Vanishing of the first reduced cohomology with values in an L^p-representation.

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Abstract

We prove that the first reduced cohomology with values in a mixing L^p-representation, $1 < p < \infty$, vanishes for a class of amenable groups including connected amenable Lie groups. In particular this solves for this class of amenable groups a conjecture of Gromov saying that every finitely generated amenable group has no first reduced ℓ^p-cohomology. As a byproduct, we prove a conjecture by Pansu. Namely, the first reduced L^p-cohomology on homogeneous, closed at infinity, Riemannian manifolds vanishes. We also prove that a Gromov hyperbolic geodesic metric measure space with bounded geometry admitting a bi-Lipschitz embedded 3-regular tree has non-trivial first reduced L^p-cohomology for large enough p. Combining our results with those of Pansu, we characterize Gromov hyperbolic homogeneous manifolds: these are the ones having non-zero first reduced L^p-cohomology for some $1 < p < \infty$.

1 Introduction

1.1 A weak generalization of a result of Delorme.

In [Del], Delorme proved the following deep result: every connected solvable Lie groups has the property that every weakly mixing[1] unitary representation π has trivial first reduced cohomology, i.e. $\overline{H}^1(G, \pi) \neq 0$. This was recently extended to connected amenable Lie groups, see [Ma, Theorem 3.3], and to a large class of amenable groups including polycyclic groups by Shalom [Sh]. Shalom also proves that this property, that he calls Property H_{FD}, is invariant under quasi-isometry between amenable discrete groups. Property H_{FD} has nice implications in various contexts. For instance, Shalom shows that an amenable finitely generated group

[1] A unitary representation is called weakly mixing if it contains no finite dimensional subrepresentation.

with Property H_{FD} has a finite index subgroup with infinite abelianization [Sh, Theorem 4.3.1]. In [CTV1], we prove [CTV1, Theorem 4.3] that an amenable finitely generated group with Property H_{FD} cannot quasi-isometrically embed into a Hilbert space unless it is virtually abelian.

It is interesting and natural to extend the definition of Property H_{FD} to isometric representations of groups on certain classes of Banach spaces.

In this paper, we prove that a weak version of Property H_{FD}, also invariant under quasi-isometry, holds for isometric L^p-representations of a large class of amenable groups including connected amenable Lie groups and polycyclic groups: for $1 < p < \infty$, every *strongly mixing* isometric L^p-representation π has trivial first reduced cohomology (see Section 2 for a precise statement).

1.2 L^p-cohomology.

The L^p-cohomology (for p not necessarily equal to 2) of a Riemannian manifold has been introduced by Gol'dshtein, Kuz'minov, and Shvedov in [GKS]. It has been intensively studied by Pansu [Pa2, Pa3, Pa6] in the context of homogeneous Riemannian manifolds and by Gromov [Gro2] for discrete metric spaces and groups. The L^p-cohomology is invariant under quasi-isometry in degree one [HS]. But in higher degree, the quasi-isometry invariance requires some additional properties, like for instance the uniform contractibility of the space [Gro2] (see also [BP, Pa6]). Most authors focus on the first reduced L^p-cohomology since it is easier to compute and already gives a fine quasi-isometry invariant (used for instance in [B, BP]). The ℓ^2-Betti numbers of a finitely generated group, corresponding to its reduced ℓ^2-cohomology[2], have been extensively studied in all degrees by authors like Gromov, Cheeger, Gaboriau and many others. In particular, Cheeger and Gromov proved in [CG] that the reduced ℓ^2-cohomology of a finitely generated amenable group vanishes in all degrees. In [Gro2], Gromov conjectures that this should also be true for the reduced ℓ^p-cohomology. For a large class of finitely generated groups with infinite center, it is known [Gro2, K] that the reduced ℓ^p-cohomology vanishes in all degrees, for $1 < p < \infty$. The first reduced ℓ^p-cohomology for $1 < p < \infty$ is known to vanish [BMV, MV] for certain non-amenable finitely generated groups with "a lot of commutativity" (e.g. groups having a non-amenable finitely generated normal subgroup with infinite centralizer).

A consequence of our main result is to prove that the first reduced ℓ^p-cohomology, $1 < p < \infty$, vanishes for large class of finitely generated amenable groups, in-

[2] We write ℓ^p when the space is discrete.

cluding for instance polycyclic groups.

On the other hand, it is well known [Gro2] that the first reduced ℓ^p-cohomology of a Gromov hyperbolic finitely generated group is non-zero for p large enough. Although the converse is false[3] for finitely generated groups, we will see that it is true in the context of connected Lie groups. Namely, a connected Lie group has non-zero reduced first L^p-cohomology for some $1 < p < \infty$ if and only if it is Gromov hyperbolic.

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2 Main results

(The definitions of first L^p-cohomology, p-harmonic functions and of first cohomology with values in a representation are postponed to Section 4.)

Let G be a locally compact group acting by measure-preserving bijections on a measurable space equipped with an infinite measure (X, m). We say that the action is strongly mixing (or mixing) if for every measurable subset of finite measure $A \subset X$, $m(gA \cap A) \to 0$ when g leaves every compact subset of G. Let π be the corresponding continuous representation of G in $L^p(X, m)$, where $1 < p < \infty$. In this paper, we will call such a representation a mixing L^p-representation of G.

Definition 2.1. [T1] Let G be a locally compact, compactly generated group and let S be a compact generating subset of G. We say that G has Property (CF) (Controlled Følner) if there exists a sequence of compact subsets of positive measure (F_n) satisfying the following properties.

- $F_n \subset S^n$ for every n;

[3] In [CTV2] for instance, we prove that any non-amenable discrete subgroup of a semi-simple Lie group of rank one has non-trivial reduced L^p-cohomology for p large enough. On the other hand, non-cocompact lattices in $\mathrm{SO}(3,1)$ are not hyperbolic. See also [BMV] for other examples.

- there is a constant $C < \infty$ such that for every n and every $s \in S$,

$$\frac{\mu(sF_n \vartriangle F_n)}{\mu(F_n)} \leq C/n.$$

Such a sequence F_n is called a controlled Følner[4] sequence.

In [T1], we proved that following family[5] of groups are (CF).

(1) Polycyclic groups and connected amenable Lie groups;

(2) semidirect products $\mathbf{Z}[\frac{1}{mn}] \rtimes_{\frac{m}{n}} \mathbf{Z}$, with m, n co-prime integers with $|mn| \geq 2$ (if $n = 1$ this is the Baumslag-Solitar group $BS(1, m)$); semidirect products $\left(\bigoplus_{i\in I} \mathbf{Q}_{p_i}\right) \rtimes_{\frac{m}{n}} \mathbf{Z}$ with m, n co-prime integers, and $(p_i)_{i\in I}$ a finite family of primes (including ∞: $\mathbf{Q}_\infty = \mathbf{R}$)) dividing mn;

(3) wreath products $F \wr \mathbf{Z}$ for F a finite group.

Our main result is the following theorem.

