Negation

Heinrich Wansing
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chap. 18 in *The Blackwell Guide to Philosophical Logic* (p. 415-436)
ed° Lou Goble

417 / 420 Nier une phrase vs nier un terme prédicatif

While

John is pleased

affirms pleased of John,

John is not pleased

denies pleased of John. This form of negation is called *predicate negation* (or predicat denial) and is to be distinguished from *predicate term negation*. In the cas of predicat terme negation, a predicat terme is negated to obtain another predicate term. The predicate term negation of 'pleased' for instance is 'not-plesaed', and the sentence

John is not-pleased

affirms the predicate 'not-pleased' of John. If the predicate terme of a sentence is negated, this results in a *contrary* of that sentence; A pair of contrary sentences cannot both be true. Whereas a predicate term '*P*' may have many contraries, according to the neo-Aristotelian term logicians, it has exactly one logical contrary, namely 'not-*P*' (or 'non-*P*'). Among the non-logical contraries of the predicate terme 'ancient', for example, are 'medieval' and 'modern'. If the predicate of a sentence is negated, one obtains a *contradictory* of that sentence. A pair of contradictory sentences can neither both be false nor both be true. Whereas the predicate term negation of a sentence implies the predicate negation of that sentence, the converse is not true. In this sense, predicate term negation is stronger than predicate denial.

[…]

What the term logicians correctly point out is that a distinction must be drawn between predicate negation and predicate term negation.

[*commentaire nôtre* :
La *predicate term negation* est une loi singulaire sur les symboles de prédicats, mettons $P \mapsto \sim P$, et vérifie $[\sim P](a) \Rightarrow \neg[P(a)]$.
Le prédicat $\sim P$ s'interprète donc comme *un* des contraires du prédicat *P*. Par exemple, *~bleu* pourrait être *rouge* ou pourrait être *jaune*.
Ainsi, pas de tiers exclus (on peut être ni *P* ni *~P*), *a fortiori* pas de contraposition.]

426 Négation comme *fausseté*

A general definition of negation as falsity that is meant to encompass both intuitionistic and strong nefation is suggested in Wansing (1999). Suppose that a single-conclusion consequence relation $\rightarrow$ over a formal language containing a unary connective * is given. In other words, for all formulas *A*, *B* and all finite sets of formulas Δ, Γ:

Reflexivity	$\vdash A \rightarrow A$
Monotonicity	$\Gamma \rightarrow A \vdash \Gamma \cup \{B\} \rightarrow A$
Cut	$\Gamma \cup \{A\} \rightarrow B,\ \Delta \rightarrow A \vdash \Gamma \cup \Delta \rightarrow B$

A binary relation $\leftarrow$ between finite sets of formulas and single formulas is called a *single-conclusion *-refutation relation* iff for all formulas *A*, *B* and finite sets Δ, Γ of formulas:

**-reflexivity*	$\vdash {*A} \leftarrow A,\ \vdash A \leftarrow {*A}$
**-cut*	$\Delta \leftarrow A,\ \Gamma \cup \{{*A}\} \leftarrow B \vdash \Delta \cup \Gamma \leftarrow B$

Assume that $\rightarrow$ and $\leftarrow$ are given as sequent calculi. If $\rightarrow$ is single conclusion consequence relation, then * is a *negation as falsity in* $\rightarrow$ iff

(α) the relation $\leftarrow$ defined by '$\Delta \leftarrow A$ iff $\Delta \rightarrow *A$' is a single conclusion *-refutation relation
(β) for every formula A, not both $\vdash \varnothing \rightarrow A$ and $\vdash \varnothing \rightarrow *A$
(γ) there is a formula A such that not both $\vdash A \rightarrow *A$, $\vdash *A \rightarrow A$

If $\leftarrow$ is a single conclusion *-refutation relation, then * is a nefation as falsity in $\leftarrow$ iff

