DEHN FUNCTION AND ASYMPTOTIC CONES OF ABELS' GROUP

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Abstract. We prove that Abels' group over an arbitrary local field has a quadratic Dehn function. As applications, we exhibit connected Lie groups and polycyclic groups whose asymptotic cones have uncountable abelian fundamental group. We also obtain, from the case of finite characteristic, uncountably many non-quasi-isometric finitely generated solvable groups, as well as peculiar examples of fundamental groups of asymptotic cones.

1. Introduction

Let R be a commutative ring. We consider the following solvable group, introduced by Abels [Ab1].

$$A_4(R) = \left\{ \begin{pmatrix} 1 & x_{12} & x_{13} & x_{14} \\ 0 & t_{22} & x_{23} & x_{24} \\ 0 & 0 & t_{33} & x_{34} \\ 0 & 0 & 0 & 1 \end{pmatrix} : \; x_{ij} \in R; \; t_{ii} \in R^{\times}. \right\} \tag{1.1}$$

Observe that if we denote by $Z(R)$ the subgroup generated by unipotent matrices whose only nonzero off-diagonal entry is x_{14}, then $Z(R)$ is central in $A_4(R)$. Abels' initial motivation was to exhibit a finitely presented group with an infinitely generated central subgroup, namley $A_4(\mathbf{Z}[1/p])$. In particular, its quotient by the central subgroup $Z(\mathbf{Z}[1/p])$ is not finitely presented, showing that the class of finitely presented solvable groups is not stable under taking quotients.

Abels' group over local fields. Finite presentation of $A_4(\mathbf{Z}[1/p])$ is closely related to the fact that $A_4(\mathbf{Q}_p)$ is compactly presented [Ab2], motivating the study of Abels' group over local fields. By local field, we mean a nondiscrete locally compact field, endowed with its norm. Our main goal in this paper is to provide the following quantitative version of this result.

Theorem 1.1. *For every local field* $\mathbf{K}$*, the group* $A_4(\mathbf{K})$ *has a quadratic Dehn function.*

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This means that loops of length r in $A_4(\mathbf{K})$ can be filled with area in $O(r^2)$ when $r \to \infty$. This extends Abels' result [Ab1, Ab2] that $A_4(\mathbf{K})$ is compactly presented if $\mathbf{K}$ has characteristic zero; besides, Abels's result is nontrivial only when $\mathbf{K}$ is ultrametric, while Theorem 1.1 is meaningful when $\mathbf{K}$ is Archimedean too.

Recall that given a metric space (X, d) and a nonprincipal ultrafilter ω, a certain metric space $\mathrm{Cone}_\omega(X)$ can be defined as an "ultralimit" of the metric spaces $(X, \frac{1}{n}d)$ when $n \to \omega$, and is called the asymptotic cone of X along ω. (The precise definitions will be recalled in Section 2.). The bilipschitz type of $\mathrm{Cone}_\omega(X)$ is a quasi-isometry invariant of X.

By Papasoglu's theorem [Pap], a quadratic Dehn function implies that every asymptotic cone is simply connected, so we obtain

Corollary 1.2. *For every local field* $\mathbf{K}$ *and every nonprincipal ultrafilter* ω*, the asymptotic cone* $\mathrm{Cone}_\omega(A_4(\mathbf{K}))$ *is simply connected.*

Asymptotic cones and central extensions. Corollary 1.2 can be used to obtain various examples of unusual asymptotic cones, using generalities on central extensions, which we now partly describe (see Theorem 4.7 for a full version). Our main tool relates, for certain central extensions

$$1 \to Z \to G \to Q \to 1, \tag{1.2}$$

the fundamental group $\pi_1(\mathrm{Cone}_\omega(Q))$ and the group $\mathrm{Cone}_\omega(Z)$, where Z is endowed with the restriction of the metric of G. It can be viewed as an analogue of the connection between the fundamental group of a Lie group and the center of its universal cover. Notably, it implies the following

Theorem 1.3 (Corollary 4.8)**.** *Given a central extension as in (1.2), if* G, Q *are compactly generated locally compact groups,* $\mathrm{Cone}_\omega(G)$ *is simply connected and* $\mathrm{Cone}_\omega(Z)$ *is ultrametric, then*

$$\pi_1(\mathrm{Cone}_\omega(Q)) \simeq \mathrm{Cone}_\omega(Z).$$

As a direct application of Corollary 1.2 and Theorem 1.3, we deduce

Corollary 1.4 (Corollary 4.9)**.** *For every local field* $\mathbf{K}$ *and nonprincipal ultrafilter* ω*, the group* $\pi_1(\mathrm{Cone}_\omega(A_4(\mathbf{K})/Z(\mathbf{K})))$ *is an uncountable abelian group.*

Remark 1.5. If $\mathbf{K}$ is non-Archimedean, $Z(\mathbf{K})$ is not compactly generated, and it follows that $A_4(\mathbf{K})/Z(\mathbf{K})$ is not compactly presented. Similarly, if $\mathbf{K}$ is Archimedean, the exponential distortion of $Z(\mathbf{K})$ implies that the Lie group $A_4(\mathbf{K})/Z(\mathbf{K})$ has an exponential Dehn function.

Application to discrete groups. We further obtain, from a slight variant of Theorem 1.1, results concerning discrete groups. The first corollary, proved in Section 5, is the following

Corollary 1.6. *There exists a polycyclic group Λ, namely $A_4(R)/Z(R)$ if R is the ring of integers of a totally real number field of degree 3, such that Λ has an exponential Dehn function, and for every ω, the fundamental group $\pi_1(\mathrm{Cone}_\omega(\Lambda))$ is abelian and nontrivial, namely isomorphic to a $\mathbf{Q}$-vector space of continuum dimension.*

In the case of characteristic p, we prove

Corollary 1.7 (Theorem 5.1). *The group $A_4(\mathbf{F}_p[t, t^{-1}, (t-1)^{-1}])$ is finitely presented and has a quadratic Dehn function.*

The finite presentation of this group seems to be a new result. Note that $A_4(\mathbf{F}_p[t])$ is not finitely generated, while $A_4(\mathbf{F}_p[t, t^{-1}])$ is finitely generated but not finitely presented (see Remark 5.5). There were previous (substantially more complicated) examples of finitely presented solvable groups whose center contains an infinite-dimensional $\mathbf{F}_p$-vector space in [BGS, §2.4] and [Kha, §2]; those examples have an undecidable word problem, so they are not residually finite and their Dehn functions are not recursively bounded. Corollary 1.7, again combined with Theorem 1.3, has the following three corollaries.

Corollary 1.8. *There exist continuum many pairwise non-quasi-isometric solvable (actually (3-nilpotent)-by-abelian) finitely generated groups. Namely, for each prime p, such groups can be obtained as quotients of $A_4(\mathbf{F}_p[t, t^{-1}, (t-1)^{-1}])$ by central subgroups.*

To distinguish this many classes, we associate, to any metric space X, the subset of the set $\mathcal{U}_\infty(\mathbf{N})$ of nonprincipal ultrafilters on the integers

$$\nu(X) = \{\omega \in \mathcal{U}_\infty(\mathbf{N}) : \mathrm{Cone}_\omega(X) \text{ is simply connected}\};$$

the subset $\nu(X) \subset \mathcal{U}_\infty(\mathbf{N})$ is a quasi-isometry invariant of X, and we obtain the corollary by proving that ν achieves continuum many values on a certain class of groups (however, the subset $\nu(X)$ can probably not be arbitrary).

Corollary 1.8 is proved in §6.B, as well as the following one, which relies on similar ideas.

Corollary 1.9. *There exists a (3-nilpotent)-by-abelian finitely generated group R, namely a suitable central quotient of $A_4(\mathbf{F}_p[t, t^{-1}, (t-1)^{-1}])$, for which for every* *ultrafilter ω, we have $\pi_1(\mathrm{Cone}_\omega(R))$ is isomorphic to either $\mathbf{F}_p$ (cyclic group on p elements) or is trivial, and is isomorphic to $\mathbf{F}_p$ for at least one ultrafilter.*

To put Corollary 1.9 into perspective, recall that Gromov [Gro, 2.B_1(d)] asked whether an asymptotic cone of a finitely generated group always has trivial or infinitely generated fundamental group. In [OOS], a first counterexample was given, for which for some ultrafilter (in their language: for a fixed –fastly growing– scaling sequence and all ultrafilters) the fundamental group is $\mathbf{Z}$. Here we provide the first example for which the fundamental group is finite and nontrivial. Moreover we use the scaling sequence $(1/n)$, which reads all scales, so in our example the fundamental group is finite for all ultrafilters and scaling sequences.

The first example of a finitely generated group with two non-homeomorphic asymptotic cones was obtained by Thomas and Velickovic [TV], and was improved to $2^{\aleph_0}$ non-homeomorphic asymptotic cones by Drutu and Sapir [DS]. These examples are not solvable groups, although amenable examples (satisfying no group law) appear in [OOS]. Corollary 1.9 provides the first examples of finitely generated solvable groups with two non-homeomorphic asymptotic cones. The next corollary, obtained in §6.E by a variation on the proof of the previous one, improves it to infinitely many non-homeomorphic asymptotic cones.

Corollary 1.10. *There exist a (3-nilpotent)-by-abelian finitely generated group, namely a suitable central quotient of $A_4(\mathbf{F}_p[t,t^{-1},(t-1)^{-1}])$, with at least $2^{\aleph_0}$ non-bilipschitz homeomorphic, respectively at least $\aleph_0$, non-homeomorphic, asymptotic cones.*

The fundamental group of a metric space has a natural bi-invariant pseudo-metric space structure, whose bilipschitz class is a bilipschitz invariant of the metric space, and the isomorphism in Theorem 1.3 is actually bilipschitz. This extra-feature is used in the proof of the bilipschitz statement of Corollary 1.10. Actually, this pseudo-metric structure on the fundamental group of the cone is used in a crucial way in the proof of Theorem 1.3 itself.

In view of Theorem 1.3, Corollaries 1.8, 1.9, and 1.10 are all based on a systematic study in Section 6 of the asymptotic cone of the infinite direct sum $\mathbf{F}_p^{(\mathbf{N})}$, endowed with a suitable invariant ultrametric. In particular, Theorem 6.10 provides a complete classification of these asymptotic cones up to isomorphism of topological groups.

Organization. Section 2 recalls the definition of Dehn function and asymptotic cone. Section 1.1 is devoted to the proof of Theorem 1.1. In order to carry over the results to discrete groups, a minor variant of Theorem 1.1 is proved in Section

5. Finally Section 6 deals with asymptotic cones of the metric group $\mathbf{F}_p^{(\mathbf{N})}$, whose description leads to the latter corollaries.