Theorem 1. *Let G be a group with Property (CF) and let π be a mixing L^p-representation of G. Then the first reduced cohomology of G with values in π vanishes, i.e. $\overline{H^1}(G, \pi) = 0$.*

Invariance under quasi-isometry. The proof of [Sh, Theorem 4.3.3] that Property H_{FD} is invariant under quasi-isometry can be used identically in the context of L^p-representations and replacing the hypothesis "weak mixing" by "mixing" since the induced representation of a mixing L^p-representation is also a mixing L^p-representation. As a result, we obtain that the property that $\overline{H^1}(G, \pi) = 0$ for every mixing L^p-representation is invariant under quasi-isometry between discrete amenable groups. It is also stable by passing to (and inherited by) cocompact lattices in amenable locally compact groups.

It is well known [Pu] that for finitely generated groups G, the first reduced cohomology with values in the left regular representation in $\ell^p(G)$ is isomorphic to the space $HD_p(G)$ of p-harmonic functions with gradient in ℓ^p modulo the constants. We therefore obtain the following corollary.

Corollary 2. *Let G be a discrete group with Property (CF). Then every p-harmonic function on G with gradient in ℓ^p is constant.*

[4] A controlled Følner sequence is in particular a Følner sequence, so that Property (CF) implies amenability.

[5] This family of groups also appears in [CTV1].

Using Von Neumann algebra technics, Cheeger and Gromov [CG] proved that every finitely generated amenable group G has no nonconstant harmonic function with gradient in ℓ^2, the generalization to every $1 < p < \infty$ being conjectured by Gromov.

To obtain a version of Corollary 2 for Lie groups, we prove the following result (see Theorem 5.1).

Theorem 2.2. *Let G be a connected Lie group. Then for $1 \leq p < \infty$, the first L^p-cohomology of G is topologically (canonically) isomorphic to the first cohomology with values in the right regular representation in $L^p(G)$, i.e.*

$$H_p^1(G) \simeq H^1(G, \rho_{G,p}).$$

Now, since this isomorphism induces a natural bijection

$$HD_p(G) \simeq \overline{H^1}(G, \rho_{G,p}),$$

we can state the following result that was conjectured by Pansu in [Pa3]. Recall that a Riemannian manifold is called closed at infinity if there exists a sequence of compact subsets A_n with regular boundary ∂A_n such that $\mu_{d-1}(\partial A_n)/\mu_d(A_n) \to 0$, where μ_k denotes the Riemannian measure on submanifolds of dimension k of M.

Corollary 3. *Let M be a homogeneous Riemannian manifold. If it is closed at infinity, then for every $p > 1$, every p-harmonic function on M with gradient in $L^p(TM)$ is constant. In other words, $HD_p(M) = 0$.*

Together with Pansu's results [Pa4, Théorème 1], we obtain the following dichotomy.

Theorem 4. *Let M be a homogeneous Riemannian manifold. Then the following dichotomy holds.*

- *Either M is quasi-isometric to a homogeneous Riemannian manifold with strictly negative curvature, and then there exists $p_0 \geq 1$ such that $HD_p(M) \neq 0$ if and only if $p > p_0$;*
- *or $HD_p(M) = 0$ for every $p > 1$.*

We also prove

Theorem 5. *(see Corollary 8) A homogeneous Riemannian manifold M has non-zero first reduced L^p-cohomology for some $1 < p < \infty$ if and only if it is non-elementary*[6] *Gromov hyperbolic.*

[6] By non-elementary, we mean not quasi-isometric to $\mathbf{R}$.

To prove this corollary, we need to prove that a Gromov hyperbolic Lie group has non-trivial first reduced L^p-cohomology for p large enough. This is done in Section 9. Namely, we prove a more general result.

Theorem 6. *(see Theorem 9.2) Let G be a Gromov hyperbolic metric measure space with bounded geometry having a bi-Lipschitz embedded 3-regular tree, then for p large enough, it has non-trivial first reduced L^p-cohomology.*

Corollary 8 and Pansu's contribution to Theorem 4 yield the following corollary.

Corollary 7. *A non-elementary Gromov hyperbolic homogeneous Riemannian manifold is quasi-isometric to a homogeneous Riemannian manifold with strictly negative curvature.*

(See [He] for an algebraic description of homogeneous manifolds with strictly negative curvature).

3 Organization of the paper.

In the following section, we recall three definitions of first cohomology:

- a coarse definition of the first L^p-cohomology on a general metric measure space which is due to Pansu;
- the usual definition of first L^p-cohomology on a Riemannian manifold;
- the first cohomology with values in a representation, which is defined for a locally compact group.

In Section 5, we construct a natural topological isomorphism between the L^p-cohomology of a connected Lie group G and the cohomology with values in the right regular representation of G in $L^p(G)$. We use this isomorphism to deduce Corollary 3 from Theorem 1.

The proof of Theorem 1 splits into two steps. First (see Theorem 6.1), we prove that for any locally compact compactly generated group G and any mixing L^p-representation π of G, every 1-cocycle $b \in Z^1(G,\pi)$ is *sublinear*, which means that for every compact symmetric generating subset S of G, we have

$$\|b(g)\| = o(|g|_S)$$

when $|g|_S \to \infty$, $|g|_S$ being the word length of g with respect to S. Then, we adapt to this context a remark that we made with Cornulier and Valette (see [CTV1, Proposition 3.6]): for a group with Property (CF), a 1-cocycle belongs to $\overline{B}^1(G,\pi)$ if and only if it is sublinear. The part "only if" is an easy exercise and does not require Property (CF). To prove the other implication, we consider the affine action σ of G on E associated to the 1-cocycle b and use Property (CF) to construct a sequence of almost fixed points for σ.

In Section 8, we propose a more direct approach[7] to prove Corollary 3. The interest is to provide an explicit approximation of an element of $\mathbf{D}_p(G)$ by a sequence of functions in $W^{1,p}(G)$ using a convolution-type argument.

Finally, in Section 9, we prove that a Gromov hyperbolic homogeneous manifold has non-trivial L^p-cohomology for p large enough. This section can be read independently.

4 Preliminaries

4.1 A coarse notion of first L^p-cohomology on a metric measure space

The following coarse notion of (first) L^p-cohomology is essentially due to [Pa6] (see also the chapter about L^p-cohomology in [Gro2]).

Let $X=(X,d,\mu)$ be a metric measure space, and let $p\geq 1$. For all $s>0$, we write $\Delta_s=\{(x,y)\in X^2, d(x,y)\leq s\}$.

First, let us introduce the p-Dirichlet space $\mathbf{D}_p(X)$.

- The space $D_p(X)$ is the set of measurable functions f on X such that

$$\int_{\Delta_s}|f(x)-f(y)|^p d\mu(x)d\mu(y)<\infty$$

  for every $s>0$.

- Let $\mathbf{D}_p(X)$ be the Banach space $D_p(X)/\mathbf{C}$ equipped with the norm

$$\|f\|_{D_p}=\left(\int_{\Delta_1}|f(x)-f(y)|^p\mu(x)d\mu(y)\right)^{1/p}.$$

- By a slight abuse of notation, we identify L^p with its image in $\mathbf{D}_p$.

