Coarse embeddings into a Hilbert space, Haagerup Property and Poincaré inequalities

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Abstract

We prove that a metric space does not coarsely embed into a Hilbert space if and only if it satisfies a sequence of Poincaré inequalities, which can be formulated in terms of (generalized) expanders. We also give quantitative statements, relative to the compression. In the equivariant context, our result says that a group does not have the Haagerup property if and only if it has relative property T with respect to a family of probabilities whose supports go to infinity. We give versions of this result both in terms of unitary representations, and in terms of affine isometric actions on Hilbert spaces.

1 Introduction

1.1 Obstruction to coarse embeddings

The notion of expanders has been pointed out by Gromov as an obstruction for a metric space to coarsely embed into a Hilbert space. Recall [JS] (see also [L]) that a sequence of expanders is a sequence of finite connected graphs (X_n) with bounded degree, satisfying the following Poincaré inequality for all $f \in \ell^2(X_n)$

$$\frac{1}{|X_n|^2}\sum_{x,y\in X_n}|f(x)-f(y)|^2 \leq \frac{C}{|X_n|}\sum_{x\sim y}|f(x)-f(y)|^2, \qquad (1.1)$$

for some constant $C > 0$, and whose cardinality $|X_n|$ goes to infinity when $n \to \infty$. An equivalent formulation in ℓ^p [M2] can be used to prove that expanders do not coarsely embed into L^p for any $1 \leq p < \infty$.

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It is an open problem whether a metric space with bounded geometry that does not coarsely embed into a Hilbert space admits a coarsely embedded sequence of expanders.

In this paper, we prove that a metric space (not necessarily with bounded geometry) that does not coarsely embed into a Hilbert space admits a coarsely embedded sequence of "generalized expanders". This weaker notion of expanders can be roughly described as a sequence of Poincaré inequalities with respect to finitely supported probability measures on $X \times X$. We also provide similar obstructions for coarse embeddings into families of metric spaces such as L^p, for every $1 \leq p < \infty$, uniformly convex Banach spaces, and CAT(0) spaces.

For the sake of clarity, we chose to present most of our results first in the case of Hilbert spaces. However, our characterization (see Theorem 14) of the non-existence of coarse embedding into L^p deserves some attention. Indeed, our Poincaré inequalities are not equivalent for different values of $2 \leq p < \infty$. This follows from a result of Naor and Mendel [MN] (see also [JR]) saying that L^p does not coarsely embed into L^q if $2 \leq q < p$. This is different from what happens with real expanders, as having a sequence of expanders prevents from having a coarse embedding into L^p, for any $1 \leq p < \infty$. In particular, at least without any assumption of bounded geometry, our generalized expanders cannot be replaced by actual expanders.

To conclude, let us remark that finding subspaces of L^p for some $p > 2$, with bounded geometry, which do not coarsely embed into L^2, would answer negatively the problem mentioned above.

1.2 Obstruction to Haagerup Property

A countable group is said to have the Haagerup property if it admits a proper affine action on a Hilbert space. An obstruction for an infinite countable group to have the Haagerup Property is known as Property T (also called Property FH), which says that every isometric affine action has a fixed point (or equivalently bounded orbits). A weaker obstruction is to have relative property T with respect to an infinite subset [C1, C2]. The case where this subset is a normal subgroup has been mostly considered, as it has strong consequences. On the other hand, there are examples of groups which do not have relative property T with respect to any subgroup, but have it with respect to some infinite subset [C1]. The question whether the latter property is equivalent to the negation of Haagerup Property is still open.

In this paper, we partially answer this question by showing that a countable group does not have Haagerup Property if and only if it has relative Property T with respect to a sequence of probabilities whose supports eventually leave every finite subset.

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2 Organization of the paper

Before describing the general organization of the paper, let us emphasize that the proof of the main result (stated in its most general form in Theorem 14) is only given in the last section (Section 5). The proof of Theorem 20, relative to length functions on groups, and the proofs of the quantitative statements of Section 4.3 are straightforward adaptations of the proof of Theorem 14. Therefore they will be omitted.

- In the next section, we will state our results in the Hilbert case. First, in Subsection 3.1, we state our characterization of the non-existence of a coarse embedding into a Hilbert space. In Subsection 3.2, we show that a sequence of expanders in the usual sense is also a sequence of generalized expanders in our sense. Finally, in Subsection 3.3, we state various equivalent formulations of our characterization of non-Haagerup property.

