On the L^p-distortion of finite quotients of amenable groups.

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Abstract

We study the L^p-distortion of finite quotients of amenable groups. In particular, for every $2 \leq p < \infty$, we prove that the ℓ^p-distortions of the groups $C_2 \wr C_n$ and $C_{p^n} \ltimes C_n$ are in $\Theta((\log n)^{1/p})$, and that the ℓ^p-distortion of $C_n^2 \ltimes_A \mathbf{Z}$, where A is the matrix $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ is in $\Theta((\log \log n)^{1/p})$.

1 The main results

1.1 Distortion

Let us first recall some basic definitions.

Definition 1.1.

- Let $0 < R \leq \infty$. The distortion at scale $\leq R$ of an injection between two discrete metric spaces $F : (X, d) \to (Z, d)$ is the number (possibly infinite)

$$dist_R(F) = \sup_{0<d(x,y)\leq R} \frac{d(f(x), f(y))}{d(x, y)} \cdot \sup_{0<d(x,y)\leq R} \frac{d(x, y)}{d(f(x), f(y))}.$$

  If $R = \infty$, we just denote $dist(F)$ and call it the distortion of F.

- The ℓ^p-distortion $c_p(X)$ of a finite metric space X is the infimum of all $dist_F$ over all possible injections F from X to ℓ^p.

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Let G be a finitely generated group. Let S be a symmetric finite generating subset of G. We equip G with the left-invariant word metric associated to S: $d_S(g,h) = |g^{-1}h|_S = \min\{n \in \mathbf{N}, g^{-1}h \in S^n\}$. Let (G,S) denote the associated Cayley graph of G: the set of vertices is G and two vertices g and h are joined by an edge if there is $s \in S$ such that $g = hs$. Note that the graph metric on the set of vertices on (G,S) coincides with the word metric d_S.

Let $\lambda_{G,p}$ denote the regular representation of G on $\ell^p(G)$ for every $1 \leq p \leq \infty$ (i.e. $\lambda(g)f(x) = f(g^{-1}x)$). The ℓ^p-direct sum of n copies of $\lambda_{G,p}$ will be denoted by $n\lambda_{G,p}$.

Our main results are the following theorems.

Theorem 1. *Let m be an integer ≥ 2. For all $n \in \mathbf{N}$, consider the finite lamplighter group $C_m \wr C_n = (C_m)^{C_n} \ltimes C_n$ equipped with the generating set $S = ((\pm 1_0, 0), (0, \pm 1))$, where $1_0 \in (C_m)^{C_n}$ is the characteristic function of the singleton $\{0\}$. For every $2 \leq p < \infty$, there exists $C = C(p,m) < \infty$ such that*

$$C^{-1}(\log n)^{1/p} \leq c_p(C_2 \wr C_n, S) \leq C(\log n)^{1/p}.$$

Note that the upper bound has been very recently proved for $p = 2$ by Austin, Naor, and Valette [ANV], using representation theory. The proof that we propose here is shorter and completely elementary. On the other hand, the lower bound was known (see [LNP], or Section 2).

Theorem 2. *Let m be an integer ≥ 2. For all $n \in \mathbf{N}$, consider the group $BS_{m,n} = C_{m^n} \ltimes C_n$ equipped with the generating set $S = \{(\pm 1, 0), (0, \pm 1)\}$. For every $2 \leq p < \infty$, there exists $C = C(p,m) < \infty$ such that*

$$C^{-1}(\log n)^{1/p} \leq c_p(G_n, S) \leq C(\log n)^{1/p}.$$

Theorem 3. *For all $n \in \mathbf{N}$, consider the group $SOL_n = C_n \ltimes_A C_{o(A,n)}$, where A is a matrix of $SL_2(\mathbf{Z})$ with eigenvalues of modulus different from 1, e.g. the matrix $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$, and where $o(A,n)$ denotes the order of A in $SL_2(C_n)$. Equip G with the generating set $S = \{(\pm 1, 0), (0, \pm 1)\}$. For every $2 \leq p < \infty$, there exists $C = C(p) < \infty$ such that*

$$C^{-1}(\log\log n)^{1/p} \leq c_p(G_n, S) \leq C(\log\log n)^{1/p}.$$

1.2 About the constructions

We will say that map $F : G \to E$ from a group G to a Banach space is equivariant if it is the orbit of 0 of an isometric affine action of G on E. Let σ be such an action. The equivariance of $F(g) = \sigma(g).0$ implies that $\|F(g) - F(h)\| = \|F(g^{-1}h)\|$. Hence the distortion at scale $\leq R$ of F is just given by

$$dist_R(F) = \sup_{0<|g|_S\leq R} \frac{|g|_S}{\|F(g)\|} \cdot \sup_{0<|g|_S\leq R} \frac{\|F(g)\|}{|g|_S}.$$