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Contents

2. Dehn function, asymptotic cones

2.A. **Dehn function.** Let G be a locally compact group, generated by some compact symmetric subset S. It is *compactly presented* if for some r, the set $\mathcal{R}_r$ of relations of length at most r between elements of S generates the set of all relations, i.e. G admits the presentation $\langle S|\mathcal{R}_r\rangle$.

If this holds, and if w is a relation, i.e. an element of the kernel K_S of the natural homomorphism $F_S \to G$ (where F_S is the free group over S), the area $\alpha(w)$ of w is defined as the length of w with respect to the union of F_S-conjugates of $\mathcal{R}_r$. The Dehn function is then defined as

$$\delta(n) = \sup\{\alpha(w) : \; w \in K_S, \; |w|_S \leq n\}.$$

This function takes finite values. Although it depends on S and r, its asymptotic behavior only depends on G. By convention, if G is not compactly presented, we say it has an identically infinite Dehn function. Any two quasi-isometric locally compact compactly generated groups have asymptotically equivalent Dehn functions.

2.B. **Asymptotic cone.** Let (X, d) be a metric space and ω a nonprincipal ultrafilter. Define

$$\text{Precone}(X) = \{x = (x_n) : x_n \in X, \limsup d(x_n, x_0)/n < \infty\}.$$

Define the pseudo-distance

$$d_\omega(x,y) = \lim_\omega \frac{1}{n} d(x_n, y_n);$$

and $\mathrm{Cone}_\omega(X)$ as the Hausdorffication of $\mathrm{Precone}(X)$, endowed with the resulting distance d_ω.

If $X = G$ is a group and d is left-invariant, then $\mathrm{Precone}(G)$ (written $\mathrm{Precone}(G,d)$ if needed) is obviously a group, d_ω is a left-invariant pseudodistance, and $\mathrm{Cone}_\omega(G) = \mathrm{Precone}(G)/H$, where H is the subgroup of elements at pseudodistance 0 from the identity. It is not normal in general. However in case d is bi-invariant (e.g. when G is abelian), it is normal and $\mathrm{Cone}_\omega(G,d)$ is then naturally a group endowed with a bi-invariant distance.

Taking asymptotic cones is a functor between metric spaces, mapping large-scale Lipschitz maps to Lipschitz maps and identifying maps at bounded distance. In particular, it maps quasi-isometries to bilipschitz homeomorphisms.

3. Proof of Theorem 1.1

Fix a local field $\mathbf{K}$ and write $G = A_4(\mathbf{K})$; let us show that G has a quadratic Dehn function. The proof below partly uses some arguments borrowed from the metabelian case [CT], which apply to several subgroups of G (see Lemma 3.3), as well as Gromov's trick (see §3.C). However, the presence of a distorted center entails bringing in several new arguments, gathered in §3.D.

3.A. **Generation of** G**.** Fix $t \in \mathbf{K}$ with $|t| > 1$. Let $G \subset A_4(\mathbf{K})$ be the subgroup of elements whose diagonal $(1, t_{22}, t_{33}, 1)$ consists of elements of the form $(1, t^{n_2}, t^{n_3}, 1)$ with $(n_2, n_3) \in Z^2$. So G is closed and cocompact in $A_4(\mathbf{K})$ and we shall therefore stick to G.

Let D be the set of diagonal elements in G and let T be the set of diagonal elements as above with $(n_1, n_2) \in \{(\pm 1, 0), (0, \pm 1)\}$. Let U_{ij} be the set of upper unipotent elements with all upper coefficients vanishing except u_{ij}, and U_{ij}^1 the set of elements in U_{ij} with $|u_{ij}| \le 1$.

Define $S = T \cup W$, where $W = \bigcup_{1 \le i < j \le 4, (i,j) \neq (1,4)} U_{ij}^1$. This is a compact, symmetric subset of G. We begin by the easy

Lemma 3.1. *The subset S generates G.*

Proof. Observe that T generates D, and that for all (i,j) with $i < j$ and $(i,j) \neq (1,4)$, D and U_{ij}^1 generates DU_{ij}. Moreover $U_{14} \subset [U_{12}, U_{24}]$. So the subgroup generated by S contains D and U_{ij} for all $i < j$. These clearly generate G. □

Actually, we need a more quantified statement. Let $[T]$ be the set of words in T, and $|\cdot|_T$ the word length in $[T]$.

Lemma 3.2. *There exists a constant $C > 0$ such that every element in the n-ball S^n can be written as*

$$d \prod_{i=1}^{9} t_i s_i t_i^{-1}, \tag{3.1}$$

where $d, t_i \in [T]$, $s_i \in W$, and $|c|_T \leq n$, $|t_i|_T| \leq Cn$.

Proof. Start from an element g in G. It can be written in a unique way $du_{12}u_{23}u_{34}u_{13}u_{24}u_{14}$ with $d \in D$, $u_{ij} \in U_{ij}$. In turn, u_{14} can be uniquely written as $[v_{12}, v_{24}]$, where the $(1,2)$-entry of $v_{12} \in U_{12}$ is 1 and $v_{24} \in U_{24}$.

If we assume that $|g|_T \leq n$, then clearly $|d|_T \leq n$, and there exists a universal constant $C > 1$ (only depending on the norm of t such that the nonzero upper-diagonal entry of the u_{ij} or v_{24} is at most C^n. So each u_{ij} can be written as $\gamma_{ij} s_{ij} \gamma_{ij}^{-1}$, where $\gamma_{ij} \in [T]$ has length $\leq C'n$ an $s_{ij} \in U_{ij}^1$. Similarly v_{24} can be written this way. (Actually, $C' = 1$ works if $\mathbf{K}$ is ultrametric.) □

3.B. **Subgroups of G with contracting elements.** An easy way to prove quadratic filling is the use of elements whose action by conjugation on the unipotent part is contracting. Although G itself does not contain such elements, we will show that it contains large enough such subgroups. More precisely, the proof that G has a quadratic Dehn function (implicitely) consists in showing that G is an amalgamated product of finitely many subgroups containing contractions (Abels used a similar strategy to show that G is compactly presented).

Lemma 3.3. *Let G_1 (resp. G_2, resp. G_3, resp. G_4) be the subgroup of matrices in G such that $x_{14} = x_{24} = x_{34} = 0$ (resp. $x_{12} = x_{13} = x_{14} = 0$, resp. $x_{12} = x_{23} = x_{34} = x_{14} = 0$, resp. $x_{13} = x_{14} = x_{23} = x_{24} = 0$). Then all G_i have quadratic Dehn functions.*

Proof. All the verification are based on a standard contraction argument. Let us start by G_1. Observe that in G_1, the left conjugation by the diagonal matrix $q = (1, t, t^2, 1)$ maps S into itself, so is 1-Lipschitz on (G, d_S). Thus the right multiplication by q^{-1} is 1-Lipschitz as well; it is moreover of bounded displacement (as any right multiplication), actually 3. Moreover it is contractive on the unipotent part. Since the 9-ball has nonempty interior, there exists a constant C such that for any n, if $B(n)$ is the n-ball around 1 in G, for every $g \in B(n)$, gq^{Cn} is at distance at most 9 from its projection to D.

So starting from a loop of size n, Cn successive right multiplications by q homotope it to a loop of the same size, in so that each element is at distance at most 9 from its projection to D. Each of these multiplications has cost $\le 3n$, provided that for every $s \in S$, the relator $qsq^{-1}s'^{-1}$ is in $\mathcal{R}$, where s' is the unique element of S represented by qsq^{-1} (which holds if all relations of length 8 are relators). This loop can be homotoped to its projection to D, provided all relations of size 20=9+1+9+1 are relators. Finally this loop in D has quadratic area (provided the obvious commuting relator of $D \simeq \mathbf{Z}^2$ is a relator).

By symmetry, G_2 is similar.

Finally, G_3 is also similar, using instead of q the diagonal element $q' = (1, t^{-1}, t, 1)$, whose left conjugation contracts both U_{13} and U_{24}, and q'^{-1} works for G_4. □

3.C. **Gromov's trick.** Let $\mathcal{R}(C, n)$ be the set of null-homotopic words of the form

$$\prod_{i=1}^{27} t_i s_i t_i^{-1},$$

where $t_i \in [T]$, $|t_i| \le Cn$, and $s_i \in W$.

Proposition 3.4. *Let M be large enough, and define the set of relators as the set of all words in S of length $\le M$, that represent the trivial element of G. For every C, there exists C' such that every word in $\mathcal{R}(C, n)$ has area $\le C'n^2$.*

By Lemmas 3.1 and 3.2, as well as Gromov's trick [CT, Prop. 4.3], to show that G has a quadratic Dehn function, it is enough to show that null-homotopic words that are concatenation of three words of the form (3.1) have quadratic area. By an obvious conjugation and using that $\mathbf{Z}^2$ has a quadratic Dehn function, this follows from Proposition 3.4, which we now proceed to prove.

3.D. **Proof of Proposition 3.4.** To simplify the exposition, we will adopt the following convenient language: the phrase "the word w can be replaced by the word w', with a quadratic cost", means that there are two universal constants C_1, C_2 such that $\ell(w') \le C_1 \ell(w)$, and $\alpha(w^{-1}w') \le C_2 \ell(w)^2$. This will be denoted, for short by: $w \rightsquigarrow_2 w'$.

Let $J = \{(i,j), 1 \le i < j \le 4, (i,j) \neq (1,4)\}$. For $(i,j) \in J$, let e_{ij}^x be the elementary matrix with entry (i,j) equal to x, and fix a word $\hat{e}_{ij}^x$ of minimal length in $T \cup U_{ij}^1$ representing e_{ij}^x.

Using Lemma 3.3 (or the even easier observation that DU_{ij} has a quadratic Dehn function for all (i,j)), Proposition 3.4 reduces to proving that, given $c > 1$, words of the form

$$w = \prod_{k=1}^{27} \hat{e}_{i_k j_k}^{x_k} \quad (\sup_k |x_k| \leq c^n) \tag{3.2}$$

have area in $O_c(n^2)$.

Lemma 3.3 has the following immediate consequence.

Claim 3.5. *We have*

(1) $[\hat{e}_{12}^u, \hat{e}_{23}^v] \rightsquigarrow_2 \hat{e}_{13}^{uv}$ *and* $\hat{e}_{13}^u \rightsquigarrow_2 [\hat{e}_{12}^1, \hat{e}_{23}^u]$ *;*
(2) $[\hat{e}_{23}^u, \hat{e}_{34}^v] \rightsquigarrow_2 \hat{e}_{2,4}^{uv}$ *and* $\hat{e}_{24}^u \rightsquigarrow_2 [\hat{e}_{23}^u, \hat{e}_{34}^1]$*;*
(3) $[\hat{e}_{23}^u, \hat{e}_{24}^v] \rightsquigarrow_2 1$*;*
(4) $[\hat{e}_{23}^u, \hat{e}_{13}^v] \rightsquigarrow_2 1$.

