Local limits of multi-type Galton-Watson trees and applications to random maps

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Local limits of graphs

Balls

Let (X, ρ) be a rooted graph (possibly with some additional structure).

For $k \in \mathbb{Z}_+$, we let $B_{X,\rho}(k)$ be the ball of radius k centered at ρ:

$$B_{X,\rho}(k) = \Big\{ x \in X, d(x, \rho) \leqslant k \Big\},$$

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In the case of a tree T, we will use the notation $T_{\leqslant k}$.

Local convergence

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A sequence (X_n, ρ_n) converges to (X, ρ) (which may be an infinite graph) if, and only if, for any $k \in \mathbb{Z}_+$, we have

$$B_{X_n,\rho_n}(k) = B_{X,\rho}(k)$$

for n large enough.

Local metric

Local convergence corresponds to the following metric:

$$d\Big((X,\rho),(X',\rho')\Big) = \frac{1}{1+\sup\left\{k : B_{X,\rho}(k) = B_{X',\rho'}(k)\right\}}.$$

Simply said, two graphs are close if their balls of some large enough radius are equal.

Convergence in distribution

A random sequence (X_n, ρ_n) converges in distribution to (X, ρ) if, for any deterministic (Y, σ), we have

$$\mathbb{P}\Big(B_{X_n,\rho_n}(k) = (Y,\sigma)\Big) \longrightarrow \mathbb{P}\Big(B_{X,\rho}(k) = (Y,\sigma)\Big)$$

A well-known example

Theorem (Kennedy 75, Kesten 86)

- *Let μ be a probability distribution on $\mathbb{N}$ with mean 1 (critical) and such that $\mu(1) \neq 1$ (non-degenerate).*
- *Let T be a Galton-Watson tree with offspring distribution μ.*
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Then

$$T_n \xrightarrow{(d)} \widehat{T}$$

where $\widehat{T}$ is an infinite tree which we interpret as T conditioned to survive.

A well-known example

- Actually one must take n in $1 + d\mathbb{N}$ where d is the gcd of the support of μ.
- The distribution of $\widehat{T}$ can be obtained from that of T with a size-biasing method

$$\mathbb{E}[f(\widehat{T}_{\leqslant n})] = \mathbb{E}[Z_n f(T_{\leqslant n})]$$

where Z_n is the number of vertices with height n.

Multi-type Galton-Watson trees

Multi-type rooted plane trees

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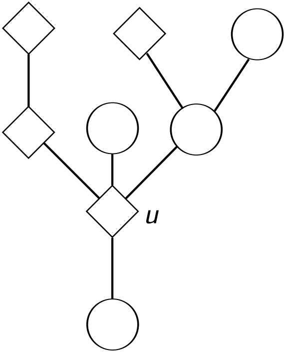

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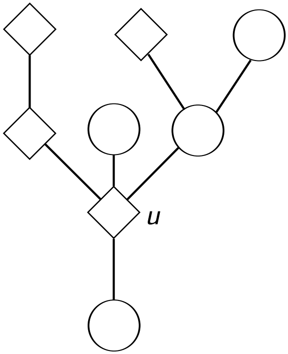

For any vertex u of a tree t, we let $w_t(u)$ be the ordered list of the types of its children. Here, $w_t(u) = (2, 1, 1)$.

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- *The ordered type-list of the children of an individual with type i has distribution $\zeta^{(i)}$.*
- *The individuals of a same generation are all independent from one another.*

Non-degenerescence and irreducibility

We assume from now on that ζ is non-degenerate, in the sense that there is at least one $i \in \{1, \dots, K\}$ such that

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We also assume that ζ is irreducible, which means that, whatever the type of the root, one has a non-zero probability of finding any other type in the tree.

Criticality

Given two types i et j, we call $m_{i,j}$ the average number of children of type j amongst the offspring of a person of type i. We are interested in the *mean matrix*

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We say that ζ is *critical* if the spectral radius of M is 1. The Perron-Frobenius theorem then tells us that there exists a unique (up to multiplicative constants) vector $\mathbf{b} = (b_i)_{i \in \{1,\dots,K\}}$ such that

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b_i should be thought of as the "mass" of individuals of type i.

