A NOTION OF GEOMETRIC COMPLEXITY AND ITS APPLICATION TO TOPOLOGICAL RIGIDITY

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Abstract. We introduce a geometric invariant, called finite decomposition complexity (FDC), to study topological rigidity of manifolds. We prove for instance that if the fundamental group of a compact aspherical manifold M has FDC, and if N is homotopy equivalent to M, then $M \times \mathbb{R}^n$ is homeomorphic to $N \times \mathbb{R}^n$, for n large enough. This statement is known as the stable Borel conjecture. On the other hand, we show that the class of FDC groups includes all countable subgroups of $\mathrm{GL}(n, K)$, for any field K.

1. Introduction

We introduce the geometric concept of finite decomposition complexity to study questions concerning the topological rigidity of manifolds. Roughly speaking, a metric space has finite decomposition complexity when there is an algorithm to decompose the space into simpler, more manageable pieces in an asymptotic way. The precise definition, presented in Section 2, is inspired by the property of finite asymptotic dimension of Gromov [G], of which it is a far reaching generalization [GTY].

While the property of finite decomposition complexity is flexible – the class of countable groups having finite decomposition complexity includes all linear groups (over a field with arbitrary characteristic), all hyperbolic groups and all elementary amenable groups and is closed under various operations [GTY] – it is a powerful tool for studying topological rigidity – we shall see, for example, that if the fundamental group of a closed aspherical manifold has finite decomposition complexity then its universal cover is boundedly rigid, and the manifold itself is stably rigid.

Topological rigidity. A closed manifold M is *rigid* if every homotopy equivalence between M and another closed manifold is homotopic to a homeomorphism. The Borel conjecture asserts the rigidity of closed aspherical manifolds. Many important results on the Borel conjecture have been obtained by Farrell and Jones [FJ1, FJ2, FJ3, FJ4], and more recently Bartels and Lück [BL]. These results are proved by studying dynamical properties of actions of the fundamental group of M.

Our approach to rigidity questions is different – we shall focus not on the dynamical properties but rather on the large scale geometry of the fundamental group. As a natural

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byproduct, we prove the *bounded Borel conjecture*, a 'large-scale geometric' version of the Borel conjecture. Our principal result in this direction is the following theorem.

Theorem. *The bounded Borel isomorphism conjecture and the bounded Farrell-Jones L-theory isomorphism conjecture hold for a metric space with bounded geometry and finite decomposition complexity.*

Our first application concerns *bounded rigidity* of universal covers of closed aspherical manifolds.

Bounded Rigidity Theorem. *Let M be a closed aspherical manifold of dimension at least five whose fundamental group has finite decomposition complexity (as a metric space with a word metric). For every closed manifold N and homotopy equivalence $M \to N$ the corresponding bounded homotopy equivalence of universal covers is boundedly homotopic to a bounded homeomorphism.*

The universal covers of M an N as in the statement are, in particular, homeomorphic. The conclusion is actually much stronger – being *boundedly* homeomorphic means that the homeomorphism is at the same time a coarse equivalence. We defer discussion of the relevant notions concerning the *bounded category* to Section 4. See, in particular Theorem 4.3.2, of which the previous result is a special case.

Davis has given examples of aspherical manifolds whose universal covers are not homeomorphic to the Euclidean space [D]. These examples satisfy the hypothesis of the previous theorem.

A closed manifold M is *stably rigid* if there exists an n such that for every closed manifold N and every homotopy equivalence $M \to N$, the product with the identity $M \times \mathbb{R}^n \to N \times \mathbb{R}^n$ is homotopic to a homeomorphism. The *stable Borel conjecture* asserts that closed aspherical manifolds are stably rigid. The first result on the stable Borel conjecture is due to Farrell and Hsiang [FH] who proved that non positively curved Riemannian manifolds are stably rigid. Our second application is the following theorem (see Corollary 4.3.3).

Stable Rigidity Theorem. *A closed aspherical manifold whose fundamental group has finite decomposition complexity is stably rigid.*

Groups with finite decomposition complexity. We consider countable groups equipped with a proper left-invariant metric. Recall that every countable group admits such a metric, and that any two such metrics are coarsely equivalent. As finite decomposition complexity is a coarse invariant, the statement that a countable group has finite decomposition complexity is independent of the choice of metric. Our next result summarizes the main examples of groups having finite decomposition complexity, and thus to which our rigidity results apply. We shall focus exclusively on the case of linear groups in this article – proofs of FDC for the remaining classes of groups in the theorem are found in [GTY]. For the statement, recall that a Lie group is *almost connected* if it has finitely many connected components.

Theorem. *The collection of countable groups having finite decomposition complexity contains all countable linear groups (over a field of arbitrary characteristic), all countable subgroups of an almost connected Lie group, all hyperbolic groups and all elementary amenable groups.*

The geometry of a *discrete* subgroup of, for example, a connected semisimple Lie group such as $\mathrm{SL}(n,\mathbb{R})$ reflects the geometry of the ambient Lie group. In this case, the theorem follows from the well-known result that such groups have finite asymptotic dimension. The difficulty in the theorem concerns the case of *non-discrete* or even *dense* subgroups whose geometry exhibits little apparent relationship to the geometry of the ambient group. An interesting example to which our theorem applies is $\mathrm{SL}(n,\mathbb{Z}[\pi])$, which has infinite asymptotic dimension (here, $\pi = 3.14\ldots$). Nevertheless, in the case of positive characteristic we have the following result.

Theorem. *A finitely generated linear group over a field of positive characteristic has finite asymptotic dimension.*

Combined, we obtain rigidity results for all countable linear groups, greatly extending an earlier theorem of Ji [J] proving the stable Borel conjecture for a special class of linear groups with finite asymptotic dimension – namely, subgroups of $\mathrm{GL}(n,K)$ for a global field K, for example when $K=\mathbb{Q}$.

We refer to [GTY] for further results about the class of groups having FDC. Let us only mention here that it includes all hyperbolic groups and all elementary amenable groups and is closed under under taking subgroups, extensions, free amalgamated products, HNN extensions, and direct unions.

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2. Decomposition complexity

Our proofs of the isomorphism conjectures will be based on Mayer-Vietoris arguments – we shall apply a *large-scale* version of an appropriate Mayer-Vietoris sequence to prove that an assembly map is an isomorphism. To carry out this idea, we shall decompose a given metric space as a union of two subspaces, which are *simpler* than the original. Roughly, *simpler* is interpreted to mean that each subspace is itself a union of spaces at a pairwise distance large enough that proving the isomorphism for the subspace amounts to proving the isomorphism for these constituent pieces 'uniformly'. Further, this basic decomposition step shall be iterated a finite number of times, until we reach a bounded family. This is the idea behind finite decomposition complexity.

2.1. **Definition of FDC.** We shall need to formulate our notion of finite decomposition complexity not for a single metric space, but rather for a *metric family*, a (countable) family of metric spaces which we shall denote by $\mathcal{X} = \{X\}$; throughout we view a single metric space as a family containing a single element.

In order to streamline our definitions we introduce some terminology and notation for manipulating decompositions of metric spaces and metric families. A collection of subspaces $\{ Z_i \}$ of a metric space Z is *r-disjoint* if for all $i \neq j$ we have $d(Z_i, Z_j) \geq r$. To express the idea that Z is the union of subspaces Z_i, and that the collection of these subspaces is r-disjoint we write

$$Z = \bigsqcup_{r-disjoint} Z_i.$$

A family of of metric spaces $\{ Z_i \}$ is *bounded* if there is a uniform bound on the diameter of the individual Z_i:

$$\sup \operatorname{diam}(Z_i) < \infty.$$

2.1.1. **Definition.** A metric family $\mathcal{X}$ is *r-decomposable* over a metric family $\mathcal{Y}$ if every $X \in \mathcal{X}$ admits an *r-decomposition*

$$X = X_0 \cup X_1, \quad X_i = \bigsqcup_{r-disjoint} X_{ij},$$

where each $X_{ij} \in \mathcal{Y}$. We introduce the notation $\mathcal{X} \xrightarrow{r} \mathcal{Y}$ to indicate that $\mathcal{X}$ is r-decomposable over $\mathcal{Y}$.

2.1.2. **Definition.** Let $\mathfrak{A}$ be a collection[1] of metric families. A metric family $\mathcal{X}$ is *decomposable* over $\mathfrak{A}$ if, for every $r > 0$, there exists a metric family $\mathcal{Y} \in \mathfrak{A}$ and an r-decomposition of $\mathcal{X}$ over $\mathcal{Y}$. The collection $\mathfrak{A}$ is *stable under decomposition* if every metric family which decomposes over $\mathfrak{A}$ actually belongs to $\mathfrak{A}$.

2.1.3. **Definition.** The collection $\mathfrak{D}$ of metric families with finite decomposition complexity is the minimal collection of metric families containing the bounded metric families and stable under decomposition. We abbreviate membership in $\mathfrak{D}$ by saying that a metric family in $\mathfrak{D}$ has FDC.

2.2. **Equivalent formulations of FDC.** We shall present two equivalent descriptions of the collection of families having finite decomposition complexity. We shall be deliberately terse, referring the reader to the companion paper [GTY] for a fuller discussion. The first description, organized around the *metric decomposition game*, provides valuable intuition into FDC. The game has two players, a challenger and a defender, and begins with a metric family. The objective of the defender is to successfully decompose the spaces comprising the family, whereas the challenger attempts to obstruct the decomposition.

Suppose $\mathcal{X} = \mathcal{Y}_0$ is the starting family. The game begins with the challenger requesting, for some natural number r_1, an r_1-decomposition of $\mathcal{Y}_0$. The defender responds by exhibiting a r_1-decomposition of $\mathcal{Y}_0$ over a new metric family $\mathcal{Y}_1$. Subsequent turns are analogous: the challenger asserts an r_{i+1} and the defender responds by exhibiting an r_{i+1}-decomposition of $\mathcal{Y}_i$ over a metric family $\mathcal{Y}_{i+1}$.

[1]While we generally prefer the term 'collection' to 'class', we do not mean to imply that a collection of metric families is a *set* of metric families. We shall not belabor the associated set-theoretic complications.

The defender has a *winning strategy* if, roughly speaking, they can produce decompositions ending in a bounded family *no matter what choices the attacker makes.* In this case, we say that the metric family $\mathcal{X}$ *admits a decomposition strategy*.

The second description, which serves mainly to establish notation we require later, is based on converting the notion of decomposability into a heirarchy. We define, for each ordinal α, a collection of metric families according to the following prescriptions:

(1) Let $\mathfrak{D}_0$ be the collection of bounded families:
$$\mathfrak{D}_0 = \{\, \mathcal{X} \colon \mathcal{X} \text{ is bounded} \,\}.$$
(2) If α is an ordinal greater than 0, let $\mathfrak{D}_\alpha$ be the collection of metric families decomposable over $\cup_{\beta<\alpha}\mathfrak{D}_\beta$:
$$\mathfrak{D}_\alpha = \{\, \mathcal{X} \colon \forall\, r\ \exists\, \beta < \alpha\ \exists\, \mathcal{Y} \in \mathfrak{D}_\beta \text{ such that } \mathcal{X} \xrightarrow{r} \mathcal{Y} \,\}.$$

For future use, we introduce the notation $\mathfrak{D}_{\text{fin}}$ (respectively $\mathfrak{D}_{\alpha+\text{fin}}$) for the union of the $\mathfrak{D}_n$ (respectively $\mathfrak{D}_{\alpha+n}$), over $n \in \mathbb{N}$. For the proof of the following theorem characterizing those metric families having FDC we refer to [GTY, Theorems 2.2.2 and 2.2.3].

2.2.1. **Theorem.** *The following statements concerning a metric family $\mathcal{X}$ are equivalent:*

(1) *$\mathcal{X}$ has finite decomposition complexity;*
(2) *$\mathcal{X}$ admits a decomposition strategy;*
(3) *there exists a countable ordinal α such that $\mathcal{X} \in \mathfrak{D}_\alpha$.*

2.2.2. **Example.** One checks (by induction) that $\mathbb{Z}^n \in \mathfrak{D}_n$, so that $\oplus\mathbb{Z} \in \mathfrak{D}_\omega$. Since $\mathbb{Z} \wr \mathbb{Z}$ is an extension of $\oplus\mathbb{Z}$ by $\mathbb{Z}$, one checks (by fibering) that $\mathbb{Z} \wr \mathbb{Z} \in \mathfrak{D}_{\omega+1}$ (compare [GTY, Remark 3.1.6]). Let now $G = \oplus G_n$, where $G_n = (\ldots((\mathbb{Z} \wr \mathbb{Z}) \wr \mathbb{Z})\ldots) \wr \mathbb{Z}$, the wreath product of n copies of $\mathbb{Z}$. Then $G \in \mathfrak{D}_{\omega^2}$. It remains open whether $G \in \mathfrak{D}_\alpha$ for some $\alpha < \omega^2$.

3. Linear groups have FDC

The wreath product $\mathbb{Z}\wr\mathbb{Z}$ can be realized as a subgroup of $G = \mathrm{SL}(2, \mathbb{Z}[X, X^{-1}])$; concretely as the subgroup comprised of all matrices of the form
$$\begin{pmatrix} X^n & p(X^2) \\ 0 & X^{-n} \end{pmatrix},$$
where $n \in \mathbb{Z}$ and p is a Laurent polynomial in the variable X^2 with $\mathbb{Z}$ coefficients. The set of matrices of this form but with $n = 0$, is an infinite rank free abelian subgroup. Thus, neither $\mathbb{Z} \wr \mathbb{Z}$ nor G has finite asymptotic dimension, so that neither belongs to $\mathfrak{D}_{\text{fin}}$ [GTY, Theorem 4.1]. On the other hand, a straightforward application of fibering shows that $\mathbb{Z}/p\mathbb{Z} \wr \mathbb{Z}$ *has* finite asymptotic dimension [BD, DS] – it remains possible that $\mathrm{SL}(2, \mathbb{Z}/p\mathbb{Z}[X, X^{-1}])$ does as well. These considerations show that the following theorem is optimal.

3.0.1. **Theorem.** *If a countable group admits a faithful, finite dimensional representation* (*as matrices over a field of arbitrary characteristic*)*, then it has finite decomposition complexity. Precisely, let G be a finitely generated subgroup of $GL(n, K)$, where K is a field. If K*

has characteristic zero then $G \in \mathfrak{D}_{\omega+fin}$; if K has positive characteristic then G has finite asymptotic dimension.

3.1. **Preliminaries on fields.** The proof of Theorem 3.0.1 relies on a strengthening of the notion of discrete embeddability introduced earlier by Guentner, Higson and Weinberger [GHW]. A *norm*[2] on a field K is a map $\gamma : K \to [0,\infty)$ satisfying, for all x, $y \in K$

(1) $\gamma(x) = 0 \Leftrightarrow x = 0$
(2) $\gamma(xy) = \gamma(x)\gamma(y)$
(3) $\gamma(x+y) \leq \gamma(x) + \gamma(y)$

A norm obtained as the restriction of the usual absolute value on $\mathbb{C}$ via a field embedding $K \to \mathbb{C}$ is *archimedean*. A norm satisfying the stronger *ultra-metric inequality*

(4) $\gamma(x+y) \leq \max\{\gamma(x), \gamma(y)\}$

in place of the triangle inequality (3) is *non-archimedean*. If in addition the range of γ on $K^\times$ is a discrete subgroup of the multiplicative group $(0,\infty)$ the norm is *discrete*.

