Hurwitz numbers and the algebra of split permutations

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Abstract. We interpret one-part Hurwitz numbers as structure coefficients of the subalgebra of invariants of an algebra of *split permutations*, which is built on the model of the Ivanov-Kerov algebra of partial permutations. The computation of Hurwitz numbers is then reduced to the diagonalization of a matrix whose entries are indexed by *multipartitions* and whose eigenvalues are known. The described algorithm has almost genus-free complexity.

Résumé. Nous interprétons les nombres de Hurwitz simples comme constantes de structure de la sous-algèbre des invariants d'une algèbre de *permutations scindées*, qui est construite sur le modèle de l'algèbre d'Ivanov-Kerov des permutations partielles. Nous ramenons ainsi le calcul des nombres de Hurwitz à la diagonalisation d'une matrice dont les entrées sont indexées par les *multipartitions*, et dont l'ensemble des valeurs propres est connu. L'algorithme obtenu est de complexité indépendante du genre.

Keywords: Hurwitz numbers, multipartitions, symmetric group.

In the end of the nineteenth century, Hurwitz asked the following: given a permutation $\sigma \in \mathfrak{S}_n$, in how many ways can one factorise σ in a product of a given number of transpositions that generate a transitive subgroup of $\mathfrak{S}_n$? When σ is a cycle and the number of transpositions is minimal, the answer has been known since Hurwitz himself ([Hur02]). However, when σ lies in a more complex conjugacy class, the computation of these Hurwitz numbers when n is large remains an open problem, which has known a renewal of interest in the late contexts of the study of moduli space of curves (see [ELSV01], [FP00], [OP01], [OP02]) and of the 2-dimensional gravity models (see [Wit91] and [Zvo05]). Abstract formulas stemming from the Gromov-Witten theory, in particular the ELSV formula, provide explicit expressions for the spheric and toric genus. One also has recurrence formulas ([OP01, p. 100-101]) that theoretically give all Hurwitz numbers, but are computationally cumbersome. On the other hand, the asymptotics of simple Hurwitz numbers when n goes to ∞ is almost completly known, see [Zvo04]. However, as far as we know, no closed and efficient formula has yet been found for general Hurwitz numbers.

We describe in this paper a natural way to compute Hurwitz numbers of one partition as structure coefficients of an algebra of *split permutations* (see section 2) reminding that of Ivanov and Kerov's *partial permutations*, *cf.* [IK99]. More precisely, the simple Hurwitz numbers are certain coefficients involved in the powers of the class of *split transpositions*, see Proposition 1. In the subalgebra of invariants, the multiplication by this class of transpositions turns out to be diagonalisable with known eigenvalues, namely, the contents of the multipartitions of size n (section 4). These results allow an easy computation of the simple Hurwitz numbers, and our algorithm has complexity $O(C^n)$ for some constant C; this is far better than the $\binom{n}{2}^{2n}$ corresponding to the raw computation of the products of all lists of $2n$ transpositions in $\mathfrak{S}_n$ — $2n$ being the typical number involved in the computation of Hurwitz numbers $H_n^g(\lambda)$.

If one omits the transitivity condition in the enumeration of factorisations in transpositions, one obtains the *disconnected* Hurwitz numbers, and they are merely a specialization of the *Frobenius formula*, which relates

- the number of factorisations of the unit element in a finite group whose factors lie in prescribed conjugacy classes,
- and the values of the irreducible characters of the group on these conjugacy classes,

see Appendix in [LZ04]. Then, if one studies the orbits of the subgroup generated by the transpositions of a factorisation of the identity in $\mathfrak{S}_n$, an inclusion-exclusion principle on set partitions yields a explicit formula for *connected* Hurwitz numbers. This formula involves many character values, and consequently can't be used for efficient computations; however, we found out that computing the structure coefficients of the powers of the class of split transpositions eventually yields the same formula, and with all symmetries being very naturally handled thanks to the invariant algebra, see Theorem 8.

To conclude this introduction, let us mention a very comfortable aspect of our algorithm: the diagonalizing of the class of n-sized split tranpositions (for a given n) gives a straightforward and *simultaneous* access to *all* one-part Hurwitz numbers of degree n — *i.e.*, for any partition λ and for any genus g. For example, it becomes easy to compute all digits of $H_{10}^{100}((3,3))$:

$$\begin{gathered}782097979460992214693804083332536583893351107785781024934173669372784194209718926379837 10\\7556058252242150177257334037305183802786325756492053941931828934914673377950313339378216 4\\0050299563299234996840635265275525532966015938390900645713106800708017285165485106027722 1\\48550228252877233219200354868567157363538695639946611186972400140456314720 0000\end{gathered}$$

and the computation of the other Hurwitz numbers of order $n = 10$ is then almost instantaneous.

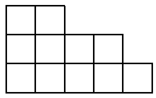
1 Combinatorial background

Let us describe the basic combinatorial objects that will be used throughout this paper, and fix some notations. The cardinality of a set S will be denoted by $|S|$, $\#S$ or $\mathrm{card}\, S$.

1.1 Partitions, Young diagrams, contents

A *partition* of a positive integer n is a finite non-increasing sequence $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_l)$ of integers λ_i (the *parts* of the partition) whose sum equals n (the *size* of the partition, also denoted $|\lambda|$). The number l of parts is called the *length* of λ and is denoted by $\ell(\lambda)$. If k is an integer in $[\![1, n]\!] := \{1, \ldots, n\}$, the *multiplicity* of k in λ is the number of parts λ_i equalling k, and will be denoted $m_k(\lambda)$. Then, a partition λ can be written multiplicatively as $\lambda = 1^{m_1(\lambda)}2^{m_2(\lambda)}\cdots s^{m_s(\lambda)}$, with $|\lambda| = \sum_{k\geq 1} k\, m_k(\lambda)$.