- In Section 4, we state a more general version of our main result in order to characterize the non-(coarse)-embeddability into various classes of metric spaces, such as L^p-spaces. This section is divided into three main subsections. Subsection 4.1 introduces the notion of sheaves of metrics in order to study coarse embeddings into various classes of metric spaces. We then state our main theorem. Subsection 4.2 is just an adaptation of the definitions and statements of Subsection 4.1 for groups, replacing metrics by length functions (or left-invariant metrics). Finally, in Subsection 4.3, we give quantitative statements relative to the compression of coarse embeddings into different classes of metric spaces.

- The last section is dedicated to the proof of our main result, namely Theorem 14.

3 Statement of results in the Hilbert case

In this section, we state our main results concerning embeddings into a Hilbert space. In Section 4, using a slightly more sophisticated vocabulary, we generalize to other geometries.

3.1 Coarse embeddability into Hilbert spaces and generalized expanders

Let $\mathcal{H}$ denote a separable infinite dimensional Hilbert space. We denote by $|v|$ the norm of a vector in $\mathcal{H}$. Let $X = (X, d)$ be a metric space. For all $r \geq 0$, denote

$$\Delta_r(X) = \{(x, y) \in X^2, d(x, y) \geq r\}.$$

In this paper, we prove that a metric space that does not coarsely embed into a Hilbert space contains in a weak sense a sequence of expanders. Precisely, following the idea of [T, Section 4.2], let us define

Definition 1.

- Let K and r be positive numbers. A finite metric space is called a *generalized (K, r)-expander* if there exists a symmetric probability measure μ supported on $\Delta_r(X)$ with the following property. For every map $F : X \to \mathcal{H}$ satisfying $\|F(x) - F(y)\| \leq d(x, y)$ for all $(x, y) \in \Delta_1(X)$, we have

$$\mathbf{Var}_\mu(F) := \sum_{x,y} \|F(x) - F(y)\|^2 \mu(x, y) \leq K^2. \tag{3.1}$$

- A sequence of finite metric spaces (X_n) is called a *sequence of generalized K-expander* if for every $n \in \mathbf{N}$, X_n is a (K, r_n)-expanders, where $r_n \to \infty$.

Remark 2. Note that (3.1) says that F sends certain pairs at distance at least r_n in X_n to pairs at distance at most K.

Recall that a family of metric spaces $(X_i)_{i \in I}$ coarsely embeds into a metric space Y if there exists a family (F_i) of uniformly coarse embeddings of X_i into Y, i.e. if there are two increasing, unbounded functions ρ_- and ρ_+ such that

$$\rho_-(d(x, y)) \leq d(F_i(x), F_i(y)) \leq \rho_+(d(x, y)), \forall x, y \in X_i, \forall i \in I. \tag{3.2}$$

Proposition 3. *A sequence of generalized expanders (X_n) does not coarsely embed into a Hilbert space.*

Proof: Let $K > 0$ and for all $n \in \mathbf{N}$, let X_n is a (K, r_n)-expander, with $r_n \to \infty$. For every $n \in \mathbf{N}$, let F_n be a map from $X_n \to \mathcal{H}$, and that there exists an increasing function ρ_+ such that $|F_n(x) - F_n(y)| \leq \rho_+(d(x,y)), \forall x, y \in X_n, \forall n \in \mathbf{N}$. As observed in [CTV, Lemmas 2.4 and 3.11], if a metric space (or a family of metric spaces) coarsely embeds into a Hilbert space, we can always assume that the function ρ_+ goes arbitrarily slowly to infinity (this follows from a result of Bochner and Schoenberg [Sch, Theorem 8]). So in particular, we can assume that $\rho_+(t) \leq t$, $\forall t \geq 1$. But then, (3.1) tells us that pairs of points of X_n, which are at distance $\geq r_n$ are sent by F_n at distance less than K. As $r_n \to \infty$, this implies that any increasing function ρ_- satisfying

$$\rho_-(d(x,y)) \leq |F_n(x) - F_n(y)|, \ \forall x, y \in X_n, \forall n \in \mathbf{N}$$

would have to be $\leq K$. ■

Our main result is the following theorem (which is a particular case of Corollary 17).