3.1.1. **Definition.** A field K is strongly discretely embeddable (for short SDE) if for every finitely generated subring A of K there exists a finite set N_A of discrete norms on K, and countable set M_A of archimedean norms on K with the following property: for every real number k there exists a finite subset $F_A(k)$ of M_A such that for every $s > 0$ the set

$$\mathcal{B}_A(k,s) = \{\, a \in A \colon \forall \gamma \in N_A \;\; \gamma(a) \leq e^k \text{ and } \forall \gamma \in F_A(k) \;\; \gamma(a) \leq s \,\}$$

is finite.

3.1.2. **Remark.** A field of positive characteristic admits no archimedean norms. In particular, a field of nonzero characteristic is strongly discretely embeddable if and only if for every finitely generated subring A there exists a finite set N_A of (discrete) norms such that for every $k \in \mathbb{N}$ the set

$$\mathcal{B}_A(k) = \{\, a \in A \colon \forall \gamma \in N_A \; \gamma(a) \leq e^k \,\}$$

is finite.

3.1.3. **Proposition.** *A finitely generated field is strongly discretely embeddable.*

Strong discrete embeddability is stronger than, and formally similar to discrete embeddability introduced in [GHW]. Exploiting the similarity, we shall prove the proposition by adapting the proof of [GHW, Theorem 2.2]. (It is also possible to give an alternate proof based on [AS, Proposition 1.2], and relying on Noether's normalization theorem). The proof comprises three lemmas.

3.1.4. **Lemma.** *Finite fields and the field of rational numbers are strongly discretely embeddable.*

[2]Guentner-Higson-Weinberger use the term valuation.

Proof. The assertion is obvious for finite fields. A finitely generated subring of $A \subset \mathbb{Q}$ has the form $A = \mathbb{Z}[1/n]$, for some positive integer n. Let N_A contain the (discrete) p-adic norms associated to the (finitely many) prime divisors of n, and let M_A consist solely of the archimedean norm coming from the inclusion $\mathbb{Q} \subset \mathbb{C}$. We leave to the reader to verify that these choices satisfy Definition 3.1.1. $\square$

3.1.5. **Lemma.** *Strong discrete embeddability is stable under the formation of simple transcendental extensions.*

Proof. Refining the proof of the corresponding result [GHW, Lemma 2.2], we shall show that the field of rational functions over a (countable) SDE field is itself SDE. Let K be an SDE field and let B be a finitely generated subring of $K(X)$. There exist monic prime polynomials $Q_1, \dots, Q_m \in K[X]$ and a finitely generated subring A of K such that $B \subset A[X][Q_1^{-1}, \dots, Q_m^{-1}]$. According to Definition 3.1.1, applied to the subring A of K, we obtain (finitely many) discrete norms N_A, and (countably many) archimedean norms M_A.

Let N_B be the following (finite) set of discrete norms on $K(X)$:

(1) the elements of N_A extended to $K(X)$;
(2) the norm $\gamma_\infty(P/Q) = e^{deg(P)-deg(Q)}$;
(3) the norms $\gamma_{Q_i}(PQ_i^l) = e^{-l}$ where $\gcd(Q_i, P) = 1$ and $l \in \mathbb{Z}$ (there are m norms of this type, one for each $i = 1, \dots, m$).

Each of the archimedean norms $\gamma \in M_A$ arises from an embedding of fields $\phi_\gamma : K \to \mathbb{C}$. Let $t_0, t_1, \dots$ be a countable family of distinct transcendentals in $\mathbb{C}$ that are *not* in the (countable!) subfield of $\mathbb{C}$ generated by the images of these embeddings. Each embedding ϕ_γ extends to an embedding $K(X) \to \mathbb{C}$ by sending X to t_i; we denote the corresponding norm on $K(X)$ by γ_i. Let

$$M_B = \{\, \gamma_i \colon \gamma \in M_A \text{ and } i = 0, 1, \dots \,\},$$

a countable set of archimedean norms on $K(X)$.

We shall show that N_B and M_B satisfy the condition in Definition 3.1.1. For this, let $k > 0$ be given. An element of $\mathcal{B}_B(k)$ necessarily has the form

$$\frac{P}{Q} = \frac{P}{Q_1^{n_1} \dots Q_m^{n_m}}, \tag{3.1}$$

where $n_1, \dots, n_m$ are $\le k$, so that also $\deg P \le k' = k\,(1 + \sum \deg Q_i)$ – here we are using the norms in N_B of types (2) ad (3) above. In particular, the set of possible denominators Q is finite; denote it by $\mathcal{Q}_k$. Set

$$k'' = k + \log \max\{\, \gamma(Q) \colon Q \in \mathcal{Q}_k, \gamma \in N_B \,\}$$

(actually, taking the maximum over $\gamma \in N_B$ of type (1) would suffice). Summarizing, an element of $\mathcal{B}_B(k)$ has the form (3.1) in which Q belongs to the finite set $\mathcal{Q}_k$, the degree of P is at most k' and all coefficients of P belong to $\mathcal{B}_A(k'')$ – the last assertion follows from the formula for the extension of an element of N_A to an element of N_B of type (1), see the proof of [GHW, Lemma 2.2].

Define a finite set of archimedean norms on $K(X)$ by

$$F_B(k) = \{ \gamma_i \in M_B \colon \gamma \in F_A(k'') \text{ and } i = 0, \ldots, k' \}$$

Let now $s > 0$; it remains to show that $\mathcal{B}_B(k,s)$ is finite. We claim that an element of $\mathcal{B}_B(k,s)$ satisfies, in addition to the conditions outlined above for membership in $\mathcal{B}_B(k)$, the following condition: there exists an s'' such that for every norm $\gamma \in F_A(k'')$ the value of γ on each coefficient of P is at most s''; in other words, form some s'' the coefficients of P belong to $\mathcal{B}_A(k'', s'')$. If indeed this is the case, the proof is complete – $\mathcal{B}_A(k'', s'')$ is a finite set, so only finitely many polynomials P can appear in (3.1) which, combined with our remarks above concludes the proof.

It remains to prove the existence of s''. Let

$$s' = s \cdot \max\{ \gamma(Q) \colon Q \in \mathcal{Q}_k,\ \gamma \in F_B(k) \}$$

so that for an element of $\mathcal{B}_B(k,s)$ written in the form (3.1) we have $\gamma_i(P) \leq s'$ for every $\gamma \in F_A(k'')$ and $i = 0, \ldots, k'$. Now, the linear transformation

$$P \longmapsto (P(t_0), \ldots, P(t_{k'})), \qquad \mathbb{C}[X]_{k'} \to \bigoplus_0^{k'} \mathbb{C}$$

is invertible; here $\mathbb{C}[X]_{k'}$ denotes the vector space of polynomials of degree at most k'. The condition that $\gamma_i(P) \leq s'$ for every $i = 0, \ldots, k'$ and $\gamma \in F_A(k'')$ means that the polynomials $\phi_\gamma(P)$ belong to the subset of the domain mapping into the compact subset of the range defined by the requirement that the absolute value of each entry is at most s'. This is a compact set so that there is an s'' such that the absolute value of the coeffecients of the polynomials $\phi_\gamma(P)$ are bounded by s''; in other words, they belong to $\mathcal{B}_A(k'', s'')$ as required. □

3.1.6. **Lemma.** *Strong discrete embeddability is stable under the formation of finite extensions.*

Proof. We shall show that a finite extension of an SDE field is SDE. The proof is essentially the proof of [GHW, Lemma 2.3], but with careful bookkeeping.

Let L be a finite extension of an SDE field K. As a subfield of an SDE field is itself SDE we may, enlarging L as necessary, assume that L is a finite normal extension of K. Let B be a finitely generated subring of L. Fix a basis of the K-vector space L and let A be a finitely generated subring of K containing the matrix entries of each element of B, viewed as a K-linear transformation of L.

According to Definition 3.1.1 applied to the subring A of K, we obtain (finitely many) discrete norms N_A and (countably many) archimedean norms M_A. Now, every discrete norm on K admits at least one extension to a discrete norm on L; a similar statement applies to archimedean norms. See [L, Chapter 12]. Moreover, the finite group $\mathrm{Aut}_K(L)$ of K-automorphisms of L acts on the set of extensions of each individual norm on K.

Let N_B be a (finite) set of discrete norms on L comprising exactly one $\mathrm{Aut}_K(L)$-orbit of extensions of each norm in N_A; let M_B be a (countable) set of archimedean norms on L

defined similarly with respect to M_A. Finally, for each k let

$$F_B(k) = \{\, \gamma \in M_B \colon \gamma \text{ extends a norm in } F_A(k') \,\};$$

here $k' = \max\{\, |f(x_0, \ldots, x_n)| \,\}$, where n is the degree of the extension and the maximum is over all elementary symmetric functions f and all tuples of real numbers $x_0, \ldots, x_n$ each of which has absolute value at most k. Each $F_B(k)$ is a finite set of archimedean norms invariant under the action of $\mathrm{Aut}_K(L)$.

Let k and $s > 0$ be given. We must show that $\mathcal{B}_B(k, s)$ is finite. But indeed, the argument in [GHW] shows that the coefficients of the characteristic polynomial of each element of $\mathcal{B}_B(k, s)$, again viewed as a K-linear transformation of L, belong to the finite set $\mathcal{B}_A(k', s')$ where s' is defined in terms of s as k' was in terms of k. Thus, every element of $\mathcal{B}_B(k, s)$ is the root of one of finitely many polynomials and $\mathcal{B}_B(k, s)$ is itself finite. □

3.2. **The general linear group.** Let γ be a norm on a field K. Following Guentner, Higson and Weinberger define a (pseudo)-length function ℓ_γ on $\mathrm{GL}(n, K)$ as follows: if γ is non-archimedean

$$\ell_\gamma(g) = \log \max_{ij} \{\, \gamma(g_{ij}), \gamma(g^{ij}) \,\}, \tag{3.2}$$

where g_{ij} and g^{ij} are the matrix coefficients of g and g^{-1}, respectively; if γ is archimedian, arising from an embedding $K \hookrightarrow \mathbb{C}$ then

$$\ell_\gamma(g) = \log \max\{\, \|g\|, \|g^{-1}\| \,\}, \tag{3.3}$$

where $\|g\|$ is the norm of g viewed as an element of $\mathrm{GL}(n, \mathbb{C})$, and similarly for g^{-1}. The following proposition is central to our discussion of linear groups.

3.2.1. **Proposition.** *Let γ be an archimedean or a discrete norm on a field K. The group $GL(n, K)$, equipped with the (left-invariant pseudo-)metric induced by ℓ_γ, is in $\mathfrak{D}_{fin}$.*

Sketch of Proof. In the archimedean case $\mathrm{GL}(n, K) \subset \mathrm{GL}(n, \mathbb{C})$ as a metric subspace so that the result follows from the corresponding result for $\mathrm{GL}(n, \mathbb{C})$. For $\mathrm{GL}(n, \mathbb{C})$ standard arguments apply, once we see that the length function (3.3) is continuous and proper – $\mathrm{GL}(n, \mathbb{C})$ is coarsely equivalent to the subgroup of all upper triangular matrices and a fibering argument based on [GTY, Theorem 3.1.4] show thats this solvable group has FDC.

The discrete case is more subtle, primarily because we do not assume that K is locally compact. In this case the result is due to Matsnev [Ma]. We shall present a simplified version of his proof elsewhere [GTY]. □

3.3. **Finite decomposition complexity.** The proof of Theorem 3.0.1 is easily reduced to the special case $G = \mathrm{GL}(n, A)$, where A a finitely generated domain with fraction field K. Indeed, suppose K is a field and G is a finitely generated subgroup of $\mathrm{GL}(n, K)$. The subring of K generated by the matrix entries of a finite generating set for G is a finitely generated domain A, we have $G \subset \mathrm{GL}(n, A)$, and may replace K by the (finitely generated) fraction field of A. The strategy behind our proof is to embed $\mathrm{GL}(n, A)$ into the product of several copies of $\mathrm{GL}(n, K)$ equipped with metrics associated to various norms. The proof rests on a permanence property summarized in the following lemma.

3.3.1. **Lemma.** *Let G be a countable discrete group. Suppose there exists a (pseudo-)length function ℓ' on G with the following properties:*

(1) *G is in $\mathfrak{D}_{fin}$ with respect to the associated (pseudo-)metric d'*
(2) *$\forall\, r > 0\ \exists\, \ell_r$, a (pseudo-)length function on G, for which*
    (i) *G is in $\mathfrak{D}_{fin}$ with respect to the associated (pseudo-)metric d_r,*
    (ii) *ℓ_r is proper when restricted to $B_{\ell'}(r)$.*

Then G has finite decomposition complexity, and indeed $G \in \mathfrak{D}_{\omega+fin}$.

Condition (ii) in the lemma means precisely that $B_{\ell_r}(s) \cap B_{\ell'}(r)$ is finite for every $s > 0$.

Proof. Fix a proper length function ℓ on G, with associated metric d. By [GTY, Proposition 3.2.3], applied to the action of G on the metric space (G, d'), it suffices to show that for every $r > 0$ the ball $B_{\ell'}(r)$ is in $\mathfrak{D}_{\mathrm{fin}}$ when equipped with the metric d.

Let $r > 0$. Obtain ℓ_{2r} as in the statement. The ball $B_{\ell'}(r)$ is in $\mathfrak{D}_{\mathrm{fin}}$ with respect to the metric d_{2r}. Thus, it remains to show that the metrics d and d_{2r} on $B_{\ell'}(r)$ are coarsely equivalent.

Since ℓ-balls in G are finite, we easily see that for every s there exists s' such that if $d(g, h) \leq s$ then $d_{2r}(g, h) \leq s'$; this holds for every g and $h \in G$. Conversely, for every s the set $B_{\ell'}(2r) \cap B_{\ell_{2r}}(s)$ is finite by assumption, and we obtain s' such that for every g in this set $\ell(g) \leq s'$. If now g and $h \in B_{\ell'}(r)$ are such that $d_{2r}(g, h) \leq s$ then $g^{-1}h \in B_{\ell'}(2r)$ and

$$d(g, h) = \ell(g^{-1}h) \leq s'.$$

□

Proof of Theorem 3.0.1. Let A be a finitely generated domain, K the fraction field of A and $G = \mathrm{GL}(n, A)$. (We have previously reduced the theorem to this case.) Obtain a finite family $N_A = \{\, \gamma_1, \dots, \gamma_q \,\}$ of discrete norms on K as in the definition of strong discrete embeddability. For each norm γ_i we have the corresponding length function ℓ_{γ_i} and metric on $\mathrm{GL}(n, K)$ defined as in (3.2). Define a length function on G by

$$\ell' = \ell_{\gamma_1} + \cdots + \ell_{\gamma_q}.$$

Thus, G is metrized so that the diagonal embedding

$$G \hookrightarrow \mathrm{GL}(n, K) \times \cdots \times \mathrm{GL}(n, K)$$

is an isometry when the i^{th} factor in the product is equipped with the metric associated to the norm γ_i and the product is given the sum metric. Equipped with this metric G is in $\mathfrak{D}_{\mathrm{fin}}$ by Proposition 3.2.1, and [GTY, Remark 3.1.5]. To apply the lemma, we shall study the balls $B_{\ell'}(r)$ of the identity in G.