Claim 3.6. *We have*

(1) *for any* $(i,j) \in J$, $\hat{e}_{ij}^u \hat{e}_{ij}^v \rightsquigarrow_2 \hat{e}_{ij}^{u+v}$ *and* $(\hat{e}_{ij}^u)^{-1} \rightsquigarrow_2 \hat{e}_{ij}^{-u}$*;*
(2) $[\hat{e}_{12}^u, \hat{e}_{34}^v] \rightsquigarrow_2 1$*;*
(3) $[\hat{e}_{13}^u, \hat{e}_{24}^v] \rightsquigarrow_2 1$*;*
(4) $[\hat{e}_{34}^u, \hat{e}_{24}^v] \rightsquigarrow_2 1$*;*
(5) $[\hat{e}_{12}^u, \hat{e}_{13}^v] \rightsquigarrow_2 1$.

Claim 3.7. *For every relation w as in (3.2), we have*

$$w \rightsquigarrow_2 w_1 w_2, \quad \text{where } w_1 = \hat{e}_{13}^{u_1} \hat{e}_{34}^{v_1} \dots \hat{e}_{13}^{u_q} \hat{e}_{34}^{v_q}, \quad \text{and } w_2 = \hat{e}_{12}^{u'_1} \hat{e}_{24}^{v'_1} \dots \hat{e}_{12}^{u'_{q'}} \hat{e}_{24}^{v'_{q'}}, \tag{3.3}$$

with $q + q' \leq 27$.

Proof. In w, we can, using Claim 3.5, shuffle all subwords of the form $\hat{e}_{23}^{x_k}$ to the left. Some subwords of the form $\hat{e}_{13}^y$ or $\hat{e}_{24}^y$ appear: at most $27^2 = 729$ (although we can do much better). We can now aggregate all terms $\hat{e}_{23}^{x_k}$ to a single term $\hat{e}_{23}^x$, with quadratic cost by Claim 3.6(1); since w is null-homotopic, necessarily $x = 0$, so we got rid of all elements $\hat{e}_{23}^{x_k}$.

Next, using Claim 3.6, we can shuffle all subwords ($\hat{e}_{12}^{x_k}$ or $\hat{e}_{24}^{x_k}$) to the left, since they commute with quadratic cost with the subwords of the form ($\hat{e}_{13}^{x_k}$ or $\hat{e}_{34}^{x_k}$). We can aggregate when necessary consecutive subwords of the form $\hat{e}_{ij}^{x_k}$ using Claim 3.6(1). Since there are at most 27 subwords of the form $\hat{e}_{12}^{x_k}$ or $\hat{e}_{34}^{x_k}$ (or 12 instead of 27 with some little effort), the claim is proved. □

We are led to reduce words as in (3.3). We first need the following general formula. For commutators, we use the convention

$$[x, y] = x^{-1} y^{-1} x y.$$

In any group and for any elements $x_1, \dots y_k$, we have, denoting $x_{ij} = x_i x_{i+1} \dots x_j$, (if $i \le j$) and defining similarly y_{ij}

$$x_1 y_1 \dots x_k y_k = x_{1k} y_{1k} [y_{1k}, x_{2k}][x_{2k}, y_{2k}][y_{2k}, x_{3k}] \dots [x_{(k-1)k}, y_{(k-1)k}][y_{(k-1)k}, x_{kk}][x_{kk}, y_{kk}] \tag{3.4}$$

Thus, given w as in (3.3), and using Claim 3.6(1) several times, it can be reduced, with quadratic cost, to a word of the form

$$\hat{e}_{13}^x \hat{e}_{12}^y \hat{e}_{34}^z \hat{e}_{24}^t \prod_{i=1}^{52} [\hat{e}_{13}^{x_i} \hat{e}_{12}^{y_i}, \hat{e}_{34}^{z_i} \hat{e}_{24}^{t_i}]^{(-1)^i},$$

(here $52 = 2(27-1)$, observing that the number of brackets in (3.4) is $2(k-1)$). By projection, we have $x = y = z = t = 0$. Reshuffling as we did in the proof of Claim 3.7, we obtain, with quadratic cost, the word

$$\prod_{i=1}^{52} [\hat{e}_{13}^{x_i}, \hat{e}_{34}^{z_i}]^{(-1)^i} \prod_{i=1}^{52} [\hat{e}_{12}^{y_i}, \hat{e}_{24}^{t_i}]^{(-1)^i}. \tag{3.5}$$

We have proved

Claim 3.8. *Every relation w as in (3.2), can be reduced with quadratic cost to a relation as in (3.5).*

We now recall the following formula due to Hall, valid in any group.

$$[a^b, [b, c]] \cdot [b^c, [c, a]] \cdot [c^a, [a, b]] = 1, \tag{3.6}$$

where $a^b = b^{-1}ab$. We also recall the simpler formula

$$[ab, c] = [a, c]^b \, [b, c]. \tag{3.7}$$

Claim 3.9. *We have*

- $[\hat{e}_{13}^x, \hat{e}_{34}^y] \rightsquigarrow_2 [\hat{e}_{12}^1, \hat{e}_{24}^{xy}]$
- $[\hat{e}_{12}^x, \hat{e}_{24}^y] \rightsquigarrow_2 [\hat{e}_{13}^1, \hat{e}_{34}^{xy}]$

Proof. Both verifications are similar, so we only prove the first one. Define $(a, b, c) = (\hat{e}_{12}^1, \hat{e}_{23}^x, \hat{e}_{34}^y)$. First observe that using Claim 3.5, $\hat{e}_{13}^x \rightsquigarrow_2 [a, b]$, so

$$[\hat{e}_{13}^x, \hat{e}_{34}^y] \rightsquigarrow_2 [[a, b], c] = [c, [a, b]]^{-1}.$$

Since $[c, a] \rightsquigarrow_2 1$, we have $[b^c, [c, a]] \rightsquigarrow_2 1$ and $[c, [a, b]] \rightsquigarrow_2 [c^a, [a, b]]$; thus applying Hall's formula (3.6) to (a, b, c) we get

$$[c, [a, b]]^{-1} \rightsquigarrow_2 [a^b, [b, c]].$$

On the other hand, using (3.7),

$$[a^b, [b, c]] = [[b, a^{-1}]a, [b, c]] = [[b, a^{-1}], [b, c]]^a \cdot [a, [b, c]].$$

Thus using Claim 3.5 and Claim 3.6(3), we get

$$[a^b, [b, c]] \rightsquigarrow_2 [\hat{e}_{13}^x, \hat{e}_{24}^{xy}]^a \cdot [\hat{e}_{12}^1, \hat{e}_{24}^{xy}] \rightsquigarrow_2 [\hat{e}_{12}^1, \hat{e}_{24}^{xy}].$$

So we proved $[\hat{e}_{13}^x, \hat{e}_{34}^y] \rightsquigarrow_2 [\hat{e}_{12}^1, \hat{e}_{24}^{xy}]$. □

Define $\hat{e}_{14}^r = [\hat{e}_{12}, \hat{e}_{24}^r]$.

Claim 3.10. *We have* $[\hat{e}_{14}^r, \hat{e}_{24}^x] \rightsquigarrow_2 1$.

Proof. We have $[\hat{e}_{14}^r, \hat{e}_{24}^x] \rightsquigarrow_2 [[\hat{e}_{13}^1, \hat{e}_{34}^r], \hat{e}_{24}^x]$. Since $[\hat{e}_{13}^1, \hat{e}_{24}^x] \rightsquigarrow_2 1$ and $[\hat{e}_{34}^r, \hat{e}_{24}^x] \rightsquigarrow_2 1$, it follows that $[[\hat{e}_{13}^1, \hat{e}_{34}^r], \hat{e}_{24}^x] \rightsquigarrow_2 1$. □

In the following claim, we use the identities, true in an arbitrary group: $[a, b] = [b, a^{-1}]^a$ and $[a, bc] = [a, c][a, b]^c$.

Claim 3.11. *We have* $(\hat{e}_{14}^r)^{-1} \rightsquigarrow_2 \hat{e}_{14}^{-r}$ *and* $\hat{e}_{14}^r \hat{e}_{14}^s \rightsquigarrow_2 \hat{e}_{14}^{r+s}$.

Proof.

$$(\hat{e}_{14}^r)^{-1} = [\hat{e}_{12}^1, \hat{e}_{24}^r]^{-1} = [\hat{e}_{24}^r, \hat{e}_{12}^1] = [\hat{e}_{12}^1, (\hat{e}_{24}^r)^{-1}]^{\hat{e}_{24}^r}$$

Therefore, by Claims 3.6(1) and 3.10,

$$(\hat{e}_{14}^r)^{-1} \rightsquigarrow_2 [\hat{e}_{12}^1, \hat{e}_{24}^{-r}]^{\hat{e}_{24}^r} \rightsquigarrow_2 [\hat{e}_{12}^1, \hat{e}_{24}^{-r}] = \hat{e}_{14}^{-r}.$$

For the addition, using Claim 3.6(1) and Claim 3.10

$$\hat{e}_{14}^{r+s} = [\hat{e}_{12}^1, \hat{e}_{24}^{r+s}] \rightsquigarrow_2 [\hat{e}_{12}^1, \hat{e}_{24}^r \hat{e}_{24}^s] = [\hat{e}_{12}^1, \hat{e}_{24}^s][\hat{e}_{12}^1, \hat{e}_{24}^r]^{\hat{e}_{24}^s} = \hat{e}_{14}^s (\hat{e}_{14}^r)^{\hat{e}_{14}^s} \rightsquigarrow_2 \hat{e}_{14}^s \hat{e}_{14}^r.$$ □

Conclusion of the proof of Proposition 3.4. By Claim 3.8, we start from a word as in (3.5). By Claim 3.9, it can be reduced with quadratic cost to a word of the form $\prod_{i=1}^{104} (\hat{e}_{14}^{r_i})^{(-1)^i}$. The inverse reduction in Claim 3.11 reduces this word to $\prod_{i=1}^{104} \hat{e}_{14}^{(-1)^i r_i}$. The second one reduces it to $\hat{e}_{14}^s$, with $s = \sum (-1)^i r_i$. Since this is a null-homotopic word, $s = 0$ and we are done. □

4. Asymptotic cones and central extensions

4.A. **Topology on the fundamental group.** In order to determine the fundamental group of some asymptotic cones, it will be useful to equip it with a group topology, and actually better, with a bi-invariant metric. We will see two possible choices for such a metric, both being potentially interesting as they provide more refined quasi-isometry invariants than the fundamental group alone. We shall use them in order to state and prove Theorem 4.7 (which holds for both choices of metric).