Theorem 4. *A metric space does not coarsely embed into a Hilbert space if and only if it has a coarsely-embedded sequence of generalized expanders.*

3.2 Comparison with the usual notion of expanders

The usual definition of an expander is a sequence of finite connected graphs (X_n) with degree $\leq k$, satisfying the following Poincaré inequality for all $f \in \ell^2(X_n)$

$$\frac{1}{|X_n|^2} \sum_{x,y \in X_n} |f(x) - f(y)|^2 \leq \frac{C}{|X_n|} \sum_{x \sim y} |f(x) - f(y)|^2, \tag{3.3}$$

for some constant $C > 0$, and whose cardinality $|X_n|$ goes to infinity when $n \to \infty$. If ν_n denote the uniform measure on $X_n \times X_n$, this can be rewritten as

$$\mathbf{Var}_{\nu_n}(f) \leq \frac{C}{|X_n|} \sum_{x \sim y} |f(x) - f(y)|^2.$$

Now, assuming that f is 1-Lipschitz, we have

$$\mathbf{Var}_{\nu_n}(f) \leq kC,$$

To obtain condition (3.1), we need to replace ν_n by a probability supported far away from the diagonal. To do that, we just notice that at least half of the mass of ν_n is actually supported on Δ_{r_n}, with $r_n = \log_k(|X_n|/2)$. Indeed, if r

is some positive number, the number of pairs of $X_n \times X_n$ which are at distance $\leq r$ is at most $k^r|X_n|$. Hence the proportion of such pairs is $\leq k^r/|X_n|$, and the statement follows. Therefore, renormalizing the restriction of ν_n to Δ_{r_n}, we obtain a probability μ_n satisfying $\mathbf{Var}_{\mu_n}(f) \leq 2\mathbf{Var}_{\nu_n}(f) \leq 2kC$. Hence, we have proved

Proposition 5. *A sequence of expanders satisfying (3.3) with constant C, is a sequence of generalized K-expanders, with $K = (2kC)^{1/2}$.* ■

3.3 Haagerup property and relative property T with respect to a family of probabilities

Recall that a countable group has the Haagerup Property if it acts metrically properly by affine isometries on a Hilbert space. It is well known that this is equivalent to saying that G has a proper Hilbert length (see Section 4 for the definition of Hilbert length). On the other hand [C1] a group has relative Property FH with respect to an infinite subset Ω if every Hilbert length on G is bounded in restriction to Ω.

Definition 6. Let G be a countable group equipped with a proper length function L_0. Let (μ_n) be a sequence of probability measures on G. We say that G has *relative property FH with respect to* (μ_n) if there exists $K > 0$ such that for every Hilbert length L satisfying

$$L(g) \leq L_0(g), \quad \forall g \in G,$$

and for every $n \in \mathbf{N}$,

$$\mathbf{E}_{\mu_n}(L^2) := \sum_{g \in G} L^2(g)\mu_n(g) \leq K.$$

It is easy to see that this definition does not depend on L_0 (see [CTV, Lemmas 2.4 and 3.11]).

Note that having relative property FH with respect to an infinite subset $\Omega = \{a_1, a_2, \ldots\}$ corresponds to having relative Property FH with respect to (μ_n), where μ_n is the Dirac measure at a_n, for every $n \in \mathbf{N}$.

Theorem 7. *A countable group G does not have the Haagerup Property if and only if it has relative Property FH with respect to a sequence of symmetric probability measures (μ_n), such that for all $n \in \mathbf{N}$, μ_n is supported on a finite subset of $\{g, L_0(g) \geq n\}$.*

Recall that an equivalent formulation of the Haagerup Property (actually the original one) is as follows: there exists a sequence (ϕ_k) of positive definite functions on the group such that $\lim_{k\to\infty}\phi_k(g) = 1$ for all $g \in G$, and $\lim_{g\to\infty}\phi_k(g) = 0$ for all $k \in \mathbf{N}$ (in terms of unitary representations, it says that there exists a C_0 unitary representation with almost-invariant vectors).

An obvious obstruction to the Haagerup Property is [C1] relative property T with respect to an unbounded subset Ω: every sequence (ϕ_k) of positive definite function on G converging to 1 pointwise, converges uniformly in restriction to Ω. In [C1], it is actually proved that relative Property T with respect to Ω is equivalent to relative Property FH with respect to Ω. Let us introduce the following definition.

