Hilbert, Logicism, and Mathematical Existence

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5 Logicism & Dedekind !

The path-breaking contributions of Frege were of very little influence until Russell and the Göttigen people (Hilbert, Zermelo) drew attention to them. In the 1890s logicism was a foundational conception associated primarily with the name of Dedekind. This thesis comes as a surprise because it goes against historigraphic usage, but one can find a great deal of evidence in its favor.

6 Sets are logic(al)

Most readers are puzzled to find that the concepts of set and mapping were simply regarded as logical ones. With regards to the concept of set, this was a usual view in the late 19th century, well represented for instance with Schröder, Peano and Russell. As Russell said […], logic consists of three parts: the theory of propositions, the theory of classes, and the theory of relations. […] – Dedekind's logicism was based on the latter two.

8 Dedekind's existence of real numbers: purely logical

In [Dedekind's] work on the real numbers, he emphasizes (following Riemann, 1868) that, is space has a real existence, "it need *not* necessarily be continuous" […]. But even if we knew for sure that it is not continuous, nothing could prevent us from "making it continuous in thought" by the introduction of new elements. The creation of Dedekind had in mind is strictly regulated by the laws of logic, as the continuous domain of the real numbers is won through "the purely logical process of building up the science of numbers" […]. Thus considered, the 1872 passage contains the kernel of Dedekind's position regarding methematical existence. The real numbers exist in a purely logical sense, not in an ontological sens: they exist "in thought," but may not correspond to physical reality. The mathematician can, by the use of pure logic, introduce continuous spaces or the set of real numbers. And he can do so in the strong sense that logic warrants such a step, including a warranty that non contradiction will emerge.

10 Ideal and real existence

Mathematical existence in the tradition of Dedekind and Hilbert is logical admissibility in the realm of pure thought. It is merely "ideal existence," as Zermleo would aptly say many years later […]. For Dedekind, to show that simply infinite sets exist was not to ptrove that there are infinitely many objects in the world – infinite sets exist in our realm of thoughts, in the *Gedankenwelt*. In a similar way, Hilbert proved using arithmetic that both Euclidean and non-Euclidean geometries "exist" mathematically, even though only one of these incompatible systems can be true of the physical world. Thus mathematical exsitence has a peculiar character, being far removed from the kind of existence we intend fot the objects of scientific theories. We might refer to this by distinguishing purely logical existence (*ideal* existence) from ontological existence (*real* existence).

11-13 / 15 Sets = logical framework

it is crucial to realize that *these early axiom systems [produced by Hilbert] has the theory of sets as their basis: set theory was taken to belong to the logical framwork* underlying the axioms. This, of course, was natural for a logicist.

[…]

In *Grundlagen der* Geometrie, Hilbert gives axioms for eht eelements [Dinge] of three sets [Systme], conventionnaly called points, straights and planes. The basic terminology is Dedekind's, but the important issue here is the following: Hilbert's axioms may deal with relations and operations between the elements, or just as well with conditions on *sets* of elements. This difference, very significant in the eyes of a modern logician, was there immaterial, because both are implicitely regarded as elementeray logical methods. The same happens with Hilbert's axiom system for the reals: one stars with a “system” of “things” and defines axiomatically relations and operations between them, including an unequivocally set-theoretic condition, the notorious axiom of completeness […]. This is similar to the way in which Dedekind proceeded while characterising structures such as number fields (1871), the ordered and complete field of the real numbers (1872), or the structure of the natural numbers (1888). In a moment we shall have a closer look at thiis feature.

[…]

Hilbert felt free to formulate axioms postulating conditions on sets of elements; to formalize them, one needs to quantify over sets of elements. The conspicuous example of this trait is the famous Axioms of Completness [*Vollständigkeit*] that Hilbert included first in his axiomatization of the real nimbers (1900), and then in subsequent editions fo the *Grundlagen der Geometrie*.

13 Dedekind's methodology is axiomatic

The great difference between Hilbert and Dedekind is terminological: while the former speaks loudly and clearly of “axioms” in the modern sense, the latter refrains completely from using that word. The rationale for this option was philosophical, having to do with the fact that he still used the word in the old meaning (like Frege) and, even more importantly, with his logictis beliefs. But this should not obscure the fact that the modern axiomatic methodology is clearly present in Dedekind's foundational work, and in his algebraic and number-theoretic work – whether we talk of “axioms” or “conditions” [*Bedingungen*] should be immaterial.

16/17 Hilbert's *maximal* structure vs Dedekind's *minimal* chain

when we require the fields to be ordered and Archemedean, the maximality condition seem enough to characterize univocally the set of real numbers, with topological completeness emerging as a by-product of maximality. This seems to have been a key realization for Hilbert at the time, especially because he was interested in having a completeness axiom that would not entail the Archimedean property; this kind of independence was important for his axiomatic work.

[…]

Hilbert's axiom: $\forall T$ (if T is an Archemedan ordered field and $S \subset T$, then $S = T$);
Dedekind's 47 $\forall K$ (if K is closed under φ and $A \subset K$, then $A_0 \subset K$).

The diffrence in the consequen (identity in one case, inclusion in the other) clearly expresses the diffrence between a maximality and a minimality condition.

And this is also the source of the ambiguity inherent in Hilbert's axiom. With Dedekind's procedure, we re dealing with an intersection and all chains K are included in a certain set S; when we wrote $\forall K$ above, we meant ‘all subsets K of S’ – we could have written $\forall K \subset S$. The ambient space is supposed to be given and, subsequently, problems of ‘eistence’ are relatively well defined. But when Hilbert says $\forall T$, it seems that the implicit domain can only be the universal set **V**, for his sets T can only be supposed to be living in **V**; we should write $\forall T \subset \mathbf{V}$.

This is why, with Hilbert, one gets the feeling of some ambiguity. His axiom system seems to presuppose the universal set **V** in an essential way, as Dedekind did not.