Let X be a topological space with base-point x_0 with a basis of neighbourhoods $\mathcal{V}$. A naive way to define a topology on $\pi_1(X, x_0)$ is as follows. For every $V \in \mathcal{V}$, define K_V as the set of elements representable as a loop with image in V; this is a subgroup of $\pi_1(X,x_0)$. However, there is, in general, no group topology on $\pi_1(X,x_0)$ such that $(K_V)_{V\in\mathcal{V}}$: indeed, it is not necessarily true (e.g. if $X = \mathbf{R}^2 - \mathbf{Q}^2$) that if g is fixed and $g_i \to 1$, then $gg_ig^{-1} \to 1$ for this topology.

A natural solution is simply to replace K_V by its normal closure. In other words, define L_V as the set of elements in $\pi_1(X,x_0)$ that can be represented by a finite product $\prod_{i=1}^k c_i\gamma_i c_i^{-1}$, where c_i,γ_i are loops based at x_0 and γ_i has image in V. Clearly L_V is a normal subgroup in $\pi_1(X,x_0)$, and $L_V \cap L_W \supset L_{V\cap W}$ for all $V,W \in \mathcal{V}$. It follows that the cosets of L_V, for $V \in \mathcal{V}$, form a basis of open (actually clopen) sets for a topology on $\pi_1(X,x_0)$, which is a group topology.

Equivalently, $\pi_1(X,x_0)$ is endowed with the topology induced by the homomorphic mapping into $\prod_{V\in\mathcal{V}} \pi_1(X,x_0)/L_V$, each $\pi_1(X,x_0)/L_V$ being discrete and the product being endowed with the product topology. Then $(X,x_0) \mapsto \pi_1(X,x_0)$ is a functor from the category of pointed topological spaces to the category of topological group. In particular, any two homeomorphic pointed topological spaces have their fundamental group isomorphic as topological groups.

If the topology of X is defined by a metric d, this topology is pseudo-metrizable, where the pseudo-distance of a loop γ to the identity is defined as $\inf\{\varepsilon > 0 : \gamma \in L_{B(\varepsilon)}\}$, where $B(\varepsilon)$ is the closed ε-ball around x_0. This pseudo-distance is bi-invariant and satisfies the ultrametric inequality. Then $(X,x_0) \mapsto \pi_1(X,x_0)$ is a functor from the category of pointed metric spaces with pointed isometric (resp. Lipschitz) maps, to the category of pseudo-metric groups. In particular, any two isometric (resp. bilipschitz) pointed metric spaces have their fundamental groups isometrically (resp. bilipschitz) isomorphic as pseudo-metric groups.

4.B. **Central subgroups and liftings.** Let G be a locally compact compactly generated group and Z a closed central subgroup. Fix an ultrafilter ω once and for all. Assume that $\mathrm{Cone}_\omega(Z)$ is totally disconnected, where Z is always endowed with the word metric from G.

If X is a metric space with base-point x_0, denote by $\mathcal{P}(X)$ the set of paths in X based at x_0, i.e. of continuous bounded maps from $\mathbf{R}_+$ to X mapping 0 to x_0 (in the sequel since the considered metric spaces will be homogeneous, the choice of x_0 won't matter and therefore will be kept implicit). This is a metric space with the sup distance.

There is an obvious 1-Lipschitz map $\psi : \mathcal{P}(\mathrm{Cone}_\omega(G)) \to \mathcal{P}(\mathrm{Cone}_\omega(G/Z))$. As Z is abelian, $\mathrm{Cone}_\omega(Z)$ is a topological abelian group in the natural way; moreover Z being central, the action of Z on G by (left) multiplication induces an action of $\mathrm{Cone}_\omega(Z)$ on $\mathrm{Cone}_\omega(G)$ such that $\mathrm{Cone}_\omega(G/Z)$ indentifies with the set of $\mathrm{Cone}_\omega(Z)$-orbits under this action[1]. Moreover, and once again because Z is central, for every $x \in \mathrm{Cone}_\omega(G)$ and $z, z' \in \mathrm{Cone}_\omega(Z)$, we have

$$d(z, z') = d(zx, z'x).$$

In particular the action is free.

The fundamental observation is the following proposition.

Proposition 4.1. *If* $\mathrm{Cone}_\omega(Z)$ *is ultrametric, then the above map* ψ *is a* $(1/3, 1)$*-bilipschitz homeomorphism.*

Thus we have to lift paths from $\mathrm{Cone}_\omega(G/Z)$ to $\mathrm{Cone}_\omega(G)$. The easier part is uniqueness, i.e. injectivity of u.

Let $X \subset \mathrm{Cone}_\omega(G)^2$ be the graph of the equivalence relation of the action of $\mathrm{Cone}_\omega(Z)$. Then there is a map $s : X \to \mathrm{Cone}_\omega(Z)$ mapping (x, y) to the unique z such that $zx = y$. This map is Lipschitz: indeed if $s(x, y) = z$ and $s(x', y') = z'$, then

$$d(z, z') = d(zx, z'x) \leq d(zx, z'x') + d(z'x', z'x) = d(y, y') + d(x, x').$$

We can now prove

Lemma 4.2. *The map* ψ *is injective.*

Proof. Assume that $\psi(u) = \psi(v)$. This means that $(u(t), v(t)) \in X$ for all t. So there is a well-defined continuous map $\sigma : t \mapsto s(u(t), v(t))$ with values in $\mathrm{Cone}_\omega(Z)$, with $\sigma(0) = 1$. As the latter is assumed to be totally disconnected, the map σ has to be constant, hence equal to 1, i.e. $u = v$. □

Lemma 4.3. *The map* ψ *has dense image.*

To prove this we first need the following lemma.

Lemma 4.4. *Let* G *be group endowed with a word metric with respect to some generating subset* S*. Let* N *be a closed normal subgroup, and endow* G/N *with the word metric with respect to the image of* S*. Fix* $\varepsilon > 0$*. Consider* $x \in \mathrm{Cone}_\omega(G)$ *and* $y, y' \in \mathrm{Cone}_\omega(G/N)$ *satisfying* $d(y, y') \leq \varepsilon$ *and* $p(x) = y$*, where* p *is the natural projection. Then there exists* $x' \in \mathrm{Cone}_\omega(G)$ *with* $d(x, x') \leq \varepsilon$ *and* $p(x') = y'$*.*

Proof. By homogeneity, we can suppose that $x = 1$. Write y' as a sequence (y_n) with $\lim_\omega d(y_n, 1) \leq 1$, and lift y_n to an element x_n of G with the same

[1] It turns out that this identification holds as metric spaces: the distance on $\mathrm{Cone}_\omega(G/Z)$ coincides with the distance between $\mathrm{Cone}_\omega(Z)$-orbits in $\mathrm{Cone}_\omega(G)$.

word length. Then (x_n) defines an element x' of $\mathrm{Cone}_\omega(G)$ with the required properties. □

Proof of Lemma 4.3. Let v be an element of $\mathcal{P}(\mathrm{Cone}_\omega(G/Z))$ and fix $\varepsilon > 0$. There exists a sequence $0 = t_0 < t_1 < \dots$ tending to infinity such that every segment $[t_i, t_{i+1}]$ is mapped by v to a set of diameter at most ε. By applying inductively Lemma 4.4, there exists a sequence (x_i) in $\mathrm{Cone}_\omega(G)$ such that $x_0 = 1$, $p(x_i) = v(t_i)$ and $d(x_i, x_{i+1}) \leq \varepsilon$ for all i. As $\mathrm{Cone}_\omega(G)$ is geodesic, we can find a continuous function $u : \mathbf{R}_+ \to \mathrm{Cone}_\omega(G)$ such that $u(t_i) = x_i$ for all i and u is geodesic on every segment $[t_i, t_{i+1}]$ (in the sense that for some constant $c_i \geq 0$ and all t, t' in this segment, $d(u(t), u(t')) = c_i|t - t'|$). Then $d(p \circ u, v) \leq 2\varepsilon$, and observe that $\psi(u) = p \circ u$ by definition of ψ. □

Let $\mathcal{P}_g$ denote the set of elements u of $\mathcal{P}(\mathrm{Cone}_\omega(G))$ such that there exists $0 = t_0 < t_1 < \dots$ tending to infinity such that u is geodesic in restriction to each $[t_i, t_{i+1}]$ and such that the projection

$$p : \mathrm{Cone}_\omega(G) \to \mathrm{Cone}_\omega(G/Z)$$

is isometric in restriction to $u([t_i, t_{i+1}])$. We summarize this as: the sequence $0 = t_0 < t_1 < \dots$ is u-good; if moreover $d(u(t_i), u(t_{i+1})) \leq \varepsilon$ for all i, we call it (u, ε)-good; note that given $\varepsilon > 0$, any u-good sequence can be refined to a (u, ε)-good sequence. The proof of Lemma 4.3 actually shows that $\psi(\mathcal{P}_g)$ is dense in $\mathcal{P}(\mathrm{Cone}_\omega(G/Z))$.

Lemma 4.5. *Suppose that* $\mathrm{Cone}_\omega(Z)$ *is ultrametric. Then the map* ψ^{-1} *is 3-Lipschitz on* $\psi(\mathcal{P}_g)$.

Proof. Let u, v belong to $\mathcal{P}_g$ with $d(p \circ u, p \circ v) = \sigma$. Fix $\varepsilon > 0$. Consider a sequence $0 = t_0 < t_1 < \dots$ which is both (u, ε) and (v, ε)-good. By Lemma 4.4, there exists w_i such that $p(w_i) = p(u(t_i))$ and $d(w_i, v(t_i)) \leq \sigma$ (we choose $w_0 = 1$). Set $z_i = s(u(t_i), w_i)$, i.e. $w_i = z_i u(t_i)$. Then

$$d(z_i, z_{i+1}) = d(z_i u(t_i), z_{i+1} u(t_i)) \leq d(z_i u(t_i), v(t_i)) + d(v(t_i), v(t_{i+1})) +$$

$$d(v(t_{i+1}), z_{i+1} u(t_{i+1})) + d(z_{i+1} u(t_{i+1}), z_{i+1} u(t_i))$$

$$\leq \sigma + \varepsilon + \sigma + \varepsilon.$$

As $\mathrm{Cone}_\omega(Z)$ is ultrametric, we obtain that $d(1, z_i) \leq 2\sigma + 2\varepsilon$ for all i. So $d(w_i, u(t_i)) \leq 2\sigma + 2\varepsilon$ for all i, and so $d(u(t_i), v(t_i)) \leq 3\sigma + 2\varepsilon$ for all i. Accordingly, $d(u(t), v(t)) \leq 3\sigma + 3\varepsilon$ for all t. As ε is arbitrary, we obtain $d(u(t), v(t)) \leq 3\sigma$. □

Proof of Proposition 4.1. It follows from Lemma 4.5 that ψ^{-1} extends to a 3-Lipschitz map φ defined on the closure of $\psi(\mathcal{P}_g)$, which is by Lemma 4.3 all of $\mathcal{P}(\mathrm{Cone}_\omega(G/Z))$; moreover by density and continuity, $\psi \circ \varphi$ is the identity on $\mathcal{P}(\mathrm{Cone}_\omega(G/Z))$. So ψ is surjective and is duly $(1/3, 1)$-bilipschitz. □

If X is a metric space with base-point x_0, denote by $\mathcal{L}(X)$ the set of continuous loops based on x_0. It can be naturally viewed as a closed metric subspace of $\mathcal{P}(X)$, by extending all function as constant equal to the base point beyond 1. In particular, all the above can be applied.