Definition 4.1. The first L^p-cohomology of X is the space

$$H_p^1(X)=\mathbf{D}_p(X)/L^p(X),$$

[7]However, the ingredients are the same: sublinearity of cocycles, and existence of a controlled Følner sequence.

and the first reduced L^p-cohomology of X is the space

$$\overline{H^1}_p(X) = \mathbf{D}_p(X)/\overline{L^p(X)}^{\mathbf{D}_p(X)}.$$

Definition 4.2. (1-geodesic spaces) We say that a metric space $X = (X, d)$ is 1-geodesic if for every two points $x, y \in X$, there exists a sequence of points $x = x_1, \dots x_m = y$, satisfying

- $d(x, y) = d(x_1, x_2) + \dots + d(x_{m-1}, x_m)$,
- for all $1 \leq i \leq m-1$, $d(x_i, x_{i+1}) \leq 1$.

Remark 4.3. Let X and Y be two 1-geodesic metric measure spaces with bounded geometry in the sense of [Pa6]. Then it follows from [Pa6] that if X and Y are quasi-isometric, then $H_p^1(X) \simeq H_p^1(Y)$ and $\overline{H^1}_p(X) \simeq \overline{H^1}_p(Y)$.

Example 4.4. Let G be a locally compact compactly generated group, and let S be a symmetric compact generating set. Then the word metric on G associated to S,

$$d_S(g, h) = \in \{n \in \mathbf{N}, g^{-1}h \in S^n\},$$

defines a 1-geodesic left-invariant metric on G. Moreover, one checks easily two such metrics (associated to different S) are bilipschitz equivalent. Hence, by Pansu's result, the first L^p-cohomology of (G, μ, d_S) does not depend on the choice of S.

Definition 4.5. (coarse notion of p-harmonic functions) Let $f \in D_p(X)$ and assume that $p > 1$. The p-Laplacian[8] of f is

$$\Delta_p f(x) = \frac{1}{V(x,1)} \int_{d(x,y) \leq 1} |f(x) - f(y)|^{p-2}(f(x) - f(y)) d\mu(y),$$

where $V(x, 1)$ is the volume of the closed ball $B(x, 1)$. A function $f \in D_p(X)$ is called p-harmonic if $\Delta_p f = 0$. Equivalently, the p-harmonic functions are the minimizers of the variational integral

$$\int_{\Delta_1} |f(x) - f(y)|^p d\mu(x) d\mu(y).$$

Definition 4.6. We say that X satisfies a Liouville D_p-Property if every p-harmonic function on X is constant.

[8]Here we define a coarse p-Laplacian at scale 1: see [T2, Section 2.2] for a more general definition.

As $\mathbf{D}_p(X)$ is a strictly convex, reflexive Banach space, every $f \in \mathbf{D}_p(X)$ admits a unique projection $\tilde{f}$ on the closed subspace $\overline{L^p(X)}$ such that $d(f, \tilde{f}) = d(f, \overline{L^p(X)})$. One can easily check that $f - \tilde{f}$ is p-harmonic. In conclusion, the reduced cohomology class of $f \in \mathbf{D}_p(X)$ admits a unique p-harmonic representant modulo the constants. We therefore obtain

Proposition 4.7. *A metric measure space X has Liouville D_p-Property if and only if $\overline{H_p}^1(X) = 0$.*

4.2 First L^p-cohomology on a Riemannian manifold

Let M be Riemannian manifold, equipped with its Riemannian measure m. Let $1 \leq p < \infty$.

Let us first define, in this differentiable context, the p-Dirichlet space $\mathbf{D}_p$.

- Let D_p be the vector space of continuous functions whose gradient is (in the sense of distributions) in $L^p(TM)$.
- Equip $D_p(M)$ with a pseudo-norm $\|f\|_{D_p} = \|\nabla f\|_p$, which induces a norm on $D_p(M)$ modulo the constants. Denote by $\mathbf{D}_p(M)$ the completion of this normed vector space.
- Write $W^{1,p}(M) = L^p(M) \cap D_p(M)$. By a slight abuse of notation, we identify $W^{1,p}(M)$ with its image in $\mathbf{D}_p(M)$.

Definition 4.8. The first L^p-cohomology of M is the quotient space

$$H_p^1(M) = \mathbf{D}_p(M)/W^{1,p}(M),$$

and the first reduced L^p-cohomology of M is the quotient

$$\overline{H_p}^1(M) = \mathbf{D}_p(M)/\overline{W^{1,p}(M)},$$

where $\overline{W^{1,p}(M)}$ is the closure of $W^{1,p}(M)$ in the Banach space $\mathbf{D}_p(M)$.

Definition 4.9. (p-harmonic functions) A function $f \in D_p(M)$ is called p-harmonic if it is a weak solution of

$$\operatorname{div}(|\nabla f|^{p-2} \nabla f) = 0,$$

that is,

$$\int_M \langle |\nabla f|^{p-2} \nabla f, \nabla \varphi \rangle dm = 0,$$

for every $\varphi \in C_0^\infty$. Equivalently, p-harmonic functions are the minimizers of the variational integral

$$\int_M |\nabla f|^p dm.$$

Definition 4.10. We say that M satisfies a Liouville D_p-Property if every p-harmonic function on M is constant.

As $\mathbf{D}_p(M)$ is a strictly convex, reflexive Banach space, every $f \in \mathbf{D}_p(M)$ admits a unique projection $\tilde{f}$ on the closed subspace $\overline{W^{1,p}(M)}$ such that $d(f,\tilde{f}) = d(f, \overline{W^{1,p}(M)})$. One can easily check that $f - \tilde{f}$ is p-harmonic. In conclusion, the reduced cohomology class of $f \in \mathbf{D}_p(M)$ admits a unique p-harmonic representant modulo the constants. Hence, we get the following well-known fact.

Proposition 4.11. *A Riemannian manifold M has Liouville D_p-Property if and only if $\overline{H_p}^1(M) = 0$.*

Remark 4.12. In [Pa6], Pansu proves (in particular) that if a Riemannian manifold has bounded geometry (which is satisfied by a homogeneous manifold), then the first L^p-cohomology defined as above is topologically isomorphic to its coarse version defined at the previous section. In particular, the Liouville D_p-Property is invariant under quasi-isometry between Riemannian manifolds with bounded geometry.

4.3 First cohomology with values in a representation

Let G be a locally compact group, and π a continuous linear representation on a Banach space $E = E_\pi$. The space $Z^1(G,\pi)$ is defined as the set of continuous functions $b : G \to E$ satisfying, for all g, h in G, the 1-cocycle condition $b(gh) = \pi(g)b(h) + b(g)$. Observe that, given a continuous function $b : G \to E$, the condition $b \in Z^1(G,\pi)$ is equivalent to saying that G acts by affine transformations on E by $\alpha(g)v = \pi(g)v + b(g)$. The space $Z^1(G,\pi)$ is endowed with the topology of uniform convergence on compact subsets.

The subspace of coboundaries $B^1(G,\pi)$ is the subspace (not necessarily closed) of $Z^1(G,\pi)$ consisting of functions of the form $g \mapsto v - \pi(g)v$ for some $v \in E$. In terms of affine actions, $B^1(G,\pi)$ is the subspace of affine actions fixing a point.

