Externally definable sets in NIP theories

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Let $A \subset \mathcal{U}$ be any set (big or small).

An *externally definable* subset of A is a $\phi(A; \bar{b}) \subseteq A^k$, $\bar{b} \in \mathcal{U}$.

An *internally definable* subset of A is a $\psi(A; \bar{d}) \subseteq A^k$, $\bar{d} \in A$.

Problem

Can we understand externally definable sets in terms of internally definable ones?

Stable T

Ideal situation: any externally definable subset of A is internally definable.
(A is *stably-embedded*.)

DLO

An externally definable set of M is a finite union of convex subsets.
$\Longrightarrow$ tame

NIP

According to the philosophy that *NIP* is "stable + DLO", it should be tame too.

From now on, T is NIP.

Theorem (Honest definitions)

Let $A \subset \mathcal{U}$ and $\phi(\bar{x}; \bar{b}) \in L(\mathcal{U})$.
There is $\psi(\bar{x}; \bar{z})$ such that for any finite $A_0 \subseteq \phi(A; \bar{b})$, we can find $\bar{d} \in A$ such that

$$A_0 \subseteq \psi(A; \bar{d}) \subseteq \phi(A; \bar{b}).$$

Another way to say the same thing.

Let $L_{\mathbf{P}} = L \cup \{\mathbf{P}(x)\}$.
Let $A \subseteq M$, (M, A): expansion of M to $L_{\mathbf{P}}$ setting $\mathbf{P}(M) = A$.

Theorem (Honest definitions)

Let $A \subset M$ and $\phi(\bar{x}; \bar{b}) \in L(M)$.
There is an extension $(M, A) \prec (M', A')$ and $\psi(\bar{x}; \bar{d}) \in L(A')$ such that:

$$\psi(A; \bar{d}) = \phi(A; \bar{b})$$
$$\psi(A'; \bar{d}) \subseteq \phi(A'; \bar{b})$$

Corollary (Weak stable-embeddedness, Guingona)

Let $A \subset M$ and $\phi(\bar{x}; \bar{b}) \in L(M)$.
There is an extension $(M, A) \prec (M', A')$ and $\psi(\bar{x}; \bar{d}) \in L(A')$ such that:

$$\psi(A; \bar{d}) = \phi(A; \bar{b})$$

Examples

$A = I$, a small indiscernible sequence.
Assume (for simplicity) that I is ordered by some $\emptyset$-definable $<_I$ and the order is dense Dedekind complete.

Theorem (Baldwin-Benedikt)

I is stably embedded.

Proof.

Let $(M, I) \prec (M', I')$. Then I' is an indiscernible sequence.
By weak stable-embeddedness, it is enough to consider the case of some $\phi(I; \bar{b})$, $\bar{b} \in I'$.
By indiscernability, the set $\phi(I'; \bar{b})$ is definable using $=$ and the ordering $<_I$. We conclude by Dedekind completeness. $\square$

Examples

$A{=}M$, a model.
Let M^{Sh} be the expansion of M obtained by adding a predicate for every externally definable subset of M^k.

Theorem (Shelah)

M^{Sh} has elimination of quantifiers and is NIP.

Proof.

Let $M \prec N$ and $\phi(x_1, x_2; \bar{b}) \in L(N)$. Take $(N, M) \prec (N', M')$ and $\psi(x_1, x_2; \bar{d}) \in L(M')$ such that:

$$\psi(A; \bar{d}) = \phi(A; \bar{b})$$
$$\psi(A'; \bar{d}) \subseteq \phi(A'; \bar{b})$$

Let $\theta(x_1; \bar{d}) = (\exists x_2)\psi(x_1, x_2; \bar{d})$. Then $\theta(M; \bar{d})$ coincides with the first projection of $\phi(M; \bar{b})$. $\square$

Expansions

We consider the following situation: M is NIP, we name some subset $A \subset M$ by a new predicate $\mathbf{P}(x)$.