Take again $G, Z \ldots$ as above and keep assuming that $\mathrm{Cone}_\omega(Z)$ is ultrametric. If $u \in \mathcal{L}(\mathrm{Cone}_\omega(G/Z))$, define $\mu(u) = \varphi(u)(1)$. This defines a 3-Lipschitz map

$$\mu : \mathcal{L}(\mathrm{Cone}_\omega(G/Z)) \to \mathrm{Cone}_\omega(Z).$$

In particular, being continuous and mapping to a totally disconnected space, it factors through a map

$$\tilde{\mu} : \pi_1(\mathrm{Cone}_\omega(G/Z)) \to \mathrm{Cone}_\omega(Z).$$

Clearly, $\mu = 1$ in restriction to $\psi(\mathcal{L}(\mathrm{Cone}_\omega(G)))$. Therefore the following composition is trivial

$$\pi_1(\mathrm{Cone}_\omega(G)) \to \pi_1(\mathrm{Cone}_\omega(G/Z)) \xrightarrow{\tilde{\mu}} \mathrm{Cone}_\omega(Z).$$

The map $\tilde{\mu}$ is surjective: this is a trivial consequence of the path-connectedness of $\mathrm{Cone}_\omega(G)$.

The lifting map $\varphi = \psi^{-1}$ allows to lift homotopies and as a direct consequence we get the injectivity of the map $\pi_1(\mathrm{Cone}_\omega(G)) \to \pi_1(\mathrm{Cone}_\omega(G/Z))$.

Finally, if u is in the kernel of $\tilde{\mu}$, then this means that $\varphi(\mu)$ is a loop of which u is the image. So we get an exact sequence of groups

$$1 \to \pi_1(\mathrm{Cone}_\omega(G)) \to \pi_1(\mathrm{Cone}_\omega(G/Z)) \xrightarrow{\tilde{\mu}} \mathrm{Cone}_\omega(Z) \to 1. \tag{4.1}$$

It is actually, in a reasonable sense, an exact sequence in the context of metric groups with Lipschitz maps.

Definition 4.6. Given three metric groups (groups endowed with left-invariant pseudometrics), we call an exact sequence

$$1 \to N \xrightarrow{\iota} G \xrightarrow{p} Q \to 1$$

Lipschitz-exact[2] if ι is Lipschitz, and there exist constants $C, C' > 0$ such that

$$C d(g, \mathrm{Ker}(p)) \leq d(1, p(g)) \leq C' d(g, \mathrm{Ker}(p)) \tag{4.2}$$

[2]We do not require ι to be a bilipschitz embedding, so this could be called "right Lipschitz-exact exact sequence"; however we shall not use this stronger notion of being Lipschitz-exact.

for all $g \in G$.

Inequality (4.2) says that the distance between elements in Q is bi-Lipschitz equivalent to the distance between corresponding N-cosets in G. In particular the right-hand inequality in (4.2) means that p is C'-Lipschitz. Observe that if N is trivial, (4.2) just means that $p : G \to Q$ is a bilipschitz isomorphism.

In our case, the exact sequence (4.1) is Lipschitz-exact with constants 1 and 3. The right-hand inequality follows from the combined facts that Z is commutative (hence conjugations disappear in the image) and ultrametric (so that a large product of small loops is still small).

To check the non-trivial left-hand case, take $u \in \pi_1(\mathrm{Cone}_\omega(G/Z))$. Fixing a representing element in $\mathcal{L}(\mathrm{Cone}_\omega(G/Z))$, lift it (through φ), extend it to a closed loop via a geodesic of length $d(1, \tilde{\mu}(u))$, and take the image by p. We get an element of $\mathrm{Ker}(\tilde{\mu})$ at distance $\leq d(1, \tilde{\mu}(u))$ of u.

Finally we get

Theorem 4.7. *Let G be a locally compact, compactly generated group and Z a closed, central subgroup. Endow G with a word length with respect to a compact generating subset and let Z be endowed with the restriction of this word length. Given a nonprincipal ultrafilter ω, assume that $\mathrm{Cone}_\omega(Z)$ is ultrametric. Then the sequence*

$$1 \to \pi_1(\mathrm{Cone}_\omega(G)) \to \pi_1(\mathrm{Cone}_\omega(G/Z)) \xrightarrow{\tilde{\mu}} \mathrm{Cone}_\omega(Z) \to 1$$

is a Lipschitz-exact sequence of metric groups.

Corollary 4.8. *If $\mathrm{Cone}_\omega(G)$ is simply connected, then*

$$\pi_1(\mathrm{Cone}_\omega(G/Z)) \xrightarrow{\tilde{\mu}} \mathrm{Cone}_\omega(Z)$$

is a bilipschitz isomorphism of metric groups.

4.C. **Ultrametric on Abels' group.** Let $\mathbf{K}$ be a local field. On $\mathrm{SL}_d(\mathbf{K})$, define a left-invariant pseudometric $d(g,h) = \ell(g^{-1}h)$, where ℓ is the length defined as follows

- If $\mathbf{K}$ is ultrametric, $\ell(A) = \sup_{i,j} \log |A_{ij}|$;
- if $\mathbf{K}$ is Archimedean, $\ell(1) = 0$ and $\ell(A) = \sup_{i,j} \log |A_{ij}| + C$, where C is a large enough constant ($C \geq \log d$ works).

This length is equivalent to the word length with respect to a compact generating subset. Moreover, the embedding $A_4(\mathbf{K}) \subset \mathrm{SL}_4(\mathbf{K})$ is quasi-isometric, therefore we can endow $A_4(\mathbf{K})$ (or any cocompact lattice therein) with the restriction of this distance, which is equivalent to the word distance.

We immediately see that this distance is ultrametric in restriction to $Z(\mathbf{K}) \subset A_4(\mathbf{K})$ if $\mathbf{K}$ is ultrametric, and quasi-ultrametric in the case of $\mathbf{K}$ Archimedean, namely satisfies $d(x,z) \leq \max(d(x,y), d(y,z)) + \log(2)$. Thus, in all cases, $\mathrm{Cone}_\omega(Z(\mathbf{K}))$ is ultrametric.

Thus, Corollary 4.8 can be applied along with Theorem 1.1. This yields Corollary 1.4.

Corollary 4.9. *The group $G_{\mathbf{K}} = A_4(\mathbf{K})/Z(\mathbf{K})$ has an asymptotic cone with abelian uncountable fundamental group. Precisely, the fundamental group of $\mathrm{Cone}_\omega(G_{\mathbf{K}})$ is isomorphic, as an abstract group, with $\mathbf{F}^{(\mathbf{R})}$ (direct sum of continuum copies of $\mathbf{F}$), where $\mathbf{F}$ is the prime field of the same characteristic as $\mathbf{K}$ ($\mathbf{Q}$ or $\mathbf{F}_p$).*

Actually, the fundamental group is isomorphic, as a *topological* group, to $(\mathbf{F}^{(\kappa_1)})^{\kappa_0}$, a countable product of the direct sum of continuum copies of $\mathbf{F}$. This is established in Theorem 6.10 when $\mathbf{K}$ has characteristic p, but the proof carries over the remaining case $\mathbf{F} = \mathbf{Q}$.

5. Examples with lattices

Let R be either $\mathbf{F}_p[t, t^{-1}, (t-1)^{-1}]$, or the ring of integers of a totally real number field of degree 3.

Theorem 5.1. *The group $A_4(R)$ is finitely presented and has a quadratic Dehn function.*

The proof consists of embedding R as a cocompact lattice in a certain larger group.

There is a natural cocompact embedding $R \subset \mathbf{K}$, where $\mathbf{K}$ is the locally compact ring $\mathbf{R}^3$ or $\mathbf{F}_p(\!(t)\!)^3$. Note that $A_4(R)$ is not cocompact in $A_4(\mathbf{K})$, because $R^\times$ is not cocompact in $\mathbf{K}^\times$. However $R^\times$ is cocompact in $\mathbf{K}_1^\times$, the closed subgroup of elements in $\mathbf{K}^\times$ for which the multiplication preserves the Haar measure in $\mathbf{K}$. Let $A_4(\mathbf{K})_1$ be the set of elements of $A_4(1)$, both of whose diagonal entries are in $\mathbf{K}_1^\times$, so $A_4(R)$ is cocompact in $A_4(\mathbf{K})_1$. Theorem 5.1 follows from

Theorem 5.2. *$A_4(\mathbf{K})_1$ has a quadratic Dehn function.*

On the proof. The proof is strictly analogous to that of A_4 of a local field, so we do not repeat the technical details. The only difference lies in the proof of Claims 3.5 and 3.6. It could be proved by using the natural generalization of Lemma 3.3. However this would be more difficult as there is no "contracting element" in the subgroups involved, because of the restriction on the determinant of the diagonal elements. Recall that $\mathbf{K}$ is now a product of three local fields, say $\mathbf{K} = \mathbf{K}^1 \times \mathbf{K}^2 \times \mathbf{K}^3$. Define $J' = J \times \{(1, 2, 3)\}$, and let, for $(i, j, m) \in J'$, U_{ijm} be the set of elements in $U_{ij}(\mathbf{K})$ whose (i, j)-entry is in $\mathbf{K}^m$.

The claims have to be written as

Claim 5.3. *For all $(i, j, m) \in J'$, we have*

- $\hat{e}^x_{ijm}\hat{e}^y_{ijm} \rightsquigarrow_2 \hat{e}^{x+y}_{ijm}$; $(\hat{e}^x_{ijm})^{-1} \rightsquigarrow_2 \hat{e}^{-x}_{ijm}$;
- *if* $(k, \ell, n) \in J'$ *with* $(j, m) \neq (k, n)$, $[\hat{e}^x_{ijm}, \hat{e}^y_{k\ell n}] \rightsquigarrow_2 1$;
- *if* $(j, k, m) \in J'$ *and* $(i, k) \neq (1, 4)$, $[\hat{e}^x_{ijm}, \hat{e}^y_{jkm}] \rightsquigarrow_2 \hat{e}^{xy}_{ikm}$.