The first cohomology space of π is defined as the quotient space

$$H^1(G,\pi) = Z^1(G,\pi)/B^1(G,\pi).$$

The first *reduced* cohomology space of π is defined as the quotient space

$$\overline{H^1}(G,\pi) = Z^1(G,\pi)/\overline{B^1}(G,\pi),$$

where $\overline{B^1}(G,\pi)$ is the closure of $B^1(G,\pi)$ in $Z^1(G,\pi)$ for the topology of uniform convergence on compact subsets. In terms of affine actions, $\overline{B^1}(G,\pi)$ is the space of actions σ having almost fixed points, i.e. for every $\varepsilon > 0$ and every compact subset K of G, there exists a vector $v \in E$ such that for every $g \in K$,

$$\|\sigma(g)v - v\| \leq \varepsilon.$$

If G is compactly generated and if S is a compact generating set, then this is equivalent to the existence of a sequence of almost fixed points, i.e. a sequence v_n of vectors satisfying

$$\lim_{n\to\infty} \sup_{s\in S} \|\sigma(s)v_n - v_n\| = 0.$$

5 L^p-cohomology and affine actions on $L^p(G)$.

Let G be a locally compact group equipped with a left-invariant Haar measure. Let G act on $L^p(G)$ by right translations, which defines a representation $\rho_{G,p}$ defined by

$$\rho_{G,p}(g)f(x) = f(xg) \quad \forall f \in L^p(G).$$

Note that this representation is isometric if and only if G is unimodular, in which case $\rho_{G,p}$ is isomorphic to the left regular representation $\lambda_{G,p}$. In particular, in this case, the corresponding first reduced cohomologies are the same.

Now suppose that the group G is also compactly generated and equipped with a word metric d_S associated to a compact symmetric generating subset S. In this section, we prove that the first cohomology with values in the regular L^p-representation $\rho_{G,p}$ is topologically isomorphic to the first L^p-cohomology $H^1_p(G)$ (here, we mean the coarse version, see Section 4.1). By the result of Pansu mentioned in Remark 4.12, if G is a connected Lie group equipped with left-invariant Riemannian metric m, we can also identify $H^1(G,\rho_{G,p})$ with the first L^p-cohomology on (G,m) (see Section 4.2). We also obtain a direct proof of this fact.

We consider here the two following contexts: where G is a compactly generated locally compact group equipped with a length function d_S; or G is a connected Lie group, equipped with a left-invariant Riemannian metric.

Consider the linear map $J:\ \mathbf{D}_p(G) \to Z^1(G,\rho_{G,p})$ defined by

$$J(f)(g) = b(g) = f - \rho_{G,p}(g)f.$$

J is clearly well defined and induces a linear map $HJ:\ H^1_p(G) \to H^1(G,\rho_{G,p})$.

Theorem 5.1. *For $1 \leq p < \infty$, the canonical map $HJ: \ H^1_p(G) \to H^1(G, \rho_{G,p})$ is an isomorphism of topological vector spaces.*

Let us start with a lemma.

Lemma 5.2. *Let $1 \leq p < \infty$ and $b \in Z^1(G, \rho_{G,p})$. Then there exists a 1-cocycle c in the cohomology class of b such that*

1. *the map $G \times G \to \mathbf{C}: \ (g, x) \mapsto c(g)(x)$ is continuous;*
2. *the continuous map $f(x) = c(x^{-1})(x)$ satisfies $c(g) = f - \rho_{G,p}(g)f$;*
3. *moreover if G is a Riemannian connected Lie group, then c can be chosen such that f lies in $D_p(G)$ (and in $C^\infty(G)$).*

Proof of the lemma. Note that a cocycle b always satisfies $b(1) = 0$. Let ψ be a continuous, compactly supported probability density on G. We define $c \in Z^1(G, \rho_{G,p})$ by

$$c(g) = \int_G b(gh)\psi(h)dh - \int_G b(h)\psi(h)dh = \int_G b(h)(\psi(g^{-1}h) - \psi(h))dh.$$

We have

$$\begin{aligned} c(gg') &= \int_G b(gg'h)\psi(h)dh - \int_G b(h)\psi(h)dh \\ &= \rho_{G,p}(g)\int_G b(g'h)\psi(h)dh + \int_G b(g)\psi(h)dh - \int_G b(h)\psi(h)dh \end{aligned}$$

But note that

$$\begin{aligned} \int_G b(g)\psi(h)dh &= \int_G b(ghh^{-1})\psi(h)dh \\ &= \rho_{G,p}(g)\int_G \rho_{G,p}(h)b(h^{-1})\psi(h)dh + \int_G b(gh)\psi(h)dh \\ &= -\rho_{G,p}(g)\int_G b(h)\psi(h)dh + \int_G b(gh)\psi(h)dh. \end{aligned}$$

So we obtain

$$\begin{aligned} c(gg') &= \rho_{G,p}(g)\left(\int_G b(g'h)\psi(h)dh - \int_G b(h)\psi(h)dh\right) + \int_G b(gh)\psi(h)dh - \int_G b(h)\psi(h)dh \\ &= \rho_{G,p}(g)c(g') + c(g). \end{aligned}$$

So c is a cocycle.

Let us check that c belongs to the cohomology class of b. Using the cocycle relation, we have

$$\begin{aligned} c(g) &= \int_G (\rho_{G,p}(g)b(h) + b(g))\psi(h)dh - \int_G b(h)\psi(h)dh \\ &= b(g) + \int_G (\rho_{G,p}(g)b(h) - b(h))\psi(h)dh \\ &= b(g) + \rho_{G,p}(g)\int_G b(h)\psi(h)dh - \int_G b(h)\psi(h)dh. \end{aligned}$$

But since $\int_G b(h)\psi(h)dh \in L^p(G)$, we deduce that c belongs to the cohomology class of b.

Now, let us prove that $(g,x) \mapsto c(g)(x)$ is continuous. It is easy to see from the definition of c that $g \mapsto c(g)(x)$ is defined and continuous for almost every x: fix such a point x_0. We conclude remarking that the cocycle relation implies

$$c(g)(x_0 x) = c(xg)(x_0) - c(g)(x_0).$$

Now we can define $f(x) = c(x^{-1})(x) = -c(g)(1)$ and again the cocycle relation for c implies that $c(g) = f - \rho_{G,p}(g)f$.

Finally, assume that G is a Lie group and choose a smooth ψ. The function $\hat{\psi}$ defined by

$$\hat{\psi}(g) = \psi(g^{-1})$$

is also smooth and compactly supported. We have

$$c(g)(x) = f(x) - f(xg) = \int_G b(h)(x)(\hat{\psi}(h^{-1}g) - \hat{\psi}(h^{-1}))dh.$$

Hence, f is differentiable and

$$\nabla f(x) = -\int_G b(h)(x)(\nabla\hat{\psi})(h^{-1}))dh,$$

and so $\nabla f \in L^p(TG)$. ■

Proof of Theorem 5.1. The last statement of the lemma implies that HJ is surjective. The injectivity follows immediately from the fact that f is determined up to a constant by its associated cocycle $b = I(f)$.