Let $r = e^k$. Obtain a family of archimedean norms $F_A(k)$ as in the definition of strong discrete embeddability. For each we have the corresponding length function and metric on

$\mathrm{GL}(n, K)$ defined as in (3.3). Define a length function on G by

$$\ell_r = \sum_{\gamma \in F_A(k)} \ell_\gamma.$$

Thus, G is metrized so that the diagonal embedding

$$G \hookrightarrow \mathrm{GL}(n, K) \times \cdots \times \mathrm{GL}(n, K)$$

is an isometry when each factor in the product is equipped with the metric associated to the corresponding norm γ, and the product is given the sum metric. Equipped with this metric G is in $\mathfrak{D}_{\text{fin}}$ by Proposition 3.2.1, and [GTY, Remark 3.1.5]. To apply the lemma, we shall study the balls $B_{\ell'}(r)$ of the identity in G.

It remains only to show that for every $s > 0$ the set $B_{\ell_r}(s) \cap B_{\ell'}(r)$ is finite. Suppose g is in this set. From the definitions of the length functions it follows that the entries of g and g^{-1} satisfy inequalities

$$\gamma(g_{ij}) \leq r, \quad \gamma(g^{ij}) \leq r,$$

for $\gamma \in N_A$, and also the inequalities

$$\gamma(g_{ij}) \leq s, \quad \gamma(g^{ij}) \leq s,$$

for $\gamma \in F_A(k)$. But, these norms were chosen according to the definition of strong discrete embeddability, so that the subset of those elements of A satisfying these inequalities is finite. In particular, the number of matrices containing only these elements as their entries is finite and the proof of the general case is complete. Further, in the case of positive characteristic, there are no archimedean norms and the above inequalities show that $B_{\ell'}(r)$ is already finite for every r. In this case, we conclude that G belongs to $\mathfrak{D}_{\text{fin}}$ so that by [GTY, Theorem 4.1] it has finite asymptotic dimension. $\square$

4. Decomposition Complexity and Topological Rigidity

This section is organized into two parts. In the first part we shall state two essential results, Theorems 4.1.2 and 4.1.3, the proofs of which are defered to later sections. In the second part we shall discuss applications to topological rigidity. We shall begin by describing the bounded category, a natural framework in which to discuss bounded rigidity. We shall then state and prove our results concerning the bounded Borel and bounded Farrell-Jones L-theory isomorphism conjectures for spaces with finite decomposition complexity, Theorems 4.3.1 and 4.4.1, respectively. Finally, from these we deduce concrete applications to topological rigidity.

4.1. **Two main results.** Throughout, we shall work with a metric space Γ having *bounded geometry*: for every $r > 0$ there exists $N = N(r)$ such that every ball of radius r contains at most N elements. In several places the weaker hypothesis of *local finiteness* would suffice: every ball contains finitely many elements.

4.1.1. **Definition.** For $d \geq 0$ we define the Rips complex $P_d(\Gamma)$ to be the simplicial polyhedron with vertex set Γ, and in which a finite subset $\{\gamma_0, \dots, \gamma_n\} \subseteq \Gamma$ spans a simplex precisely when $d(\gamma_i, \gamma_j) \leq d$ for all $0 \leq i, j \leq n$.

If Γ has bounded geometry the Rips complex is finite dimensional, with dimension bounded by $N(d) - 1$; if Γ is merely locally finite the Rips complex $P_d(\Gamma)$ is a locally finite simplicial complex.

There are in general several ways to equip the Rips complex with a metric. The *simplicial metric* is the metric induced by the (pseudo) Riemannian metric whose restriction to each n-simplex is the Riemannian metric obtained by identifying the n-simplex with the standard simplex in the Euclidean space $\mathbb{R}^n$. By convention, the distance between points in different connected components of $P_d(\Gamma)$ is infinite. Equipped with the simplicial metric the Rips complex is a *geodesic space* in the sense that every two points (at finite distance) are joined by a geodesic path.

Our first essential result is a vanishing result for the Whitehead and algebraic K-theory groups. To state the result we introduce the following notation: for a locally compact metric space X and for each $\delta \geq 0$ and $i \geq 0$ the δ-controlled locally finite Whitehead group is denoted $Wh^{\delta}_{1-i}(X)$; the δ-controlled reduced locally finite algebraic K-theory group is denoted $\widetilde{K}^{\delta}_{-i}(X)$. Both groups are defined in [RY1].[3]

We then define, for each $i \geq 0$, the bounded locally finite Whitehead group, and bounded reduced locally finite algebraic K-theory group as follows:

$$\begin{aligned} Wh^{bdd}_{1-i}(P_d(\Gamma)) &= \lim_{\delta \to \infty} Wh^{\delta}_{1-i}(P_d(\Gamma)) \\ \tilde{K}^{bdd}_{-i}(P_d(\Gamma)) &= \lim_{\delta \to \infty} \tilde{K}^{\delta}_{-i}(P_d(\Gamma)). \end{aligned}$$

4.1.2. **Theorem.** *Let Γ be a bounded geometry metric space. If Γ has finite decomposition complexity then, for each $i \geq 0$,*

$$\begin{aligned} \lim_{d \to \infty} \widetilde{K}^{bdd}_{-i}(P_d(\Gamma)) &= 0 \\ \lim_{d \to \infty} Wh^{bdd}_{1-i}(P_d(\Gamma)) &= 0 \end{aligned}$$

Our second essential result asserts that an appropriate assembly map is an isomorphism. To state the result we introduce the following notation: $\mathbb{L}(e)$ denotes the simply connected surgery spectrum with $\pi_n(\mathbb{L}(e)) = L_n(\mathbb{Z}\{e\})$; $L^{bdd}_n(X)$ denotes the bounded, locally finite and free L-theory of the locally compact metric space X. Recall that $L^{bdd}_n(X)$ is defined using locally finite, free geometric modules and that a geometric module is locally finite if its support is locally finite. More precisely, for a locally compact metric space X and for each $\delta \geq 0$ and $n \geq 0$ the δ-controlled locally finite and free L-group in degree n is denoted

[3] The group we denote $\widetilde{K}^{\delta}_0(X)$ is the group $\widetilde{K}_0(X, p_X, 0, \delta)$ defined on page 14 of [RY1], taking p_X to be the identity map $X \to X$; our $\widetilde{K}^{\delta}_{-i}(X)$ is then defined to be $\widetilde{K}^{\delta}_0(X \times \mathbb{R}^i)$. The group we denote $Wh^{\delta}(X)$ is the group $Wh(X, p_X, 1, \delta)$ defined on page 22 of [RY1], where again p_X is the identity map $X \to X$; $Wh^{\delta}_{1-i}(X)$ is then defined to be $Wh^{\delta}(X \times \mathbb{R}^i)$.

$L_n^\delta(X)$. This group is defined in [RY2].[4] We then define the bounded locally finite L-group as follows:

$$L_n^{bdd}(P_d(\Gamma)) = \lim_{\delta\to\infty} L_n^\delta(P_d(\Gamma)).$$

4.1.3. **Theorem.** *Let Γ be a metric space with bounded geometry and finite decomposition complexity. The assembly map*

$$A : \lim_{d\to\infty} H_n(P_d(\Gamma), \mathbb{L}(e)) \to \lim_{d\to\infty} L_n^{bdd}(P_d(\Gamma))$$

is an isomorphism.

In the statement, the domain of assembly is the locally finite homology of the Rips complex with spectrum $\mathbb{L}(e)$, and the range is the bounded, locally finite and free L-theory of the same Rips complex.

4.2. **The bounded category.** Being invariant under coarse equivalence, finite decomposition complexity is well-adapted to a topological setting where the geometry appears only 'at large scale' and the topological properties are 'uniformly' locally trivial. These ideas are formalized in the *bounded category.*

A *coarse metric manifold* is a topological manifold M equipped with a continuous (pseudo-) metric in which balls are precompact. Although Riemannian manifolds, equipped with the path length metric, are motivating examples of coarse metric manifolds, we want to make clear that our definition entails *no* assumption on the metric at 'small scale' and that the manifold M is *not* assumed to be smooth. A continuous map $f : M \to N$, between two coarse metric manifolds is *bounded* if there exists a coarse equivalence $\phi : N \to M$ and a constant $K > 0$ such that $d(x, \phi \circ f(x)) \leq K$ for all $x \in M$. Coarse metric manifolds and bounded continuous maps comprise the *bounded category*.[5]

Before discussing rigidity in the bounded category, we must introduce appropriate notions of homeomorphism and homotopy. A *bounded homeomorphism* between coarse metric manifolds is a map $M \to N$ which is simultaneously a homeomorphism and a coarse equivalence. These are the isomorphisms in the bounded category.

Two bounded continuous maps f, $g : M \to N$ are *boundedly homotopic* if there exists a bounded homotopy between them; in other words, if there exists a continuous map $F : M \times [0,1] \to N$, for which $F(0,\cdot) = f$, $F(1,\cdot) = g$ and for which the family $(F(t,\cdot))_{t\in[0,1]}$ is bounded (uniformly in t, in the obvious sense). A bounded continuous map $f : M \to N$ is a *bounded homotopy equivalence* if there exists a bounded continuous map $g : N \to M$ such that the compositions $f \circ g$ and $g \circ f$ are boundedly homotopic to the identity.

[4]The group we denote $L_n^\delta(X)$ corresponds to the δ-controlled locally finite and free L-theory group $L_n^{\delta,\delta}(X; p_X, \mathbb{Z})$ in [RY2], where again p_X is the identity map $X \to X$.

[5]In [CFY], the authors give an essentially equivalent definition of the bounded category in which an auxiliary metric space X is introduced. An object is a pair (M, p) where $p : M \to X$ has precompact preimages. To obtain a coarse metric manifold, one must merely pull back the metric from X to M.

4.2.1. **Definition.** A coarse metric manifold M is *boundedly rigid* if the following condition holds: every bounded homotopy equivalence $M \to N$ to another coarse metric manifold is boundedly homotopic to a (bounded) homeomorphism.

A coarse metric manifold M is *uniformly contractible* if for every $r > 0$, there exists $R \geq r$ such that every ball in M with radius r is contractible to a point within the larger ball of radius R and the same center. Uniform contractibility is invariant under bounded homotopy equivalence.

A coarse metric manifold has *bounded geometry* if there exists $r > 0$ with the following property: for every $R > 0$ there exists $N > 0$ such that every ball of radius R is covered by N or fewer balls of radius r.[6]

Perhaps the most important, and motivating, example of a coarse metric manifold is the universal cover $\tilde{M}$ of a closed (topological) manifold M. To realize the structure of a coarse metric manifold on $\tilde{M}$ we can equip it with a continuous Γ-invariant pseudo-metric in which balls are precompact, where Γ is the fundamental group of M. Equipped with such a pseudo-metric, $\tilde{M}$ is coarsely equivalent to Γ.

4.3. **Application to bounded rigidity.** The *bounded Borel isomorphism conjecture* asserts that an appropriate assembly map is an isomorphism. Precisely this conjecture asserts that for a locally finite metric space Γ the assembly map

$$A : \lim_{d\to\infty} H_n(P_d(\Gamma), \mathbb{L}(e)) \to \lim_{d\to\infty} L_n^{bdd,s}(P_d(\Gamma)) \tag{4.1}$$

is an isomorphism: as in the previous section, the domain of assembly is the locally finite homology of the Rips complex of Γ with spectrum $\mathbb{L}(e)$, the simply connected surgery spectrum with $\pi_n(\mathbb{L}(e)) = L_n^s(\mathbb{Z}\{e\}) = L_n(\mathbb{Z}\{e\})$; the range of assembly is the bounded simple L-theory of the Rips complex of Γ defined using locally finite free geometric modules.

4.3.1. **Theorem.** *The bounded Borel isomorphism conjecture is true for metric spaces with bounded geometry and finite decomposition complexity.*

Proof. By Theorem 4.1.2 and the Ranicki-Rothenberg sequence in the controlled setting [FP], we have

$$\lim_{d\to\infty} L^{bdd,s}(P_d(\Gamma)) \cong \lim_{d\to\infty} L^{bdd}(P_d(\Gamma)).$$

The result now follows from Theorem 4.1.3. □

The bounded Borel isomorphism conjecture has strong topological implications. Our principal result in this direction is the following theorem.

4.3.2. **Theorem** (Bounded Rigidity Theorem)**.** *A uniformly contractible coarse metric manifold with bounded geometry, finite decomposition complexity, and dimension at least five is boundedly rigid.*

[6]Traditionally, a Riemannian manifold is said to have bounded geometry if its curvature is bounded and its radius of injectivity is bounded away from zero. Such local conditions are known to imply our global condition.

4.3.3. **Corollary.** *Let M be a closed aspherical manifold of dimension at least five whose fundamental group has finite decomposition complexity (as a metric space with a word metric). For every closed manifold N and homotopy equivalence $M \to N$ the corresponding bounded homotopy equivalence of universal covers is boundedly homotopic to a homeomorphism.*

Proof. The universal cover of a closed manifold has bounded geometry as a coarse metric manifold. Further, the universal cover of a closed aspherical manifold is uniformly contractible as a coarse metric manifold. Thus, the previous theorem applies. □

Let M be a coarse metric manifold. A *net* in M is a metric subspace $\Gamma \subset M$ which is both *uniformly discrete* – the distance between distinct points of Γ is bounded uniformly away from zero – and *coarsely dense* in M – for some $C > 0$, every ball $B(x, C)$ in M intersects Γ. Clearly, the inclusion of a net into M is a coarse equivalence, so that any two nets are coarsely equivalent. If M has bounded geometry (as a coarse metric manifold) then any net in M has bounded geometry (as a discrete metric space).

4.3.4. **Proposition.** *Let M be a uniformly contractible coarse metric manifold having bounded geometry and dimension at least five. Let Γ be a net in M. The assembly map (4.1) of the bounded Borel isomorphism conjecture for Γ identifies with the assembly map for M:*

$$A : H_n(M, \mathbb{L}(e)) \to L_n^{bdd,s}(M). \tag{4.2}$$

Precisely, there are isomorphisms

$$H_n(M, \mathbb{L}(e)) \cong \lim_{d\to\infty} H_n(P_d(\Gamma), \mathbb{L}(e)) \quad \textit{and} \quad L_n^{bdd,s}(M) \cong \lim_{d\to\infty} L^{bdd,s}(P_d(\Gamma))$$

commuting with the assembly maps.

4.3.5. **Remark.** The bounded geometry condition is essential here; Dranishnikov, Ferry and Weinberger have constructed an example of a uniformly contractible manifold M for which the first asserted isomorphism fails [DFW].

The proof of the above proposition will follow the standard arguments, based on the following lemma; see [HR, Section 3]. For the statement define a *coarse metric CW-space* to be a CW-complex equipped with a continuous (pseudo-)metric in which balls are relatively compact, and in which the cells have uniformly bounded diameter. The latter property can always be achieved by refining the CW-structure. The straightforward proof of the next lemma is left to the reader.