Definition 8. Let G be a countable group. Let (μ_n) be a sequence of probability measures on G. We say that G has *relative property T with respect to* (μ_n) if every sequence of positive definite function (ϕ_k) that pointwise converges to 1, satisfies that $\lim_{k\to\infty}\mathbf{E}_{\mu_n}(\phi_k) = 1$ uniformly with respect to $n \in \mathbf{N}$.

We have the following theorem

Theorem 9. *A countable group G does not have the Haagerup Property if and only if it has relative Property T with respect to a sequence of symmetric probability measures (μ_n), such that for all $n \in \mathbf{N}$, μ_n is supported on a finite subset of $\{g, L_0(g) \geq n\}$.*

Proof: It is clear that relative Property T with respect to a sequence of probabilities whose supports go to infinity violates the Haagerup Property. So what we need to prove is the converse, namely, that the negation of Haagerup Property implies relative Property T with respect to some (μ_n). By Theorem 7, it is enough to prove that relative Property FH implies relative Property T, which is a straightforward adaptation of the proof of [AW, Theorem 3]. ■

4 A more general setting, and quantitative statements

In this section, we switch to a slightly different point of view. The statements we want to prove are of the following form: a metric space X cannot coarsely embed into some class of spaces $\mathcal{M}$ if and only if it satisfies some sequence of Poincaré inequalities. It is worth noting that these inequalities consist essentially in a comparison between metrics on X. Namely, we compare the original metric on X with all the pull-back metrics obtained from maps to metric spaces of $\mathcal{M}$. Let us be more precise.

Let X be a set. A pseudo-metric on X is a function: $\sigma : X^2 \to \mathbf{R}_+$ such that $\sigma(x, y) = \sigma(y, x)$, $\sigma(x, y) \leq \sigma(x, z) + \sigma(z, y)$, and $\sigma(x, x) = 0$, for all $x, y, z \in X$. In the sequel, a pseudo-metric will simply be called a metric.

If (Y, d) is a metric space and $F : X \to Y$ is a map, then we can consider the pull-back metric $\sigma_F(x, y) = d(F(x), F(y))$, for all $x, y \in X$. Such metrics are called Y-metrics on X. More generally, if $\mathcal{M}$ is a class of metric spaces, a $\mathcal{M}$-metric on X is a Y-metric for some $Y \in \mathcal{M}$.

4.1 Sheaves of metrics

Assume here that $X = (X, d)$ is a metric space. A metric σ on X is called coarse if there exist two increasing unbounded functions ρ_-, ρ_+ such that, for all $x, y \in X$,

$$\rho_-(d(x, y)) \leq \sigma(x, y) \leq \rho_+(d(x, y)).$$

Note that if $\sigma = \sigma_F$ is a Y-metric associated to a map $F : X \to Y$, then σ_F is coarse if and only if F is a coarse embedding.

Definition 10. A *sheaf of metrics* on a set X is a collection of pairs (σ, Ω), where Ω is a subset of X, and σ is a metric defined on Ω. If Ω is an subset of X, we denote by $\mathcal{F}(\Omega)$ the set of pairs $(\sigma, \Omega) \in \mathcal{F}$.

We also assume that the restriction is well-defined from $\mathcal{F}(\Omega)$ to $\mathcal{F}(\Omega')$ for every $\Omega' \subset \Omega$ (which is satisfied by the sheave of $\mathcal{M}$-metrics for some $\mathcal{M}$).

One checks easily that squares of Hilbert metrics, and more generally p-powers of L^p-metrics form a convex cone of the space of real-valued functions on X^2. This is in fact a crucial remark for what follows.

Definition 11. Let X be a set. A sheaf $\mathcal{F}$ of metrics on X is called *p-admissible* (for some $p > 0$) if for every Ω, the following hold.

(i) The set of σ^p, where $\sigma \in \mathcal{F}(\Omega)$ forms a convex cone of the space of functions on Ω^2.

(ii) $\mathcal{F}(\Omega)$ is closed for the topology of pointwise convergence.