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Problem: what can we mean by "large"?

- → The total number of vertices
- → The number of vertices of one fixed type.
- → We will take a general approach:

$$|T|_\gamma = \sum_{i=1}^{K} \gamma_i \, \#_i(T)$$

for some integer weights $(\gamma_i, i \in \{1, \dots, K\})$.

Convergence to an infinite multi-type tree

The convergence theorem

Take a non-degenerate, irreducible and critical ζ. Consider a tree T with ordered offspring distribution ζ with root of type i a.s. and, for $n \in \mathbb{N}$, take a version T_n of T conditioned on $|T|_\gamma = n$. Assume moreover *one* of the two following conditions:

- There exists j such that $\gamma_k = \mathbf{1}_{k=j}$ for all k. (only count one type)
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Theorem

As n tends to infinity,

$$T_n \xrightarrow{(d)} \widehat{T},$$

where $\widehat{T}$ is an infinite multi-type tree.

A few details

- T_n is only defined if the probability of T to have n vertices of type 1 is positive. We therefore restrict ourselves to such n, which amounts to considering a subset of $\mathbb{N}$ of the form $\alpha_i + d\mathbb{N}$, where d and α which depend on ζ, γ.

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- The distribution of $\widehat{T}$ is given by a generalized size-bias procedure:
$$E\Big[f\big(\widehat{T}_{\leqslant k}\big)\Big] = E\Big[Z_k f\big(T_{\leqslant k}\big)\Big]$$
where Z_k is the "size" of the k-th generation.

$$Z_k = \frac{1}{b_i}\sum_{j=1}^{K} b_j \#\{\text{vertices of } T \text{ with type } j \text{ and height } k\}$$

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- Conditionally on the offspring of an element of the spine being w, the next element of the spine will be the i-th child with probability proportional to b_{w_i}.
- Outside of the spine, we use the original offspring distribution ζ.

Key point of the proof

The essential ingredient of the proof is a study of the asymptotics of the distribution of $|F|_\gamma$, where F is a Galton-Watson *forest* with the same offspring distribution.

Lemma

Let $\mathbf{w} \in \mathbb{W}$ and consider a forest F of independent GW trees, where, for every term w_i of $\mathbf{w}$, there is a tree with root of type w_i. Then, for any integer p, we have

$$\mathbb{P}\Big(|F|_\gamma = \alpha_{\mathbf{w}} + dn\Big) \underset{n\to\infty}{\sim} x_n \sum_{i=1}^{|\mathbf{w}|} b_{w_i},$$

where $\alpha_{\mathbf{w}} = \sum_i \alpha_{w_i}$ and x_n is a "reference" sequence, given for example by $x_n = \frac{1}{b_1}\mathbb{P}\Big(|T|_\gamma = \alpha_1 + dn\Big)$ where T is our usual tree, with root of type 1.

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The lemma is proved differently depending on which case we are in:

- If we count only vertices of one type, then we use ratio theorems for random walks and many involved liminf/limsup arguments.
- If ζ has exponential moments, then we can obtain explicit asymptotics (of order $n^{-3/2}$) with the help of analytic combinatorics.

Other work

Recent related result by Abraham, Delmas and Guo (ArXiv 2015):

- Assume aperiodicity
- Let T_n be a version of T conditioned on the number of vertices of each type: the event

$$\Big\{\#_1(T) = k_1(n), \#_2(T) = k_2(n), \dots, \#_K(T) = k_K(n)\Big\}$$

where, for all types i, $\frac{k_i(n)}{\sum_j k_j(n)} \underset{n\to\infty}{\to} a_i$, and $\mathbf{a}$ is the *left* eigenvector of the mean offspring matrix.

Then T_n converges to $\widehat{T}$ in distribution

Open question...

