Convergence of bivariate Markov chains to multi-type self-similar processes, and applications to scaling limits of some random trees

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Joint work with Bénédicte Haas

Introduction

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Notation : $\Big((X_n^{(i)}(k), J_n^{(i)}(k)), k \geqslant 0 \Big)$ is a version of the process starting at (n, i).

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Basic principle : assume that the macroscopic jumps are rare, *i.e.* that there exists $\gamma > 0$ such that

$$\mathbb{P}[X_n^{(i)}(1) \leqslant (1-\varepsilon)n] \underset{n\to\infty}{\sim} c_\varepsilon^{(i)} n^{-\gamma}, \ \forall \varepsilon > 0, \forall i \in \{1, \dots, \kappa\}$$

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Then

$$\Big(\frac{X_n^{(i)}(\lfloor n^\gamma t \rfloor)}{n}, t \geqslant 0\Big) \xrightarrow{(d)} (X_\infty^{(i)}(t), t \geqslant 0)$$

where $X_\infty^{(i)}$ is some kind of self-similar Markov process.

Previous work

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- Bertoin & Kortchemski (2015) : still monotype case, but X is not assumed to be nonincreasing.

The monotype case

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where :

- ξ is a Lévy process.

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- α is a real parameter called the *index of self-similarity*.

The result

Theorem (Haas-Miermont)

Assume that, for any continuous function f on $[0,1]$,

$$n^\gamma \mathbb{E}\Big[(1-\frac{X_n(1)}{n})f(\frac{X_n(1)}{n})\Big] \xrightarrow[n\to\infty]{(d)} \int_{[0,1]} f(x)\mathrm{d}\mu(x),$$

where $\gamma > 0$ and μ is a finite and nontrivial measure on $[0,1]$.

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where $\gamma > 0$ and μ is a finite and nontrivial measure on $[0,1]$. Then

$$\Big(\frac{X_n(\lfloor n^{\gamma}t\rfloor)}{n}, t \geqslant 0\Big) \xrightarrow{(d)} (X_\infty(t), t \geqslant 0)$$

where X_∞ is a pssMp with index of self-similarity γ, and the underlying Lévy process ξ is a subordinator with Laplace exponent ψ such that

$$\psi(\lambda) = \mu(\{0\}) + \mu(\{1\})\lambda + \int_{(0,1)} (1-x^{\lambda})\frac{\mathrm{d}\mu(x)}{1-x}.$$

The result

The underlying topology is the Skorokhod topology on the space of càdlàg functions on $[0, \infty)$.

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and

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then

$$\left(\frac{X_n(\lfloor n^{\gamma} \cdot \rfloor)}{n}, \frac{A_n}{n}\right) \xrightarrow{(d)} (X_\infty(\cdot), \sigma).$$

The limiting processes : Markov Additive Processes and their Lamperti transforms

Markov Additive Processes (MAP)

Let $((\xi_t, K_t), t \geqslant 0)$ be a Markov process on $\mathbb{R} \times \{1, \ldots, \kappa\}$, such that the ξ component is nondecreasing. We write $\mathbb{P}_{(x,i)}$ for its distribution when starting at a point (x, i),

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$$\Big((\xi_{t+s} - \xi_t, K_{t+s}), s \geqslant 0\Big) \text{ has distribution } \mathbb{P}_{0,K_t}.$$

In practice - parametrisation

We think of (ξ, K) as a "typed subordinator" :

- K is a continuous-time Markov chain on $\{1, \ldots, \kappa\}$, with transition rates $(\lambda_{i,j})_{(i,j)\in\{1,\ldots,\kappa\}^2, i\neq j}$.

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- On the intervals of constancy of K, ξ is a subordinator with Laplace exponent ψ_i.
- Each jump of K induces a jump of ξ, and we call $B_{i,j}$ its distribution if we jump from i to $j \neq i$.