4.3.6. **Lemma.** *Let X be a uniformly contractible coarse metric finite dimensional CW-space. Suppose that X admits a bounded geometry net Γ. For every sufficiently large $d > 0$ there exist continuous coarse equivalences*

$$f_d : X \to P_d(\Gamma) \quad \textit{and} \quad g_d : P_d(\Gamma) \to X$$

with the following properties:

(1) *$g_d \circ f_d$ is boundedly homotopic to the identity map of X;*

(2) *$i_{dd'} \circ f_d \circ g_d$ is boundedly homotopic to the inclusion $i_{dd'} : P_d(\Gamma) \to P_{d'}(\Gamma)$, for $d' > d$ sufficiently large.*

Proof of Proposition 4.3.4. A topological manifold of dimension at least five admits the structure of a CW-complex [KS].[7] Thus a coarse metric manifold of dimension at least five is a coarse metric CW-space and the lemma applies. □

Proof of Theorem 4.3.2. Let M be as in the statement. Let N be another coarse metric manifold and suppose that N is boundedly homotopy equivalent to M. According to the bounded surgery exact sequence [FP], the bounded Borel isomorphism conjecture for M implies that N is homeomorphic to M, assuming that $\dim M \geq 5$. □

4.4. **Application to stable rigidity.** The *bounded Farrell-Jones L-theory isomorphism conjecture* asserts that a certain assembly map is an isomorphism. Precisely this conjecture asserts that for a locally finite metric space Γ the assembly map

$$A : \lim_{d\to\infty} H_n(P_d(\Gamma), \mathbb{L}(e)) \to \lim_{d\to\infty} L_n^{bdd,<-\infty>}(P_d(\Gamma))$$

is an isomorphism. Here, for a metric space X and natural number n, we define $L_n^{bdd,<-\infty>}(X)$ to be the direct limit of the bounded locally finite and free L-groups $L_n^{bdd}(X \times \mathbb{R}^k)$ with the maps given by crossing with $\mathbb{R}$. Recall that $\mathbb{L}(e)$, the (simply) connected surgery spectrum, satisfies $\pi_n(\mathbb{L}(e)) = L_n^{<-\infty>}(\mathbb{Z}\{e\})$.

4.4.1. **Theorem.** *The bounded Farrell-Jones L-theory isomorphism conjecture is true for metric spaces spaces with bounded geometry and finite decomposition complexity.*

Proof. Immediate from Theorem 4.1.3 and from the observation that if X has finite decomposition complexity, then so does $X \times \mathbb{R}^n$ for all n. □

The bounded Farrell-Jones L-theory isomorphism conjecture has implications to question of stable rigidity. Let M be a closed, aspherical manifold. By the arguments presented in the previous section, the bounded Farrell-Jones L-theory isomorphism conjectures for the universal cover of M and for the fundamental group of M are equivalent. According to the *descent principle* they imply the *integral Novikov conjecture* – a detailed argument is contained in the proof of [CP, Theorem 5.5].

Recall now from the introduction that a closed manifold M is *stably rigid* if there exists a natural number n with the following property: for every closed manifold N and every homotopoy equivalence $M \to N$ the map $M \times \mathbb{R}^n \to N \times \mathbb{R}^n$ is homotopic to a homeomorphism. The *stable Borel conjecture* asserts that closed aspherical manifolds are stably rigid. The fact that the integral Novikov conjecture implies the stable Borel conjecture was stated without proof in [FP]; for a detailed treatment see [J, Proposition 2.8]. From this discussion, and our previous results, we conclude:

[7] This is the only point at which we require the dimension to be ≥ 5 – the question of whether a manifold admits the structure of a CW-complex remains open in low dimensions. One could give an alternative proof of Proposition 4.3.4 using a Mayer-Vietoris argument, which would allow us to remove the dimension restriction.

4.4.2. **Theorem.** *The stable Borel conjecture holds for closed aspherical manifolds whose fundamental groups have finite decomposition complexity.* □

5. Vanishing theorem

We devote this section to the proof of Theorem 4.1.2, our vanishing result for the bounded Whitehead and bounded reduced lower algebraic K-theory groups. In view of the definitions, we obtain Theorem 4.1.2 as an immediate consequence of the following result:

5.1. **Theorem.** *Let Γ be a locally finite metric space with bounded geometry and finite decomposition complexity. The controlled locally finite Whitehead group and the controlled reduced locally finite algebraic K-theory group vanish asymptotically. Precisely, given $i \geq 0$, $\delta > 1$ and $a > 1$ there exists $b > 1$ such that, for any $Z \subset \Gamma$ the natural homomorphisms:*

$$Wh^{\delta}_{1-i}(P_a(Z)) \to Wh^{\delta}_{1-i}(P_b(Z)) \tag{5.1}$$

$$\widetilde{K}^{\delta}_{-i}(P_a(Z)) \to \widetilde{K}^{\delta}_{-i}(P_b(Z)) \tag{5.2}$$

are zero. Here $P_a(\Gamma)$ is equipped with the simplicial metric and $P_a(Z) \subset P_a(\Gamma)$ with the subspace metric (and similarly for $P_b(Z)$). The constant b depends only on i, δ, a and Γ, and not on Z.

5.2. **Remark.** To emphasize the dependence among the various constants and metric families we shall encounter we shall write, for example, $f = f(g, h)$ when f depends on g and h; if additionally $g = g(p, q)$ and $h = h(q, r)$ we write $f = f(g, h) = f(p, q, r)$.

In preparation for the proof of Theorem 5.1 we formalize the notion of a vanishing family: a collection $\mathcal{F}$ of metric subspaces of Γ is a *vanishing family* if for every $i \geq 0$, $\delta > 1$, $a > 1$, $t > 1$ and $p \geq 0$ there exists $b > 1$ such that for every $X \in \mathcal{F}$ and every $Z \subset N_t(X)$ the homomorphisms

$$Wh^{\delta}_{1-i}(P_a(Z) \times T^p) \to Wh^{\delta}_{1-i}(P_b(Z) \times T^p) \tag{5.3}$$

$$\widetilde{K}^{\delta}_{-i}(P_a(Z) \times T^p) \to \widetilde{K}^{\delta}_{-i}(P_b(Z) \times T^p) \tag{5.4}$$

are zero, where $N_t(X)$ is the t-neighborhood of X in Γ, i.e. $N_t(X) = \{y \in \Gamma : d(y, X) \leq t\}$. Here, T^p is the p-dimensional torus with the standard Riemannian metric of diameter one. Note that $b = b(i, p, t, a, \delta, \mathcal{F})$. We denote the collection of vanishing families by $\mathfrak{V}$.

Observe that in the definition of vanishing family we have not specified the metric to be used on $P_a(Z)$ and $P_b(Z)$. Indeed, this was intentional as we shall need to employ *two* different metrics in the proof of Theorem 5.1. The first is the *simplicial metric* on $P_a(Z)$ and the second is the *subspace metric* inherited from $P_a(\Gamma)$. Similarly we consider the simplicial and subspace metrics on $P_b(\Gamma)$.

5.3. **Proposition.** *The notion of vanishing family is independent of the choice of metric on $P_a(Z)$ and $P_b(Z)$.*

Proof. The subspace metric is always smaller than the simplicial metric. Consequently there is a hierarchy among the four (*a priori* different) definitions of vanishing family. The weakest version of vanishing states:

> For every a (5.3) and (5.4) are zero for sufficiently large b, when $P_a(Z)$ is equiped with the simplicial metric and $P_b(Z)$ with the subspace metric;

whereas the strongest version states:

> For every a (5.3) and (5.4) are zero for sufficiently large b, when $P_a(Z)$ is equiped with the subspace metric and $P_b(Z)$ with the simplicial metric.

It suffices to show that the weak version of vanishing implies the strong version. We shall focus on the Whitehead groups (the case of the K-groups being similar). Suppose that Z is a vanishing family in the weak sense. We shall show that, for sufficiently large a' depending on a and δ, there exist maps

$$Wh^{\delta}_{1-i}(P^{\mathrm{sub}}_a(Z) \times T^p)) \to Wh^{\delta}_{1-i}(P^{\mathrm{sim}}_{a'}(Z) \times T^p)); \tag{5.5}$$

here, and below, the superscript makes clear which metric is to be employed, either the subspace or the simplicial. Assuming this for the moment, the proof of the proposition is completed by considering the diagram

$$\begin{tikzcd} Wh^{\delta}_{1-i}(P^{\mathrm{sub}}_a(Z) \times T^p) \arrow[r] \arrow[d] & Wh^{\delta}_{1-i}(P^{\mathrm{sim}}_b(Z) \times T^p) \\ Wh^{\delta}_{1-i}(P^{\mathrm{sim}}_{a'}(Z) \times T^p) \arrow[r] & Wh^{\delta}_{1-i}(P^{\mathrm{sub}}_{b'}(Z) \times T^p); \arrow[u] \end{tikzcd}$$

given a we choose a' to ensure existence of the left hand vertical map as in (5.5); according to the weak version of vanishing we choose b' so that the bottom horizontal map is zero; finally, we choose b to ensure existence of the right hand vertical map as in (5.5).

It remains to verify the existence of the maps (5.5). This follows from the following two observations. First, for a' sufficiently large, the inclusion

$$P^{\mathrm{sub}}_a(Z) \to P^{\mathrm{sim}}_{a'}(Z)$$

is 1-*Lipschitz at scale* 100δ – meaning that whenever $x, y \in P^{\mathrm{sub}}_a(Z)$ satisfy $d(x,y) \leq 100\delta$ then the distance between x and y in $P^{\mathrm{sim}}_{a'}(Z)$ is not greater than their distance in $P^{\mathrm{sub}}_a(Z)$. Indeed, choose $a' \geq a$ to be large enough such that any pair of points of $P^{\mathrm{sub}}_a(Z)$ at distance less than 100δ lie in a common simplex in $P_{a'}(Z)$ – this is possible because the map $P_a(\Gamma) \to \Gamma$ associating to a point some vertex of the smallest simplex containing it is uniformly expansive. Now, the first map in the composition

$$P^{\mathrm{sub}}_a(Z) \to P^{\mathrm{sub}}_{a'}(Z) \to P^{\mathrm{sim}}_{a'}(Z)$$

is contractive. The second map is isometric *for pairs of points in a simplex* – the subspace and simplicial metrics on $P_a(Z)$ coincide for pairs of points belonging to a common simplex, essentially because each simplex is a convex subspace of $P_a(\Gamma)$.

Second, the δ-controlled Whitehead groups are independent of the behavior of the metric at scales much larger than δ. More precisely, an injection $X \to Y$ which is 1-*Lipschitz at scale* 100δ induces a map $Wh^{\delta}_{1-i}(X) \to Wh^{\delta}_{1-i}(Y)$. This follows from the definitions of these groups [RY1]. $\square$

Proof of Theorem 5.1. Assuming that Γ has finite decomposition complexity we shall prove that the collection of vanishing families contains the bounded families and, using a controlled Mayer-Vietoris argument based on part (5) of Theorem B.1 (proved in [RY1]), is closed under decomposability. We thereby conclude that the family $\{\,\Gamma\,\}$ is a vanishing family and the theorem follows.

A uniformly bounded family of subspaces of Γ is a vanishing family, as we conclude from the following facts:

(1) If a subspace $Y \subset \Gamma$ has diameter at most b for some $b \geq 0$, then $P_b(Z)$ is Lipschitz homotopy equivalent to a point (with Lipschitz constant one); indeed the same is true for any larger b.
(2) If two metric spaces P and Q are Lipschitz homotopy equivalent (with Lipschitz constant one) then $Wh^{\delta}_{1-i}(P)$ is isomorphic to $Wh^{\delta}_{1-i}(Q)$, and similarly $\widetilde{K}^{\delta}_{-i}(P)$ is isomorphic to $\widetilde{K}^{\delta}_{-i}(Q)$.
(3) By the choice of the Riemannian metric on T^p and the assumption $\delta > 1$, $Wh^{\delta}(T^p)$ and $\tilde{K}^{\delta}_{-i}(T^p)$ vanish for each $p \geq 0$.

Now, let $\mathcal{F}$ be a family of subspaces of Γ and assume that $\mathcal{F}$ is decomposable over the collection of vanishing families. We must show that $\mathcal{F}$ is a vanishing family; precisely, there exists $b = b(i, p, t, a, \delta, \mathcal{F})$ such that for every $X \in \mathcal{F}$ and every $Z \subset N_t(X)$ the maps (5.3) and (5.4) are zero.

Set $r = r(t, a, \delta, \lambda)$ sufficiently large, to be specified later. Obtain an r-decomposition of $\mathcal{F}$ over a vanishing family $\mathcal{G} = \mathcal{G}(r, \mathcal{F})$. Let $X \in \mathcal{F}$. We obtain a decomposition:

$$X = A \cup B, \qquad A = \bigsqcup_{r} A_i, \quad B = \bigsqcup_{r} B_j,$$

for which all A_i and $B_j \in \mathcal{G}$. Let Z be a subset of the neighborhood of radius t of X inside Γ. *From now on, all the neighborhoods will be taken inside* Z. Setting $C_i = N_{t+a}(A_i)$ and $D_j = N_{t+a}(B_j)$ we obtain an analogous decomposition:

$$Z = C \cup D, \qquad C = \bigsqcup_{r-2(t+a)} C_i, \quad D = \bigsqcup_{r-2(t+a)} D_j.$$

Denote $\mathcal{C} = \{\,C_i\,\}$ and $\mathcal{D} = \{\,D_j\,\}$. By the separation hypothesis we have $r - 2(t + a) > a$ so that $P_a(\mathcal{C}) = P_a(C)$ and $P_a(\mathcal{D}) = P_a(D)$. Further, $P_a(Z) = P_a(C) \cup P_a(D) = P_a(\mathcal{C} \cup \mathcal{D})$. We intend to compare the Mayer-Vietoris sequence of this pair of subspaces of $P_a(Z)$ to a Mayer-Vietoris sequence for certain subspaces of an appropriate relative Rips complex. We enlarge the intersection $\mathcal{C} \cap \mathcal{D} = \{ C_i \cap D_j \}$ by setting

$$\begin{aligned} W &= N_{a\beta\lambda\delta}(C) \cap N_{a\beta\lambda\delta}(D) \\ &= (N_{a\beta\lambda\delta}(C) \cap D) \cup (C \cap N_{a\beta\lambda\delta}(D)) \\ &= \bigsqcup_{r-2(t+a\beta\lambda\delta)} W_{ij}, \end{aligned}$$

and

$$W_{ij} = N_{a\beta\lambda\delta}(C_i) \cap N_{a\beta\lambda\delta}(D_j),$$

and where β is the constant appearing in Lemma A.3.4. Observe that $C_i \cap D_j \subset W_{ij}$, so that denoting $\mathcal{W} = \{ W_{ij} \}$ we have $\mathcal{C} \cap \mathcal{D} \subset \mathcal{W}$. Provided $a \leq b$ we have a commuting diagram

$$\begin{tikzcd} Wh^\delta(P_a(\mathcal{C} \cup \mathcal{D})) \arrow[r] \arrow[d] & \widetilde{K}_0^{\lambda\delta}(N_{\lambda\delta}(P_a(\mathcal{C} \cap \mathcal{D}))) \arrow[d] \\ Wh^\delta(P_{ab}(\mathcal{C} \cup \mathcal{D}, \mathcal{W})) \arrow[r] & \widetilde{K}_0^{\lambda\delta}(N_{\lambda\delta}(P_b(\mathcal{W}))). \end{tikzcd} \tag{5.6}$$

The horizontal maps are boundary maps in controlled Mayer-Vietoris sequences in Appendix B: in the top row the neighborhood is taken in $P_a(\mathcal{C} \cup \mathcal{D})$, and all spaces are given the subspace metric from $P_a(Z)$; in the bottom row the neighborhood is taken in $P_{ab}(\mathcal{C} \cup \mathcal{D}, \mathcal{W})$, and all spaces are given the subspace metric from $P_{ab}(Z, W)$. The vertical maps are induced from the proper contraction $P_a(Z) \to P_{ab}(Z, W)$. In fact, the right hand vertical map factors as the composite

$$N_{\lambda\delta}(P_a(\mathcal{C} \cap \mathcal{D})) \subset P_a(\mathcal{W}) \to P_b(\mathcal{W}) \subset N_{\lambda\delta}(P_b(\mathcal{W})); \tag{5.7}$$

in which the first two spaces are subspaces of $P_a(\mathcal{C} \cup \mathcal{D}) \subset P_a(Z)$ and the last two are subspaces of $P_{ab}(\mathcal{C} \cup \mathcal{D}, \mathcal{W}) \subset P_{ab}(Z, W)$. The first inclusion in (5.7) follows from

$$\begin{aligned} N_{\lambda\delta}(P_a(\mathcal{C} \cap \mathcal{D})) &= \bigcup_{i,j} N_{\lambda\delta}(P_a(C_i \cap D_j)) \\ &\subset \bigcup_{i,j} P_a(N_{a\beta\lambda\delta}(C_i) \cap N_{a\beta\lambda\delta}(D_j)) \\ &\subset \bigcup_{i,j} P_a(W_{ij}) = P_a(\mathcal{W}), \end{aligned}$$

where we have applied Lemma A.3.4 of the appendix for the first inclusion – keep in mind that the neighborhoods on the first line are taken in $P_a(\mathcal{C} \cup \mathcal{D})$.