A partition λ is usually represented by its *Young diagram*, which consists in $\ell(\lambda)$ lines of piled squares with λ_j squares in the j-th line. So for instance, the Young diagram of the partition $(5, 4, 2)$ is:



The *content* of a case (x, y) in a Young diagram is the integer $x - y$, and the *content* $c(\lambda)$ of a partition is the sum of the contents of all the cases of the associated Young diagram. Thus, the content of $(5, 4, 2)$ is $4 + 3 + 2 + 1 + 0 + 2 + 1 + 0 - 1 - 1 - 2 = 9$. A n-sized *standard tableau* is a n-sized Young diagram whose cases are filled each with an integer of $[\![1, n]\!]$, so that the corresponding sequences on each line and on each column increase.

In the following, $\lambda \vdash n$ means that λ is a partition of n, and $\mathfrak{Y}_n$ is the set of n-sized Young diagrams. One can totally order the set of all partitions $\mathfrak{Y} = \bigsqcup_{n=0}^{\infty} \mathfrak{Y}_n$ by setting $\lambda \leq \mu$ if and only if $|\lambda| < |\mu|$, or $|\lambda| = |\mu|$ and $\lambda \leq_{\text{lexico}} \mu$. It is well-known that the number of n-sized partitions satisfies the asymptotic formula of Hardy and Ramanujan

$$\operatorname{card} \mathfrak{Y}_n \cong \frac{\mathrm{e}^{\pi\sqrt{2n/3}}}{4\sqrt{3}\,n}, \tag{1}$$

see [FS09, VIII. 6]. In particuliar, $\#\mathfrak{Y}_n$ is always smaller than $A^{\sqrt{n}}$ for some constant $A > 0$.

It is well-known that the partitions of size n parametrize the conjugacy classes of $\mathfrak{S}_n$. So, the *type* $t(\sigma)$ of a permutation σ is the partition obtained by ordering the lengths of its cycles (including singletons), and two permutations are conjugated if and only if they have same type. When $\lambda \vdash n$, we shall denote by C_λ both the set of permutations of $\mathfrak{S}_n$ whose type is λ, and the sum $\sum_{t(\sigma)=\lambda} \sigma$ of such elements in the group algebra $\mathbb{Q}[\mathfrak{S}_n]$. We also set $\widetilde{C}_\lambda := C_\lambda / \operatorname{card} C_\lambda$ for the *normalized* conjugacy class of type λ, and $z_\lambda = n!/|C_\lambda| = \prod_{i\geq 1} i^{m_i(\lambda)}\, m_i(\lambda)!$.

Finally, if λ is a partition, the *completed* partition (with respect to a given positive integer $n \geq |\lambda|$) is $\overline{\lambda} := 1^{n-|\lambda|+m_1(\lambda)}2^{m_2(\lambda)}\cdots s^{m_s(\lambda)}$; hence, one has added parts of size 1 so that $|\overline{\lambda}| = n$. On the other hand, the *ramification* of a partition λ is the integer $r(\lambda) := |\lambda| - \ell(\lambda) = \sum_{i\geq 1}(\lambda_i - 1)$, and its *signature* is $\varepsilon(\lambda) := (-1)^{r(\lambda)}$, which equals $\varepsilon(\sigma)$ for any $\sigma \in C_\lambda$. These two quantities are conserved when completing the partition λ.

1.2 Set partitions, Young subgroups, irreducible modules of $\mathbb{Q}[\mathfrak{S}_n]$

A *set partition* $\pi = \pi_1 \sqcup \cdots \sqcup \pi_p$ of a set S is the set of the parts of a partition of S (the parts are therefore *un*ordered). For instance, the cycles of a permutation $\sigma \in \mathfrak{S}_n$ gives a set partition $\operatorname{orb}\sigma$ of $[\![1, n]\!]$. The *profile* of a set partition $\pi = \pi_1 \sqcup \cdots \sqcup \pi_p$ is the partition obtained by ordering the sizes $|\pi_i|$ of the parts of π.

The *refinement order* on the set $\mathfrak{P}_n$ of set partitions of $[\![1,n]\!]$ is defined by setting $\pi \leq \pi'$ if and only if each part π_i is included in a part π'_j; equivalently, each π'_j is a union of π_i. Thus, $[\![1,n]\!]$ is the greatest (coarsest) element of $\mathfrak{P}_n$ and $\{1\} \sqcup \{2\} \sqcup \cdots \sqcup \{n\}$ is the smallest (finest) one. Since $(\mathfrak{P}_n, \leq)$ is a finite distributive lattice, there is a Möbius function $\mu : \mathfrak{P}_n \times \mathfrak{P}_n \to \mathbb{Z}$ satisfying the Rota inversion formula ([Rot64]): for any function f on $\mathfrak{P}_n$ taking values in an abelian group, if one defines $f^*(\pi) := \sum_{\pi' \geq \pi} f(\pi')$ for all $\pi \in \mathfrak{P}_n$, then $f(\pi) = \sum_{\pi' \geq \pi} \mu(\pi,\pi')\, f^*(\pi')$.

The *Young subgroup* of a set partition $\pi \in \mathfrak{P}_n$ is defined by $\mathfrak{S}_\pi := \prod_i \mathfrak{S}_{\pi_i}$. Then, for any n-sized standard tableau T, the *Young idempotent* e_T is defined by the product in $\mathbb{Q}[\mathfrak{S}_n]$ of two factors: the sum of all elements in the Young subgroup associated to the rows of T and the alternating sum of all elements in the Young subgroup associated to the colums of T. The $\mathfrak{S}_n$-modules $\mathbb{Q}[\mathfrak{S}_n]\, e_T$ are irreducible and, for same-shaped tableaux T, are all isomorphic one to another, whereas two different-shaped tableaux lead to non-isomorphic modules ([JK81]). For any partition $\lambda \vdash n$, the *Specht module* of type λ is any of the $\mathbb{Q}[\mathfrak{S}_n]\, e_T$'s where T is a λ-shaped tableau. It is denoted V_λ, and the character of the representation V_λ will be denoted χ^λ. If one sets $e_\lambda := \sum_{\text{shape}(T)=\lambda} e_T$, then the e_λ's are central idempotents (up to some scalars) that sum up to 1, whence a decomposition in blocks $\mathbb{Q}[\mathfrak{S}_n] = \bigoplus_{\lambda \vdash n} E_\lambda$, where $E_\lambda := e_\lambda\, \mathbb{Q}[\mathfrak{S}_n]$. The projection on the block E_λ will be denoted pr_λ.

