Scaling limits of multi-type Markov branching trees

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Joint work with Bénédicte Haas.

Based on

- *Scaling limits of k-ary growing trees.* AIHP, 2015.
- *Bivariate Markov chains converging to Lamperti transform Markov Additive Processes.* SPA, 2018.
- *On the exponential functional of Markov Additive Processes, and applications to multi-type self-similar fragmentation processes and trees.* To appear in ALEA.
- *Scaling limits of multi-type Markov branching trees.* In preparation.

Scaling limits of k-ary growing trees

Construction

Fix an integer $k \geqslant 2$. We define a sequence $(T_n(k), n \geqslant 0)$ of random k-ary trees by the following recursion:

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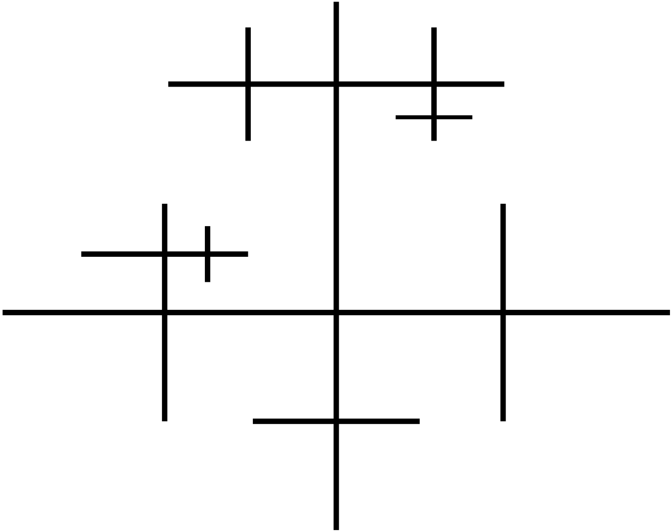
- given $T_n(k)$, to make $T_{n+1}(k)$, choose uniformly at random one of its edges, add a new vertex in the middle, thus splitting this edge in two, and then add $k-1$ new edges starting from the new vertex.

After a few steps: $T_{10}(3)$



A few observations

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- $k=2$: the algorithm constructs uniform binary trees (Rémy, 1985). It is then well-known (Aldous, 1991, 1993) that, when rescaled by $\sqrt{n}$, the tree $T_n(2)$ converges in distribution to a scalar multiple of Aldous' Brownian continuum random tree.

Scaling limit

Theorem (Haas-S.,2015)

We have the following convergence as n tends to infinity

$$\frac{1}{n^{1/k}} T_n(k) \xrightarrow{\mathbb{P}} \mathcal{T}_k.$$

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(measure on $T_n(k)$ = uniform measure on the leaves)

Branching structure

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Let for $n \geqslant 1$:

- $T_n^1, \ldots, T_n^k$ be the k subtrees rooted at the first node of $T_n(k)$.
- $X_n^1, \ldots, X_n^k$ be their number of internal nodes. (sorted nonincreasingly)

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- $T_n^1, \ldots, T_n^k$ are independent.
- For all i, T_n^i has the same distribution as $T_{X_n^i}(k)$.

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- If e is the "first" edge of $T_{n-1}(k)$ then $(T^1_n, \dots, T^k_n) = (T_{n-1}(k), T_0(k), \dots, T_0(k))$. They are independent since $T_0(k)$ is deterministic.

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This essentially ends the proof.