Applying the induction hypothesis we claim that for sufficiently large b the right hand vertical map in (5.6) is zero. Indeed, the components $W_{ij} \in \mathcal{W}$ are contained in the neighborhoods $N_{t+a\beta\lambda\delta}(A_i)$ (and also of $N_{t+a\beta\lambda\delta}(B_j)$) and we can apply the hypothesis with appropriate choices of the parameters: $t' = t + a\beta\lambda\delta$, $\delta' = \lambda\delta$, $a' = a$, etc. In detail,

$$\widetilde{K}_0^{\lambda\delta}(P_a(\mathcal{W})) \xrightarrow{\cong} \prod \widetilde{K}_0^{\lambda\delta}(P_a(W_{ij})) \xrightarrow{0} \prod \widetilde{K}_0^{\lambda\delta}(P_b(W_{ij})) \longrightarrow \widetilde{K}_0^{\lambda\delta}(P_b(\mathcal{W}));$$

as the spaces $P_a(W_{ij})$ and $P_a(\mathcal{W})$ are given the subspace metric from $P_a(Z)$ and the individual W_{ij} are well-separated, the first map is an isomorphism by Lemma A.3.5 (which guarantees that the various $P_a(W_{ij})$ are separated by at least $\lambda\delta$); the spaces $P_b(W_{ij})$ are given the simplicial metric and the middle map is 0 for sufficiently large b by hypothesis; the space $P_b(\mathcal{W})$ is given the subspace metric from $P_{ab}(Z,\mathcal{W})$ and the last map is induced by proper contractions $P_b(W_{ij}) \subset P_b(\mathcal{W})$ onto disjoint subspaces.

Having chosen $b = b(i,p,t',a',\delta',\mathcal{G})$ we extend the diagram (5.6) to incorporate the relax-control map for the bottom sequence:

(5.8)

$$\begin{tikzcd}
 & Wh^{\delta}(P_a(\mathcal{C}\cup\mathcal{D})) \arrow[d] & \\
 & Wh^{\delta}(P_{ab}(\mathcal{C}\cup\mathcal{D},\mathcal{W})) \arrow[r] \arrow[d, "relax"] & \widetilde{K}_0^{\lambda\delta}(N_{\lambda\delta}(P_b(\mathcal{W}))) \\
\left.\begin{matrix} Wh^{\lambda^2\delta}(P_{ab}(\mathcal{C},\mathcal{W})\cup N_{\lambda\delta}(P_b(\mathcal{W}))) \\ \oplus \\ Wh^{\lambda^2\delta}(P_{ab}(\mathcal{D},\mathcal{W})\cup N_{\lambda\delta}(P_b(\mathcal{W})))\end{matrix}\right\} \arrow[r] & Wh^{\lambda^2\delta}(P_{ab}(\mathcal{C}\cup\mathcal{D},\mathcal{W})) &
\end{tikzcd}$$

We conclude from the above discussion and the controlled Mayer-Vietoris sequence that the image of $Wh^{\delta}(P_a(\mathcal{C}\cup\mathcal{D}))$ under the composite of the two vertical maps is contained in the image of the bottom horizontal map. It remains to apply the induction hypothesis to $\mathcal{C}$ and $\mathcal{D}$. The case of $\mathcal{D}$ being analogous, we concentrate on $\mathcal{C}$ and shall show that for sufficiently large $c \geq b$ the composite

$$P_{ab}(\mathcal{C},\mathcal{W}) \cup N_{\lambda\delta}(P_b(\mathcal{W})) \subset P_{ab}(\mathcal{C}\cup\mathcal{D},\mathcal{W}) \to P_b(Z) \to P_c(Z),$$

in which the arrows are induced by proper contractions $P_{ab}(Z,W) \to P_b(Z) \to P_c(Z)$ is zero on the $\lambda^2\delta$-controlled Whitehead group. We have, as subspaces of $P_{ab}(\mathcal{C}\cup\mathcal{D},\mathcal{W}) \subset P_{ab}(Z,W)$,

$$P_{ab}(\mathcal{C},\mathcal{W}) \cup N_{\lambda\delta}(P_b(\mathcal{W})) = \bigcup_i \left(P_a(C_i) \cup \bigcup_j N_{\lambda\delta}(P_b(W_{ij})) \right), \tag{5.9}$$

in which the spaces comprising the union over i are well-separated by Lemma A.3.5 (which guarantees $\lambda^2\delta$-separation). Further, for fixed i and j we have

$$N_{\lambda\delta}(P_b(W_{ij})) \subset P_{ab}(N_{a\beta\lambda\delta}(W_{ij}), W_{ij})) \to P_b(N_{a\beta\lambda\delta}(W_{ij})) \subset P_b(N_{2a\beta\lambda\delta}(C_i)),$$

where we have applied Lemma A.3.4 for the first containment (we point out that this is one of the places where the notion of relative Rips complex is important), and the arrow represents the assertion that the space on its left maps to the space on its right under the proper contraction $P_{ab}(Z,W) \to P_b(Z)$. Accordingly, for each fixed i we have

$$P_a(C_i) \cup \bigcup_j P_b(N_{a\beta\lambda\delta}(W_{ij})) \to P_b(N_{2a\beta\lambda\delta}(C_i)),$$

where the arrow is interpreted as above. Now, we apply our induction hypothesis a second time, with appropriate choices of the parameters: $t'' = t + 2a\beta\lambda\delta$, $\delta'' = \lambda^2\delta$, $a'' = b$, etc, noting that $N_{2a\beta\lambda\delta}(C_i) \subset N_{t+2a\beta\lambda\delta}(A_i)$. We get $c = c(i,p,t'',a'',\delta'',\mathcal{G})$, and analyze

$$\begin{aligned}
Wh^{\lambda^2\delta}(P_{ab}(\mathcal{C},\mathcal{W}) \cup N_{\lambda\delta}(P_b(\mathcal{W})) &\cong \prod Wh^{\lambda^2\delta}\left(P_a(C_i) \cup \bigcup_j P_b(N_{a\beta\lambda\delta}(W_{ij}))\right) \\
&\to \prod Wh^{\lambda^2\delta}(P_b(N_{2a\beta\lambda\delta}(C_i))) \\
&\to \prod Wh^{\lambda^2\delta}(P_c(N_{2a\beta\lambda\delta}(C_i))) \\
&\to Wh^{\lambda^2\delta}(P_c(Z)) \\
&\to Wh^{\delta}(P_{\lambda^2 c}(Z))
\end{aligned}$$

the $\cong$ follows from the well-separatedness in (5.9); the spaces $P_c(N_{2a\lambda\delta}(C_i))$ are given the simplicial metrics, and the second arrow is 0; the fourth arrow is induced from inclusion of disjoint subspaces of $P_c(Z)$. The last arrow follows from the definition of the controlled Whitehead groups. Checking the dependence of the constant c we find $c = c(i,p,t,a,\lambda,\delta,\mathcal{F})$ as required. $\square$

6. Assembly isomorphism

We devote this section to the proof of Theorem 4.1.3, which asserts that assembly is an isomorphism for spaces having finite decomposition complexity. In view of the definitions, we obtain Theorem 4.1.3 as an immediate consequence of the following result:

6.1. **Theorem.** *Let Γ be a locally finite metric space with bounded geometry and finite decomposition complexity. Assembly for Γ is an asymptotic isomorphism. Precisely, given $n \geq 0$, $\delta > 1$ and $a > 1$ there exists $b = b(a,\delta,n) \geq a$ such that, for any $Z \subset \Gamma$,*

1. *the kernel of $H_n(P_a(Z)) \to L^\delta_n(P_a(Z))$ is mapped to zero in $H_n(P_b(Z))$;*
2. *the image of $L^\delta_n(P_a(Z)) \to L^\delta_n(P_b(Z))$ is contained in the image of $H_n(P_b(Z)) \to L^\delta_n(P_b(Z)$.*

We shall refer to condition (2) in the statement as *asymptotic surjectivity* and to condition (1) as *asymptotic injectivity*.

Before turning to the proof we pause to outline the strategy. The proof consists essentially of a quantitative version of the five lemma, which we shall prove using the controlled Mayer-Vietoris sequence in L-theory, precisely parts (4) and (5) of Theorem B.2. Borrowing the notation from the previous section, consider the following diagram, which again does not make sense in the controlled setting and must be loosely interpreted:

$$
\begin{tikzcd}
H_n(P_a(\mathcal{C})) \oplus H_n(P_a(\mathcal{D})) \arrow[r] \arrow[d] & L_n(P_a(\mathcal{C})) \oplus L_n(P_a(\mathcal{D})) \arrow[d] \\
H_n(P_a(\mathcal{C} \cup \mathcal{D})) \arrow[r] \arrow[d] & L_n(P_a(\mathcal{C} \cup \mathcal{D})) \arrow[d] \\
H_{n-1}(P_a(\mathcal{C} \cap \mathcal{D})) \arrow[r] \arrow[d] & L_{n-1}(P_a(\mathcal{C} \cap \mathcal{D})) \arrow[d] \\
H_{n-1}(P_a(\mathcal{C})) \oplus H_{n-1}(P_a(\mathcal{D})) \arrow[r] & L_{n-1}(P_a(\mathcal{C})) \oplus L_{n-1}(P_a(\mathcal{D})).
\end{tikzcd}
\tag{6.1}
$$

In the diagram, the vertical exact sequences are portions of appropriate Mayer-Vietoris sequences; the horizontal maps are the assembly maps. The induction hypothesis applies to the first, third and fourth rows; we are to prove that the second horizontal map is an (asymptotic) isomorphism. In the proof below, we shall concentrate on (asymptotic) surjectivity – a simple diagram chase reveals that this follows (asymptotic) surjectivity of rows one and three and (asymptotic) injectivity of row four.

In the proof below, to help the reader follow our trajectory we shall adopt the following conventions: x, y and z will be used for elements in the bounded L-theory for unions, intersections and direct sums, respectively; x', y', z' will be used for elements in the corresponding homology groups.

As preparation for the proof we introduce the notion of an L-isomorphism family: a collection $\mathcal{F}$ of metric subspaces of Γ is an *L-isomorphism family* if for every $n \geq 0$, $\delta > 1$, $a > 1$, and $t > 1$ there exists $b = b(a, \delta, t, n) > 1$ such that for every $X \in \mathcal{F}$ and every $Z \subset N_t(X)$ the assertions (1) and (2) of the theorem are satisfied. As was the case for vanishing families the notion of an L-isomorphism family is not sensitive to the choice of metric on $P_a(Z)$ and $P_b(Z)$. Compare Proposition 5.3 – the proof in the present situation is based on the same argument.

Finally, the proof employs both the relative Rips complex, $P_{ab}(\mathcal{C}, \mathcal{W})$ and the scaled Rips complex, $P_{abm}(\mathcal{C}, \mathcal{W})$ – see Definition A.1.1 and Definition A.1.2, respectively, and also Section A.2.

Proof. The proof will be much more condensed than the proof of Theorem 5.1 which we presented in some detail; while the present proof is not technically more difficult, it is somewhat longer.

We proceed as in the proof of Theorem 5.1. Assuming Γ has finite decomposition complexity we shall show that the collection of families that are both vanishing families and L-isomorphism families contains the bounded families, and is closed under decomposability. We thereby conclude that the family $\{\, \Gamma \,\}$ is an isomorphism family, and the theorem follows.

The case of bounded families is handled by the following facts:

(1) If a subspace $Y \subset \Gamma$ has diameter at most b for some $b \geq 0$, then $P_b(Z)$ is Lipschitz homotopy equivalent to a point (with Lipschitz constant one); indeed the same is true for any larger b.
(2) If two metric spaces P and Q are Lipschitz homotopy equivalent (with Lipschitz constant one) then $L^\delta_n(P)$ is isomorphic to $L^\delta_n(Q)$.

Now, let $\mathcal{F}$ be a family of subspaces of Γ, and assume $\mathcal{F}$ is decomposable over the collection of families that are both vanishing and L-isomorphism families. It follows from the proof of Theorem 5.1 that $\mathcal{F}$ itself is a vanishing family and we are to prove that $\mathcal{F}$ is an L-isomorphism family. We shall concentrate on proving asymptotic surjectivity; asymptotic injectivity can be proved in essentially the same manner.

Set $r = r(t, a, \delta, \lambda)$ sufficiently large, to be specified later – precisely, when a union below is called *well-separated*, this will mean for a sufficiently good choice of r, and the reader will verify that this choice depends only on the parameters t, a, δ and λ. Obtain an r-decomposition of $\mathcal{F}$ over an L-isomorphism (and vanishing) family $\mathcal{G} = \mathcal{G}(r, \mathcal{F})$.