1.3 Multipartitions and their symmetries

A *multipartition* is a set $\Lambda = \{\lambda^i\}$ of partitions λ^i, or equivalently an ordered list $\Lambda = [\lambda^1 \geq \lambda^2 \geq \cdots]$ of partitions. The *size* of a multipartition $\Lambda = \{\lambda^i\}$ is the sum $\sum |\lambda^i|$, and the number of partitions λ^i equalling a given partition λ will be denoted $m_\lambda(\Lambda)$. In order to harmonize the definitions to be seen, a n-sized multipartition (where $n \geq 1$ is an integer) will also be called a *split partition* of n. Their set will be denoted $\mathfrak{Y}^s_n$, and we shall write $\Lambda \models n$ to say that Λ is a n-sized split partition.

The *profile* of a split partition $\Lambda = [\lambda^1 \geq \cdots \geq \lambda^p]$ is the partition $|\Lambda| := (|\lambda^1| \geq \cdots \geq |\lambda^p|)$. If λ is a partition of an integer $k \leq n$, one defines the *maximally split partition* of λ by the split partition

$$\lambda^s := \left[(\lambda_1) \geq (\lambda_2) \geq \cdots \geq (\lambda_{\ell(\lambda)}) \geq (1) \geq (1) \geq \cdots \geq (1)\right]. \tag{2}$$

So for instance, $(3,2,2,1)^s = [(3),(2),(2),(1)^{n-7}]$ for any $n \geq 8$. For a given $n \geq 1$, set $p\,(n)$ and $sp\,(n)$ for the numbers of n-sized partitions and split partitions respectively. The sequence of $sp\,(n)$ goes like

$$1, 3, 6, 14, 27, 58, 111, 223, 424, 817 \ldots \tag{3}$$

Clustering n-sized partitions according to their profile gives the majoration

$$sp(n) \leq \sum_{\lambda \vdash n} \prod_i p(\lambda_i) \leq \sum_{\lambda \vdash n} \prod_i A^{\sqrt{\lambda_i}} \leq \sum_{\lambda \vdash n} \prod_i A^{\lambda_i} = A^n \sum_{\lambda \vdash n} 1 \leq A^{n+\sqrt{n}}. \tag{4}$$

Therefore, the number of n-sized split partitions is smaller than B^n for some n-free constant $B > 0$.

A *symmetry* (or *automorphism*) of a split partition $\Lambda = [\lambda^1 \geq \lambda^2 \geq \cdots \geq \lambda^p]$ is an element of the set of words $\{[\tau(\lambda^1)\tau(\lambda^2)\ldots\tau(\lambda^p)]\}_\tau$ that are permutations of the word $[\lambda^1\lambda^2\ldots\lambda^p]$, and such that

$|\tau(\lambda^1)| \geq |\tau(\lambda^2)| \geq \cdots \geq |\tau(\lambda^p)|$. The set of such symmetries will be denoted $\operatorname{Aut}\Lambda$. For instance, the split partition $\Lambda = \ydiagram{1,2}\,, \ydiagram{2}\,, \ydiagram{2}\,, \ydiagram{1,1}$ has three symmetries corresponding to the reorderings of the three 2-sized parts. More generally, choosing a symmetry of Λ amounts to choosing for each size $k \geq 1$ a permutation of the k-sized partitions λ^i modulo the permutations of the same λ^i's, whence the formula

$$|\operatorname{Aut}\Lambda| = \prod_{k\geq 1} \binom{m_k(|\Lambda|)}{\{m_\lambda(\Lambda)\}_{|\lambda|=k}} = \frac{\prod_{k\geq 1} m_k(|\Lambda|)\,!}{\prod_{\lambda\vdash n} m_\lambda(\Lambda)!}. \tag{5}$$

1.4 Hurwitz numbers

Let n and k be positive integers. A *constellation* ([LZ04, Chapter 1]) of degree n and length k is a family of k permutations in $\mathfrak{S}_n$ generating a group which acts transitively on $[\![1, n]\!]$, and whose product is the identity. The *type* of a constellation $\{\sigma_i\}_i$ is the family of the types of the σ_i, and its *genus* is the integer $g := g' + 1$ defined by the Riemann-Hurwitz formula $2g' + 2n = \sum_i r(\sigma_i)$. By using a monodromy argument, one can show that constellations correspond to marked ramified coverings of the sphere; then, the genus of the constellation defined as above equals the genus of the ramified covering, whence the terminology.

Let $n, g, k \geq 0$ be integers and $\lambda^1, \ldots, \lambda^k$ be partitions of sizes less than n. We denote by t the integer such that $2g' + 2n = \sum_i r\left(\lambda^i\right) + t$; in other words, one adds t transpositions in order to obtain a constellation of genus g. The Hurwitz numbers are defined by(i):

$$H_n^g(\lambda^1, \ldots, \lambda^k) := \frac{1}{n!}\operatorname{card}\Big\{\text{constellations } (\sigma_1, \ldots, \sigma_k, \tau_1, \ldots, \tau_t) \in (\mathfrak{S}_n)^{k+t} \;\Big|\; \substack{t(\sigma_i)=\overline{\lambda^i} \\ t(\tau_j)=\overline{2}}\Big\} \tag{6}$$

As previously mentioned, these numbers also count (up to topological equivalence) some marked ramified coverings of the sphere. If one forgets the transitivity condition, finding the disconnected Hurwitz number $H_n^{g\bullet}(\lambda^1, \ldots, \lambda^k)$ amounts to computing the coefficient of the unit element in the product $C_{\overline{\lambda^1}} \cdots C_{\overline{\lambda^k}} \left(C_{\overline{2}}\right)^t$ in the algebra $\mathbb{Q}[\mathfrak{S}_n]$. We will add some structure to the latter algebra to take into account the orbits of the generated group, and more precisely to ensure its being transitive.