(iii) Let (U_i) be a family of finite subsets whose union is Ω, satisfying that for all $i, j \in I$, there exists k such that $U_i \cup U_j \subset U_k$. Let (σ_i, U_i) be a compatible family of sections, in the sense that σ_i and σ_j coincide on the intersection $U_i \cap U_j$. Then there exists a section $\sigma \in \mathcal{F}(\Omega)$, whose restriction to every U_i is σ_i. In other words, $\mathcal{F}(\Omega)$ is the inverse limit of the $\mathcal{F}(U_i)$.

Proposition 12. *Let X be a set, and let $\mathcal{M}$ be a class of metric spaces which is closed under ultra-limits. Then the sheaf of $\mathcal{M}$-metrics on (subsets of) X satisfies conditions (ii) and (iii) of Definition 11.*

Proof: That X satisfies (ii) is trivial. Let $(U_i)_{i\in I}$ be as in (iii). Note that we can assume that they all contain a point $o \in \Omega$. $\sigma_i \in \mathcal{F}(U_i)$, choose $Y_i \in \mathcal{M}$ and $F_i : U_i \to Y_i$ be such that $\sigma_i(x,y) = d(F_i(x), F_i(y))$ for all $x, y \in U_i$. In every Y_i, take $y_i = F_i(o)$ for the origin. Fix a non-principal ultra-filter $\mathcal{U}$ on I. Now, the limit F of the F_i is well defined from

$$\Omega = \bigcup_{i\in I} U_i \to \lim_{\mathcal{U}} (Y_i, y_i),$$

and $\sigma(x,y) = d(F(x), F(y))$ satisfies the third condition of Definition 11. ■

As a consequence of the proposition, we get the following examples.

Examples 13.

- for $p \geq 1$, the sheaf of L^p-metrics is p-admissible [H].
- Let $c > 0$ and $1 < p < \infty$. The class $\mathcal{M}_{c,p}$ of (c,p)-uniformly convex Banach spaces, is the class of uniformly convex Banach spaces whose moduli of convexity satisfy $\delta(t) \geq ct^p$. The sheaf of $\mathcal{M}_{c,p}$-metrics is p-admissible.
- The sheaf of CAT(0)-metrics is 2-admissible.

Theorem 14. *Let X be a metric space. Let $\mathcal{F}$ be a p-admissible sheaf of metrics on X. Then there exists no coarse metric in $\mathcal{F}(X)$ if and only if for every function $\rho_+ : \mathbf{R}_+ \to \mathbf{R}_+$, there exist $K > 0$, and a sequence of symmetric probability measures (μ_n) with the following properties*

- *for every $n \in \mathbf{N}$, μ_n is supported on a finite subset A_n of $\Delta_n(X)$;*
- *for every $n \in \mathbf{N}$ and every $\sigma \in \mathcal{F}(A_n)$ satisfying*

$$\sigma(x,y) \leq \rho_+(d(x,y)), \quad \forall (x,y) \in A_n,$$

one has

$$\mathbf{E}_{\mu_n}(\sigma^p) \leq K^p. \tag{4.1}$$

Remark 15. Note that this theorem characterizes metric spaces that do not coarsely embed into L^p-spaces, $CAT(0)$-spaces, uniformly convex Banach spaces... Indeed, by a theorem of Pisier [Pi], any uniformly convex Banach space is isomorphic to a (c,p)-uniformly convex Banach space for some $1 < p < \infty$ and $c > 0$. We can also avoid to use this deep theorem by defining ϕ-admissible sheafs of metrics for any non-decreasing convex function ϕ, and by adapting the proof of Theorem 14 to this slightly more general setting.

Generalizing the case of Hilbert spaces, the previous theorem can be reformulated in terms of generalized expanders. Let $\mathcal{M}$ be a class of metric spaces.

Definition 16. A *sequence of* $(\mathcal{M}, p)$*-valued generalized expanders* is a sequence of finite metric spaces (X_n) satisfying the following property. For every function $\rho_+ : \mathbf{R}_+ \to \mathbf{R}_+$, there exist $K > 0$, and a sequence $r_n \to \infty$ such that each X_n carries a symmetric probability measure μ_n satisfying