All the groups involved in the main theorems are of the form $G = N \ltimes A$ where A is a finite cyclic group. To prove an upper bound on $c_p(G)$, our general approach is to construct an embedding $F = F_1 \oplus^{\ell^p} F_2$, where F_1 is the orbit of 0 of an affine action σ_1 of G, whose linear part is $K\lambda_{G,p}$ (for some $K \in \mathbf{N}$), and such that for $R = \mathrm{Diam}(N)$, we have

$$dist_R(F_1) \approx (\log R)^{1/p}.$$

More precisely, for $F_{m,n}$ and $BS_{m,n}$ (resp. for $SOL_{A,n}$), we will need $K \approx \log(mn)$ (resp. $K \approx \log\log n$) copies of $\lambda_{G,p}$.

For $G = F_{m,n}$ or $BS_{m,n}$, we can take $F = F_1$ since $\mathrm{Diam}(N) \approx \mathrm{Diam}(G) \approx n$ (see Proposition 3.1). But, for $G = SOL_{A,n}$, we have $\mathrm{Diam}(N) \approx \log n$, which can be much less than $\mathrm{Diam}(G) \approx o(A,n)$. Hence, the solution in this case is to add some map $F_2 : G/N \approx C_{o(A,n)} \to \ell^p$ with a bounded distortion (for instance, take the orbit of 0 under the action of $C_{o(A,n)}$ on $\mathbf{R}^2$ such that 1 acts by rotation of center $(o(A,n), 0)$ and angle $2\pi/o(A,n)$).

Note that Theorem 3 also holds for the group $C_n \ltimes_A \mathbf{Z}$, in which case we can take an action of $\mathbf{Z}$ by translations on $\mathbf{R}$ to embed the quotient with bounded distortion (i.e. for F_2).

2 Upper bounds on the distortion

Let $1 \leq p \leq \infty$. Recall [T1] that the left-ℓ^p-isoperimetric profile in balls of (G, S) is defined by

$$J_{G,S,p}(n) = \sup_{\mathrm{Supp}(f)\subset B(1,n)} \frac{\|f\|_p}{\sup_{s\in S} \|\lambda(s)f - f\|_p},$$

where $B(1, n)$ denotes the open ball of radius n and center 1 in (G, S). For convenience, we will

Our main result in [T1] consisted in showing that a lower bound on the isoperimetric profile can be used to construct metrically proper affine isometric actions of G on $\ell^p(G)$ whose compressions satisfy lower bounds which are optimal in certain cases. Here, we will use it to produce upper bounds on the ℓ^p-distortion of finite groups.

On the other hand, as explained in [T2], if $X = (G, d_S)$ is a Cayley graph, then the inequality $J_{p,G} \geq J$ for some non-decreasing function $J : \mathbf{R}_+ \to \mathbf{R}_+$ implies Property A(J,p) (see [T2, Definition 4.1]) for the space X (if the group G is amenable, a standard average argument actually shows that this is an equivalence). So in a large extend, the results of the present paper are easy consequences of the method explained in [T2].

A crucial remark is that $J_{G,S,p}$ is a local quantity, and hence behaves well under quotients. Namely, we recall the following easy fact.

Proposition 2.1. *(for a proof, see [T3, Theorem 4.2]) Let $\pi : G \to Q$ be a surjective homomorphism between two finitely generated groups and let S be a symmetric generating subset of G. Then*

$$J_{G,S,p} \leq J_{Q,\pi(S),p}.$$

Our main technical tool is the following proposition, which is an analogue of [T2, Proposition 4.5]. For the convenience of the reader, we give its relatively short proof in Section 4.