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However, some subcritical trees do not converge to an infinite spine tree but instead display *condensation*: in the limit the spine is finite and has an infinite degree vertex at the end.

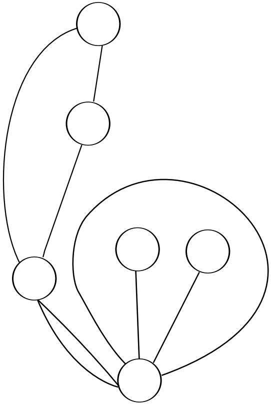
Will this also happen in the multi-type setting?

Boltzmann random maps

Planar maps

- Proper embedding of a connected graph in the sphere

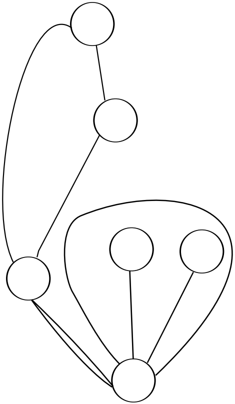
Planar maps



Planar maps

- Proper embedding of a connected graph in the sphere
- Taken up to orientation-preserving homeomorphisms of the sphere.

Planar maps



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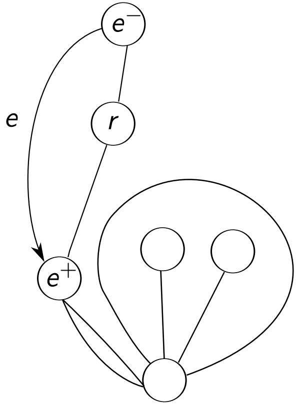
- There has been much recent interest in the study of the geometry of large random maps.
- *Scaling* limits: rescale the map to make it converge to a continuous metric space, typically the Brownian map... (Le Gall 2013, Miermont 2013...)
- *Local* convergence of maps to infinite maps: triangulations (Angel & Schramm 2002), quadrangulations (Krikun 2005)

Rooted and pointed maps

We will actually consider triples (m, e, r), where m is a map, e is a selected oriented edge called the root edge, and r is a selected vertex. We call such a triple a *rooted and pointed* map.



Sign of a rooted and pointed map

Note that we always have $|d(e^+, r) - d(e^-, r)| \leqslant 1$. We say that (m, e, r) is:

- positive if $d(e^+, r) = d(e^-, r) + 1$
- null if $d(e^+, r) = d(e^-, r)$
- negative if $d(e^+, r) = d(e^-, r) - 1$

Boltzmann distributions

Take a sequence of weights $\mathbf{q} = (q_n)_{n \geqslant 1}$ and define the weight of a map (m, e, r) by

$$W_{\mathbf{q}}(m, e, r) = \prod_{f \in \mathcal{F}_m} q_{\deg(f)}.$$

If the sum $Z_{\mathbf{q}}$ of the weights of all the maps (m, e, r) is finite, we say that $\mathbf{q}$ is admissible and renormalize $W_{\mathbf{q}}(m, e, r)$ into a probability measure

$$B_{\mathbf{q}}(m, e, r) = \frac{W_{\mathbf{q}}(m, e, r)}{Z_{\mathbf{q}}}.$$

Boltzmann distributions

Let us also conditioned versions of $B_{\mathbf{q}}$ where the map is conditioned to be positive or null:

$$B_{\mathbf{q}}^{+}(m, e, r) = \frac{W_{\mathbf{q}}(m, e, r)}{Z_{\mathbf{q}}^{+}}$$

and

$$B_{\mathbf{q}}^{0}(m, e, r) = \frac{W_{\mathbf{q}}(m, e, r)}{Z_{\mathbf{q}}^{0}}$$

where $Z_{\mathbf{q}}^{+}$ et $Z_{\mathbf{q}}^{0}$ are two well-chosen constants.

How to study maps

Random maps are often to complicated to study directly...

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A common method is to describe them as transforms of *decorated trees*. Here we use special trees called *mobiles*.