Problem

Give sufficient conditions for the pair (M, A) to be NIP.

Results/special cases established by Berenstein, Boxall, Dolich, Günaydin, Hieronymi, Onshuus.

Definition

An $L_{\mathbf{P}}$-formula is *bounded* if it is of the form

$$(\forall x_1 \in \mathbf{P})(\exists x_2 \in \mathbf{P}) \cdots (\forall x_n \in \mathbf{P})\phi(\bar{x}; \bar{y}),$$

where $\phi(\bar{x}; \bar{y})$ in an L-formula.

We say that the theory of (M, A) is *bounded* if all $L_{\mathbf{P}}$-formulas are equivalent to a bounded one.

Theorem

Assume that M is NIP, A_{ind} is NIP and the theory of the pair (M, A) is bounded, then (M, A) is NIP.

Corollary

If M is NIP, $A \prec M$ and the theory of (M, A) is bounded, then (M, A) is NIP.

Uniformity

Theorem (Uniformity of honest definitions)

Let $\phi(\bar{x};\bar{y}) \in L$. Then there is some $\psi(\bar{x};\bar{z}) \in L$ such that: For every $A \subset \mathcal{U}$, and $\bar{b} \in \mathcal{U}$, for every finite $A_0 \subseteq \phi(A;\bar{b})$, there is $\bar{d} \in A$ such that

$$A_0 \subseteq \psi(A;\bar{d}) \subseteq \phi(A;\bar{b}).$$

Corollary (UDTFS)

Let $\phi(\bar{x};\bar{y}) \in L$, then there is $\psi(\bar{x};\bar{z}) \in L$ such that for every finite set A and every $\bar{b} \in \mathcal{U}$, there is $\bar{d} \in A$ with

$$\phi(A;\bar{b}) = \psi(A;\bar{d}).$$

Remark: We have to assume that the full theory is NIP. The UDTFS conjecture is still open for an NIP formula $\phi(\bar{x};\bar{y})$ in a (possibly) independent theory.

The proof uses compactness and a theorem of Alon-Kleitman and Matousek:

(p,q)-Theorem (special case)

If $\phi(x;\bar{y})$ in NIP and $q<\omega$ is big enough, then there is N such that for any finite family $\mathcal{B}=\{\phi(x;\bar{b}_i) : i<n\}$ if any q sets from $\mathcal{B}$ intersect, then there is an N-point set in $\mathcal{U}$ intersecting all sets of $\mathcal{B}$.

Distal theories

Distal theories are “completely unstable” NIP theories.

Definition

An NIP theory T is *distal* if for every indiscernible sequence $I + b + J$ (I and J infinite sequences) and set A, if

$$I + J \text{ is indiscernible over } A,$$

then

$$I + b + J \text{ is indiscernible over } A.$$

Examples

O-minimal theories, $\mathbb{Q}_p$ are distal.

Theorem (Strong honest definitions for distal theories)

*Let $\phi(\bar{x};\bar{y}) \in L$. Then there is $\psi(\bar{x};\bar{z}) \in L$ such that: for any $\bar{b} \in \mathcal{U}^{|\bar{y}|}$ and finite $A_0 \subseteq \phi(\bar{x};\bar{b})$,
there is $\bar{d} \in A_0$ such that:*

$$A_0 \subseteq \psi(\bar{x};\bar{d}),$$
$$\psi(\bar{x};\bar{d}) \to \phi(\bar{x};\bar{b}).$$

Corollary (UDTFS for distal theories)

Let $\phi(\bar{x};\bar{y}) \in L$, there there is $\theta(\bar{x};\bar{z}) \in L$ and N such that for any $\bar{b} \in \mathcal{U}^{|\bar{y}|}$ and finite $A \subset \mathcal{U}$, there is some $A_0 \subseteq A$ of size $\leq N$ with:

$$\mathrm{tp}_\theta(\bar{b}/A_0) \vdash \mathrm{tp}_\phi(\bar{b}/A).$$

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