2 The algebra of split permutations and its invariant subalgebra

2.1 The algebra of split permutations $\mathscr{B}_n$

A *split permutation* of order n is a couple (σ, π) that consists in a set partition $\pi \in \mathfrak{P}_n$ together with a permutation σ lying in $\mathfrak{S}_\pi$, which amounts to saying $\operatorname{orb}\sigma \leq \pi$ for the refinement order. The set $\mathfrak{S}_n^s$ of split permutations of order n has cardinality

$$\operatorname{card}\mathfrak{S}_n^s = \sum_{\pi\in\mathfrak{P}_n} \prod_i |\pi_i|! = \sum_{\sigma\in\mathfrak{S}_n} B_{\text{\#cycles of }\sigma} = \sum_{\lambda\vdash n} |C_\lambda|\, B_{\ell(\lambda)} \tag{7}$$

(i) In the litterature, one may find a normalising factor to remember the parts of size 1 that were needed to complete each partition λ^i. This factor equals the product of the binomials $\binom{n-|\lambda^i|+m_1(\lambda^i)}{m_1(\lambda^i)}$.

where B_k is the k-th Bell number and corresponds to merging some cycles of σ to get π. The sequence of $|\mathfrak{S}_n^s|$ with $n \geq 1$ goes like

$$1, 3, 13, 73, 501, 4051, 37633, 394353, 4596553, 58941091 \ldots \tag{8}$$

The sets $\mathfrak{S}_n$ and $\mathfrak{P}_n$ being monoids for (respectively) the composition and the supremum $\vee$, the set of split permutations has a natural monoid structure given by $(\sigma, \pi)(\sigma', \pi') := (\sigma\sigma', \pi \vee \pi')$. So, for instance, if $s = ((1,2)(3,4), \{1,2,6\} \sqcup \{3,4\} \sqcup \{5\})$ and $t = ((2,1,6), \{1,2,6\} \sqcup \{4,5\} \sqcup \{3\})$, then

$$s \cdot t = ((1,6)(3,4), \{1,2,6\} \sqcup \{3,4,5\}) \in \mathfrak{S}_6^s. \tag{9}$$

To check why $\mathfrak{S}_n^s$ is indeed a submonoid of $\mathfrak{S}_n \times \mathfrak{P}_n$, one need to check why $\pi \vee \pi'$ is coarser than $\mathrm{orb}\, \sigma\sigma'$ when $\mathrm{orb}\, \sigma \leq \pi$ and $\mathrm{orb}\, \sigma' \leq \pi'$. For $k = \sigma(j)$ and $j = \sigma'(i)$, one knows that k and j (resp. j and i) are both in a part of π (resp. π'), hence k and i are both in a part of $\pi \vee \pi'$, *q.e.d.*

One can therefore consider the algebra $\mathscr{B}_n := \mathbb{Q}[\mathfrak{S}_n^s]$ of the monoid $\mathfrak{S}_n^s$. Notice that our construction is essentially the same as the one of [IK99], except that the distributive lattice that fibers $\mathfrak{S}_n$ is the lattice of set partitions, instead of the hypercube lattice of subsets[(ii)]. The letter $\mathscr{B}$ suggests that we will rather consider an algebra $\mathscr{A}_n$; indeed, we will define $\mathscr{A}_n$ as being the invariant subalgebra of $\mathscr{B}_n$ under a group action.

2.2 The subalgebra of invariants $\mathscr{A}_n$

The symmetric group $\mathfrak{S}_n$ acts on $\mathfrak{S}_n$ by conjugation and on $\mathfrak{P}_n$ by taking the images of the parts. One has therefore a product action on $\mathfrak{S}_n \times \mathfrak{P}_n$ given by

$$\rho \cdot (\sigma, \pi) := \left(\rho\, \sigma\, \rho^{-1}, \rho(\pi)\right), \tag{10}$$

which stabilizes $\mathfrak{S}_n^s$ and is distributive with respect to the law of the monoid $\mathfrak{S}_n^s$. Two split permutations (σ, π) and (σ', π') are conjugate under this action if and only if π and π' have same profile and if there is a size-preserving correspondance $\pi_i \leftrightarrow \pi'_i$ between the parts of π and those of π' such that $\sigma_{|\pi_i}$ and $\sigma'_{|\pi'_i}$ have same type for all i. Therefore, conjugacy classes in $\mathfrak{S}_n^s$ are labelled by n-sized split partitions. The conjugacy class corresponding to a split permutation Λ will be denoted C_Λ, with the same abuse of notation as for the C_λ's. Thus,

$$\mathscr{A}_n := \mathbb{Q}\,[\mathfrak{S}_n^s]^{\,\mathfrak{S}_n} = \bigoplus_{\Lambda \models n} \mathbb{Q}\, C_\Lambda. \tag{11}$$

The projection $\mathrm{pr}_{\mathscr{A}_n}$ on $\mathscr{A}_n$ sends an element of $\mathscr{B}_n$ to the mean of its conjugates: if an x in $\mathfrak{S}_n^s$ has type Λ, then $\mathrm{pr}_{\mathcal{A}_n}(x)$ equals the normalised class $\widetilde{C}_\Lambda := C_\Lambda / |C_\Lambda|$. Since the action on $\mathfrak{S}_n^s$ is distributive, the projection $\mathrm{pr}_{\mathscr{A}_n}$ is a morphism of $\mathscr{A}_n$-modules.

Remark. Recall that the invariant subalgebra in $\mathbb{Q}[\mathfrak{S}_n]$ is exactly the center of $\mathbb{Q}[\mathfrak{S}_n]$. Similarly, its counterpart $\mathscr{A}_n$ (in the algebera $\mathscr{B}_n$) can be shown to be a commutative subalgebra of $\mathscr{B}_n$.