Theorem 4. *Let $X = (G, S)$ be a finite Cayley graph such that $J_{G,S,p}(r) \geq J(r)$ when $r \leq R$, for some $R \leq Diam(G)/2$. Then, there exists an affine isometric action σ of G on such that*

- *the linear part of σ is the ℓ^p-direct sum of $K = [\log R]$ regular representations of G in $\ell^p(G)$.*
- *The orbit of 0 induces an injection $F : G \to \bigoplus_{k=0}^{K-1} \ell^p(G)$ such that*

$$dist_R(F) \leq 2\left(2\int_2^{R/2}\left(\frac{t}{J(t)}\right)^p \frac{dt}{t}\right)^{1/p}.$$

In particular, if $J(t) = t/C$, then

$$dist_R(F) \leq 2C\left(2\log(R/2)\right)^{1/p}.$$

Corollary 2.2. *Assume that G_n has diameter $\leq n$ and that $J_{G,p}(t) \geq t/C$, then, $c_p(G_n) \leq 2C\left(2\log(n/4)\right)^{1/p}$.*

On the other hand, we have proved in [T1] that the following finitely generated groups satisfy $J_p(t) \geq t/C$ for some $C < \infty$ and for all $1 \leq p < \infty$.

- the lamplighter group $L_m = C_m \wr \mathbf{Z}$;

- solvable Baumslag-Solitar groups $BS_m = \mathbf{Z}[1/m] \ltimes \mathbf{Z}$ for all $m \in \mathbf{N}$, where $n \in \mathbf{Z}$ acts by multiplication by m^n;
- polycyclic groups. Here, we will focus on the following example: $SOL_A = \mathbf{Z}^2 \ltimes_A \mathbf{Z}$ where A is a matrix of $\mathrm{SL}_2(\mathbf{Z})$ with eigenvalues of modulus different from 1, e.g. the matrix $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$.

Note that respectively $L_{m,n}$, $BS_{m,n}$ and $SOL_{A,n}$ are quotients of L_m, BS_m and SOL_A.

3 Proofs of the main theorems

3.1 Upper bounds

Thanks to Corollary 2.2, the upper bounds in Theorems 1, 2 and 3 follow from the following upper bounds on the diameters of the groups $L_{m,n}$, $BS_{m,n}$ and $SOL_{A,n}$ (for the latter, see the discussion in Section 1.2).

Proposition 3.1. *We have*

(i) $Diam(L_{m,n}) \leq (m+3)n$;

(ii) $Diam(BS_{m,n}) \leq (m+1)n$;

(iii) Let $N_n \simeq C_n^2$ *be the kernel of* $SOL_{A,n} \to C_{o(A,n)}$. *Then, with the distance on* N_n *induced by the word distance on* $SOL_{A,n}$, *we have* $Diam(N_n) \leq c \log n$ *for some* $c = c(A) > 0$.

Proof: For (i), see [Pa]. For (ii), note that every element of C_{m^n} can be written as

$$\sum_{i=0}^{n-1} a_i m^i = a_0 + m(a_1 + m(a_2 + \ldots) \ldots),$$

where $0 \leq a_i \leq m-1$. Finally, (iii) follows from the following well known lemma. ■

Lemma 3.2. *Let* $N \sim \mathbf{Z}^2$ *be the kernel of* $SOL_A \to \mathbf{Z}$. *For all* $r \geq 1$, *denote by* $B_{N,SOL_A}(r)$ *(resp.* $B_N(r)$*), the ball of radius* r *for the metric on* N *induced by the word length on* SOL_A *(resp. for the usual metric on* $\mathbf{Z}^2$*). There exists some* $\alpha = \alpha(A) < \infty$ *such that*

$$B_N(1, e^{r/\alpha}) \subset B_{N,SOL_A}(r) \leq B_N(1, e^{\alpha r}).$$

Proof: Note that SOL_A embeds as a co-compact lattice in the connected solvable Lie group $G = \mathbf{R}^2 \ltimes_A R$, such that N maps on a (co-compact) lattice of $\tilde{N} = \mathbf{R}^2$. The lemma follows from the fact that $\tilde{N}$ is the exponential radical of G (Guivarc'h [G] was the first one to introduce and to study the exponential radical of a connected solvable Lie group, without actually naming it, and this was rediscovered by Osin [O]). ■

3.2 Lower bounds

To obtain the lower bound on the distortion, we will need the following notion of relative girth.