Lamperti transforms

Let $\alpha \in \mathbb{R}$, we call the Lamperti transform of (ξ, K) the process (X, J) defined by

$$X_t = \mathrm{e}^{-\xi_{\tau(t)}}, \quad J_t = K_{\tau(t)},$$

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Note that, if $\alpha > 0$, the death time $\sigma = \inf\{t \geqslant 0 : X(t) = 0\} = \int_0^\infty \mathrm{e}^{-\alpha \xi_s} \mathrm{d}s$ is finite. (X, J) is càdlàg on $[0, \sigma)$, but J does not have a limit at σ.

Main results

Three regimes

The nature of the limiting process depends on the rate of change of the type J : assume that there is some $\beta \geqslant 0$ such that

$$\forall j \neq i, \mathbb{P}[J_n^{(i)}(1) = j] \sim p_{i,j} n^{-\beta}.$$

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- Solo regime : if $\beta > \gamma$, the type does not change in the limit, and $X_\infty^{(i)}$ is a pssMp which depends on i.

Critical regime : assumption $(\mathsf{H}_{\mathrm{cr}})$

Assume that, for all $i, j \in \{1, \ldots, \kappa\}$, there exists finite measures $\mu^{(i,j)}$ on $(0,1]$, such that for all continuous functions $f : [0,1] \to \mathbb{R}$,

$$n^\gamma \mathbb{E}\Big[\Big(1 - \frac{X_n^{(i)}(1)}{n}\mathbf{1}_{\{j=i\}}\Big) f\Big(\frac{X_n^{(i)}(1)}{n}\Big)\Big] \xrightarrow[n\to\infty]{(d)} \int_{[0,1]} f(x)\mathrm{d}\mu^{(i,j)}(x),$$

Moreover, for all $i \in \{1, \ldots, \kappa\}$, at least one of the measure $\mu^{(i,j)}, j \in \{1, \ldots, \kappa\}$ is not trivial.

Critical regime : main convergence

Theorem (Haas-S.)

Assume $(\mathsf{H}_{\mathrm{cr}})$. *Then, for all* $i \in \{1, \ldots, \kappa\}$,

$$\Big(\frac{X_n^{(i)}(\lfloor n^\gamma \cdot \rfloor)}{n}\Big) \xrightarrow{(d)} X_\infty^{(i)}(\cdot),$$

where $X_\infty^{(i)}$ *is the first component of a Lamperti MAP with the following parameters :*

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where $X_\infty^{(i)}$ *is the first component of a Lamperti MAP with the following parameters :*

- *The self-similarity index is* γ.
- $\psi_i(\lambda) = \mu^{(i,i)}(\{0\}) + \mu^{(i,i)}(\{1\})\lambda + \int_{(0,1)} (1 - x^\lambda) \frac{\mathrm{d}\mu^{(i,i)}(x)}{1-x}$.
- $\lambda_{i,j} B_{i,j} = \mu^{(i,j)} \circ (-\log)^{-1}$.
- *The initial type is* i.

Critical regime : death times

Theorem (Haas-S.)

Assume moreover that, for all $i \in \{1, \ldots, \kappa\}$*, there exists* j *such that* $\mu^{(i,j)}([0,1)) > 0$.

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Assume moreover that, for all $i \in \{1, \ldots, \kappa\}$, there exists j such that $\mu^{(i,j)}([0,1)) > 0$.

Then, calling $A_n^{(i)}$ the death time of $X_n^{(i)}$ and $\sigma^{(i)}$ that of $X_\infty(i)$, we have, jointly with the previous convergence,

$$\frac{A_n^{(i)}}{n} \xrightarrow{(d)} \sigma^{(i)}.$$

Mixing regime : assumption $(\mathsf{H}_{\mathrm{mix}})$

Assume that there exists $\beta \geqslant 0$ with $\beta < \gamma$ for which :