- μ_n is supported on $\Delta_{r_n}(X_n)$;
- for all maps F from X_n to a metric space $(Y, \sigma) \in \mathcal{M}$, satisfying

$$\sigma(F(x), F(y)) \leq \rho_+(d(x, y)) \quad \forall (x, y) \in \Delta_1(X_n),$$

  we have

$$\sum_{x,y \in X_n} \sigma(F(x), F(y))^p \mu_n(x, y) \leq K^p. \tag{4.2}$$

Corollary 17. *Let* $\mathcal{M}$ *be a class of metric spaces such that the corresponding sheafs are* p*-admissible for some* $1 \leq p < \infty$*. Then a metric space* X *does not coarsely embed into any element of* $\mathcal{M}$ *if and only if it has a coarsely embedded sequence of* $(\mathcal{M}, p)$*-valued generalized expanders.* ■

4.2 The invariant setting

If G is a countable discrete group, a length function on G is a function $L : G \to \mathbf{R}_+$ satisfying $L(1) = 0$, $L(gh) \leq L(g) + L(h)$, and $L(g^{-1}) = L(g)$ for all $g, h \in G$. Clearly, a length function on G gives rise to a left-invariant metric $\sigma_L(g, h) = L(g^{-1}h)$. Conversely, given a left-invariant metric σ, we define a length function by $L(g) = \sigma(1, g)$.

Let X be a metric space. An X-length L is a length whose associated left-invariant metric σ_L is an X-metric. We can define sheaves of length functions as we defined sheaves of metrics.

Definition 18. A *sheaf of length functions* $\mathcal{F}$ on a group G is a family of pairs (Ω, L), where Ω is a symmetric neighborhood of 1, and $L : \Omega \to \mathbf{R}_+$ satisfying $L(1) = 0$, $L(gh) \leq L(g) + L(h)$, $L(g^{-1}) = L(g)$ for all $g, h \in \Omega$ such that $gh \in \Omega$.

Note that a sheaf of lengths naturally induces a sheaf of "locally invariant" metrics on G by the relation $\sigma(g, h) = L(g^{-1}h)$ (whenever this is well defined). We will say that a sheaf of lengths is p-admissible if so is the corresponding sheaf of metrics.

Example 19. If $\mathcal{M}$ is a class of metric spaces, the sheaf of $\mathcal{M}$-lengths on G is the set of (Ω, L) as above, where $L(g) = \sigma(1, g)$ for some $\mathcal{M}$-metric σ defined on Ω^2, satisfying $\sigma(hg, hg') = \sigma(g, g')$ for all $g, g', h \in G$ such that $g, g', hg, hg' \in \Omega$.

Let G equipped with a length function L_0 (one might think of a finitely generated group equipped with a word metric). A length L on G is called coarse (relative to L_0) if σ_L is a coarse metric on (G, σ_{L_0}).

It is easy to see that the proof of Theorem 14 can be formulated with length functions instead of metrics, which yields the following result

Theorem 20. *Let G be a group equipped with a length function L_0. Let $\mathcal{F}$ be a p-admissible sheaf of lengths on G. Then there exists no coarse length in $\mathcal{F}(X)$ if and only if for every function $\rho_+ : \mathbf{R}_+ \to \mathbf{R}_+$, there exist $K > 0$, and a sequence of symmetric probability measures (μ_n) with the following properties*

- *for every $n \in \mathbf{N}$, μ_n is supported on a finite subset A_n of $\{g, L_0(g) \geq n\}$;*
- *for every $n \in \mathbf{N}$ and every $L \in \mathcal{F}(A_n)$ satisfying*

$$L(g) \leq \rho_+(L_0(g)), \quad \forall g \in A_n,$$

  one has

$$\mathbf{E}_{\mu_n}(L^p) \leq K^p. \tag{4.3}$$

4.3 Quantitative statements

Let $\mathcal{F}$ be a sheaf of metrics on X. The $\mathcal{F}$-compression rate of X, denoted by $R_{\mathcal{F}}(X)$ is the supremum of all $\alpha > 0$ such that there exists $\sigma \in \mathcal{F}(X)$ satisfying $d(x, y)^\alpha \leq \sigma(x, y) \leq d(x, y)$, for $d(x, y)$ large enough. The Hilbert compression rate, usually denoted by $R(X)$, has been introduced in [GK] and studied by many authors since then as it provides an interesting quasi-isometry invariant of finitely generated groups.