Let $X \in \mathcal{F}$. Let Z, $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{W}$ be as in the proof of Theorem 5.1. Let $x \in L^\delta_n(P_a(\mathcal{C}) \cup P_a(\mathcal{D}))$. We need to prove that x is in the image of the assembly map up to increasing a.
Step 1. Using the well-separatedness of $\mathcal{W}$, and the vanishing assumption for the family $\mathcal{W}$, we can find $b = b(a, \delta, t, n)$ such that the map

$$\widetilde{K}_0^{\lambda_n \delta}(P_a(\mathcal{W})) \to \widetilde{K}_0^{\lambda_n \delta}(P_b(\mathcal{W})) \tag{6.2}$$

is zero. This allows us to consider the boundary map

$$\partial : \ L^\delta_n(P_a(\mathcal{C} \cup \mathcal{D})) \to L^{\lambda_n \delta}_{n-1}(P_b(\mathcal{W})),$$

where ∂ is the boundary map in Theorem B.2 of Appendix B and $P_b(\mathcal{W})$ is seen as a subspace of $P_{ab}(Z, \mathcal{W})$.
Step 2. Lemma A.3.5 implies that $P_b(\mathcal{W})$ is well separated, as a subspace of $P_{ab}(Z, \mathcal{W})$. Hence

$$L^{\lambda_n \delta}_{n-1}(P_b(\mathcal{W})) \cong \prod_{i,j} L^{\lambda_n \delta}_{n-1}(P_b(W_{ij})).$$

Hence, by the surjectivity assumption for $\mathcal{W}$, there exists $c = c(a, \delta, n, t) \geq b$ and $y' \in H_{n-1}(P_c(\mathcal{W}))$ mapping to (the image of) x in $L^{\lambda_n \delta}_{n-1}(P_c(\mathcal{W}))$, which we will simply write $A(y') = \partial(x)$.
Step 3. By Theorem B.2 in Appendix B, part (5), and (6.2) (using that $c \geq b$), we have $i_* \circ \partial = 0$ in

$$L^\delta_n(P_a(\mathcal{C} \cup \mathcal{D})) \xrightarrow{\partial} L^{\lambda_n \delta}_{n-1}(P_c(\mathcal{W})) \xrightarrow{i_*} L^{\lambda_n \delta}_{n-1}(P_{ac}(\mathcal{C}, \mathcal{W})) \oplus L^{\lambda_n \delta}_{n-1}(P_{ac}(\mathcal{D}, \mathcal{W})).$$

In particular, $i_* \circ \partial(x) = i_* \circ A(y') = 0$. Considering the following commutative diagram

$$\begin{tikzcd} H_{n-1}(P_c(\mathcal{W})) \arrow[r, "i_*"] \arrow[d, "A"] & H_{n-1}(P_{ac}(\mathcal{C},\mathcal{W})) \oplus H_{n-1}(P_{ac}(\mathcal{D},\mathcal{W})) \arrow[d, "A"] \\ L^{\lambda_n\delta}_{n-1}(P_c(\mathcal{W})) \arrow[r, "i_*"] & L^{\lambda_n\delta}_{n-1}(P_{ac}(\mathcal{C},\mathcal{W})) \oplus L^{\lambda_n\delta}_{n-1}(P_{ac}(\mathcal{D},\mathcal{W})), \end{tikzcd}$$

we deduce $A \circ i_*(y') = 0$.

Step 4. By the injectivity assumption for $\mathcal{W}$, there exists $d = d(a,\delta,n,t) \geq c$ such that the map

$$H_{n-1}(P_{ac}(\mathcal{C},\mathcal{W})) \oplus H_{n-1}(P_{ac}(\mathcal{D},\mathcal{W})) \to H_{n-1}(P_d(\mathcal{C},\mathcal{W})) \oplus H_{n-1}(P_d(\mathcal{D},\mathcal{W}))$$

sends $i_*(y')$ to 0.

Step 5. By exactness of the sequence

$$H_n(P_d(\mathcal{C}\cup\mathcal{D},\mathcal{W})) \xrightarrow{\partial} H_{n-1}(P_d(\mathcal{W})) \xrightarrow{i_*} H_{n-1}(P_d(\mathcal{C},\mathcal{W})) \oplus H_{n-1}(P_d(\mathcal{D},\mathcal{W})),$$

there exists $x' \in H_n(P_d(\mathcal{C}\cup\mathcal{D},\mathcal{W}))$ such that $y' = \partial(x')$.

Step 6. If m is large enough, the metric subfamily $P_d(\mathcal{W})$ of $P_{adm}(\mathcal{C}\cup\mathcal{D},\mathcal{W})$ is well-separated by Lemma A.3.5. Hence,

$$\widetilde{K}_0^{\lambda_n^2\delta}(N_{\lambda_n^2\delta}(P_d(\mathcal{W}))) \cong \prod_{i,j} \widetilde{K}_0^{\lambda_n^2\delta}(N_{\lambda_n^2\delta}(P_d(W'_{ij}))).$$

On the other hand, by Lemma A.3.6, when m is large enough, $N_{\lambda_n^2\delta}(P_d(\mathcal{W}))$ is 2-Lipschitz homotopy equivalent to a subset of $P_d(\mathcal{W}')$ (just take the homotopy equivalence F of Lemma A.3.6, restricted to V, which in our case is $N_{\lambda_n^2\delta}(P_d(\mathcal{W})))$ where $\mathcal{W}' = N_{a\beta\lambda_n^2\delta}(\mathcal{W})$ (β is as in Lemma A.3.6) and $P_d(\mathcal{W}')$ is viewed as subspace of $P_{adm}(\mathcal{C}\cup\mathcal{D},\mathcal{W}')$. Hence there exists $e = e(a,\delta,n,t)$ such that[8]

$$\widetilde{K}_0^{\lambda_n\delta}(N_{\lambda_n^2\delta}(P_d(\mathcal{W}))) \longrightarrow \widetilde{K}_0^{2\lambda_n\delta}(P_d(\mathcal{W}')) \xrightarrow{0} \widetilde{K}_0^{\lambda_n\delta}(P_e(\mathcal{W}')). \tag{6.3}$$

We can thus define the boundary map

$$L_n^{\lambda_n\delta}(P_{adm}(\mathcal{C}\cup\mathcal{D},\mathcal{W})) \xrightarrow{\partial} L_{n-1}^{\lambda_n^2\delta}(P_e(\mathcal{W}')).$$

Step 7. Remember that $P_d(\mathcal{C}\cup\mathcal{D},\mathcal{W})$ and $P_{adm}(\mathcal{C}\cup\mathcal{D},\mathcal{W})$ are the same topological space equipped with two different metrics. Considering the following commutative diagram,

$$\begin{tikzcd} H_n(P_d(\mathcal{C}\cup\mathcal{D},\mathcal{W})) \arrow[r, "\partial"] \arrow[d, "A"] & H_{n-1}(P_d(\mathcal{W})) \arrow[d, "A"] \\ L_n^{\lambda_n\delta}(P_{adm}(\mathcal{C}\cup\mathcal{D},\mathcal{W})) \arrow[r, "\partial"] & L_{n-1}^{\lambda_n^2\delta}(P_e(\mathcal{W}')), \end{tikzcd}$$

[8] as up to increasing e, one can change $2\lambda_n\delta$ to $\lambda_n\delta$ in the right-hand term.

we obtain $\partial \circ A(x') = A \circ \partial(x') = A(y') = \partial(x)$. In other words, $\partial(x - A(x'))) = 0$ in $L_{n-1}^{\lambda_n^2\delta}(P_e(\mathcal{W}'))$. Up to replacing x by $x - A(x')$, we can therefore suppose that $\partial(x) = 0$.
Step 8. Applying part (4) of Theorem B.2 with

$$\begin{array}{ccc}
 & & L_n^{\lambda_n\delta}(P_{adm}(\mathcal{C}\cup\mathcal{D},\mathcal{W})) \xrightarrow{\partial} L_{n-1}^{\lambda_n^2\delta}(\mathcal{V}) \\
 & & \downarrow \\
L_n^{\lambda_n^3\delta}(P_{aem}(\mathcal{C},\mathcal{W}')\cup\mathcal{V}) \oplus L_n^{\lambda_n^3\delta}(P_{aem}(\mathcal{D},\mathcal{W}')\cup\mathcal{V}) & \xrightarrow{j_*} & L_n^{\lambda_n^3\delta}(P_{aem}(\mathcal{C}\cup\mathcal{D},\mathcal{W}')) \\
\downarrow & & \downarrow \\
L_n^{2\lambda_n^3\delta}(P_{aem}(\mathcal{C},\mathcal{W}')) \oplus L_n^{2\lambda_n^3\delta}(P_{aem}(\mathcal{D},\mathcal{W}')) & \xrightarrow{j_*} & L_n^{2\lambda_n^3\delta}(P_{aem}(\mathcal{C}\cup\mathcal{D},\mathcal{W}')),
\end{array}$$

where $\mathcal{V}$ is the $\beta\lambda_n^2\delta$-neighborhood of $P_d(\mathcal{W}')$ in $P_{adm}(\mathcal{C}\cup\mathcal{D},\mathcal{W})$. The lower part of the diagram follows from the Lipschitz-homotopy lemma (see Lemma A.3.6). Together with (6.3), we deduce the existence of z such that $x = j_*(z)$, where j_* is the map defined above.
Step 9. We have $P_{aem}(\mathcal{C},\mathcal{W}')) = \bigcup_i P_{aem}(C_i, \cup_j W'_{ij}))$, where the union over i is well-separated provided m was chosen large enough. Moreover, since $\mathcal{W}' \subset N_{2a\beta\lambda_n^2\delta}(\mathcal{C}\cap\mathcal{D})$, we have the following contractive inclusion

$$P_{aem}(C_i, \cup_j W'_{ij})) \subset P_e(N_{2a\beta\lambda_n^2}(C_i)).$$

We therefore get a map

$$L_n^{2\lambda_n^3\delta}(P_{aem}(\mathcal{C},\mathcal{W}')) \to \prod_i L_n^{2\lambda_n^3\delta}(P_e(N_{2a\beta\lambda_n^2}(C_i))).$$

The similar statement is true for $\mathcal{D}$.
Step 10. By the surjectivity assumption applied to the families $\mathcal{C}$ and $\mathcal{D}$, there exists $f = f(a,\delta,n,t)$ such that the range of

$$L_n^{2\lambda_n^3\delta}(P_e(N_{2a\beta\lambda_n^3}(C_i))) \to L_n^{2\lambda_n^3\delta}(P_f(N_{2a\beta\lambda_n^3}(C_i)))$$

is contained in the range of

$$H_n(P_f(N_{2a\beta\lambda_n^3}(C_i))) \to L_n^{2\lambda_n^3\delta}(P_f(N_{2a\beta\lambda_n^3}(C_i))),$$

and similarly for D_i, for all i. Hence there exists z' in

$$\prod_i \left(H_n(P_f(N_{2a\beta\lambda_n^2}(C_i))) \oplus H_n(P_f(N_{2a\beta\lambda_n^2}(D_i)))\right) \cong H_n(P_f(N_{2a\beta\lambda_n^2}(\mathcal{C}))) \oplus H_n(P_f(N_{2a\beta\lambda_n^2}(\mathcal{D})))$$

such that $A(z') = z$ where z is identified with its image through the map

$$\prod_i \left(L_n^{2\lambda_n^3\delta}(P_f(N_{2a\beta\lambda_n^2}(C_i))) \oplus L_n^{2\lambda_n^3\delta}(P_f(N_{2a\beta\lambda_n^2}(D_i))\right) \to L_n^{2\lambda_n^3\delta}(P_f(N_{2a\beta\lambda_n^2}(\mathcal{C}))) \oplus L_n^{2\lambda_n^3\delta}(P_f(N_{2a\beta\lambda_n^3}(\mathcal{D}))).$$

Step 11. Finally we use the commutative diagram

$$\begin{tikzcd} H_n(P_f(N_{2a\beta\lambda_n^2}(\mathcal{C}))) \oplus H_n(P_f(N_{2a\beta\lambda_n^2}(\mathcal{D}))) \arrow[r, "j_*"] \arrow[d, "A"] & H_n(P_f(Z)) \arrow[d, "A"] \\ L_n^{2\lambda_n^3\delta}(P_f(N_{2a\beta\lambda_n^2}(\mathcal{C}))) \oplus L_n^{2\lambda_n^3\delta}(P_f(N_{2a\beta\lambda_n^2}(\mathcal{D}))) \arrow[r, "j_*"] & L_n^{2\lambda_n^3\delta}(P_f(Z)), \end{tikzcd}$$

to get $x = j_*(z) = j_*(A(z')) = A(j_*(z'))$, viewed in $L_n^{2\lambda_n^3\delta}(P_f(Z))$. The first two equalities following from steps 8 and 10. We have therefore proved that x is in the range of

$$H_n(P_f(Z)) \to L_n^{2\lambda_n^3\delta}(P_f(Z)),$$

which is enough to conclude, as up to increasing f, we can replace $2\lambda_n^3\delta$ by δ in the right-hand term. □

Appendix A. Variations on the Rips complex

In this appendix, we introduce the *relative Rips complex* and the *scaled* (*relative*) *Rips complex* and prove several useful results about their geometry. These complexes, and the assorted technical results presented here, play a crucial role in the proofs of Theorems 4.1.2 and 4.1.3. The appendix is designed to be read independently and, in spite of their technical nature, we believe that the results presented may be useful in other contexts.

The appendix is organized as follows. In the first subsection, we shall introduce the *relative Rips complex* and the *scaled Rips complex*. In the second, we extend the definitions to the setting of metric families, relevant for the proofs Theorems 4.1.2 and 4.1.3. The final subsection contains a collection of lemmas, also necessary for the proofs of Theorems 4.1.2 and 4.1.3. While we shall state and prove the lemmas in the context of metric spaces they generalize immediately to the context of metric families.

Throughout, Γ is a locally finite metric space with the property that $d(x, y) \geq 1$ for each pair of distinct points x and $y \in \Gamma$. The Rips complex was defined previously (see Definition 4.1.1 and the surrounding discussion).

A.1. **The relative Rips complex and the scaled Rips complex.** In this subsection, we shall introduce the *relative Rips complex* and the *scaled Rips complex*. These play important roles in the proofs of Theorems 4.1.2 and 4.1.3, respectively. The purpose of these complexes is to selectively rescale parts of the ambient space while maintaining the separation between them.

A.1.1. **Definition.** Let Σ be a subset of Γ. For $1 \leq a \leq b$ we define the relative Rips complex $P_{ab}(\Gamma, \Sigma)$ to be the simplicial polyhedron with vertex set Γ and in which a finite subset $\{ \gamma_0, \ldots, \gamma_n \}$ spans a simplex if one of the following conditions hold:

(1) $d(\gamma_i, \gamma_j) \leq a$ for all i and j;
(2) $d(\gamma_i, \gamma_j) \leq b$ for all i, j, and $\gamma_i \in \Sigma$ for all i.

The relative Rips complex is equipped with the simplicial metric.

If C is a subspace of Γ, then $P_d(C)$ is, in a natural way, a *subset* of $P_d(\Gamma)$. When $P_d(C)$ and $P_d(\Gamma)$ are equipped with the simplicial metric, the inclusion $P_d(C) \subset P_d(\Gamma)$ is contractive. Observe that $P_d(C)$ carries, in addition to the simplicial metric, a subspace metric inherited from $P_d(\Gamma)$. If $C \subset \Gamma$ and $W \subset \Sigma$ we have inclusions of sets

$$P_a(C) \subset P_{ab}(\Gamma, \Sigma), \quad P_b(W) \subset P_{ab}(\Gamma, \Sigma).$$

If $P_b(W)$ is equipped with the intrinsic metric the second inclusion is contractive; the analogous statement is generally false if $P_b(W)$ is equipped with the subspace metric inherited from $P_b(\Gamma)$. Similar remarks apply for $P_a(C)$.