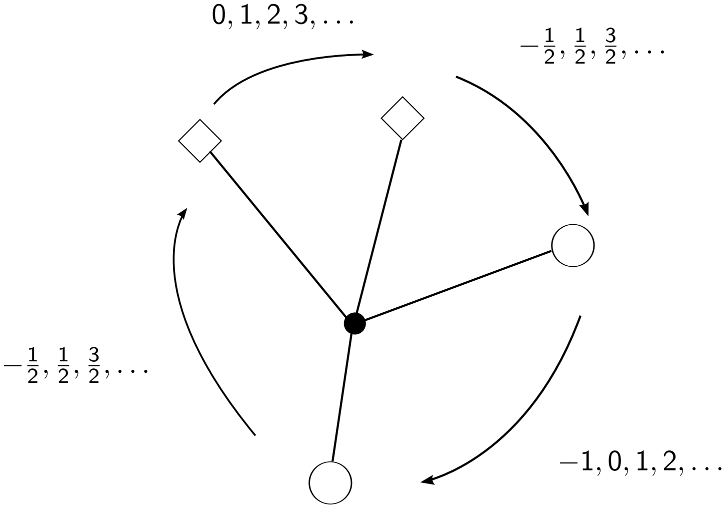
Mobiles

A mobile is a tree with three types (actually four) of vertices $\circ$ (1), $\diamond$ (2) et $\bullet$ (3 and 4), satisfying a few properties.

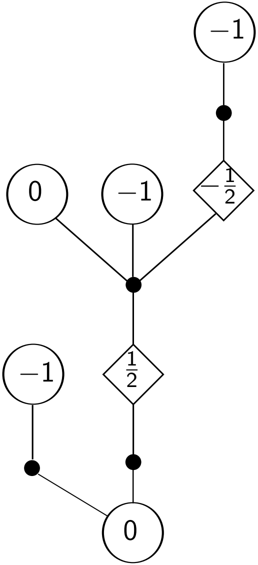
- Vertices of type $\circ$ and $\diamond$ are on even generations, while vertices of type $\bullet$ are on odd generations.
- Vertices of type $\diamond$ have exactly two neighbours (which have type $\bullet$).
- Vertices of type $\circ$ have integer labels (0 for the root) while vertices of type $\diamond$ have labels in $\mathbb{Z} + 1/2$ ($1/2$ for the root).

We split $\bullet$ vertices into two types 3 et 4, depending on whether the parent has type $\circ$ or $\diamond$.

The labels also must satisfy this condition:



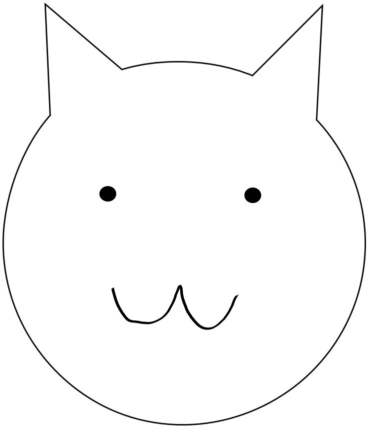
A mobile



The Bouttier-Di Francesco-Guitter bijection

First, we need a friend.

The Bouttier-Di Francesco-Guitter bijection



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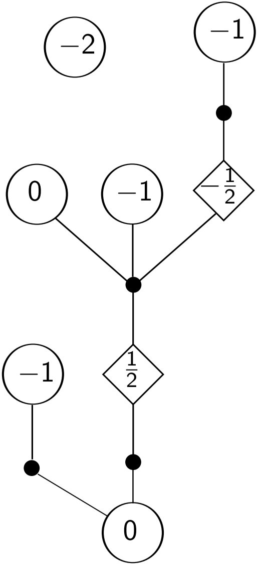