(i) There exist finite measures $(\mu^{(i)}, i \in \{1, \ldots, \kappa\})$ on $[0,1]$, such that, for all continuous functions $f : [0,1] \to \mathbb{R}$,

$$n^{\gamma}\mathbb{E}\left[f\left(\frac{X_n^{(i)}(1)}{n}\right)\left(1 - \frac{X_n^{(i)}(1)}{n}\right)\right] \underset{n\to\infty}{\longrightarrow} \int_{[0,1]} f(x)\mathrm{d}\mu^{(i)}(x).$$

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(ii) Moreover, there exists a Q-matrix $Q = (q_{i,j})_{i,j\in\{1,\ldots,\kappa\}}$ having a unique irreducible component, such that, for all types $i \neq j$

$$n^{\beta}\mathbb{P}[J_n^{(i)}(1) = j] \underset{n\to\infty}{\sim} n^{-\beta} q_{i,j}$$

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We call $\pi = (\pi_i)_{i\in\{1,\ldots,\kappa\}}$ the irreducible distribution on $\{1, \ldots, \kappa\}$ associated to Q.

Mixing regime : main convergence

Theorem (Haas-S.)

Assume $(\mathsf{H}_{\mathrm{mix}})$*. Then, for all* $i \in \{1, \ldots, \kappa\}$*,*

$$\Big(\frac{X_n^{(i)}(\lfloor n^\gamma \cdot \rfloor)}{n}\Big) \xrightarrow{(d)} (X_\infty(\cdot)),$$

where X_∞ *is a pssMp with the following parameters :*

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Mixing regime : death times

Theorem (Haas-S.)

Assume now that all the measures $\mu^{(i)}$ are nonzero.

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Assume now that all the measures $\mu^{(i)}$ are nonzero.

Then, with similar notation to earlier, jointly with the previous convergence,

$$\frac{A_n^{(i)}}{n} \xrightarrow{(d)} \sigma.$$

Solo regime : assumption $(\mathrm{H}_{\mathrm{sol}})$

We fix a type i and assume the following :

(i) There exists a nontrivial finite measure $\mu^{(i)}$ on $[0,1]$, such that, for all continuous functions $f : [0,1] \to \mathbb{R}$,

$$n^{\gamma}\mathbb{E}\left[f\left(\frac{X_n^{(i)}(1)}{n}\right)\left(1-\frac{X_n^{(i)}(1)}{n}\right)\right] \underset{n\to\infty}{\longrightarrow} \int_{[0,1]} f(x)\mathrm{d}\mu^{(i)}(x).$$

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(ii) Moreover,

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Assume moreover to $(\mathsf{H}_{\rm sol})$ *that, for some* $a<1$ *and for all types* j*,*

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Assume moreover to $(\mathsf{H}_{\mathrm{sol}})$ *that, for some* $a < 1$ *and for all types* j,

$$\liminf_{n\to\infty} n^{-\gamma}\mathbb{P}[X_n^{(j)}(1) \leqslant an] > 0.$$

Then, with similar notation to earlier, jointly with the previous convergence,

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Multi-type Markov branching trees

Definition

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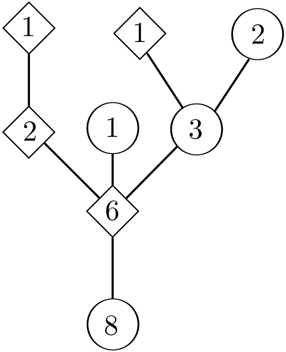

Definition

- Offspring distribution : for all n and i, $q_n^{(i)}$ is the distribution of the offspring of an individual of size n and type i. It is a probability measure on

$$\overline{\mathcal{P}}_n = \Big\{ ((n_1,t_1),\ldots,(n_k,t_k)) : \; n_1 \geqslant n_2 \geqslant n_k, \; \sum n_i \leqslant n, \; t_i \in \{1,\ldots,\kappa\} \Big\}$$

- We call $T_n^{(i)}$ a version of the tree with an ancestor of size n and type i
- Sometimes it is convenient for the tree to be *planted* : an extra untyped vertex is added below the root
- The monotype trees were studied by Haas and Miermont (2012)

Example 1 : Contioned multi-type Galton-Watson trees

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- the total number of vertices. The size of a vertex is then the number of vertices in the descending subtree.