A slight modification of the proof of Theorem 14 yields

Theorem 21. *Let X be a metric space. Let $\mathcal{F}$ be a p-admissible sheaf of metrics on X. The $\mathcal{F}$-compression rate of X is at most α if and only if for all $\beta > \alpha$, there exist $K > 0$, and for every $n \in N$, a symmetric, finitely supported probability measure μ_n on $\Delta_n(X)$, with the following property: for every $\sigma \in \mathcal{F}(X)$ satisfying*

$$\sigma(x, y) \leq d(x, y), \quad \forall (x, y) \in \Delta_1(X),$$

one has

$$\mathbf{E}_{\mu_n}(\sigma^p) \leq (Kn^\beta)^p.$$

Assume that $X = G$ is a finitely generated group equipped with a word metric, denoted by $|g| = |g|_S$, associated to a finite symmetric generating subset S. Theorem 21 becomes

Theorem 22. *Let $\mathcal{F}$ be a p-admissible sheaf of length functions on G. The $\mathcal{F}$-compression rate of G is at most α if and only if fro all $\beta > \alpha$, there exist $K > 0$, and for every $n \in \mathbf{N}$, a symmetric, finitely supported probability measure μ_n on $\{g, |g| \geq n\}$ with the following property: for every $L \in \mathcal{F}(G)$ satisfying*

$$L(g) \leq |g|, \quad \forall g \in G$$

one has

$$\mathbf{E}_{\mu_n}(L^p) \leq (K|g|^\beta)^p.$$

5 Proof of Theorem 14

The "if" part is obvious, as the condition (4.3) roughly says that the sequence μ_n selects pairs (x, y) of arbitrarily distant points in X for which $\sigma(x, y) \leq K$.

Let X be a metric space, and let $\mathcal{F}$ be a p-admissible sheaf of metrics on X. We assume that $\mathcal{F}(X)$ contains no coarse metric. Let $\rho_+ : \mathbf{R}_+ \to \mathbf{R}_+$ be an increasing unbounded function.

Our first step is the next lemma. Let $\mathcal{A}$ be the set of finite subsets of X containing a distinguished point o.

Lemma 23. *Assume that there exists a function $T : \mathbf{R}_+ \to \mathbf{R}_+$ with the following property: for all $U \in \mathcal{A}$, and all $K > 0$, there exists $\sigma \in \mathcal{F}(U)$ such that*

$$\sigma(x, y) \leq \rho_+(d(x, y)), \quad \forall x, y \in U$$

and

$$\sigma(x, y) \geq K$$

for all $(x, y) \in \Delta_{T(K)}(U)$. Then there exists a coarse element in $\mathcal{F}(X)$.

Proof: As $\mathcal{F}$ is p-admissible, up to taking a pointwise limit with respect to an ultrafilter on $\mathcal{A}$, we can assume that for all $K > 0$, there exists $\sigma \in \mathcal{F}(X)$ such that

$$\sigma(x, y) \leq \rho_+(d(x, y)), \quad \forall x, y \in X$$

and

$$\sigma(x, y) \geq K$$

for all $(x, y) \in \Delta_{T(K)}(X)$.

Let K_n be an increasing sequence satisfying

$$\sum_{n=1}^{\infty} \frac{1}{K_n^p} \leq 1,$$

and let T be as in the lemma. Now take σ_n as above, and define

$$\sigma(x,y) = \left(\sum_{n \geq 1} (\sigma_n(x,y)/K_n)^p \right)^{1/p}$$

for all $x, y \in X$. The fact that $\sigma \in \mathcal{F}(X)$ follows from the fact that $\mathcal{F}$ is p-admissible. Moreover, we have

$$\rho_-(d(x,y)) \leq \sigma(x,y) \leq \rho_+(d(x,y)),$$

for all $x, y \in X$, where

$$\rho_-(t) = (\text{card}\{n, T(K_n) \leq t\})^{1/p}.$$

Clearly, $\rho_-(t) \to \infty$ when $t \to \infty$, so we are done. ■

The second step of the proof is an adaptation of the proof of [M1, Proposition 15.5.2]. Suppose that there is no coarse element in $\mathcal{F}(X)$.

By the lemma, there exists a number K_0 with the following property. For all T_0, there exist $U \in A$ such that for all $\sigma \in \mathcal{F}(U)$ satisfying

$$\sigma(x,y) \leq \rho_+(d(x,y)), \tag{5.1}$$

there are two points x, y in U such that $T_0 \leq d(x,y)$ and

$$\sigma(x,y) < K_0.$$

Note that we can take T_0 such that $\rho_+(T_0) \geq K_0$.