It is proved by showing that each of these relations hold in a smaller subgroup with a quadratic Dehn function, using the following substitute for Lemma 3.3:

Lemma 5.4. *For all $(i, j, m), (k, \ell, n) \in J$, such that $(j, m) \neq (k, n)$, $DU_{ijm}U_{k\ell n}$ has a quadratic Dehn function; for each $(i, j, m), (j, k, m) \in J$ with $(i, k) \neq (1, 4)$, $DU_{ijm}U_{jkm}U_{ikm}$ has a quadratic Dehn function.*

Proof (sketched). The point is, for each of these groups, to find in D a contracting element. Most cases were already considered in the proof of Lemma 3.3. The only remaining ones $(1, i, m)$ and $(i, 4, n)$ with $m \neq n$ and $i \in \{2, 3\}$. To do it, it is enough to find an element in $\mathbf{K}_1^\times$ contracting $\mathbf{K}^m$ and dilating $\mathbf{K}^n$. This is well-known and was already done in [CT]. □

Proof of Corollary 1.6. The proof of Corollary 4.9, without changes, proves that $\pi_1(\mathrm{Cone}_\omega(A_4(\mathbf{R}^3)_1/Z(\mathbf{R}^3)))$ is isomorphic, as an abstract group, to $\mathbf{Q}^{(\mathbf{R})}$. Since Λ is a lattice in $A_4(\mathbf{R}^3)_1/Z(\mathbf{R}^3)$, the same result holds for $\pi_1(\mathrm{Cone}_\omega(\Lambda))$.

By [Gro, Corollary 3.F'_5], every connected solvable Lie group has an at most exponential Dehn function, so every polycyclic group has an at most exponential Dehn function. Since $Z(\mathbf{R}^3)$ is central and exponentially distorted, for every path γ of linear size in $A_4(\mathbf{R}^3)_1$ joining the identity to an element of exponential size in $Z(\mathbf{R}^3)$, the image of γ in $A_4(\mathbf{R}^3)_1/Z(\mathbf{R}^3)$ has (at least) exponential area. Thus $A_4(\mathbf{R}^3)_1/Z(\mathbf{R}^3)$, and therefore Λ as well, has an at least exponential Dehn function. Finally, we conclude that Λ has an exactly exponential Dehn function. □

Corollary 1.7 follows from Theorem 5.2, because $A_4(\mathbf{F}_p[t, t^{-1}, (t-1)^{-1}])$ is a cocompact lattice in $A_4(\mathbf{F}_p(\!(t)\!)^3)_1$.

Remark 5.5. The group $A_4(\mathbf{F}_p[t])$ is not finitely generated. Indeed, consider its subgroup H of matrices (a_{ij}) satisfying $a_{22} = 1$. Since the group of invertible elements in $\mathbf{F}_p[t]$ is reduced to $\mathbf{F}_p^\times$, H has finite index (namely, $p-1$) in $A_4(\mathbf{F}_p[t])$. There is a surjective homomorphism $H \to \mathbf{F}_p[t]$, mapping a matrix as above to a_{12}. Since $\mathbf{F}_p[t]$ is not finitely generated as a group, it follows that H is not finitely generated, nor is its overgroup of finite index $A_4(\mathbf{F}_p[t])$.

From similar but more elaborate reasons, $A_4(\mathbf{F}_p[t,t^{-1}])$ is not finitely presented (its finite generation is an easy exercise). Indeed, define L as those matrices for which a_{ij} is some power of t. Since the group of invertible elements in $\mathbf{F}_p[t,t^{-1}]$ consists of ut^k for $u \in \mathbf{F}_p^\times$ and $k \in \mathbf{Z}$, L has index $p-1$ in $A_4(\mathbf{F}_p[t,t^{-1}])$. There is a homomorphism ϕ from L to $\mathrm{GL}_2(\mathbf{F}_p[t,t^{-1}])$, mapping a matrix to its 2×2 northwest block, so that

$$\phi(L) = \left\{ \begin{pmatrix} 1 & Q \\ 0 & t^k \end{pmatrix} :\ Q \in \mathbf{F}_p[t,t^{-1}], k \in \mathbf{Z} \right\}.$$

The latter group is isomorphic to the lamplighter group $\mathbf{F}_p \wr \mathbf{Z}$ and is not a quotient of a finitely presented solvable group [Bau, BS]. So L is not finitely presented, nor is its overgroup of finite index $A_4(\mathbf{F}_p[t,t^{-1}])$.

6. Cones of subgroups of $\mathbf{F}_p^{(\mathbf{N})}$

Let $\mathbf{N}$ be the set of positive integers (so $0 \notin \mathbf{N}$). Consider the group $\mathbf{F}_p^{(\mathbf{N})}$ (with basis $(\delta_n)_{n\in\mathbf{N}}$), with left-invariant ultradistance defined by the length $|u| = \sup\{n : u_n \neq 0\}$. Each subgroup is endowed with the induced distance. Observe that the ball of radius n has cardinality p^n.

For every subset J of $\mathbf{N}$, we endow the subgroup $\mathbf{F}_p^{(J)}$ with the induced metric.

The purpose of this section is to study the topological groups $\mathrm{Cone}_\omega\left(\mathbf{F}_p^{(J)}\right)$ in terms of properties of the subset J and of the ultraproduct ω. This analysis culminates in a classifications of these cones up to continuous isomorphism (see Theorem 6.10).

The relevance of this study comes from Corollary 4.8: indeed, since $\mathbf{F}_p^{(I)}$ can be viewed as a bilipschitz embedded central subgroup of $A_4(\mathbf{F}_p[t,t^{-1},(t-1)^{-1}])$, there is an isomorphism

$$\mathrm{Cone}_\omega\left(\mathbf{F}_p^{(J)}\right) \simeq \pi_1\left(\mathrm{Cone}\left(\ A_4(\mathbf{F}_p[t,t^{-1},(t-1)^{-1}])/\mathbf{F}_p^{(J)}\ \right)\right) \tag{6.1}$$

6.A. Cones of $\mathbf{F}_p^{(\mathbf{N})}$.

Lemma 6.1. *Let X be a metric space and x_0 a base-point. Let c_n be the minimal number of closed n-balls needed to cover the closed $2n$-ball around x_0. If ω is a nonprincipal ultrafilter and $\lim_\omega c_n = \infty$, then $\mathrm{Cone}_\omega(X)$ has continuum cardinality.*

Proof. Let C_n be a set of c_n points in the $2n$-ball, at mutual distance at least n. There is a natural map from the ultraproduct $\prod^\omega C_n \to \mathrm{Cone}_\omega(X)$, which maps any two distinct points at distance at least 1. If we show that $\prod^\omega C_n$ has continuum cardinality, we are done. To check the latter (which is well-known), since it is no longer related to the original metric on C_n, we can now identify C_n with the subset $\{0, 1/c_n, \dots, (c_n-1)/c_n\}$ of $[0,1]$, and map any $u \in \prod^\omega C_n$ to $\phi(u) = \lim_\omega u_n \in [0,1]$. Then this map $\prod_\omega C_n \to [0,1]$ is easily shown to be surjective. □

Lemma 6.2. *If X is a separable metric space, then* $\mathrm{Cone}_\omega(X)$ *and* $\pi_1(\mathrm{Cone}_\omega(X))$ *have cardinality at most continuum.*

Proof. If D is a countable dense subset, every element of $\mathrm{Cone}_\omega(X)$ is determined by a sequence in D, whence the conclusion. Also, every loop $\mathrm{Cone}_\omega(X)$ is determined by a continuous map from a dense subset C of the circle, to $\mathrm{Cone}_\omega(X)$. Since $(2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0}$, we are done. □

Lemma 6.3. *For every nonprincipal ultrafilter,* $\mathrm{Cone}_\omega\left(\mathbf{F}_p^{(\mathbf{N})}\right)$ *is isomorphic, as an abstract group, to a $\mathbf{F}_p$-vector space of continuum dimension.*

Proof. It is obviously of p-torsion, so it is enough to check it has continuum cardinality. This follows from Lemma 6, observing that the cardinality of a ball of radius $2n$ is p^n times larger than the cardinality of the ball of radius n. □

6.B. **Subgroups with cone of finite rank.** If $x \in \mathbf{N}$, define

$$\sigma_J(x) = \inf_{y \in J} |\log(x/y)|.$$

It measures the "multiplicative distance" from x to J.

Lemma 6.4. *Let J be a subset of $\mathbf{N}$ and ω a nonprincipal ultrafilter. Then* $\mathrm{Cone}_\omega\left(\mathbf{F}_p^{(J)}\right)$ *is trivial if and only if* $\lim_{n\to\omega} \sigma_J(n) = \infty$.

Proof. Suppose that $\lim_{n\to\omega} \sigma_J(n) = \ell < \infty$. So there is $K \in \omega$ with $\sigma_J(n) \le 2\ell$ for all $n \in K$. For all $n \in K$, let $m(n)$ be an element of K with $|\log(n/m(n))| \le 2\ell$, in other words, $m(n) \in [e^{-2\ell}n, e^{2\ell}n]$. Define $u(n) \in \mathbf{F}_p^{(J)}$ as follows: $u(n) = \delta_{m(n)}$ if $n \in J$ and $u(n) = 0$ if $n \notin J$. So $|u(n)| \le e^{2\ell}n$ for all n, so that $(u(n))$ defines an element of $\mathrm{Cone}_\omega\left(\mathbf{F}_p^{(J)}\right)$; moreover its norm is at least $e^{-2\ell}$ so it is a nonzero element.

Conversely, assume that $\mathrm{Cone}_\omega\left(\mathbf{F}_p^{(J)}\right)$ is nontrivial; let $(u(n))$ be a nontrivial element, so $|u(n)| \le Cn$ for all n and $\lim_\omega |u(n)|/n = c > 0$. Define $K =$

$\{n \geq 1 : |u(n)| \geq cn/2$, so $K \in \omega$. For $n \in K$, $|u(n)| \in J$ and thus $\sigma_J(n) \leq \max(|\log(c/2)|, |\log(C))$, thus $\lim_\omega \sigma_J < \infty$. □

We need to refine the function σ. Suppose that J is infinite (otherwise $\mathbf{F}_p^{(J)}$ is bounded and all its asymptotic cones are points). For $n \in \mathbf{N}$, let $\phi_{J,n}$ be a bijection $\mathbf{N} \to J$ such that $\sigma_J^{(m)}(n) := (|\log(n/\phi_{J,n}(m))|)$ is nondecreasing with respect to m. Thus $\sigma_J^{(1)}(n)$ is the multiplicative distance from m to J; if a closest point m in J from n is removed, $\sigma_J^{(2)}(n)$ is the multiplicative distance from n to $J - \{m\}$, and so on. The following lemma extends Lemma 6.4.