Definition 3.3. Let $\pi : G \to Q$ be a surjective homomorphism between two finitely generated groups and let S be a symmetric generating subset of G. Denote by $X = (G, S)$ and $Y = (H, \pi(S))$. The relative girth $g(Y, X)$ of Y with respect to X is the maximum integer $n \in \mathbf{N}$ such that a ball of radius n in Y is isometric to a ball of radius n in X.

Recall [Bou] that the rooted binary tree T_n of dept n satisfies $c_p(T_n) \geq c(\log n)^{1/p}$ for all $2 \leq p < \infty$ and for some constant $c > 0$. The following remark follows trivially from this result and from the definition of relative girth.

Proposition 3.4. *We keep the notation of the previous definition. Assume that X contains a bi-Lipschitz embedded 3-regular tree. Then there exists some $c > 0$ such that $c_p(Y) \geq c(\log g(X, Y))^{1/p}$.* ■

On the other hand, the groups L_m, BS_m and SOL_A are solvable non-virtually nilpotent. Hence by [CT], they admit a bi-Lipschitz embedded 3-regular tree (for the lamplighter, see also [LPP]). So to prove the lower bounds of Theorems 1, 2 and 3, we just need to find convenient lower bounds for the relative girths, which is done by the following proposition.

Proposition 3.5. *We have*

(i) $g(L_{m,n}, L_m) \geq n$;

(ii) $g(BS_{m,n}, BS_m) \geq n$;

(iii) $g(SOL_{A,n}, SOL_A) \geq c \log n$ *for some* $c = c(A) > 0$.

Proof: The only non-trivial case, (iii), follows from Lemma 3.2. ■

4 Proof of Theorem 4

Let f_0 be the dirac at 1, and for every integer $1 \leq k \leq K$, choose a function $f_k \in \ell^p(G)$ such that

- the support of f_k is contained in the ball $B(1, 2^k)$,
- $\|f_k\|_p \geq J(2^k)$
- $\sup_{s\in S} \|\lambda(s) f_k - f_k\|_p \leq 1$

For all $v = (v_k)_{1\leq k\leq n} \in K\ell^p(G)$ and all $g \in G$, define

$$\sigma(g)v = \bigoplus_k^{\ell^p} (\lambda(g)v_k + F_k)$$

where

$$F_k(g) = \left(\frac{2^k}{J(2^k)}\right)(f_k - \lambda(g)f_k).$$

Now consider the map $F = \bigoplus^{\ell^p} b_k : G \to K\ell^p(G)$. For all $g \in G$, we have

$$\begin{aligned}
\|F(g)\|_p &= \|b(g)\|_p \\
&\leq \left(\sum_{k=0}^{n} \left(\frac{2^k}{J(2^k)}\right)^p \|\lambda(g)f_k - f_k\|_p^p\right)^{1/p} \\
&\leq \left(\sum_{k=0}^{n} \left(\frac{2^k}{J(2^k)}\right)^p\right)^{1/p} \\
&\leq |g|_S \left(\int_1^{\mathrm{Diam}(G)/2} \left(\frac{t}{J(t/2)}\right)^p \frac{dt}{t}\right)^{1/p} \\
&= 2^{2/p}|g|_S \left(\int_1^{\mathrm{Diam}(X)/4} \left(\frac{t}{J(t)}\right)^p \frac{dt}{t}\right)^{1/p}.
\end{aligned}$$

On the other hand, since f_k is supported in $B(1, 2^k)$, if $|g|_S \geq 2.2^k$, then the supports of f_k and $\lambda(g)f_k$ are disjoint. Thus,

$$\begin{aligned}
\|F(g)\|_p &= \|b(g)\|_p \\
&\geq \|b_k\|_p \\
&= 2^{1/p}\frac{2^k}{J(2^k)}\|f_k\|_p \\
&\geq 2^{1/p}2^k,
\end{aligned}$$

whenever $d_S(x, y) \geq 2.2^k$. To conclude, we have to consider the case when $g \in S \setminus \{1\}$. But as f_0 is a dirac at 1, $\|F(g)\|_p \geq 1$. So we are done. ■

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