Consider the two following convex subsets of $\ell^2(U^2)$. Let C_1 be the set of functions $\phi : U^2 \to \mathbf{R}_+$ satisfying

$$\phi(x,y) \leq \rho_+(d(x,y))^p, \quad \forall (x,y) \in U^2,$$

and

$$\phi(x,y) \geq K_0^p, \quad \forall (x,y) \in \Delta_{T_0}(U). \tag{5.2}$$

Let C_2 be the set of σ^p, where $\sigma \in \mathcal{F}(U)$ satisfies

$$\sigma(x,y) \leq \rho_+(d(x,y)).$$

The previous reformulation of the lemma implies that these two convex subsets are disjoint. We have even better. For every subset V of a vector space E, we denote

$$\mathbf{R}_+V = \{tv,\ t \in \mathbf{R}_+, v \in V\}.$$

Lemma 24. *The cones $\mathbf{R}_+C_1$ and $\mathbf{R}_+C_2$ intersect only at $\{0\}$. Moreover, $\{0\}$ is extremal in both cones.*

Proof: The fact that $\{0\}$ is extremal is just a consequence of the fact that the two cones only contain non-negative functions. Let $t > 0$ and let $\phi \in C_2 \setminus \{0\}$. We want to prove that $t\phi$ does not belong to C_1. By the first condition of p-admissibility, there exists $\sigma_t \in \mathcal{F}(U)$ such that $t\phi = \sigma_t^p$. Moreover, if $t\phi$ also satisfies (5.2), then σ_t satisfies (5.1), so $t\phi \in C_2$, and hence it cannot be in C_1. ■

Hence by Hahn-Banach's theorem, there exists a vector $u \in \ell^2(U^2)$ such that

$$\langle \phi, u \rangle > 0,$$

for all non-zero $\phi \in C_1$ and

$$\langle \phi, u \rangle < 0,$$

for all non-zero $\phi \in C_2$.

Let $u_+ = \max\{u, 0\}$ and $u_- = \max\{-u, 0\}$. One sees that u_+ is non-zero in restriction to $\Delta_{T_0}(U)$ by applying the first inequality to the function

$$\phi(x,y) = \begin{cases} 0, & \text{if } d(x,y) < T_0; \\ K_0^p, & \text{otherwise.} \end{cases}$$

Now, apply the first inequality to the function

$$\phi(x,y) = \begin{cases} K_0^p, & \text{if } u(x,y) > 0, \text{ and } d(x,y) \geq T_0; \\ 0, & \text{if } u(x,y) > 0, \text{ and } d(x,y) < T_0; \\ \rho_+(d(x,y))^p, & \text{otherwise.} \end{cases}$$

Note that this is possible as T_0 has been chosen such that $\rho_+(T_0) \geq K_0$. We get

$$\sum_{x,y} \rho_+(d(x,y))^p u_-(x,y) \leq K_0^p \sum_{(x,y)\in\Delta_{T_0}(U)} u_+(x,y).$$

On the other hand, if $\phi \in C_2$, i.e. $\phi(x,y) = \sigma(x,y)^p$, then using the second inequality,

$$\begin{aligned} \sum_{(x,y)\in\Delta_{T_0}(B(o,T_0))} \sigma(x,y)^p u_+(x,y) &\leq \sum_{(x,y)\in B(o,T_0)^2} \sigma(x,y)^p u_+(x,y) \\ &\leq \sum_{x,y} \sigma(x,y)^p u_-(x,y) \\ &\leq \sum_{x,y} \rho_+(d(x,y))^p u_-(x,y). \end{aligned}$$

Now, combining these two inequalities, we get

$$\sum_{x,y} \sigma(x,y)^p \frac{u_+(x,y)}{\sum_{(x,y)\in\Delta_{T_0}(U)} u_+(x,y)} \leq K_0^p.$$

So the theorem follows by taking the probability measure on $\Delta_{T_0}(U)$ defined by

$$\mu(x,y) = \frac{u_+(x,y)}{\sum_{(x,y)\in\Delta_{T_0}(U)} u_+(x,y)}. \blacksquare$$

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