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Specifically, letting $Y_n^i = \frac{X_n^i}{n}$, we have

$$n^{1/k}\mathbb{E}[(1 - Y_n^1)f(Y_n^1, \ldots, Y_n^k)] \to \int_{\mathcal{S}_k}(1 - x_1)f(x_1, \ldots, x_k)d\nu_k(\mathbf{x})$$

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$$\mathrm{d}\nu_k(\mathbf{x}) = \frac{(k-1)!}{k(\Gamma(\frac{1}{k}))^{(k-1)}} \prod_{i=1}^{k} x_i^{1/k-1} \left(\sum_{i=1}^{k} \frac{1}{1 - s_i} \right) \mathbf{1}_{\{x_1 \geqslant x_2 \geqslant \ldots \geqslant x_k\}} \mathrm{d}\mathbf{x}$$

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By a theorem of Haas and Miermont ('12), this implies the convergence in distribution of $n^{-1/k} T_n(k)$ to $\mathcal{T}_k$, where $\mathcal{T}_k$ is the *self-similar fragmentation tree with self-similarity index* $-1/k$ *and dislocation measure* ν_k.

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Informally, $\mathcal{T}_k$ is the family tree of a branching population model where:

- individuals are characterised by their mass $x \in (0, 1]$
- for $\mathbf{s} \in \mathcal{S}_k$ and $x > 0$, an individual with mass x splits into k individuals with masses $xs_1, \ldots, xs_k$ at rate $x^{-1/k}\mathrm{d}\nu_k(\mathbf{s})$.

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- instead of adding a "star" at each step, add a more complex figure. This is interesting!

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If τ is a star with $k-1$ vertices, we get the previous construction.

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But it is Markov branching if we take more information into account and enter the *multi-type* world.

Multi-type Markov branching trees

Definition

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Consider a discrete branching population model where:

- Individuals are characterised by their *size* $n \in \mathbb{N}$, and their *type* $i \in \{1, \dots, K\}$.
- The sum of the sizes of the offspring of one individual with size $n \in \mathbb{N}$ is at most n.

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This model needs a set of *offspring distributions* which give, for all $n \in \mathbb{N}$ and $i \in \{1, \dots, K\}$, give the distribution of the sizes and types of the children of a (n, i) individual.

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Under this notation, the sequence $T_n^{(i)}$ is (planted) Markov branching.

Example II: Conditioned multi-type Galton-Watson trees

Consider a K-type Galton-Watson population. We will condition it on its number of individuals with a specific type, say 1.

- Let $T_n^{(i)}$ be the family tree when we start with an ancestor of type i, and conditioned on having n vertices of type 1.

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This then forms a Markov Branching sequence.

The scaling limit candidates: multi-type fragmentation trees

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The distribution of such a process is characterised by the *index of self-similarity* $\alpha < 0$ and the *dislocation measures* $\nu_i, i \in \{1, \dots, K\}$.

Dislocation measures

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The dislocation measures are measures on the simplex-like space $\overline{\mathcal{S}}^{\downarrow}$ which is the space of sequences $(\mathbf{s}, \mathbf{i}) = \big((s_n, i_n), n \in \mathbb{N}\big)$ such that

- $\sum_n s_n = 1$ and for all n, $s_n \geqslant 0$
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We call $\mathcal{T}^{(i)}_{\alpha,\nu}$ the tree with self-similarity index α and set of dislocation measures (ν_j), when starting with an ancestor with characteristics $(1, i)$.

Our scaling limit results

The general principle

Let $(T_n^{(i)})$ be a K-type family of Markov branching trees, $(X_n^{(i)}, J_n^{(i)})$ = size and type of the largest individual in the first generation of $T_n^{(i)}$.

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Then

$$\frac{1}{n^\gamma} T_n^{(i)} \xrightarrow[n\to\infty]{(d)} \mathcal{T}^{(i)}$$

where $\mathcal{T}^{(i)}$ is a fragmentation tree with index of self-similarity $-\gamma$ and which is:

- K-type if $\beta = \gamma$
- monotype is $\beta < \gamma$.

Scaling limits I: to a K-type fragmentation tree

Theorem (Haas-S.,18+)

Let $(X_n^{(i)}(k), J_n^{(i)}(k))$ *be the sizes and types of the first generation in* $T_n^{(i)}$*, sorted in decreasing lexicographical ordering. Let also* $Y_n^{(i)}(k) = \frac{X_n^{(i)}(k)}{n}$.