We now have to prove that the isomorphism HJ is a topological isomorphism. This is immediate in the context of the coarse L^p-cohomology. Let us prove it for a Riemannian connected Lie group. Let S be a compact generating subset of G and define a norm on $Z^1(G, \rho_{G,p})$ by

$$\|b\| = \sup_{s \in S} \|b(s)\|_p.$$

Let ψ be a smooth, compactly generated probability density on G as in the proof of Lemma 5.2. Denote

$$f * \hat{\psi}(x) = \int_G f(k)\hat{\psi}(k^{-1}x)dk = \int_G f(xh)\psi(h)dh = \int_G f(k)\psi(x^{-1}k)dk.$$

We have

Lemma 5.3. *There exists a constant $C < \infty$ such that for every $f \in \mathbf{D}_p(G)$,*

$$C^{-1}\|f * \hat{\psi}\|_{\mathbf{D}_p} \leq \|J(f)\| \leq C\|f\|_{\mathbf{D}_p}.$$

Proof of the lemma. First, one checks easily that if b is the cocycle associated to f, then the regularized cocycle c constructed in the proof of Lemma 5.2 is associated to $f * \psi$.

We have

$$\begin{aligned}
\nabla(f * \hat{\psi})(x) &= \int f(k)\nabla\hat{\psi}(k^{-1}x)dk \\
&= \int (f(k) - f(x))\nabla\hat{\psi}(k^{-1}x)dk \\
&= \int (f(xh) - f(x))\nabla\hat{\psi}(h^{-1})dh
\end{aligned}$$

So

$$\begin{aligned}
\|\nabla(f * \hat{\psi})\|_p &\leq \sup_{h \in \mathrm{Supp}(\hat{\psi})} \int |f(xh) - f(x)|^p \|\nabla\hat{\psi}\|_\infty^p dx \\
&= \sup_{h \in \mathrm{Supp}(\hat{\psi})} \|b(h)\|^p \|\nabla\hat{\psi}\|_\infty^p,
\end{aligned}$$

which proves the left-hand inequality of Lemma 5.3. Let $g \in G$ and $\gamma : [0, d(1,g)] \to G$ be a geodesic between 1 and g. For any $f \in \mathbf{D}_p(G)$ and $x \in G$, we have

$$(f - \rho_{G,p}(g)f)(x) = f(x) - f(xg) = \int_0^{d(1,g)} \nabla f(x) \cdot \gamma'(t)dt.$$

So we deduce that

$$\|f - \rho_{G,p}(g)f\|_p \leq d(1,g)\|\nabla f\|_p,$$

which proves the right-hand inequality of Lemma 5.3. ■

Continuity of HJ follows from continuity of J which is an immediate consequence of Lemma 5.3.

Let us prove that the inverse of HJ is continuous. Let b_n be a sequence in $Z^1(G, \rho_{G,p})$, converging to 0 modulo $B^1(G, \rho_{G,p})$. This means that there exists a sequence a_n in $B^1(G, \rho_{G,p})$ such that $\|b_n + a_n\| \to 0$. By Lemma 5.2, we can assume that $b_n(g) = f_n - \rho_{G,p}(g)f_n$ with $f \in \mathbf{D}_p(G)$. On the other hand, $a_n = h - \rho_{G,p}(g)h$ with $h \in L^p(G)$. As compactly supported, regular[9] functions on G are dense in $L^p(G)$, we can assume that h is regular. So finally, replacing f_n by $f_n + h_n$, which is in $\mathbf{D}_p(G)$, we can assume that $J(f_n) \to 0$. Then, by Lemma 5.3, $\|f_n * \hat{\psi}\|_{\mathbf{D}_p} \to 0$. But by the proof of Lemma 5.2, $f_n * \hat{\psi}$ is in the class of L^p-cohomology of f_n. This finishes the proof of Theorem 5.1. ■

6 Sublinearity of cocycles

Theorem 6.1. *Let G be a locally compact compactly generated group and let S be a compact symmetric generating subset. Let π be a mixing L^p-representation of G. Then, every 1-cocycle $b \in Z^1(G, \pi)$ is sublinear, i.e.*

$$\|b(g)\| = o(|g|_S)$$

when $|g|_S \to \infty$, $|g|_S$ being the word length of g with respect to S.

Let $L^p(X, m)$ the L^p-space on which G acts. We will need the following lemma.

Lemma 6.2. *Let us keep the assumptions of the theorem. For any fixed $j \in \mathbf{N}$,*

$$\|\pi(g_1)v_1 + \ldots + \pi(g_j)v_j\|_p^p \to \|v_1\|_p^p + \ldots + \|v_j\|_p^p$$

when $d_S(g_k, g_l) \to \infty$ whenever $k \neq l$, uniformly with respect to $(v_1, \ldots, v_j)$ on every compact subset of $(L^p(X, m))^j$.

Proof of Lemma 6.2. First, let us prove that if the lemma holds pointwise with respect to $\overline{v} = (v_1, \ldots, v_j)$, then it holds uniformly on every compact subset K of $(L^p(X, m))^j$. Let us fix some $\varepsilon > 0$. Equip $(L^p(X, m))^j$ with the norm

$$\|\overline{v}\| = \max_i \|v_i\|_p,$$

and take a finite covering of K by balls of radius ε: $B(\overline{w}, \varepsilon)$, $\overline{w} \in W$, where W is a finite subset of K. Take $\min_{1 \leq k \neq i \leq j} d_S(g_k, g_l)$ large enough so that for any $\overline{w} \in W$, $\|\pi(g_1)v_1 + \ldots + \pi(g_j)v_j\|_p^p$ is closed to $\|v_1\|_p^p + \ldots + \|v_j\|_p^p$ up to ε. As $\pi(g)$ preserves the L^p-norm for every $g \in G$, we immediately see that for any $\overline{v}$ in K, $\|\pi(g_1)v_1 + \ldots + \pi(g_j)v_j\|_p^p$ is closed to $\|v_1\|_p^p + \ldots + \|v_j\|_p^p$ up to some ε' only depending on K, p and ε, and such that $\varepsilon' \to 0$ when $\varepsilon \to 0$.

[9] Regular here, means either continuous, or smooth if G is a Lie group.

So now, we just have to prove the lemma for $v_1, \dots, v_j$ belonging to a dense subset of $L^p(X,m)$. Thus, assume that for every $1 \leq k \leq j$, v_k is bounded and compactly supported. Let us denote by A_k the support of v_k. For every finite sequence $\overline{g} = g_1, \dots, g_j$ of elements in G, we write, for every $1 \leq i \leq j$,

- $U_{i,\overline{g}} = \left(\bigcup_{l \neq i} g_l A_l\right) \cap g_i A_i$;
- $A_{i,\overline{g}} = g_i A_i \smallsetminus U_{i,\overline{g}}$.

The key point of the proof is the following observation

Claim 6.3. *For every $1 \leq i \leq j$,*

$$m(U_{i,\overline{g}}) \to 0,$$

when the relative distance between the g_k goes to ∞.