Lemma 6.5. *Let ω be a nonprincipal ultrafilter and $J \subset \mathbf{N}$ an infinite set. Then* $\mathrm{Cone}_\omega\left(\mathbf{F}_p^{(J)}\right)$ *has rank less than q if and only if* $\lim_\omega \sigma_J^{(q)} = \infty$.

Proof. Suppose that $\lim_\omega \sigma_J^{(q)} < \infty$. For $1 \leq i \leq q$, let u_i be the sequence $u_i(n) = \sigma_J^{(i)}(n)$. We can alter u_i outside some subset in ω, to ensure that u_i has at most linear growth. Then $(u_i)_{1\leq i\leq k}$ is easily seen to be $\mathbf{F}_p$-free in $\mathrm{Cone}_\omega\left(\mathbf{F}_p^{(\mathbf{N})}\right)$. □

Proof of Corollary 1.9. Set $I = \{2^{2^n} : n \in \mathbf{N}\}$. Then it follows from Lemma 6.5 that $\mathrm{Cone}_\omega\left(\mathbf{F}_p^{(I)}\right)$ has rank less than one. If moreover $I \in \omega$ then it has rank one. So the result follows from (6.1). □

Let I be a fastly growing set of positive integers ($I = \{\alpha_m : m \geq 0\}$, where $\alpha_{m+1}/\alpha_m \to \infty$), and $(I_n)_{n\in\mathbf{N}}$ a partition of I into infinite sets. For every $n \in \mathbf{N}$, let ω_n be an ultrafilter supported by I_n.

Proposition 6.6. *The map*

$$\begin{array}{rcl} \{\text{Subsets of } \mathbf{N}\} & \to & \{\text{Subsets of } \mathbf{N}\} \\ J & \mapsto & \left\{n : \mathrm{Cone}_{\omega_n}\left(\mathbf{F}_p^{(J)}\right) \text{ is trivial}\right\} \end{array}$$

is surjective.

Proof. Write $H(J) = \mathbf{F}_p^{(J)}$. Let M be a subset of $\mathbf{N}$ and $W(M) = \bigcup_{n\in M} I_n$. Let us check that $\{n : \mathrm{Cone}_{\omega_n}(H(W(M)))$ is trivial$\}$ is exactly the complement $\mathbf{N} - M$. Indeed, if $n \in M$ then $I_n \subset W(M)$, so $\sigma_{W(M)}(m) = 1$ for all $m \in I_n$ so $\lim_{\omega_n} \sigma_{W(M)} = 1$ and $\mathrm{Cone}_{\omega_n}(H(W(M))$ is nontrivial by Lemma 6.4. Conversely, if $n \notin M$, then $\lim_{n\in I_n, n\in\infty} \sigma_{W(M)} = \infty$ (because I is a fastly growing set), so $\lim_{\omega_n} \sigma_{W(M)} = \infty$, so $\mathrm{Cone}_{\omega_n}(H(W(M))$ is trivial, again by Lemma 6.4. □

Proposition 6.6 together with (6.1) prove Corollary 1.8.

6.C. **Topological properties of the cone.**

Lemma 6.7. *Let G be a group endowed with a biinvariant complete Hausdorff ultrametric. Then G is the projective limit of G/G_r, G_r its closed r-ball, when $r \to 0$. In particular, if G_r has finite index for all $r > 0$, then G is compact.*

Proof. The fact that the distance on G is ultrametric implies that G_r is a subgroup, and its bi-invariance that G_r is normal. Hence the projective limit itself is a group, endowed with an ultra bi-invariant metric such that the closed ball of radius r consists of $(g_s)_{s>0}$ such that $g_s = 1_{G/G_s}$ for all $s > r$. This metric makes the natural continuous homomorphism ψ from G to the projective limit an isometry onto its image. But since the latter is dense and G is complete, ψ is an isomorphism of metric groups. □

We say that $J \subset \mathbf{N}$ is ω-doubling if, denoting $j_n = \#(J \cap \{1, \dots, n\})$, for all $c, C \in \mathbf{N}$ we have $\lim_\omega(j_{Cn} - j_{cn}) < \infty$, lower ω-semidoubling if for some C_0, this is true for all c and all $C \leq C_0$ and upper ω-semidoubling if for some c_0, this is true for all C and $c \geq c_0$. We say that a metric space is σ-bounded if it is a countable union of uniformly bounded subsets.

Recall that a metric space is *proper* if all closed bounded subsets are compact.

Lemma 6.8. *Let J be an infinite subset of $\mathbf{N}$. Then J is ω-doubling if and only if* $\mathrm{Cone}_\omega\left(\mathbf{F}_p^{(J)}\right)$ *is proper (and otherwise it contains a closed bounded discrete subgroup of continuum cardinality); J is lower ω-semidoubling if and only if* $\mathrm{Cone}_\omega\left(\mathbf{F}_p^{(J)}\right)$ *is locally compact; J is upper ω-semidoubling if and only if* $\mathrm{Cone}_\omega\left(\mathbf{F}_p^{(J)}\right)$ *is σ-bounded (and otherwise every ball has continuum index).*

Proof. Denote by G_r (H_r) the closed (open) r-ball in $\mathrm{Cone}_\omega\left(\mathbf{F}_p^{(J)}\right)$; both are open subgroups.

Let us first check that if G_C is compact then we have $\lim_\omega(j_{Cn} - j_{cn}) < \infty$ for all $c > 0$.

Fix $C, c > 0$ and suppose $\lim_\omega(j_{Cn} - j_{cn}) = \infty$. Let J_n be the set of elements in $J \cap [cn/2, Cn]$. Then there is an obvious homomorphism ϕ from $\prod_\omega \mathbf{F}_p^{(J_n)}$ to the C-ball in $\mathrm{Cone}_\omega\left(\mathbf{F}_p^{(\mathbf{N})}\right)$, mapping any two distinct elements at distance $\geq c/2$. Since $\lim_\omega(j_{Cn} - j_{cn}) = \infty$, $\lim_\omega \#J_n = \infty$, so we obtain a infinite, closed discrete set in the C-ball of $\mathrm{Cone}_\omega\left(\mathbf{F}_p^{(\mathbf{N})}\right)$, which is therefore not compact. So $\mathrm{Cone}_\omega\left(\mathbf{F}_p^{(J)}\right)$ is not proper; if moreover for all C there exists c such that $\lim_\omega(j_{Cn} - j_{cn}) = \infty$, then we deduce that $\mathrm{Cone}_\omega\left(\mathbf{F}_p^{(J)}\right)$ is not locally compact. Besides, if for all c

there exists C such that $\lim_\omega(j_{Cn} - j_{cn}) = \infty$, we deduce that every ball has continuum index in $\mathrm{Cone}_\omega\left(\mathbf{F}_p^{(J)}\right)$.

Let us now check an almost converse, namely assume that $\lim_\omega(j_{Cn}-j_{cn}) < \infty$, for all c then H_c is compact.

First suppose that $\lim_\omega(j_{Cn} - j_{cn}) < \infty$ for a given $c > 0$. Let J_n and ϕ be defined as above. Consider the composite homomorphism $\prod_\omega(\mathbf{F}_p^{(J_n)}) \xrightarrow{\phi} G_C \to G_C/G_c$. We claim that it its image contains H_C/H_c. Indeed, let (u_n) be in H_C; we can suppose that $|u_n| \le Cn$ for all n. Set $K_n = J \cap [0, cn/2[$. Write $u_n = v_n + w_n$ with $v_n \in \mathbf{F}_p^{(J_n)}$ and $w_n \in \mathbf{F}_p^{(K_n)}$. Then $|w_n| \le cn$, so $(u_n) = (v_n)$ modulo G_c, while clearly (v_n) is in the image of ϕ.

It follows from Lemma 6.7 that H_C is compact. If this is true for all C, we deduce that $\mathrm{Cone}_\omega\left(\mathbf{F}_p^{(J)}\right)$ is proper; if this is true for C small enough, this implies that $\mathrm{Cone}_\omega\left(\mathbf{F}_p^{(J)}\right)$ has a compact open subgroup and therefore is locally compact. If this is true for c large enough, it follows that the closed c-ball has finite index in any larger ball, so has at most countable index in $\mathrm{Cone}_\omega\left(\mathbf{F}_p^{(J)}\right)$. □

For infinite $J \subset \mathbf{N}$, $C > 1$, and $n \in \mathbf{N}$, let $\rho_C(n)$ be the smallest element in $J \cap [Cn, \infty[$; for $c < 1$ let $\lambda_c(n)$ be the largest element in $\{0\} \cup J \cap [0, cn]$.

Lemma 6.9. $\mathrm{Cone}_\omega\left(\mathbf{F}_p^{(J)}\right)$ *is bounded if and only if* $\lim_\omega \rho_C(n)/n = \infty$ *for some* $C > 1$, *and is discrete if and only if* $\lim_\omega \lambda_c(n)/n = 0$ *for some* $c < 1$.

Proof. Suppose that $\lim_\omega \rho_C(n)/n = \infty$. Then $\mathrm{Cone}_\omega\left(\mathbf{F}_p^{(J)}\right)$ is equal to its closed C-ball. Conversely if $\mathrm{Cone}_\omega\left(\mathbf{F}_p^{(J)}\right)$ is equal to its open C-ball, then $\lim_\omega \rho_C(n)/n = \infty$. The verifications are straightforward and the other equivalence is similar. □

6.D. **A classification result.** We can now classify the groups $\mathrm{Cone}_\omega\left(\mathbf{F}_p^{(J)}\right)$ modulo isomorphism of topological groups. Let κ_0 be the countable cardinal, and $\kappa_1 = 2^{\kappa_0}$ be the continuum cardinal. By $\mathbf{F}_p^{(\kappa_1)}$ we denote the direct sum of κ_1 copies of the cyclic group $\mathbf{F}_p$, with the discrete topology. By H^{κ_0} we denote the product of κ_0 copies of H, with the product topology.