(ii) In a recent work, the first author succeeded in fibering $\mathfrak{S}_n$ and its Hecke algebra by the lattice of compositions, and obtained q-analogs of some results of Faharat and Higman. It seems that such constructions — *i.e.*, fibering a (semi)-group by a (semi)-distributive lattice — can be made quite general, and in some cases, they allow to construct projective limits of objects that have natural direct limits, but no natural projections.

2.3 Structure constants in $\mathscr{A}_n$ and Hurwitz numbers

Once constructed, the algebra of split permutations allows one to gather the announced statement.

Theorem 1 *With the same notations as in the previous sections, the Hurwitz number $H_n^g\left(\lambda^1,\ldots,\lambda^k\right)$ equals*

$$\frac{1}{n!}\left[C_{[1^n]}\right]\left\{C_{(\lambda^1)^s}\cdots C_{(\lambda^k)^s}\left(C_{(2)^s}\right)^t\right\} \tag{12}$$

in the subalgebra of invariants $\mathscr{A}_n$.

Proof: The elements in C_{λ^s} are exactly the $(\sigma,\operatorname{orb}\sigma)$ for σ running over C_λ. Therefore, the coefficient of $C_{[1^n]}=(\mathrm{id},[\![1,n]\!])$ in a product $\prod_i C_{(\lambda^i)^s}$ is the number of factorisations $\prod\sigma_i=\mathrm{id}$ of type $(\overline{\lambda^i})_i$ for which $\bigvee_i\operatorname{orb}\sigma_i=[\![1,n]\!]$. So, there only remains to establish for any given set E the equivalence between the transitivity of the group generated by a (finite) family $(\sigma_i)_{i\in I}\in\mathfrak{S}_E^I$, and the equality $\bigvee_{i\in I}\operatorname{orb}\sigma_i=E$. But this is obvious once noticed that the supremum $\bigvee_{i\in I}\operatorname{orb}\sigma_i$ is the very set partition given by the transitive components of the genereted subgroup $\langle\sigma_i\rangle_{i\in I}$. □

Corollary 2 *The one-part Hurwitz number $H_n^g(\lambda)$ equals $\frac{1}{n!}\,[\widetilde{C}_{\overline{\lambda}}]\,\left\{(C_{(2)^s})^t\right\}=\frac{1}{z_\lambda}\,[C_{\overline{[\lambda]}}]\,\left\{(C_{(2)^s})^t\right\}$.*

Proof: For just one partition λ, the identity $\sigma\tau_1\cdots\tau_t=\mathrm{id}$ can be rewritten as $\sigma=\tau_t\cdots\tau_1$. Consequently, the generated subgroup $\langle\sigma,\tau_1,\ldots,\tau_t\rangle$ equals $\langle\tau_1,\ldots,\tau_t\rangle$, and therefore is transitive if and only if $\bigvee\operatorname{orb}\tau_i=[\![1,n]\!]$. So $n!\,H_n^g(\lambda)$ equals the number of terms in $(C_{(2)^s})^t$ equalling a $(\sigma,[\![1,n]\!])$ with $\sigma\in C_\lambda$. By using the projection $\mathrm{pr}_{\mathscr{A}_n}$, if one performs the computations in $\mathscr{A}_n$, then

$$n!\,H_n^g(\lambda)=[\widetilde{C}_{\overline{[\lambda]}}]\,\left\{(C_{(2)^s})^t\right\}=|C_\lambda|\times[C_{\overline{[\lambda]}}]\,\left\{(C_{(2)^s})^t\right\}, \tag{13}$$

the coefficients being taken with respect to the basis $\widetilde{C}$ in the second member, and with respect to the basis C in the third member of this identity. □

3 Structure of $\mathscr{B}_n$

Our algorithm in section 4 describes a computation of $H_n^g(\lambda)$ relying on the algebra of $\mathscr{B}_n$. In order to make all computations clear, we now carry on with the description of the structure of $\mathscr{B}_n$.

3.1 The isomorphism $\mathscr{B}_n\cong\bigoplus_{\pi\in\mathfrak{P}_n}\mathbb{Q}[\mathfrak{S}_\pi]$

As the following proposition shows, the understanding of $\mathscr{B}_n$ amounts to that of the $\mathbb{Q}[\mathfrak{S}_\pi]$'s for all set partitions π. For $\pi\in\mathfrak{P}_n$, the *forgetful morphism* $\varphi^\pi:\mathbb{Q}[\mathfrak{S}_n^s]\to\mathbb{Q}[\mathfrak{S}_\pi]$ is the unique $\mathbb{Q}$-linear map such that

$$\varphi^\pi((\sigma,\psi))=\begin{cases}\sigma\in\mathbb{Q}[\mathfrak{S}_\pi] & \text{if } \pi\geq\psi,\\ 0 & \text{otherwise,}\end{cases} \tag{14}$$

for any split permutation (σ,ψ). It is a morphism of algebras as one can easily see. Let us denote φ the sum $\sum_{\pi\in\mathfrak{P}_n}\varphi^\pi$ taking values in $\bigoplus_{\pi\in\mathfrak{P}_n}\mathbb{Q}[\mathfrak{S}_\pi]$.

Proposition 3 *The morphism φ is an isomorphism of algebras between $\mathscr{B}_n$ and $\bigoplus_{\pi\in\mathfrak{P}_n}\mathbb{Q}[\mathfrak{S}_\pi]$.*

Proof: The dimension of $\mathscr{B}_n$ is the cardinal of $\mathfrak{S}_n^s$, which we have computed as being $\sum_{\pi\in\mathfrak{P}_n}\prod_i|\pi_i|!$ obviously equal to the dimension of $\bigoplus_{\pi\in\mathfrak{P}_n}\mathbb{Q}[\mathfrak{S}_\pi]$. Therefore, it is sufficient to show that φ is surjective. Setting $\sigma_\pi := \sigma \in \mathbb{Q}[\mathfrak{S}_\pi]$ for any $(\sigma,\pi)\in\mathfrak{S}_n^s$ (the σ_π's form a basis of $\bigoplus_{\pi\in\mathfrak{P}_n}\mathbb{Q}[\mathfrak{S}_\pi]$), one has

$$\varphi\,(\sigma,\psi) = \sum_{\pi\geq\psi}\sigma_\pi \tag{15}$$

for all $\psi\in\mathfrak{P}_n$, whence by the Rota inversion formula the wanted preimages of the σ_π by φ. □

Notice that our Proposition 3 is the very analog of [IK99, Corollary 3.2], and holds in fact for general "fibrations" of a finite group by a finite lattice.