Proof of the claim. For $u, v \in L^2(G,m)$, write $\langle u, v \rangle = \int_X u(x)v(x)dm(x)$. For every $1 \leq i \leq j$,

$$\begin{aligned} m\left(\left(\bigcup_{l \neq i} g_l A_l\right) \cap g_i A_i\right) &= \left\langle \sum_{l \neq i} \pi(g_l) 1_{A_l}, \pi(g_i) 1_{A_i} \right\rangle \\ &= \sum_{l \neq i} \langle \pi(g_l) 1_{A_l}, \pi(g_i) 1_{A_i} \rangle \\ &= \sum_{l \neq i} \langle \pi(g_l^{-1} g_i) 1_{A_i}, 1_{A_l} \rangle \\ &= \sum_{l \neq i} m(g_l^{-1} g_i A_i \cap A_l) \to 0 \end{aligned}$$

by mixing property of the action. ■

Proof of the lemma. First, observe that by the claim,

$$\|\pi(g_i) v_i 1_{U_{i,\bar{g}}}\|_p^p \leq \|v_i\|_\infty^p m(U_{i,\overline{g}}) \to 0,$$

when the relative distance between the g_k goes to ∞. In other words, as $\pi(g_i)v_i = \pi(g_i)v_i 1_{A_{i,\bar{g}}} + \pi(g_i)v_i 1_{U_{i,\bar{g}}}$,

$$\|\pi(g_i)v_i 1_{A_{i,\bar{g}}} - \pi(g_i)v_i\|_p^p \to 0.$$

In particular,

$$\|\pi(g_i)v_i 1_{A_{i,\bar{g}}}\|_p^p \to \|v_i\|_p^p.$$

On the other hand, the $A_{i,\bar{g}}$ are piecewise disjoint. So finally, we have

$$\begin{aligned}\lim_{d_S(g_l,g_k)\to\infty} \|\pi(g_1)v_1+\ldots+\pi(g_j)v_j\|_p^p &= \lim_{d_S(g_l,g_k)\to\infty} \|\pi(g_1)v_1 1_{A_{1,\bar{g}}}+\ldots+\pi(g_j)v_j 1_{A_{j,\bar{g}}}\|_p^p \\ &= \lim_{d_S(g_l,g_k)\to\infty} \|\pi(g_1)v_1 1_{A_{1,\bar{g}}}\|^p+\ldots+\|\pi(g_j)v_j 1_{A_{j,\bar{g}}}\|_p^p \\ &= \|v_1\|_p^p+\ldots+\|v_j\|_p^p,\end{aligned}$$

which proves the lemma. ■

Proof of Theorem 6.1. Fix some $\varepsilon > 0$. Let $g = s_1 \ldots s_n$ be a minimal decomposition of g into a product of elements of S. Let $m \leq n$, q and $r < m$ be positive integers such that $n = qm + r$. To simplify notation, we assume $r = 1$. For $1 \leq i < j \leq n$, denote by g_j the prefix $s_1 \ldots s_j$ of g and by $g_{i,j}$ the subword $s_{i+1} \ldots s_j$ of g. Developing $b(g)$ with respect to the cocycle relation, we obtain

$$b(g) = b(s_1) + \pi(g_1)b(s_2) + \ldots + \pi(g_{n-1})b(s_n).$$

Let us put together the terms in the following way

$$\begin{aligned} b(g) = &\big[b(s_1) + \pi(g_m)b(s_{m+1}) + \ldots + \pi(g_{(q-1)m})b(s_{(q-1)m+1})\big] \\ + &\big[\pi(g_1)b(s_2) + \pi(g_{m+1})b(s_{m+2}) + \ldots + \pi(g_{(q-1)m+1})b(s_{(q-1)m+2})\big] \\ + \ldots + &[\pi(g_{m-1})b(s_m) + \pi(g_{2m-1})b(s_{2m}) + \ldots + \pi(g_{qm})b(s_{qm+1})]\end{aligned}$$

In the above decomposition of $b(g)$, consider each term between $[\cdot]$, e.g. of the form

$$\pi(g_k)b(s_{k+1}) + \ldots + \pi(g_{(q-1)m+k})b(s_{(q-1)m+k+1}) \tag{6.1}$$

for $0 \leq k \leq m-1$ (we decide that $s_0 = 1$). Note that since S is compact and π is continuous, there exists a compact subset K of E containing $b(s)$ for every $s \in S$. Clearly since $g = s_1 \ldots s_n$ is a minimal decomposition of g, the length of $g_{i,j}$ with respect to S is equal to $j - i - 1$. For $0 \leq i < j \leq q-1$ we have

$$d_S(g_{im+k}, g_{jm+k}) = |g_{im+k,jm+k}|_S = (j-i)m \geq m.$$

So by Lemma 6.2, for $m = m(q)$ large enough, the p-power of the norm of (6.1) is less than

$$\|b(s_{k+1})\|_p^p + \|b(s_{m+k+1})\|_p^p + \ldots + \|b(s_{(q-1)m+k+1})\|_p^p + 1.$$

The above term is therefore less than $2q$. Hence, we have

$$\|b(g)\|_p \leq 2mq^{1/p}.$$

So for $q \geq q_0 = (2/\varepsilon)^{p/(p-1)}$, we have

$$\|b(g)\|_p/n \leq 2q^{1-1/p} \leq \varepsilon.$$

Now, let n be larger than $m(q_0)q_0$. We have $\|b(g)\|_p/|g| \leq \varepsilon$. ■

7 Proof of Theorem 1

Theorem 1 results from Theorem 6.1 and the following result, which is an immediate generalization of [CTV1, Proposition 3.6]. For the convenience of the reader, we give its short proof.

Proposition 7.1. *Let G be a group with property (CF) and let π be a continuous isometric action of G on a Banach space E. Let b a 1-cocycle in $Z^1(G,\pi)$. Then b belongs to $\overline{B^1(G,\pi)}$ if and only if b is sublinear.*

Proof: Assume that b is sublinear.

Let (F_n) be a controlled Følner sequence in G. Define a sequence $(v_n) \in E^{\mathbf{N}}$ by

$$v_n = \frac{1}{\mu(F_n)} \int_{F_n} b(g)dg.$$

We claim that (v_n) defines a sequence of almost fixed points for the affine action σ defined by $\sigma(g)v = \pi(g)v + b(g)$. Indeed, we have

$$\begin{aligned}
\|\sigma(s)v_n - v_n\| &= \left\| \frac{1}{\mu(F_n)} \int_{F_n} \sigma(s)b(g)dg - \frac{1}{\mu(F_n)} \int_{F_n} b(g)dg \right\| \\
&= \left\| \frac{1}{\mu(F_n)} \int_{F_n} b(sg)dg - \frac{1}{\mu(F_n)} \int_{F_n} b(g)dg \right\| \\
&= \left\| \frac{1}{\mu(F_n)} \int_{s^{-1}F_n} b(g)dg - \frac{1}{\mu(F_n)} \int_{F_n} b(g)dg \right\| \\
&\leq \frac{1}{\mu(F_n)} \int_{s^{-1}F_n \Delta F_n} \|b(g)\|dg.
\end{aligned}$$

Since $F_n \subset S^n$, we obtain that

$$\|\sigma(s)v_n - v_n\| \leq \frac{C}{n} \sup_{|g|_S \leq n+1} \|b(g)\|$$

which converges to 0. This proves the non-trivial implication of Proposition 7.1. ■

8 Liouville D_p-Properties: a direct approach.

In this section, we propose a direct proof of Corollary 3. Instead of using Theorem 1 and Theorem 5.1, we reformulate the proof, only using Theorem 6.1 and [T1, Theorem 11]. The interest is to provide an explicit approximation of an element of $\mathbf{D}_p(G)$ by a sequence of functions in $W^{1,p}(G)$ using a convolution-type argument. Since Liouville D_p-Property is equivalent to the vanishing of $\overline{H_p}^1(G)$, we have to show that for every p-Dirichlet function on G, there exists a sequence of functions (f_n) in $W^{1,p}(G)$ such that the sequence $(\|\nabla(f - f_n)\|_p)$ converges to zero. Let (F_n) be a *right* controlled Følner sequence. By a standard regularization argument, we can construct for every n, a smooth 1-Lipschitz function φ_n such that

- $0 \leq \varphi_n \leq 1$;
- for every $x \in F_n$, $\varphi_n(x) = 1$;
- for every y at distance larger than 2 from F_n, $\varphi_n(y) = 0$.