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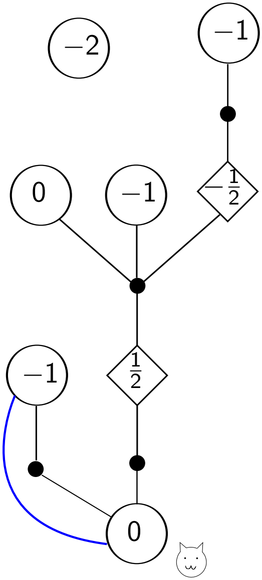

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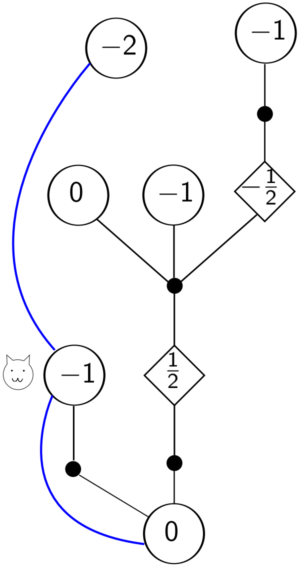

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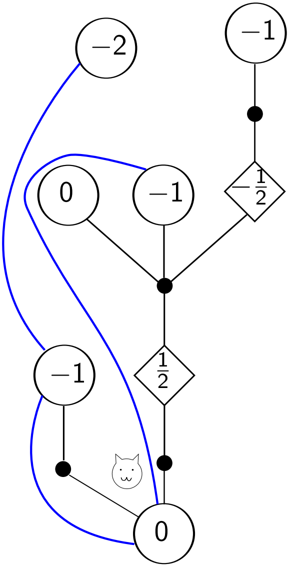

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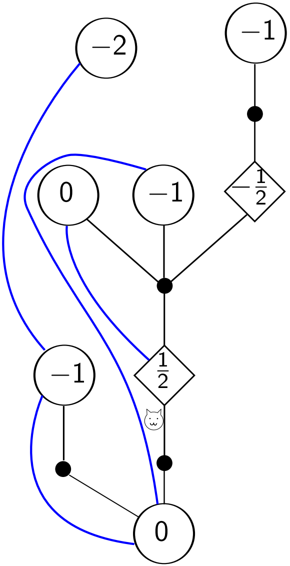

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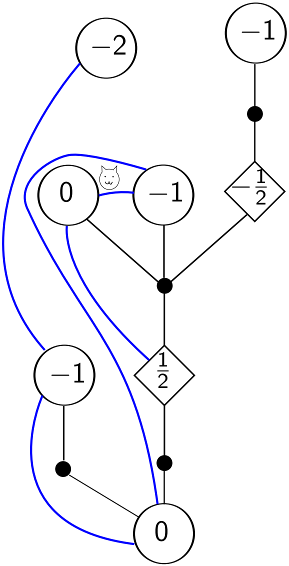

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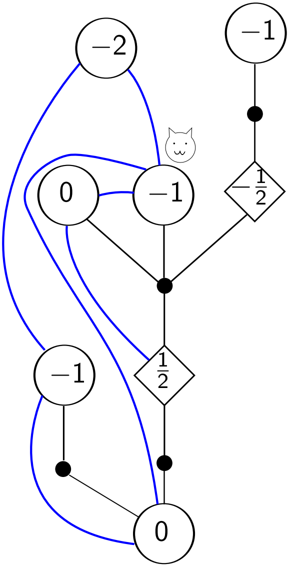

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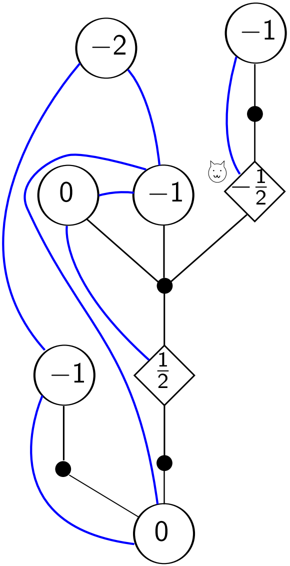

