_4(R)$.

Historically, Abels introduced the group $A_4(\mathbf{Z}[1/p])$ as a first example of a finitely presented solvable group with a infinitely presented quotient (namely $A_4(\mathbf{Z}[1/p])/Z(\mathbf{Z}[1/p])$). It is well-known that a finitely generated group admitting a infinitely generated central extension is not finitely presented [Ba72] (compare Proposition 5.3). Building on these ideas, the authors of [CT13] relate, for certain central extensions

$$1 \to Z \to G \to Q \to 1, \tag{5.2}$$

the fundamental group $\pi_1(\mathrm{Cone}_\omega(Q))$ and the group $\mathrm{Cone}_\omega(Z)$, where Z is endowed with the restriction of the metric of G.

Theorem 5.10. *For every local field* $\mathbf{K}$ *the group* $A_4(\mathbf{K})$ *has a quadratic Dehn function, and for every nonprincipal ultrafilter* ω*, the fundamental group of the cone of* $\mathcal{A}_4(\mathbf{K})/Z(\mathbf{K})$ *is an uncountable abelian group.*

It is also possible to obtain polycyclic examples with such behavior (see [CT13, Corollary 1.6]).

Allowing torsion, one has more flexibility to produce even more exotic behavior.

Theorem 5.11. *There exists a (3-nilpotent)-by-abelian finitely generated group* G*, namely a suitable central quotient of* $A_4(\mathbf{F}_p[t, t^{-1}, (t-1)^{-1}])$*, for which for every ultrafilter* ω*, we have* $\pi_1(\mathrm{Cone}_\omega(G))$ *is isomorphic to either* $\mathbf{F}_p$ *(cyclic group on* p *elements) or is trivial, and is isomorphic to* $\mathbf{F}_p$ *for at least one ultrafilter.*

In [Gr93, 2.B_1(d)] Gromov asks whether an asymptotic cone of a finitely generated group always has trivial or infinitely generated fundamental group. In [OOS09], Olshanskii, Osin and Sapir exhibited a counterexample for which the fundamental group is $\mathbf{Z}$ for some ultrafilter. Theorem 5.11 gives the first example for which the fundamental group is finite and nontrivial.

5.7. **Dependence of the asymptotic cone on the ultrafilter.** Theorem 5.11 provides the first examples of finitely generated solvable groups with two non-homeomorphic asymptotic cones.

By contrast, uniqueness of the asymptotic cone up to bi-Lipschitz homeomorphism has been established for nilpotent finitely generated groups by Pansu [Pan83], and extended to connected nilpotent Lie groups in [Bre].

Other examples of groups with unique asymptotic cone are hyperbolic groups. Recall that asymptotic cones of hyperbolic groups are isometric to metrically complete $\mathbf{R}$-trees [Gr87]. Moreover, for non-elementary hyperbolic groups, these $\mathbf{R}$-trees have valency $2^{\aleph_0}$ at every point (see [DP01] for a detailed argument). In [MNO92], it is proved that such $\mathbf{R}$-tree is unique up to isometry. This statement also applies to Lie groups admitting a left-invariant negatively curved riemannian metric. These groups have a well-understood structure and are in particular solvable [8] [Hei74].

Finally let us mention that the cone of $\mathrm{SOL}(\mathbf{K})$ for any local field can be described explicitely as the subset of a product of two $\mathbf{R}$-trees $T_1 \times T_2$ defined by the equation $b_1(x_1) + b_2(x_2) = 0$, where b_i is a Busemann fonction for the $\mathbf{R}$-tree T_i [Cor08, Section 9]. Therefore, it follows from the main result of [MNO92] that these cones are all bi-Lipschitz equivalent with each other.

[8]The simpliest example being the "affine group $ax + b$" which acts simply transitively on the hyperbolic plane via its identification with the subgroup of upper triangular matrices of $\mathrm{SL}(2, \mathbf{R})$.

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