Denote by $F_n' = \{x \in G : \ d(x, F_n) \leq 2\}$. As F_n is a controlled Følner sequence, there exists a constant $C < \infty$ such that

$$\mu(F_n' \smallsetminus F_n) \leq C\mu(F_n')/n$$

and

$$F_n' \subset B(1, Cn).$$

Define

$$p_n = \frac{\varphi_n}{\int_G \varphi_n d\mu}.$$

Note that p_n is a probability density satisfying for every $x \in X$,

$$|\nabla p_n(x)| \leq \frac{1}{\mu(F_n)}.$$

For every $f \in D_p(G)$, write $P_n f(x) = \int_X f(y) p_n(y^{-1}x) d\mu(y)$. As G is unimodular,

$$P_n f(x) = \int_X f(yx^{-1}) p_n(y^{-1}) d\mu(y).$$

We claim that $P_n f - f$ is in $W^{1,p}$. For every $g \in G$ and every $f \in D_p$, we have

$$\|f - \rho(g) f\|_p \leq d(1, g) \|\nabla f\|_p.$$

Recall that the support of p_n is included in F_n' which itself is included in $B(1, Cn)$. Thus, integrating the above inequality, we get

$$\|f - P_n f\|_p \leq Cn \|\nabla f\|_p,$$

so $f - P_n f \in L^p(G)$.

It remains to show that the sequence $(\|\nabla P_n f\|_p)$ converges to zero. We have

$$\nabla P_n f(x) = \int_G f(y) \nabla p_n(y^{-1}x) d\mu(y)$$

Since $\int_G \nabla p d\mu = 0$, we get

$$\begin{aligned}\nabla P_n f(x) &= \int_G (f(y) - f(x^{-1}))\nabla p_n(y^{-1}x)d\mu(y) \\ &= \int_G (f(yx^{-1}) - f(x^{-1}))\nabla p_n(y^{-1})d\mu(y).\end{aligned}$$

Hence,

$$\begin{aligned}\|\nabla P_n f\|_p &\leq \int_G \|\lambda(y)f - f\|_p |\nabla p_n(y^{-1})| d\mu(y) \\ &\leq \frac{1}{\mu(F_n)} \int_{F'_n \setminus F_n} \|\lambda(y)f - f\|_p d\mu(y) \\ &\leq \frac{\mu(F'_n \setminus F_n)}{F_n} \sup_{|g| \leq Cn} \|b(g)\|_p \\ &\leq \frac{C}{n} \sup_{|g| \leq Cn} \|b(g)\|_p\end{aligned}$$

where $b(g) = \lambda(g)f - f$. Note that $b \in Z^1(G, \lambda_{G,p})$. Thus, by Theorem 6.1,

$$\|\nabla P_n f\|_p \to 0.$$

This completes the proof of Corollary 3. ■

9 Non-vanishing of the first reduced L^p-cohomology on a non-elementary Gromov hyperbolic space.

Let us start with a remark about first L^p-cohomology on a metric measure space.

Remark 9.1. **(Coupling between 1-cycles and 1-cocycles)** A 1-chains on (X, d, μ) is a functions supported on $\Delta_r = \{(x, y) \in X^2, d(x, y) \leq r\}$ for some $r > 0$. The L^p-norm of a (measurable) 1-chain s is the norm

$$\left(\int_{X^2} |s(x, y)|^p d\mu(x) d\mu(y)\right)^{1/p}.$$

A 1-chain s is called a 1-cycle if $s(x, y) = s(y, x)$.

Given $f \in \mathbf{D}_p$, we define a 1-cocycle associated to f by $c(x, y) = f(x) - f(y)$, for every $(x, y) \in X^2$. Let s be a 1-cycle in L^q, with $1/p + 1/q = 1$. We can form a coupling between c and s

$$\langle c, s\rangle = \int_{X^2} c(x, y)s(x, y)d\mu(x)d\mu(y) = \int_{X^2} (f(x) - f(y))s(x, y)d\mu(x)d\mu(y).$$

Clearly, if $f \in L^p$, then as s is a cycle, we have $\langle c, s\rangle = 0$. This is again true for f in the closure of $L^p(X)$ for the norm of $\mathbf{D}_p(X)$. Hence, to prove that a 1-cocycle c is non-trivial in $\overline{H^1}_p(X)$, it is enough to find a 1-cycle in L^q whose coupling with c is non-zero.

The main result of this section is the following theorem.

Theorem 9.2. *Let X be a Gromov hyperbolic 1-geodesic metric measure space with bounded geometry having a bi-Lipschitz embedded 3-regular tree, then for p large enough, it has non-trivial first reduced L^p-cohomology.*

From this theorem, we will deduce

Corollary 8. *A homogeneous Riemannian manifold M has non-zero first reduced L^p-cohomology for some $1 < p < \infty$ if and only if it is non-elementary Gromov hyperbolic.*

Proof of Corollary 8. By Theorem 4, if M has non-zero $\overline{H}^1_p(M)$ for some $1 < p < \infty$, then being quasi-isometric to a negatively curved homogeneous manifold, it is non-elementary Gromov hyperbolic.

Conversely, let M be a Gromov hyperbolic homogeneous manifold. As M is quasi-isometric to its isometry group G, which is a Lie group with finitely many components, we can replace M by G, and assume that G is connected. If G has exponential growth, then [CT, Corollary 1.3] it has a bi-Lipschitz embedded 3-regular tree T, and hence Theorem 9.2 applies. Otherwise G has polynomial growth, and we conclude thanks to the following classical fact.

Proposition 9.3. *A non-elementary Gromov-hyperbolic connected Lie group has exponential growth.*

Proof: Let G be a connected Lie group with polynomial growth. By [Gui], G is quasi-isometric to a simply connected nilpotent group G, whose asymptotic cone [Pa1] is homeomorphic to another (graded) simply connected nilpotent Lie group with same dimension. Hence, unless G is quasi-isometric to $\mathbf{R}$, the asymptotic cone of G has dimension larger or equal than 2. But [Gro2, page 37] the asymptotic cone of a Gromov hyperbolic space is an $\mathbf{R}$-tree, and therefore has topological dimension 1. ■

Proof of Theorem 9.2. The proof contains ideas that we found in [Gro2, page 258]. Roughly speaking, we start by considering a non-trivial cycle defined on a bi-Lipschitz embedded 3-regular subtree T of X. To construct a 1-cocycle which has non-trivial reduced cohomology, we take a Lipschitz function F defined on the boundary of X, such that F is non-constant in restriction to the boundary of the subtree T. We then extend F to a function defined f on X which defines a 1-cocyle in $D_p(X)$. Coupling this cocycle with our cycle on T proves its non-triviality in $\overline{H}_p^1(X)$.