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Let $k = 1 + \#\tau$, then

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Actually the convergence is a.s. - see upcoming work by Sénizergues.

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Assume this time that, if we ignore the types,

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and that, for $j \neq i$,

$$n^{\beta}P(J_n^{(i)}(1)=j) \underset{n\to\infty}{\longrightarrow} q_{i,j}$$

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$$\frac{1}{n^{\gamma}} T_n^{(i)} \xrightarrow[n\to\infty]{(d)} \mathcal{T}_{-\gamma,\nu}.$$

where $\mathcal{T}_{-\gamma,\nu}$ is a monotype fragmentation tree.

Scaling limits II: the mixing case

The dislocation measure ν is given by a mixing of the ν_i:

$$\nu = \sum_i \chi_i \nu_i$$

where χ is the invariant measure of the matrix Q.

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Let $T_n^{(i)}$ be a Galton-Watson tree with offspring distributions (ξ_j), with root of type i, and conditioned to have n vertices of type 1.

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Theorem (Haas-S., 2018+)

There exists $C > 0$ (which does not depend on i) such that

$$\frac{1}{\sqrt{n}} T_n^{(i)} \xrightarrow{(d)} C\,\mathcal{T}_{\mathrm{Br}}$$

where $\mathcal{T}_{\mathrm{Br}}$ is the Brownian continuum random tree.

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This is an improvement of a result of Miermont ('08), which gives the same convergence under exponential moments.

Thank you!

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- S_1 follow Dir$(\frac{N_{i,1}}{k}, \dots, \frac{N_{i,p_i}}{k})$ and $I_1 = (j_{i,1}, \dots, j_{i,p_i})$
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$$\int f(\mathbf{s}, \mathbf{i}) \mathrm{d}\nu^{(i)} = \frac{\Gamma(\frac{N_i}{k})}{\Gamma(\frac{N_i - 1}{k})} \mathbb{E}[f(S_1, I_1)] + \frac{\Gamma(\frac{N_i}{k})}{k\Gamma(\frac{N_i + k - 1}{k})} \mathbb{E}[\frac{1}{1 - \max S_2} f(S_2, I_2)]$$

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And take the push-forward of $\nu^{(i)}$ by reordering.

GW technical setup

Let $m_{i,j} = \sum_{\mathbf{z} \in (\mathbb{Z}_+)^K} \zeta^{(i)}(\mathbf{z}) z_j$ be the average number of children of type j among the progeny of an individual of type i. Let

$$M = (m_{i,j})$$

be the *mean matrix*, which we assume irreducible in the Perron-Frobenius sense, and has largest eigenvalue 1 ($=$ criticality).

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Let $\mathbf{a}$ and $\mathbf{b}$ be the left and right positive eigenvectors for the eigenvalue 1, normalised such that

$$\mathbf{a} \cdot \mathbf{1} = \mathbf{a} \cdot \mathbf{b} = 1$$

GW: the normalising constant

Define the following quantities:

$$Q_{j,k}^{(i)} = \sum_{\mathbf{z} \in (\mathbb{Z}_+)^K} \zeta^{(i)}(\mathbf{z}) z_j z_k, \qquad i, j, k \in [K], j \neq k,$$

$$Q_{j,j}^{(i)} = \sum_{\mathbf{z} \in (\mathbb{Z}_+)^K} \zeta^{(i)}(\mathbf{z}) z_j (z_j - 1), \qquad i, j \in [K],$$

$$\sigma^2 = \sum_{i,j,k} a_i b_j b_k Q_{j,k}^{(i)}.$$

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Then

$$\frac{1}{\sqrt{n}} T^{(i)}_n \xrightarrow{(d)} \frac{2}{\sigma\sqrt{a_1}} \mathcal{T}_{\mathrm{Br}}$$