A.1.2. **Definition.** Let W be a subset of the metric space Γ. For $1 \leq a \leq b$ and a sequence of positive integers $\overline{m} = m_1, \ldots, m_n, \ldots$, we define the metric space $P_{ab\overline{m}}(\Gamma; W)$ to be the polyhedron $P_b(\Gamma)$ with the metric defined as follows:

(1) each simplex K spanned by a finite subset $\{\gamma_0, \gamma_1, \cdots, \gamma_n\}$ of Γ is given by the (pseudo) Riemannian metric defined inductively on n:
   (i) if K is a simplex in $P_{ab}(\Gamma; W)$, then the simplex is endowed the standard simplicial Riemannian metric;
   (ii) if K is not a simplex in $P_{ab}(\Gamma; W)$ and we have inductively defined the (pseudo) Riemannian metric g_{n-1} on its $(n-1)$-skeleton $K^{(n-1)}$, then we identify K with the cone

   $$([0, 1] \times K^{(n-1)})/(0 \times K^{(n-1)})$$

   and define a (pseudo) Riemannian metric g_n on K by:

   $$g_n = m_n^2 dt^2 + t^2 g_{n-1}$$

   for $t \in [0, 1]$.
(2) the (pseudo) Riemannian metrics on simplices of $P_{ab}(\Gamma; W)$ can be used to define the length of any piecewise smooth path in the polyhedron. For any pair of points x and y in $P_{ab\overline{m}}(\Gamma; W)$, $d(x, y)$ is defined to be the infimum of the lengths of all piecewise smooth paths in $P_{ab\overline{m}}(\Gamma; W)$ connecting x and y.

A.1.3. **Remark.** We shall actually only use the case $\overline{m} = (m, m, \ldots)$ in the proofs of Theorems 4.1.2 and 4.1.3, where we will denote $P_{ab\overline{m}}(\Gamma; W)$ by $P_{abm}(\Gamma; W)$. We however chose to introduce the more general notion since it will streamline the proofs of several results in this appendix.

A.2. **Extension of the definitions for metric families.** In this subsection, we introduce some further notations in order to deal with families of subsets of Γ instead of just one subspace at a time. In particular, we will introduce the Rips complex and the relative Rips complex for metric families. We will not treat the case of the scaled Rips complex since it is a straightforward adaptation of the case of the relative Rips complex.

For a family $\mathcal{C} = \{ C \}$ of subspaces of Γ we define

$$P_d(\mathcal{C}) = \bigcup_{C \in \mathcal{C}} P_d(C) \subset P_d(\Gamma),$$

which we shall always equip with the subspace metric. Typically, we shall employ this notation when the family $\mathcal{C}$ is disjoint. Note that if the family $\mathcal{C}$ is d-disjoint and $\tilde{C}$ is the union of the $C \in \mathcal{C}$ then

$$P_d(\mathcal{C}) = P_d(\tilde{C}).$$

If the union of families is defined naively, and the intersection of families is *defined* to be the family of intersections $\mathcal{C} \cap \mathcal{D} = \{ C \cap D : C \in \mathcal{C}, D \in \mathcal{D} \}$ we have

$$P_d(\mathcal{C} \cup \mathcal{D}) = P_d(\mathcal{C}) \cup P_d(\mathcal{D}), \quad P_d(\mathcal{C} \cap \mathcal{D}) = P_d(\mathcal{C}) \cap P_d(\mathcal{D}).$$

Just as for the standard Rips complex, we can extend the definition of the relative Rips complex to families. For families $\mathcal{C} = \{ C \}$ and $\mathcal{W} = \{ W \}$ with each $C \subset \Gamma$ and each $W \subset \Sigma$ we define

$$P_{ab}(\mathcal{C}, \mathcal{W}) = \bigcup_{C \in \mathcal{C}} P_a(C) \cup \bigcup_{W \in \mathcal{W}} P_b(W),$$

as subspaces of $P_{ab}(\Gamma, \Sigma)$. If Σ is not explicitly specified, then Σ is understood to be the union of all W in $\mathcal{W}$. In the special case $a = b$ we have $P_{aa}(\Gamma, \Sigma) = P_a(\Gamma)$ and, more generally $P_{aa}(\mathcal{C}, \mathcal{W}) = P_a(\mathcal{C} \cup \mathcal{W})$. As for the standard Rips complex, we have the elementary equalities

$$P_{ab}(\mathcal{C} \cup \mathcal{D}, \mathcal{W}) = P_{ab}(\mathcal{C}, \mathcal{W}) \cup P_{ab}(\mathcal{D}, \mathcal{W}), \quad P_{ab}(\mathcal{C} \cap \mathcal{D}, \mathcal{W}) = P_{ab}(\mathcal{C}, \mathcal{W}) \cap P_{ab}(\mathcal{D}, \mathcal{W})$$

as subspaces of $P_{ab}(\Gamma, \Sigma)$.

A.3. **A few technical results.** In this subsection, we prove a several useful results about the geometry of the (relative) Rips and scaled Rips complex. These results are important tools in the proofs of Theorems 4.1.2 and 4.1.3.

Henceforth, we assume Γ has bounded geometry.

A.3.1. **Lemma** (Comparison lemma)**.** *Let $a \geq 1$, and let $P_a(\Gamma)$ be equipped as usual with the simplicial metric. For x and $y \in \Gamma$ we have*

$$d_\Gamma(x, y) \leq a \; \alpha \; d_{P_a(\Gamma)}(x, y),$$

for some constant α depending only on the dimension of $P_a(\Gamma)$.

The proof of the above lemma is straightforward and is left to the reader.

A.3.2. **Lemma.** *[Comparison lemma for the scaled complex] Let $a \geq 1$, and let C be a subspace of Γ. There exists $\beta \geq 1$ depending only on the dimension of $P_a(\Gamma)$ such that for all $b \geq a$, there exists $M > 0$ for which*

$$d_\Gamma(x, C) \leq a \; \beta \; d(x, P_b(C)),$$

for all $x \in \Gamma$, provided $m_k \geq M$ for all k, where the distance for the right-hand term is taken in $P_{ab\overline{m}}(\Gamma, C)$.

Proof. It is enough to show that if γ is a path of length l in $P_{ab\overline{m}}(\Gamma, C)$, parametrized by its arc length with respect to the (pseudo) riemannian metric, between $x \in \Gamma$ and $P_b(C)$, then

$$d_\Gamma(x, C) \leq a\beta l. \tag{A.1}$$

We proceed by induction on n, the minimal integer such that γ is contained in the union of $P_a(\Gamma)$ and the n-skeleton of $P_{ab\overline{m}}(\Gamma, C)$. Precisely, our induction hypothesis will be the following: for all $\beta > \alpha$, where α appears in the comparison lemma for $P_a(\Gamma)$, and every path of length l contained in the union of $P_a(\Gamma)$ and the n-skeleton, there exists M such that (A.1) holds for all $\overline{m}$ such that $m_k \geq M$ for all $1 \leq k \leq n$.

Let us start with the case $n = 1$. Note that up to replacing γ by a sub-path, we can always suppose that it does not intersect C at any $t < l$. We can also suppose that if γ meets the interior of an edge not belonging to $P_a(\Gamma)$, then this edge is completely contained in γ. Hence traveling along γ means that, either we stay in $P_a(\Gamma)$, or we jump between two points in Γ, at distance $\leq b$, through an edge of length m_1. Hence choosing $M = b$, we conclude thanks to the comparison lemma in $P_a(\Gamma)$.

Now let us suppose that $n \geq 2$. Fix some $\beta_1 > \beta_2 > \alpha$ and choose an M such that the induction hypothesis applies for $\beta = \beta_2$. We assume moreover that $M \leq m_k \leq K$ for all $1 \leq k \leq n-1$, where K is some integer. Let us assume that γ meets at least a simplex Δ of dimension n which does not belong to $P_a(\Gamma)$. Let $u < v$ be such that $\gamma(t) \in \Delta$ for $u \leq t \leq v$, and γ meets the boundary of Δ at u and v. We start with two observations. Let $\Delta = ([0,1] \times \partial\Delta)/(0 \times \partial\Delta)$ and let $\eta \in (0,1)$.

First, note that if γ meets $[0, 1-\eta] \times \partial\Delta$, then $v - u \geq \eta m_n$. But the diameter of $\partial\Delta$ is less than ρK, for some ρ depending only on n. Hence we can replace the portion of γ between u and v by a path contained in $\partial\Delta$, of length $\leq \rho K \leq \rho K(v-u)/(\eta m_n)$.

Second, if γ is contained in $[1-\eta, 1] \times \partial\Delta$, then observe that the retraction of $[1-\eta, 1] \times \partial\Delta$ onto $\partial\Delta$ is a $(1-\eta)^{-1}$-Lipschitz map, and hence, projecting γ to the boundary increases its length by at most $(1-\eta)^{-1}$. Hence there exists a path γ' completely contained in the union of $P_a(\Gamma)$ and the $(n-1)$-skeleton whose length l' satisfies

$$l' \leq (\rho K/(\eta m_n))l + (1-\eta)^{-1} l.$$

Applying the induction hypothesis to γ' yields

$$d_\Gamma(x, C) \leq a\beta_2 l' \leq a\beta_2((\rho K/(\eta m_n) + (1-\eta)^{-1})l.$$

First fix η such that

$$\beta_2(1-\eta)^{-1} < \beta_1.$$

We then take $M' \geq M$ big enough so that

$$\beta_2((\rho K/(\eta m_n) + (1-\eta)^{-1}) \leq \beta_1$$

for all $m_n \geq M'$. This gives the desired inequality

$$d_\Gamma(x, C) \leq a\beta_1 l,$$

under the assumption that $M \leq m_k \leq K$ for $1 \leq k \leq n-1$, and $m_n \geq M'$. But since increasing m_k can only increase l, this inequality remains true under the condition that $m_k \geq M'$ for all $1 \leq k \leq n$. □

Next we make the following observation, from which we will immediately deduce the *neighborhood* and the *separation* lemmas below.

A.3.3. **Lemma.** *Let C be a subspace of Γ and let $\varepsilon \geq 1$ and $a \geq 1$. There exists $\beta \geq 1$ depending only on the dimension of $P_a(\Gamma)$ such that the following statements are true. Viewing $P_a(C)$ as a subspace of $P_a(\Gamma)$ we have*

$$N_\varepsilon(P_a(C)) \cap \Gamma \subset N_{a\varepsilon\beta}(C),$$

Similarly for the relative Rips complex, viewing $P_b(C)$ as a subspace of $P_{ab}(\Gamma, C)$ ($b \geq a$) we have

$$N_\varepsilon(P_b(C)) \cap \Gamma \subset N_{a\varepsilon\beta}(C).$$

Finally, for the scaled complex, viewing $P_b(C)$ as a subspace of $P_{ab\overline{m}}(\Gamma, C)$ we have

$$N_\varepsilon(P_b(C)) \cap \Gamma \subset N_{a\varepsilon\beta}(C).$$

provided that $\overline{m}$ is large enough in sense that $m_k \geq M$ for all k, where M depends only on b.

The proof of the above lemma is straightforward and is left to the reader. The following lemma is an easy consequence of the previous and is left to the reader.

A.3.4. **Lemma** (Neighborhood lemma)**.** *Let $C \subset \Gamma$, $\varepsilon \geq 1$ and $a \geq 1$. Viewing $P_a(C) \subset P_a(\Gamma)$ we have*

$$N_\varepsilon(P_a(C)) \subset P_a(N_{a\varepsilon\beta}(C)),$$

for some constant β depending only on the dimension of $P_a(\Gamma)$. Similarly for the relative Rips complex, viewing $P_b(C) \subset P_{ab}(\Gamma, C)$ ($b \geq a$) we have

$$N_\varepsilon(P_b(C)) \subset P_{ab}(N_{a\varepsilon\beta}(C), C).$$

A.3.5. **Lemma** (Separation lemma)**.** *Let $\varepsilon \geq 1$ and $a \geq 1$. If the family $\mathcal{C}$ of subsets of Γ is ε-separated, then the family $P_a(\mathcal{C})$ (resp. $P_b(\mathcal{C})$) is $\varepsilon(a\beta)^{-1}$-separated in $P_a(\Gamma)$ (resp. in $P_{ab}(\Gamma, \mathcal{C})$ for $b \geq a$, and in $P_{ab\overline{m}}(\Gamma, \mathcal{C})$ for $b \geq a$ if $\overline{m}$ is large enough), where β only depends on the dimension of $P_a(\Gamma)$.*

Proof. The first two cases are direct consequences of the neighborhood lemma above. For the scaled complex, it follows from Lemma A.3.2. □

Note that the neighborhood lemma does not apply to the scaled Rips complex. Instead, we have the following slightly weaker statement whose proof is left to the reader.

A.3.6. **Lemma** (Lipschitz homotopy lemma)**.** *Let C and W be subspaces of the metric space Γ. Let $\varepsilon \geq 1$ and $b \geq a \geq 1$. Let V be the ε-neighborhood of $P_b(W)$ in $P_{ab\overline{m}}(\Gamma, W)$, let W' be* *the $a\beta\varepsilon$-neighborhood of W in Γ, where β is the constant appearing in Lemma A.3.3. Then, for all $c \geq b$, there exist $M > 0$ and a proper continuous map*

$$F : (P_{ac\overline{m}}(C, W') \cup V) \times [0,1] \to P_{ac\overline{m}}(C, W') \cup V$$

such that

(1) *$F(\cdot, t)$ is 2-Lipschitz for all $t \in [0,1]$, provided that $m_k \geq M$ for all k,*
(2) *for each $t \in [0,1]$, $F(\cdot, t)$ restricts to the identity map on $P_{ac\overline{m}}(C, W')$,*
(3) *$F(\cdot, 0)$ is the identity map on $P_{ac\overline{m}}(C, W') \cup V$, and the image of $F(\cdot, 1)$ lies in $P_{ac\overline{m}}(C, W')$.*

Moreover, the constant M depends only on ε and the dimension of $P_c(\Gamma)$.

Appendix B. Mayer-Vietoris sequences in bounded K and L-theory

In this section, we recall from [RY1, RY2] the controlled Mayer-Vietoris sequences in K and L-theory. These are important tools in our proof of the bounded Borel conjecture for spaces with finite decomposition complexity.