Boundary at infinity of a hyperbolic space. To denote the distance between to points in X or in its boundary, we will use indifferently the notation $d(x, y)$, or the notation of Gromov $|x - y|$. Let us fix a point $o \in X$. We will denote $|x| = |x - o| = d(x, o)$.

Consider the Gromov boundary (see [Gro1, Chapter 1.8] or [GH]) of X, i.e. the set of geodesic rays issued from o up to Hausdorff equivalence.

For ε small enough, there exists [GH] a distance $|\cdot|_\varepsilon$ on $\partial_\infty X$, and $C < \infty$ such that

$$|u - v|_\varepsilon \leq \limsup_{t\to\infty} e^{-\varepsilon(v(t)|w(t))} \leq C|u - v|_\varepsilon.$$

for all $v, w \in \partial_\infty X$, where $(\cdot|\cdot)$ denotes the Gromov product, i.e.

$$(x|y) = \frac{1}{2}(|x| + |y| - |x - y|).$$

Reduction to graphs. A 1-geodesic metric measure space with bounded geometry is trivially quasi-isometric to a connected graph with bounded degree (take a maximal 1-separated net, and join its points which are at distance 1 by an edge). Hence, we can assume that X is the set of vertices of a graph with bounded degree.

A Lipschitz function on the boundary. By [Gro2, page 221], T has a cycle which has a non-zero pairing with every non-zero 1-cochain c on T supported on a single edge e. Hence, to prove that $\overline{H}_p^1(X) \neq 0$, it is enough to find an element c in $D_p(X)$ whose restriction to T is zero everywhere but on e.

The inclusion of T into X being bi-Lipschitz, it induces a homeomorphic inclusion of the boundary of T, which is a Cantor set, into the boundary of X. We therefore identify $\partial_\infty T$ with its image in $\partial_\infty X$. Consider T_1 and T_2 the two complementary subtrees of T which are separated by e. This induces a partition of the boundary $\partial_\infty T$ into two clopen non-empty subsets O_1 and O_2. As O_1 and O_2 are disjoint compact subsets of $\partial_\infty X$, they are at positive distance from one another. Hence, for $\delta > 0$ small enough, the δ-neighborhoods V_1 and V_2 of respectively O_1 and O_2 in $\partial_\infty X$ are disjoint.

Now, take a Lipschitz function F on $\partial_\infty X$ which equals 0 on V_1 and 1 on V_2.

Extension of F to all of X.

Let us first assume that every point in X is at bounded distance from a geodesic ray issued from o.

Let us define a function f on X: for every x in $X \setminus \{o\}$, we denote element by u_x a geodesic ray issued from o and passing at bounded distance, say C from x. Define

$$f(x) = F(u_x) \quad \forall x \in X \setminus \{o\}.$$

Let us prove that for p large enough, $f \in D_p(X)$. Take two elements x and y in X such that $|x-y| \leq 1$, we have

$$\begin{aligned} |u_x(t) - u_y(t)| &\leq |u_x(t) - x| + |x - u_y(t)| \\ &\leq |u_x(t) - x| + |y - u_y(t)| + |x - y| \\ &\leq |u_x(t) - x| + |y - u_y(t)| + 1. \end{aligned}$$

So for large t,

$$\begin{aligned} 2(u_x(t)|u_y(t)) &= |u_x(t)| + |u_y(t)| - |u_x(t) - u_y(t)| \\ &\geq |u_x(t)| + |u_y(t)| - |u_x(t) - x| - |y - u_y(t)| - 1 \\ &\geq |x| + |y| - 2|u_x(t) - x| - 2|y - u_y(t)| - 1 \\ &\geq |x| + |y| - 4C - 1 \\ &\geq 2|x| - 4(C+1) \end{aligned}$$

Let $K > 0$ be the Lipschitz norm of F, i.e. $K = \sup_{u \neq v \in \partial_\infty T} \frac{|F(u)-F(v)|}{|u-v|_\varepsilon}$. We have

$$\begin{aligned} |f(x) - f(y)|^p &= |F(u_x) - F(u_y)|^p \\ &\leq K^p |u_x - u_y|^p \\ &\leq K^p \limsup_{t \to \infty} e^{-p\varepsilon(u_x(t)|u_y(t))} \\ &\leq K^p e^{-p\varepsilon|x| + 2(C+1)p} \end{aligned}$$

On the other hand, as $\mu(B(o,|x|)) \leq Ce^{\lambda|x|}$ for some λ, if $p\varepsilon > \lambda$, then f is in $D_p(X)$.

Now, let us consider the values of f along T. To fix the ideas, let us assume that o is a vertex of T. We will now show that up to modifying T, we can assume that f takes the value 0 on T_1, and 1 on T_2. Hence the coupling of the corresponding cocycle c with the cycle of [Gro2, page 221] is non-zero, which implies that $\overline{H}^1_p(X) \neq 0$.

For $i = 1, 2$, take x_i a vertex of T_i. Let e_{x_i} be the edge whose one extremity is x_i and that separates o and x_i. Let T_{x_i} be the connected component of $T \setminus \{e_{x_i}\}$ contained in T_i.

The point that we need to prove is that if both x_1 and x_2 are far enough from o, f equals 0 on T_{x_1} and 1 on T_{x_2}. Then, up to replacing T_1 and T_2 by T_{x_1} and T_{x_2}, and the geodesic segment between x_1 and x_2 by a single edge (which becomes e), we are done.

Let v be a geodesic ray of T_1 emanating from o and passing through some vertex y of T_{x_1}. Let z be the corresponding element of $\partial_\infty T \subset \partial_\infty X$. Let t be a geodesic ray in X from o to z. As T is bi-Lipschitz embedded in X, v is a quasi-geodesic ray in X. Hence it stays at bounded distance, say less than C from t. In particular, t passes at distance less than C from y. So by choosing $d(o,y)$ large enough, $|u_y - z|$ can be made arbitrarily small, in particular $\leq \delta$. Hence, choosing x_1 far enough from o in T_1, we have that all u_y where $y \in T_{x_1}$ belong to V_1. Therefore $f(y) = 0$. The case $i = 2$ is similar.

Reduction to the case when every point in X is at bounded distance from a geodesic ray issued from o.

In this section, we embed X into a larger graph $\tilde{X}$ satisfying the property that every vertex is contained in a geodesic ray emanating from o.

Let Y be the graph whose set of vertices is $\mathbf{N}$ and such that n and m are joined by an edge if and only if $|n-m| = 1$. Consider the graph $\tilde{X}$ obtained by gluing a copy of Y to every vertex of X. This is done by identifying this vertex with the vertex 0 of the corresponding copy of Y. Clearly $\tilde{X}$ is a hyperbolic graph with bounded degree. It contains X as an isometrically embedded subgraph. In particular, T is bi-Lipschitz embedded into $\tilde{X}$. Finally, $\tilde{X}$ satisfies that every point in $\tilde{X}$ belongs to a geodesic ray issued from o.

Applying the above to $\tilde{X}$, we construct an element $\tilde{f}$ in $D_p(\tilde{X})$ that has a non-trivial coupling with the cycle that we considered on T. As the support of this cycle is contained in X, the restriction f of $\tilde{f}$ to X also has a non-trivial coupling with it. Moreover, f belongs to $D_p(X)$, so it defines a non-trivial cocycle in $\overline{H}^1_p(X)$. ■

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