3.2 The indecomposable blocks of $\mathscr{B}_n$

Because of Proposition 3, the algebra $\mathscr{B}_n$ is isomorphic to an essentially unique direct sum of matrix algebras (the so-called *indecomposable blocks* of the algebra); indeed, $\mathbb{Q}$ is known to be a splitting field for the symmetric groups, so this is true for the symmetric group algebras $\mathbb{Q}[\mathfrak{S}_n]$, for their tensor products $\mathbb{Q}[\mathfrak{S}_\pi]\cong\bigotimes_i\mathbb{Q}[\mathfrak{S}_{\pi_i}]$ and for any direct sum of such algebras. In this paragraph, we shall detail the decomposition of $\mathscr{B}_n$ into indecomposable blocks.

Remind notations e_λ, E_λ and pr_λ from section 1.2. Since we will deal with the symmetric groups on subsets of $[\![1,n]\!]$ (the parts π_i), we will recall the corresponding sets by an exponent, writing $e_\lambda^{\pi_i}$, $E_\lambda^{\pi_i} := \mathbb{Q}[\mathfrak{S}_{\pi_i}]\,e_\lambda^{\pi_i}$ and $\mathrm{pr}_\lambda^{\pi_i}$.

Fix a set partition $\pi\in\mathfrak{P}_n$ and a family of partitions $\overrightarrow{\lambda}=(\lambda^i)_i\in\prod_i\mathfrak{Y}_{|\pi_i|}$. We define $e^\pi_{\overrightarrow{\lambda}} := \bigotimes_i e^{\pi_i}_{\lambda^i}$; this is a central idempotent of $\bigotimes\mathbb{Q}[\mathfrak{S}_{\pi_i}]\cong\mathbb{Q}[\mathfrak{S}_\pi]$. Then, the

$$E^\pi_{\overrightarrow{\lambda}} := \bigotimes_i E^{\pi_i}_{\lambda^i} = \bigotimes_i\left(e^{\pi_i}_{\lambda^i}\,\mathbb{Q}[\mathfrak{S}_{\pi_i}]\right) = \left(\bigotimes_i e^{\pi_i}_{\lambda^i}\right)\left(\bigotimes_i\mathbb{Q}[\mathfrak{S}_{\pi_i}]\right) = e^\pi_{\overrightarrow{\lambda}}\,\mathbb{Q}[\mathfrak{S}_\pi] \tag{16}$$

are tensor products of indecomposable blocks of the algebras $\mathbb{Q}[\mathfrak{S}_{\pi_i}]$, so they are indecomposable blocks of $\mathbb{Q}[\mathfrak{S}_\pi]$. Consequently:

Proposition 4 *The isomorphism φ yields the decomposition $\mathscr{B}_n\cong\bigoplus_{\pi,\overrightarrow{\lambda}}E^\pi_{\overrightarrow{\lambda}}$, where $\pi\in\mathfrak{P}_n$, and $\overrightarrow{\lambda}$ is then choosed in $\prod_i\mathfrak{Y}_{|\pi_i|}$.*

Proof: This is obvious: $\mathscr{B}_n\cong\bigoplus_{\pi\in\mathfrak{P}_n}\mathbb{Q}[\mathfrak{S}_\pi]$, and

$$\mathbb{Q}[\mathfrak{S}_\pi]\cong\bigotimes_i\mathbb{Q}[\mathfrak{S}_{\pi_i}] = \bigotimes_i\left(\bigoplus_{\lambda^i\vdash|\pi_i|}E^{\pi_i}_{\lambda^i}\right)\cong\bigoplus_{\overrightarrow{\lambda}\in\prod_i\mathfrak{Y}_{|\pi_i|}}\left(\bigotimes_i E^{\pi_i}_{\lambda^i}\right) = \bigoplus_{\overrightarrow{\lambda}}E^\pi_{\overrightarrow{\lambda}}. \tag{17}$$

□

Remark. If one actually wants the irreducible modules of $\mathscr{B}_n$, one can easily show they are the $\bigotimes_i V^{\pi_i}_{\lambda^i}$'s. We won't use them in the following.

3.3 The symmetries of the algebra $\mathscr{B}_n$

Let us now take into account the symmetries of the blocks $E_{\vec{\lambda}}^{\pi}$. First, if π and π' are two set partitions with the same profile, then $\mathbb{Q}[\mathfrak{S}_\pi]$ and $\mathbb{Q}[\mathfrak{S}_{\pi'}]$ are isomorphic algebras, so they yield the same blocks. Then, given a profile $|\pi| \in \mathfrak{Y}_n$ and a set partition π with this profile, two blocks $E_{\vec{\lambda}}^{\pi}$ and $E_{\vec{\mu}}^{\pi}$ are isomorphic when the families of partitions are symmetries of a same split partition. Thus, the indecomposable blocks of $\mathscr{B}_n$ are classified by the n-sized split partitions. Let us denote by $E_\Lambda := E_{\vec{\lambda}}^{\pi}$ a block of type $\Lambda \models n$. Then, the previous argument shows that

$$\mathscr{B}_n \cong \bigoplus_{\Lambda \models n} \Bigg(\bigoplus_{|\pi| = |\Lambda|} E_\Lambda \Bigg)^{\oplus |\mathrm{Aut}\, \Lambda|}. \tag{18}$$