B.1. **Theorem.** *Let X be a metric space, written as the union of closed subspaces $X = A \cup B$. There exists a universal constant $\lambda > 1$ (independent of X, A and B) such that for each $\delta > 0$,*

(1) *in $Wh^\delta(A \cap B) \xrightarrow{i_*} Wh^\delta(A) \oplus Wh^\delta(B) \xrightarrow{j_*} Wh^\delta(X)$, we have $j_* i_* = 0$;*
(2) *if $N_{\lambda\delta}(A \cap B) \subset W$, then the relax-control image of the kernel of j_* in $Wh^{\lambda^2\delta}(A \cup W) \oplus Wh^{\lambda^2\delta}(B \cup W)$ is contained in the image of i_* below*

$$\begin{array}{ccc} & Wh^\delta(A) \oplus Wh^\delta(B) & \xrightarrow{j_*} Wh^\delta(X)\ , \\ & \downarrow & \\ Wh^{\lambda^2\delta}(W) \xrightarrow{i_*} & Wh^{\lambda^2\delta}(A \cup W) \oplus Wh^{\lambda^2\delta}(B \cup W) & \end{array}$$

where $N_{\lambda\delta}(A \cap B) = \{\, x \in X \colon d(x, A \cap B) \leq \lambda\delta \,\}$;

(3) *if $N_{\lambda\delta}(A \cap B) \subset W$, then in*

$$Wh^\delta(A) \oplus Wh^\delta(B) \xrightarrow{j_*} Wh^\delta(X) \xrightarrow{\partial} \tilde{K}_0^{\lambda\delta}(W),$$

we have $\partial j_* = 0$;

(4) *if $N_{\lambda\delta}(A \cap B) \subset W$, then the relax-control image of the kernel of ∂ in $Wh^{\lambda^2\delta}(X)$ is contained in the image of j_* below*

$$\begin{array}{ccc} & Wh^\delta(X) & \xrightarrow{\partial} \tilde{K}_0^{\lambda\delta}(W) \\ & \downarrow & \\ Wh^{\lambda^2\delta}(A \cup W) \oplus Wh^{\lambda^2\delta}(B \cup W) \xrightarrow{j_*} & Wh^{\lambda^2\delta}(X) & \end{array}$$

(5) *if* $N_{\lambda\delta}(A\cap B)\subset W$*, then in*

$$Wh^{\delta}(X)\xrightarrow{\partial}\tilde K_0^{\lambda\delta}(W)\xrightarrow{i_*}\tilde K_0^{\lambda\delta}(A\cup W)\oplus\tilde K_0^{\lambda\delta}(B\cup W),$$

we have $i_*\partial=0$*;*

(6) *if* $N_{\lambda\delta}(A\cap B)\subset W$*, then the relax-control image of the kernel of* i_* *in* $\tilde K_0^{\lambda^2\delta}(W)$ *is contained in the image of* ∂

$$\begin{tikzcd} & \tilde K_0^{\delta}(A\cap B) \arrow[r, "i_*"] \arrow[d] & \tilde K_0^{\delta}(A)\oplus\tilde K_0^{\delta}(B) \\ Wh^{\lambda\delta}(X) \arrow[r, "\partial"] & \tilde K_0^{\lambda^2\delta}(W) & \end{tikzcd}$$

The precise L-theory version we require is the following result where, for each metric space Y, each integer $n\ge 0$ and $\delta>0$, $L_n^{\delta}(Y)$ is the δ-controlled locally finite and free L-theory of Y [RY2]. This result is a consequence of Theorem 7.3 and Proposition 4.6 in [RY2]), Proposition 3.2 and Proposition 3.4 in [RY1].

B.2. **Theorem.** *Let* P *be a locally compact polyhedron and* P' *a subpolyhedron of* P*. Assume that* P *and* P' *are respectively given with metrics* d *and* d' *satisfying* $d(x,y)\le d'(x,y)$ *for all* x *and* y *in* P'*. Let* X *be a metric subspace of* P'*. Assume that* X *is written as the union of closed subspaces* $X=A\cup B$*. For every integer* $n\ge 2$ *there exists* $\lambda_n>1$*, which depends only on* n*, such that for each* $\delta>0$*,*

(1) *in* $L_n^{\delta}(A\cap B)\xrightarrow{i_*}L_n^{\delta}(A)\oplus L_n^{\delta}(B)\xrightarrow{j_*}L_n^{\delta}(X)$*, we have* $j_*i_*=0$*, where the metrics on* $A\cap B$*,* A*,* B *and* X *are inherited from the metric of* P'*;*

(2) *if* $N_{\lambda_n\delta}(A\cap B)\subseteq W\subseteq P$ *and the natural homomorphism from* $\tilde K_0^{\lambda_n\delta}(N_{\lambda_n\delta}(A\cap B))$ *to* $\tilde K_0^{\lambda_n\delta}(W)$ *is zero, then the relax-control image of the kernel of* j_* *in*

$$L_n^{\lambda_n^2\delta}(A\cup W)\oplus L_n^{\lambda_n^2\delta}(B\cup W)$$

is contained in the image of i_* *below*

$$\begin{tikzcd} & L_n^{\delta}(A)\oplus L_n^{\delta}(B) \arrow[r, "j_*"] \arrow[d] & L_n^{\delta}(X)\ , \\ L_n^{\lambda_n\delta}(W) \arrow[r, "i_*"] & L_n^{\lambda_n^2\delta}(A\cup W)\oplus L_n^{\lambda_n^2\delta}(B\cup W) & \end{tikzcd}$$

where $N_{\lambda_n\delta}(A\cap B)=\{\,x\in X\colon d(x,A\cap B)\le\lambda_n\delta\,\}$ *is given the metric of* P'*, the metrics on* A*,* B *and* X *are inherited from the metric of* P'*, and the metrics on* W*,* $A\cup W$ *and* $B\cup W$ *are inherited from the metric of* P*;*

(3) *if* $N_{\lambda_n\delta}(A\cap B)\subseteq W\subseteq P$ *and the natural homomorphism from* $\tilde K_0^{\lambda_n\delta}(N_{\lambda_n\delta}(A\cap B))$ *to* $\tilde K_0^{\lambda_n\delta}(W)$ *is zero, then in*

$$L_n^{\delta}(A)\oplus L_n^{\delta}(B)\xrightarrow{j_*}L_n^{\delta}(X)\xrightarrow{\partial}L_{n-1}^{\lambda_n\delta}(W),$$

we have $\partial j_* = 0$*, where* $N_{\lambda_n\delta}(A\cap B) = \{\, x \in X : d(x, A\cap B) \le \lambda_n\delta \,\}$ *is given the metric of* P'*, the metrics on* A*,* B *and* X *are inherited from the metric of* P'*, and the metric on* W *is inherited from the metric of* P*;*

(4) *if* $N_{\lambda_n\delta}(A\cap B) \subseteq W \subseteq P$ *and the natural homomorphism from* $\tilde{K}_0^{\lambda_n\delta}(N_{\lambda_n\delta}(A\cap B))$ *to* $\tilde{K}_0^{\lambda_n\delta}(W)$ *is zero, then the relax-control image of the kernel of* ∂ *in* $L_n^{\lambda_n^2\delta}(X)$ *is contained in the image of* j_* *below*

$$\begin{array}{ccc} & L_n^{\delta}(X) & \xrightarrow{\partial} L_{n-1}^{\lambda_n\delta}(W) \\ & \downarrow & \\ L_n^{\lambda_n^2\delta}(A\cup W) \oplus L_n^{\lambda_n^2\delta}(B\cup W) & \xrightarrow{j_*} L_n^{\lambda_n^2\delta}(X\cup W) & \end{array}$$

where $N_{\lambda_n\delta}(A\cap B) = \{\, x \in X : d(x, A\cap B) \le \lambda_n\delta \,\}$ *is given the metric of* P'*, the metric on* X *is inherited from the metric of* P'*, and the metrics on* W*,* $A\cup W$*,* $B\cup W$ *and* $X\cup W$ *are inherited from the metric of* P*;*

(5) *if* $N_{\lambda_n\delta}(A\cap B) \subseteq W \subseteq P$ *and the natural homomorphism from* $\tilde{K}_0^{\lambda_n\delta}(N_{\lambda_n\delta}(A\cap B))$ *to* $\tilde{K}_0^{\lambda_n\delta}(W)$ *is zero, then in*

$$L_n^{\delta}(X) \xrightarrow{\partial} L_{n-1}^{\lambda_n\delta}(W) \xrightarrow{i_*} L_{n-1}^{\lambda_n\delta}(A\cup W) \oplus L_{n-1}^{\lambda_n\delta}(B\cup W),$$

we have $i_*\partial = 0$*, where* $N_{\lambda_n\delta}(A\cap B) = \{\, x \in X : d(x, A\cap B) \le \lambda_n\delta \,\}$ *is given the metric of* P'*, the metric on* X *is inherited from the metric of* P'*, and the metrics on* W*,* $A\cup W$ *and* $B\cup W$ *are inherited from the metric of* P*;*

(6) *if* $N_{\lambda_n\delta}(A\cap B) \subseteq W \subseteq P$ *and the natural homomorphism from* $\tilde{K}_0^{\lambda_n\delta}(N_{\lambda_n\delta}(A\cap B))$ *to* $\tilde{K}_0^{\lambda_n\delta}(W)$ *is zero, then the relax-control image of the kernel of* i_* *in* $L_{n-1}^{\lambda_n^2\delta}(W)$ *is contained in the image of* ∂

$$\begin{array}{ccc} & L_{n-1}^{\delta}(A\cap B) & \xrightarrow{i_*} L_{n-1}^{\delta}(A) \oplus L_{n-1}^{\delta}(B) \\ & \downarrow & \\ L_n^{\lambda_n\delta}(X) & \xrightarrow{\partial} L_{n-1}^{\lambda_n^2\delta}(W) & \end{array}$$

where $N_{\lambda_n\delta}(A\cap B) = \{\, x \in X : d(x, A\cap B) \le \lambda_n\delta \,\}$ *is given the metric of* P'*, the metrics on* X*,* $A\cap B$*,* A *and* B *are inherited from the metric of* P'*, and the metric on* W *is inherited from the metric of* P*.*

References

[AS] R. Alperin, P. Shalen. *Linear Groups of Finite Cohomological Dimension.* Invent. Math. 66 (1982), no. 1, 89–98.

[B] A. Bartels. *Squeezing and higher algebraic K-theory.* K-Theory 28 (2003), no. 1, 19–37.

[BL] A. Bartels, W. Lück. *The Borel Conjecture for hyperbolic and CAT(0)-groups.* arXiv:0901.0442v1, 2009.

[BD1] G. Bell, A. Dranishnikov. *On asymptotic dimension of groups.* Algebr. Geom. Topol. 1 (2001), 57–71 (electronic).

[BD] G. Bell, A. Dranishnikov. *A Hurewicz-type theorem for asymptotic dimension and applications to geometric group theory.* ArXiv:math.GR/0407431, 2004.

[BR] A. Bartels, D. Rosenthal. *On the K-theory of groups with finite asymptotic dimension.* J. Reine Angew. Math. 612 (2007), 35–57.

[CG] G. Carlsson, B. Goldfarb. *The integral K-theoretic Novikov conjecture for groups with finite asymptotic dimension.* Invent. Math. 157 (2004), no. 2, 405–418.

[CP] G. Carlsson, E. Pedersen. *Controlled algebra and the Novikov conjectures for K- and L-theory.* Topology 34 (1995), no. 3, 731–758.

[CFY] S. Chang, S. Ferry, G. Yu. *Bounded rigidity of manifolds and asymptotic dimension growth.* J. K-Theory 1 (2008), no. 1, 129–144.

[D] M. Davis. *Groups generated by reflections and aspherical manifolds not covered by Euclidean space.* Ann. of Math. (2) 117 (1983), no. 2, 293–324.

[DFW] A. Dranishnikov, S. Ferry, S. Weinberger. *Large Riemannian manifolds which are flexible.* Ann. of Math. (2) 157 (2003), no. 3, 919–938.

[DS] A. Dranishnikov, J. Smith. *Asymptotic dimension of discrete groups.* Fund. Math. 189 (2006), no. 1, 27–34.

[Ei] D. Eisenbud. *Commutative algebra.* Graduate Texts in Mathematics 150, Springer, 1991.

[FH] F. T. Farrell, W. C. Hsiang. *On Novikov's conjecture for nonpositively curved manifolds.* Ann. of Math. (2) 113 (1981), no. 1, 199–209.

[FJ1] T. Farrell, L. Jones. *A topological analogue of Mostow's rigidity theorem.* J. Amer. Math. Soc. 2 (1989), no. 2, 257–370.

[FJ2] T. Farrell, L. Jones. *Classical aspherical manifolds.* CBMS Regional Conference Series in Mathematics, 75. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 1990.

[FJ3] T. Farrell, L. Jones. *Topological rigidity for compact non-positively curved manifolds.* Differential geometry: Riemannian geometry (Los Angeles, CA, 1990), 229–274, Proc. Sympos. Pure Math., 54, Part 3, Amer. Math. Soc., Providence, RI, 1993.

[FJ4] T. Farrell, L. Jones. *Rigidity for aspherical manifolds with $\pi_1 \subset GL_m(R)$.* Asian J. Math. 2 (1998), no. 2, 215–262.

[FP] S. Ferry, E. Pedersen. *Epsilon surgery theory.* Novikov conjectures, index theorems and rigidity, Vol. 2 (Oberwolfach, 1993), 167–226, London Math. Soc. Lecture Note Ser., 227, Cambridge Univ. Press, Cambridge, 1995.

[FW1] S. Ferry, S. Weinberger. *Curvature, tangentiality, and controlled topology.* Invent. Math. 105 (1991), no. 2, 401–414.

[FW2] S. Ferry, S. Weinberger. *A coarse approach to the Novikov conjecture.* Novikov conjectures, index theorems and rigidity, Vol. 1 (Oberwolfach, 1993), 147–163, London Math. Soc. Lecture Note Ser., 226, Cambridge Univ. Press, Cambridge, 1995.

[G] M. Gromov. *Asymptotic invariants of infinite groups.* Geometric group theory, Vol. 2 (Sussex, 1991), 1–295, London Math. Soc. Lecture Note Ser., 182, Cambridge Univ. Press, Cambridge, 1993.

[G] E. Guentner, *Permanence in coarse geometry.* Preprint 2010.

[GHW] E. Guentner, N. Higson, S. Weinberger. *The Novikov Conjecture for Linear Groups.* Publ. Math. Inst. Hautes Études Sci. No. 101 (2005), 243–268.

[GTY] E. Guentner, R. Tessera, G. Yu. *Discrete groups with finite decomposition complexity.* Submitted.

[HR] N. Higson, J. Roe. *On the coarse Baum-Connes conjecture, in: Novikov conjectures, index theorems and rigidity.* Vol. 2 (Oberwolfach 1993), London Math. Soc. Lect. Note Ser. 227, Cambridge Univ. Press, Cambridge (1995), 227–254.

[J] L. Ji. *The integral Novikov conjectures for linear groups containing torsion elements.* J. Topol. 1 (2008), no. 2, 306–316.

[KS] R. Kirby, L. Siebenmann. *Foundational essays on topological manifolds, smoothings, and triangulations.* With notes by John Milnor and Michael Atiyah. Annals of Mathematics Studies, No. 88. Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1977.

[L] S. Lang. *Algebra.* Addison-Wesley Publishing Co., Inc., Reading, Mass. 1965.

[Ma] D. Matsnev. *The Baum-Connes conjecture and proper group actions on affine buildings.* math.GT/0703923

[RY1] A. Ranicki, M. Yamasaki. *Controlled K-theory.* Topology Appl. 61 (1995), no. 1, 1–59.

[RY2] A. Ranicki, M. Yamasaki. *Controlled L-theory.* Exotic homology manifolds—Oberwolfach 2003, Geom. Topol. Monogr. 9 (2006), 105–153.

[Y1] G. Yu. *The Novikov conjecture for groups with finite asymptotic dimension.* Ann. of Math. (2) 147 (1998), no. 2, 325–355.

[Y2] G. Yu. *The coarse Baum-Connes conjecture for spaces which admit a uniform embedding into Hilbert space.* Invent. Math. 139 (2000), no. 1, 201–240.

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