Theorem 6.10. *Let* $J \subset \mathbf{N}$ *be a subset and* $G = \mathrm{Cone}_\omega\left(\mathbf{F}_p^{(J)}\right)$. *Then, denoting by* $\simeq$ *the isomorphy relation within topological groups*

- *If* G *is not locally compact, then* $G \simeq (\mathbf{F}_p^{(\kappa_1)})^{\kappa_0}$.
- *If* G *is discrete, then*
  - *if* G *is bounded and proper,* $G \simeq \mathbf{F}_p^n$ *for some finite* n;

  - *if G is unbounded and proper, $G \simeq \mathbf{F}_p^{(\kappa_0)}$*
  - *if G is not proper, $G \simeq \mathbf{F}_p^{(\kappa_1)}$;*
- *If G is locally compact and not discrete, then*
  - *if G is bounded and proper, $G \simeq \mathbf{F}_p^{\kappa_0}$;*
  - *if G is unbounded and proper, $G \simeq \mathbf{F}_p^{\kappa_0} \times \mathbf{F}_p^{(\kappa_0)}$*
  - *if G is not proper, $G \simeq \mathbf{F}_p^{\kappa_0} \times \mathbf{F}_p^{(\kappa_1)}$;*

The discussion whether G is locally compact, discrete, bounded, σ-bounded, and determining the value of n in case G is finite, is settled by Lemmas 6.5, 6.8, and 6.9.

Proof. Note that G is abelian of exponent p, and has cardinality $\le \kappa_1$ by Lemma 6.2. If G is discrete, it is therefore isomorphic to $\mathbf{F}_p^{(\kappa)}$ for a certain cardinal κ. The first two statements are clear, and if G is discrete and not proper, Lemma 6.8 shows that G has continuum cardinality.

If G is locally compact, as any locally compact abelian group, it has a compact subgroup K such that G/K is a Lie group; since its identity component $(G/K)^0$ has exponent p, it has to be trivial so G/K is discrete. As a $\mathbf{F}_p$-vector space, K admits a complement in G, so $G \simeq K \oplus L$ as topological group, with L discrete. Since K is compact, Pontryagin duality shows that $K \simeq \mathbf{F}_p^X$ for some set X, while $L \simeq \mathbf{F}_p^{(Y)}$. Since G is nondiscrete, X is infinite. If G is proper and bounded, it is compact, so $G \simeq \mathbf{F}_p^X$; if G is proper and unbounded, so is L, so $L \simeq \mathbf{F}_p^{(\kappa_0)}$. If G is not proper, then by Lemma 6.8 it contains a closed subgroup isomorphic to $\mathbf{F}_p^{(\kappa_1)}$, so G/K does as well, so $L \simeq \mathbf{F}_p^{(\kappa_1)}$.

Besides (still assuming G locally compact and nondiscrete), some ball G_R of G is compact, and by Lemma 6.7 is projective limit of the discrete groups G_R/G_r when $r \to 0$. Compactness implies that G_R/G_r is finite. So G_R is isomorphic to $\mathbf{F}_p^{\kappa_0}$ and therefore K as well.

Finally assume that G is not locally compact. By Lemma 6.8, for every $\mathbf{F}_p \le \infty$ there exists $0 < c < C$ such that G_C/G_c has continuum cardinality. Hence define a decreasing sequence (ε_n), tending to zero, such that $G_{\varepsilon_n}/G_{\varepsilon_{n+1}}$ has continuum cardinality for all n (and $\varepsilon_0 = \infty$). So G is the projective limit of all G/G_{ε_n}. Let L_n be a complement subgroup of G_{ε_n} in $G_{\varepsilon_{n-1}}$ (where $G_0 = G$). So

$$G = G_{\varepsilon_0} = L_1 \oplus G_{\varepsilon_1} = L_1 \oplus L_2 \oplus G_{\varepsilon_2} = \cdots \bigoplus_{i=1}^{k} L_i \oplus G_{\varepsilon_k} = \dots$$

This provides a homomorphism $\iota : G \to \prod L_i$, clearly injective. Also map a sequence $\ell = (\ell_i)$ in $\prod L_i$ to the sequence $\rho(\ell) = (\prod_{i=1}^k \ell_i)_k$, which belongs to the projective limit. This ρ is a continuous homomorphism and $\iota \circ \rho$ is the identity, so ι is a topological isomorphism. Since each L_i is isomorphic to $\mathbf{F}_p^{(\kappa_1)}$, we are done. □

Proposition 6.11. *The group* $G = \mathrm{Cone}_\omega(\mathbf{R}, \log(1 + |\cdot|))$ *is topologically isomorphic to* $(\mathbf{Q}^{(\kappa_1)})^{\kappa_0}$ *(where* $\kappa_1 = 2^{\kappa_0}$ *and* κ_0 *is countable).*

Proof. This metric on $\mathbf{R}$ satisfies the quasi-ultrametric condition

$$d(x,z) \le \max(d(x,y), d(y,z)) + \log(2),$$

so G is ultrametric; it is readily observed that the index between any two balls is continuum □

6.E. **Non-bilipschitz cones.**

Lemma 6.12. *There exists a subset* $I \subset \mathbf{N}$ *such that for every* n *and every* $J \subset \{1, \dots, n\}$*, there exists* $m \in \mathbf{N}$ *such that* $I \cap [n^{-1}m, nm] = \{jm : j \in J\}$*.*

Proof. Construct I by an obvious induction, after enumerating finite subsets of $\mathbf{N}$. □

Lemma 6.13. *Let* I *be a subset satisfying the conditions of Lemma 6.12, and view* I *as a metric space, for the distance induced by the inclusion* $I \subset \mathbf{R}$*. Then for every* $J \subset \mathbf{N}$*, there exists an ultrafilter* ω *such that* $\mathrm{Cone}_\omega(I) = \{0\} \cup J$*.*

Proof. For every n, there exists m_n such that $I \cap [n^{-1}m_n, nm_n] = \{jm : j \in J, j \le n\}$. Clearly, we can arrange (m_n) to tend to infinity. If $\{m_n : n \in \mathbf{N}\} \in \omega$, then $\mathrm{Cone}_\omega(I) = \{0\} \cup J$. □

Proposition 6.14. *Let* $I \subset \mathbf{N}$ *be a subset satisfying the conditions of Lemma 6.12. Then there are continuum many cones* $\mathrm{Cone}_\omega(\mathbf{F}_p^{(I)})$*, up to bilipschitz group isomorphism, when* ω *ranges over nonprincipal ultrafilters of* $\mathbf{N}$*. Moreover, the topological group* $\mathrm{Cone}_\omega(\mathbf{F}_p^{(I)})$ *achieves all types described in Theorem 6.10 that are proper.*

In view of (6.1), Corollary 1.10 follows from Proposition 6.14.

Proof of Proposition 6.14. If X is a metric space, define

$$D(X) = \{\log(d(x,y)) : x \ne y \in X\}.$$

If $T \subset \mathbf{R}_{\ge 0}$, define $\ell(T) = \{\log(y), y \in T - \{0\}\}$. If X, Y are bilipschitz, then $\ell(D(X))$ and $\ell(D(Y))$ are at bounded Hausdorff distance.

Denote $H(J) = \mathbf{F}_p^{(J)}$. It is straightforward that if X is an isometry-homogeneous metric space,

$$D(\mathrm{Cone}_\omega(X)) = \mathrm{Cone}_\omega(D(X)).$$

Note that for $J \subset \mathbf{N}$, we have $D(H(J)) = J$. Let $L = \{2^{2^n} : n \in \mathbf{N}\}$. Then if $J, K \subset L$ are subsets of L, if $\ell(J)$ and $\ell(K)$ are at bounded distance then J and K coincide up to a finite set.

Let I be as in Lemma 6.13, so for every $J \subset \mathbf{N}$, there exists $\omega(J)$ such that $\mathrm{Cone}_{\omega(J)}(I) = \{0\} \cup J$.

We have, for any $J \subset \mathbf{N}$,

$$D(\mathrm{Cone}_{\omega(J)}(H(I))) = \mathrm{Cone}_{\omega(J)}(D(H(I))) = \mathrm{Cone}_{\omega(J)}(I) = \{0\} \cup J,$$

so

$$\ell(D(\mathrm{Cone}_{\omega(J)}(H(I))) = \ell(J).$$

Taking continuum many $J = J_i$ so that the $\ell(J_i)$ are pairwise at infinite Hausdorff distance, we obtain that the $\mathrm{Cone}_{\omega(J_i)}(H(I))$ are pairwise non-bilipschitz.

To prove the last statement, use the following notation: in Lemma 6.12, write $m = m(J, n)$. Fix $k \geq 0$ and for $n \geq k$, set $J_n = \{1, \dots, k\} \subset \{1, \dots, n\}$. Let ω be an ultrafilter containing $\{m(J_n, n) : n \geq k\}$. Then by Lemma 6.5, $\mathrm{Cone}_\omega(H(I)) \simeq \mathbf{F}_p^k$.

Now set $K_n = \{1, \dots, n\}$ and let ω be an ultrafilter containing $\{m(K_n, n) : n \geq 1\}$. Then $\mathrm{Cone}_\omega(H(I))$ is proper by Lemma 6.8 and discrete by Lemma 6.9, so by Theorem 6.10 it is isomorphic to $\mathbf{F}_p^{(\kappa_0)}$ as a topological group.

Set $r_n = \lfloor \sqrt{n} \rfloor$. If ω is assumed to contain $\{r_n m(K_n, n)\}$, the same lemmas imply that $\mathrm{Cone}_\omega(H(I))$ is proper and unbounded, so by Theoremï¿½6.10 it is isomorphic to $\mathbf{F}_p^{\kappa_0} \times \mathbf{F}_p^{(\kappa_0)}$ as a topological group.

Now set $L_n = \{1, \dots, r_n\}$. If ω contains $\{r_n m(L_n, n) : n \geq 1\}$, the same lemmas imply that $\mathrm{Cone}_\omega(H(I))$ is proper and bounded, so by Theorem 6.10 is isomorphic to $\mathbf{F}_p^{\kappa_0}$ as a topological group. □

Actually we could require further assumptions to ensure that the three remaining cases of Theorem 6.10 appear. Namely, we would need the following three conditions, which can be implemented in an inductive construction of I (where all intervals are meant in $\mathbf{N}$)

- for every n, there exists $m \in I$ such that $I \cap [n^{-1}m, nm] = [n^{-1}m, nm]$ (to obtain $\left(\mathbf{F}_p^{(\kappa_1)}\right)^{\kappa_0}$);
- for every n, there exists $m \in I$ such that $I \cap [n^{-1}m, nm] = [m, nm]$ (to obtain $\mathbf{F}_p^{(\kappa_1)}$);

- for every n, there exists $m \in I$ such that
$$I \cap [n^{-1}m, nm] = [m, nm] \cup \{\lfloor m/k \rfloor : 1 \leq k \leq n\}$$
(to obtain $\mathbf{F}_p^{\kappa_0} \times \mathbf{F}_p^{(\kappa_1)}$).

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