(α') the relation $\rightarrow$ defined by '$\Delta \rightarrow A$ iff $\Delta \leftarrow *A$' is a single conclusion consequence relation
(β) for every formula A, not both $\vdash \varnothing \leftarrow A$ and $\vdash \varnothing \leftarrow *A$
(γ) there is a formula A such that not $\vdash A \leftarrow A$

If * satisfies both (α) and (α') for a single-conclusion consequence relation $\rightarrow$ and a single-conclusion *-refuation relation $\leftarrow$, then negation as falsity is a vehicle for either keping $\rightarrow$ and dispensing with $\leftarrow$ or keeping $\leftarrow$ and dispensing with $\rightarrow$. Then double negation introdution $A \rightarrow **A$ and double negation elimination $**A \rightarrow A$ are derivable. Cleraly, the relation $\leftarrow$ defined by (α) is a single-conclusion *-refutation relation iff * satisfies $A \rightarrow **A$

427-429 Négation comme inconsistance (Gabbay)

Gabby (1988) defines a syntactic notion of negation as inconsistency. […] The basic idea of Gabbay's definition is that the negation $*A$ of a formula A is derivable from a set of premises Γ iff some undesirable fomula B from a set of unwanted formulas $\theta*$ is derivable from Γ with A.

[…]

The unary operation * is […] sais to be a *negation (as inconsistency) in* $\rightarrow$ iff there is a non-empty set $\theta*$ of formulas [which is not the same as the set of all formulas] such that for every finite set Γ of formulas and every formula A:

$$\vdash \Gamma \rightarrow *A \quad \text{iff} \quad (\exists B \in \theta*)\ (\vdash \Gamma \cup \{A\} \rightarrow B)$$

Moreover, $\theta*$ must not contain anay theorems. If such a collection of unwanted formulas exists, it can always be chosen as $\{C \mid \vdash \varnothing \rightarrow *C\}$, since by (reflexivity), the latter set is non-empty, if * is a negation. The definition of negation as inconsistency can therefore be reformulated without appeal to $\theta*$. Namely, * is a negation as inconsistency in $\rightarrow$ iff for every finite set Γ of formulas and every formula A:

$$\vdash \Gamma \rightarrow *A \quad \text{iff} \quad \exists C\ (\vdash \varnothing \rightarrow *C\ \&\ \vdash \Gamma \cup \{A\} \rightarrow C)$$

[…]

Observation. Suppose $\rightarrow$ is [*-]consistent in the sense that for no formula A of the underlying language, both $\varnothing \rightarrow A$ and $\varnothing \rightarrow *A$ are provable. Then * is a nefgtion as inconsistency iff * satisfies contraposition as a rule, the Law of Excluded Contradiction, and double negation introduction.

[…]

Observation. Every negation as inconsistency is a negation as falsity.

431 Negation interne, fortement symétrique

A unary connective * is said to be an *internal negation* of a consequence relation $\rightarrow$ iff the relation $\rightarrow$ is closed under

$A, \Gamma \rightarrow \Delta \vdash \Gamma \rightarrow \Delta, *A$ and $\Gamma \rightarrow \Delta, A \vdash A, \Gamma \rightarrow \Delta$ [çàd si * permet de passer A d'un côté à l'autre]

The existence of an internal negation forces a consequence relation to be a multiple-conclusion relation. A single-relation consequenc relation $\rightarrow$ over a language with a unary connective * is said to be *strongly symmetric with respect to* * iff there exists a multiple-conclusion consequence relation $\rightarrow'$ defined ober the same language such that

$$\Gamma \rightarrow' A \quad \text{iff} \quad \Gamma \rightarrow A$$

and * is an internal negation for $\rightarrow'$.

[…]

if $\rightarrow$ is a consequence relation, then it is strongly symmetric with respect to * iff

(i) $A \rightarrow **A$
(ii) $**A \rightarrow A$, and
(iii) $\Gamma, A \rightarrow B$ implies $\Gamma, *B \rightarrow *A$.