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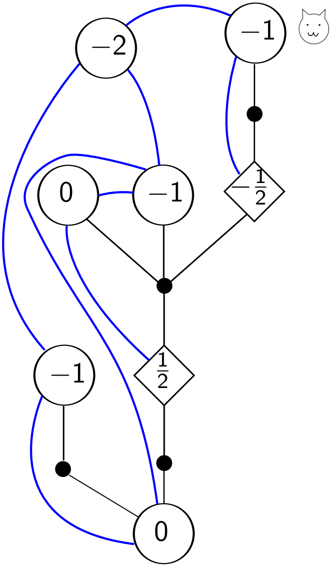

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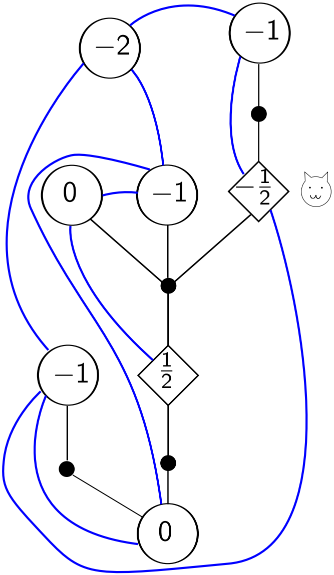

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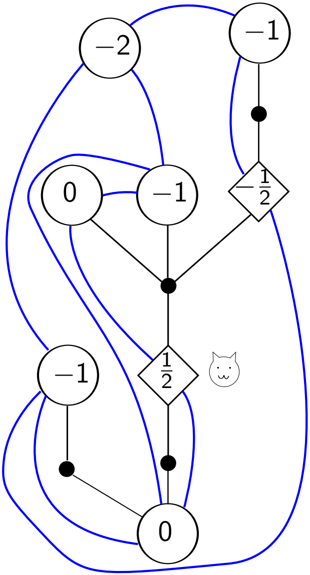

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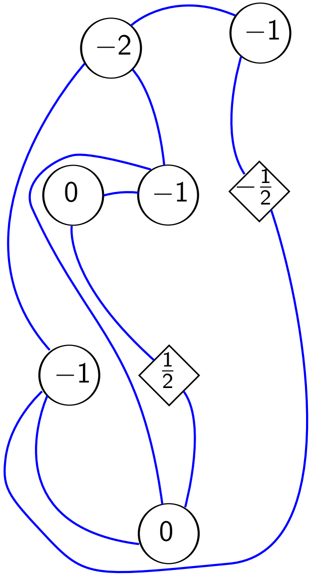

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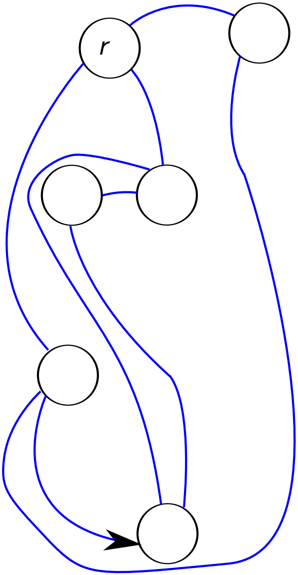

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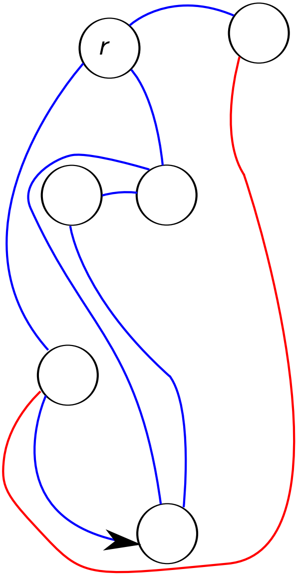