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- the number of vertices of fixed type i. The size of a vertex is then the number of vertices of type i in the descending subtree.
- more generally, we can condition on $\sum_{i=1}^{\kappa} \alpha_i |T|_i$, where $|T|_i$ is the number of vertices of type i...

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The case where S is a star is the subject of previous work (2015).

Example 2 : Recursively growing trees

This yields a (planted) Markov branching tree if :

- Each vertex of S has a type, with 1 for its root.
- The size of a vertex is the number of vertices of type 1 in its descending subtree.

(with eventual “superfluous" types which we can identify if we want to.)

The scaling limits of multi-type Markov branching trees : multi-type self-similar fragmentation trees

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We assume that the Markov branching structure is conservative : for $n \neq 1$, the sum of the sizes of the children of an individual with size n is exactly n.

Call X_n the biggest element of the first generation of a T_n. Assume that

$$\mathbb{P}[X_n \leqslant (1-\varepsilon)n] \underset{n\to\infty}{\sim} c_\varepsilon n^{-\gamma}, \ \forall \varepsilon > 0.$$

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$$\frac{1}{n^\gamma} T_n \xrightarrow{(d)} \mathcal{T}$$

where $\mathcal{T}$ is a *self-similar fragmentation tree* with explicit distribution.

Multi-type self-similar fragmentation processes and trees

Let, for $i \in \{1, \ldots, \kappa\}$, ν_i be a σ-finite measure on

$$\mathcal{S}^{\downarrow} = \Big\{ \mathbf{s} = (s_i, t_i)_{i \in \mathbb{N}} : s_1 \geqslant s_2 \geqslant \ldots \geqslant 0, \sum s_i \leqslant 1, t_i \in \{1, \ldots, \kappa\} \Big\}$$

which satisfies

$$\int_{\mathcal{S}^{\downarrow}} (1 - s_1 \mathbf{1}_{\{t_1 = i\}}) \nu^{(i)}(\mathrm{d}\mathbf{s}) < \infty.$$

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A multi-type self-similar fragmentation process with self-similarity index $\alpha \in \mathbb{R}$ and *dislocation measures* (ν_i) is a $\mathcal{S}^{\downarrow}$-valued process such that a particle (x, i) transforms into a set of particles with masses and types $x\mathbf{s} = (xs_i, t_i)_{i \in \mathbb{N}}$ at rate $x^{\alpha} \mathrm{d}\nu_i(\mathbf{s})$.

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When $\alpha < 0$, a multi-type self-similar fragmentation *tree* with the same parameters is, informally, the family tree of the above process.

Upcoming theorem

Let $X_n^{(i)}$ be the largest element of the first generation of $T_n^{(i)}$, and $J_n(i)$ its type. Assume that, for some $\gamma > 0$ and $\beta \geqslant 0$,

$$\mathbb{P}[X_n^{(i)} \leqslant (1-\varepsilon)n] \underset{n\to\infty}{\sim} c_\varepsilon^{(i)} n^{-\gamma}, \ \forall \varepsilon > 0, i \in \{1, \ldots, \kappa\}$$

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$$\frac{1}{n^\gamma} T_n^{(i)} \xrightarrow{(d)} \mathcal{T}^{(i)}$$

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- multi-type if $\beta = \gamma$.
- monotype, not depending on i if $0 \leqslant \beta < \gamma$.
- monotype, depending on i if $\beta > \gamma$.

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- We will also obtain convergence to the stable trees, at least when conditioning on the number of vertices of one type.
- For the recursively growing trees, $n^{-1/|S|}T_n$ converges in distribution, and maybe in probability, to a multi-type fragmentation tree.

Thank you for your attention