We have already computed $|\mathrm{Aut}\, \Lambda|$. Then, to choose a set partition of profile $(c_1 \geq \cdots \geq c_p)$, one has to choose the parts knowing their cardinal c_i, whence a multinomial coefficient $\binom{n}{c_1 \, \cdots \, c_p}$. However, a set partition is unordered, so one has to divide by all the reorderings of the chosen parts having same size. Hence,

$$\forall \Lambda \models n, \;\; \mathrm{card}\left\{\pi \in \mathfrak{P}_n \mid |\pi| = |\Lambda|\right\} = \binom{n}{|\lambda^1| \cdots |\lambda^p|} \frac{1}{\prod_{k \geq 1} m_k(|\Lambda|!)}. \tag{19}$$

A straightforward simplification leads with the notation $|\Lambda|! := \prod_i |\lambda^i|!$ to the following proposition:

Proposition 5 *There is an isomorphism of algebras $\mathscr{B}_n \cong \bigoplus_{\Lambda \models n} (E_\Lambda)^{\oplus b(\Lambda)}$, where the number $b(\Lambda)$ of blocks of type Λ is $b(\Lambda) = n!/(|\Lambda|! \prod_\lambda m_\lambda(\Lambda)!)$.*

4 Hurwitz numbers and the powers of the class of transpositions

We now carry on with the spectral decomposition of $C_{(2)^s}$. We show that the latter acts diagonally in $\mathscr{A}_n$ with eigenvalues the contents of all n-sized split partitions.

4.1 Describing the matrix of $C_{(2)^s}$

Let us describe the action of $C_{(2)^s}$ by multiplication on $\widetilde{C}_\Lambda$ for a given n-sized split partition Λ. We set for convenience

$$(a, b)^s := \Bigg((a, b), \{a, b\} \sqcup \bigsqcup_{c \neq a, b} \{c\} \Bigg). \tag{20}$$

By definition, $C_{(2)^s}$ is the sum of the $(a, b)^s$'s for $1 \leq a < b \leq n$. Since the product $C_{(2)^s}\, \widetilde{C}_\Lambda$ lies in $\mathscr{A}_n$, it equals its projection in $\mathscr{A}_n$, hence for any fixed (σ, π) in C_Λ the equality $C_{(2)^s}\, \widetilde{C}_\Lambda = \sum_{a<b} \mathrm{pr}_{\mathscr{A}_n}[(a, b)^s\, (\sigma, \pi)]$. One has therefore to determine the type of the products $(a, b)^s\, (\sigma, \pi)$:

1. If a and b are in cycles of lengths λ_k^i and λ_l^j in different parts π_i and π_j, then the type of the product is the split partition $\Lambda[\lambda^i \sqcup \lambda^j, \lambda_k^i + \lambda_l^j]$ obtained from Λ by replacing the two partitions λ^i and λ^j by their disjoint union and then replacing the two parts λ_k^i and λ_l^j by their sum.

2. If a and b are in cycles of lengths λ^i_k and λ^i_l in a same part λ^i, then the type of the product is the split partition $\Lambda[\lambda^i_k + \lambda^i_l]$ obtained from Λ by replacing in the partition λ^i the two parts λ^i_k and λ^i_l by their sum.

3. Finally, if a and b lie in a same cycle λ^i_k, then the looked-for type is the split partition $\Lambda[\lambda^i_k = d+d']$, where the part λ^i_k has been replaced by the two parts corresponding to the two distances d and d' between a and b in the cycle.

The following proposition comes from clustering the projections $\mathrm{pr}_{\mathscr{A}_n}[(a,b)^s\,(\sigma,\pi)]$ according to the three previous cases: for each of them, choose the part(s) then the cycle(s) and in the third case remind the symmetry between both distances in a cycle.

Proposition 6 *For any split partition $\Lambda = [\lambda^1, \ldots, \lambda^p]$, one has the decomposition:*

$$C_{(2)^s}\,\widetilde{C}_\Lambda = \sum_{\substack{1\le i<j\le p\\ 1\le k\le \ell(\lambda^i)\\ 1\le l\le \ell(\lambda^j)}} \lambda^i_k \lambda^j_l\, \widetilde{C}_{\Lambda[\lambda^i\sqcup\lambda^j,\lambda^i_k+\lambda^j_l]} + \sum_{\substack{1\le i\le p\\ 1\le k<l\le \ell(\lambda^i)}} \lambda^i_k \lambda^i_l\, \widetilde{C}_{\Lambda[\lambda^i_k+\lambda^i_l]} + \sum_{\substack{1\le i\le p\\ 1\le k\le \ell(\lambda^i)\\ 1\le d<\lambda^i_k}} \frac{\lambda^i_k}{2}\, \widetilde{C}_{\Lambda[\lambda^i_k=d+d']} \tag{21}$$

For example, the matrix of $C_{(2)^s}$ in the basis $\yng(3)$, $\yng(2,1)$, $\yng(1,1,1)$, $\yng(2)\,\yng(1)$, $\yng(1,1)\;\yng(1)$, $\yng(1)\;\yng(1)\;\yng(1)$ of $\mathscr{B}_3$ is the following 6×6-matrix:

$$\begin{pmatrix} 0 & 2 & 0 & 2 & 0 & 0\\ 3 & 0 & 3 & 0 & 2 & 0\\ 0 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 3\\ 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{22}$$

4.2 The diagonalisability and spectrum of $C_{(2)^s}$

We shall now prove that $C_{(2)^s}$ acts diagonally on $\mathscr{A}_n$. Let us first recall some basic facts about the *Jucys-Murphy* elements $J_k := \sum_{i<k}(i,k) \in \mathbb{Q}[\mathfrak{S}_n]$ defined for all $1 \le k \le n$. If one fixes n, then J_n acts as a scalar in V_λ (*a fortiori* diagonally in $\mathbb{Q}[\mathfrak{S}_n]$) by the contents of the corners (i,λ_i) of the Young diagram λ. Moreover, each eigenspace $E \subset V_\lambda$ of J_n is isomorphic over $\mathfrak{S}_{n-1}$ to $V_{\lambda\setminus(i,\lambda_i)}$, which allows one to carry on the spectral decomposition with the remaining $J_1, \ldots, J_{n-1}$. Therefore, if f is any symmetric function, then $f\,(J_1,\ldots,J_n)$ acts on E_λ by $f(\text{contents of }\lambda)$.