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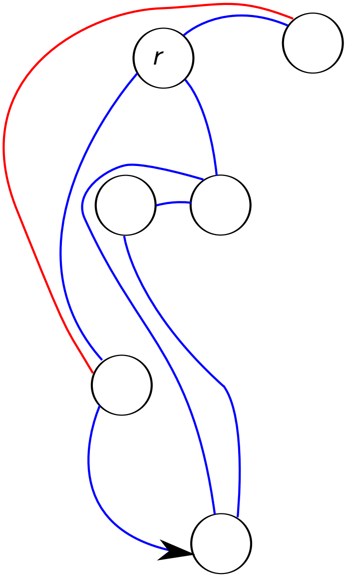

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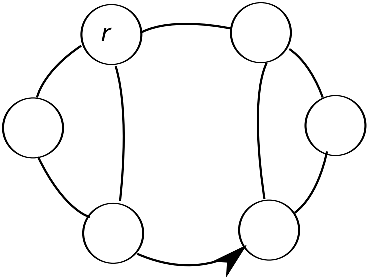

Galton-Watson mobiles and Boltzmann maps

The BDG bijection transforms well-chosen Galton-Watson trees into Boltzmann maps.

Galton-Watson mobiles and Boltzmann maps

Theorem (Miermont 06)

- *Let T^+ be a tree with offspring distribution ζ, with root $\circ$. We give its root label 0, and then label the other vertices uniformly in the set of admissible labelings.*
- *Let also T^0 be a tree with root of type $\diamond$ with two children of type 4, and where the other vertices use the offspring distribution ζ. We label the root $1/2$, the rest of the labels still being chosen uniformly.*

Then the BDG bijection sends T^+ and T^0 to maps with distribution $B^+_{\mathbf{q}}$ and $B^0_{\mathbf{q}}$.

Criticality

We say that **q** is critical if the offspring distribution ζ is critical.

We say that **q** is regular critical if ζ is critical and has small exponential moments.

Convergence to an infinite map

We take a critical weight sequence $\mathbf{q}$.

Theorem

Let (M_n, E_n, R_n) be a map with distribution $B_{\mathbf{q}}$ conditioned to have n vertices. The rooted map (M_n, E_n) then converges locally in distribution to an infinite map (M_∞, E_∞), which we call the Infinite Boltzmann Planar Map with weights $\mathbf{q}$ ($\mathbf{q}$-IBPM).

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If $\mathbf{q}$ is regular critical, then we can condition the map by its faces or edges.

Theorem

Let (M_n, E_n, R_n) be a map with distribution $B_\mathbf{q}$ conditioned to have n edges/faces. The rooted map (M_n, E_n) then converges locally in distribution to the same $\mathbf{q}$-IBPM.

A few properties of the **q**-IBPM

- (M_∞, E_∞) is a proper infinite map of the plane, in the sense that in can be embedded in such a way that all balls of finite radius only intersect a finite number of edges and and vertices.

A few properties of the **q**-IBPM

- (M_∞, E_∞) is a proper infinite map of the plane, in the sense that in can be embedded in such a way that all balls of finite radius only intersect a finite number of edges and and vertices.

- The graph M_∞ is recurrent for the simple random walk. (consequence of Gurel-Gurevich and Nachmias, 2013.)

The case of p-angulations

A p-angulation is a map where each face has degree p.

Theorem

- *Let (M_n, E_n) be a uniform rooted $2p$-angulation with n faces. Then it converges locally in distribution to an infinite $2p$-angulation.*

The case of p-angulations

A p-angulation is a map where each face has degree p.

Theorem

- *Let (M_n, E_n) be a uniform rooted $2p$-angulation with n faces. Then it converges locally in distribution to an infinite $2p$-angulation.*
- *Let (M_n, E_n) be a uniform rooted $2p$-angulation with $2n$ faces. Then it converges locally in distribution to an infinite $2p+1$-angulation.*

Uniform map

Theorem

Let (M_n, E_n) be a uniform map with n edges. Then it converges locally in distribution to an infinite map called the Uniform Infinite Planar Map (UIPM).

Other results

Björnberg and Stefànsson (2014) have proved a similar result, conditioning on the number of edges, with different assumptions:

- they are restricted only to bipartite maps
- only criticality, and not necessarily regular criticality, is needed.

Thank you!