Now, $C_2 = p_1(J_1,\ldots,J_n)$, so the action of the class C_2 by multiplication on $\mathbb{Q}[\mathfrak{S}_n]$ is the direct sum of the $c(\lambda)\,\mathrm{id}_{E_\lambda}$. More generally, if π is a set partition of $[\![1,n]\!]$, then the sum $C_2^\pi = \sum_i C_2^{\pi_i}$ of all transpositions in $\mathfrak{S}_\pi$ acts on $E^\pi_{\overrightarrow{\lambda}} = \bigotimes_i E^{\pi_i}_{\lambda^i}$ by the sum $c(\overrightarrow{\lambda}) := \sum_i c(\lambda^i)$ of all contents. Since

$$\varphi^\pi\left((a,b)^s\right) = \begin{cases} (a,b) & \text{if } a \text{ and } b \text{ are in the same part } \pi_i,\\ 0 & \text{otherwise,}\end{cases} \tag{23}$$

the class of split transpositions $C_{(2)^s}$ is sent to C_2^π by φ^π. Therefore, once sent by φ, the class $C_{(2)^s}$ acts as

$$\varphi(C_{(2)^s}\, x) = \sum_\pi \varphi^\pi(C_{(2)^s})\,\mathrm{pr}^\pi(x) = \sum_\pi C_2^\pi \left(\sum_{\overrightarrow{\lambda}} \mathrm{pr}^\pi_{\overrightarrow{\lambda}}(x) \right) = \sum_{\pi,\overrightarrow{\lambda}} c(\overrightarrow{\lambda})\,\mathrm{pr}^\pi_{\overrightarrow{\lambda}}(x) \qquad (24)$$

showing that the action of $C_{(2)^s}$ in $\mathscr{B}_n$ is diagonalisable. Moreover, $C_{(2)^s}$ stabilizes the subspace $\mathscr{A}_n$ because the latter is an algebra, which proves its diagonalisability in $\mathscr{A}_n$.

Remark. If one seeks to diagonalize C_{λ^s} for $\lambda \neq (2)$ (so as to get more-than-one-part Hurwitz numbers), one will encounter two hindrances :

1. If λ has at least two parts, the decomposition of C_λ^π on the Young subgroup π is no longer trivial.
2. The decomposition of the C_λ's as symmetric functions of the Jucys-Murphy elements generally involves n-dependant functions, hence differents actions on the Young subgroups.

Remark. Heuristically, the action of $C_{(2)^s}$ on $\mathscr{A}_n$ has for set of eigenvalues *with multiplicities* $\{c(\Lambda)\}_{\Lambda \models n}$. In particular, all contents of split partitions are involved. We did not prove this precise result, and it is not necessary to know it in order to compute Hurwitz numbers. We conjecture the following: for any given split partition Λ, the intersection of the direct sum of blocks of type Λ with $\mathscr{A}_n$ is a one-dimensional vector space.

4.3 Final description of the algorithm and the Frobenius formula

Let us finally describe the algorithm provided by the previous paragraphs:

Algorithm 7 *In order to compute a one-part Hurwitz number $H_n^g(\lambda)$, one has to:*

1. *List the split partitions of size n and write down the matrix M of $C_{(2)^s}$ acting on $\mathscr{A}_n$; this is easy thanks to Proposition 6.*
2. *Find a diagonalization basis of M; since we know* a priori *the eigenvalues, it amounts to solve linear systems of equations.*
3. *Compute the $(2n + 2g' - r(\lambda))$-th power of M, which is easy because M has been diagonalized.*

Since $|\mathfrak{Y}_n^s| = O(B^n)$, our algorithm has complexity $O(C^n)$ for some $C > 0$.

On the other hand, by writing down explicitly the projections $\mathrm{pr}^\pi_{\overrightarrow{\lambda}}$ in terms of the characters of the symmetric groups, one can easily deduce from equation 24 an abstract formula for one-part Hurwitz numbers, which turns out to be the formula one could have obtained by applying an inclusion-exclusion principle on the aforementioned Frobenius formula for disconnected Hurwitz numbers. If Λ is a split partition of size n, we set $\mathfrak{S}_\Lambda = \mathfrak{S}_{|\lambda^1|} \times \cdots \times \mathfrak{S}_{|\lambda^p|}$, and we denote

$$\dim \Lambda = \dim V_{\lambda^1} \times \cdots \times \dim V_{\lambda^p} \qquad ; \qquad \chi^\Lambda = \chi^{\lambda^1} \otimes \cdots \otimes \chi^{\lambda^p}. \qquad (25)$$

As a tensor product, χ^Λ is an irreducible character of the Young subgroup $\mathfrak{S}_\Lambda$. Finally, $m(\Lambda) = (-1)^{p-1}\,(p-1)!$ is the Möbius function between a set partition of profile $|\Lambda|$ and the coarsest set partition $[\![1, n]\!]$. With these notations and by using the Möbius inversion formula for the reciprocal of φ and the previous computations:

Theorem 8 *The one-part (connected) Hurwitz numbers are given by the abstract formula:*

$$H_n^g(\mu) = \sum_{\Lambda \in \mathfrak{Y}_n^s} c(\Lambda)^{2n+2g'-r(\mu)} \; \frac{m(\Lambda)\, b(\Lambda)\, \dim \Lambda}{n!\, |\Lambda|!} \left(\sum_{\sigma \in C_{\widetilde{\mu}} \cap \mathfrak{S}_\Lambda} \chi^\Lambda(\sigma) \right) \tag{26}$$

This last formula gives for instance $H_3^g((2)) = (9^{g+1}-1)/2$ and $H_4^g((3)) = 6^{2g+2} - 3^{2g+2}$.

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