The group structure of rational points of elliptic curves over a finite field

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Introduction

- Cryptography!
- We are interested in $E(\mathbb{F}_q)$, were E is an elliptic curve over a finite field $\mathbb{F}_q$;
- References: [Sil86; Len96; Wat69; WM71; Mil06];

Torus

- An elliptic curve $E/\mathbb{C}$ is a torus $E=\mathbb{C}/\Lambda$, where Λ is a lattice $\Lambda=\tau\mathbb{Z}+\mathbb{Z}$, $(\tau\in\mathfrak{H})$.
- Let $\wp(z,\Lambda)=\sum_{w\in\Lambda\setminus\{0_E\}}\frac{1}{(z-w)^2}-\frac{1}{w^2}$ be the Weierstrass $\wp$-function and $E_{2k}(\Lambda)=\sum_{w\in\Lambda\setminus\{0_E\}}\frac{1}{w^{2k}}$ be the (normalised) Eisenstein series of weight $2k$.
- Then $\mathbb{C}/\Lambda\to E, z\mapsto(\wp(z,\Lambda),\wp'(z,\Lambda))$ is an analytic isomorphism to the elliptic curve

$$y^2=4x^3-60E_4(\Lambda)-140E_6(\Lambda)=4x^3-g_2(\Lambda)-g_3(\Lambda).$$

- In particular the elliptic functions are rational functions in $\wp,\wp'$: $\mathbb{C}(E)=\mathbb{C}(\wp,\wp')$.
- Two elliptic curves $E=\mathbb{C}/\Lambda$ and $E'=\mathbb{C}/\Lambda'$ are isomorphic if there exists $\alpha\in\mathbb{C}^*$ such that $\Lambda=\alpha\Lambda'$;
- Two elliptic curves are isomorphic if and only if they have the same j-invariant: $j(\Lambda)=j(\Lambda')$.

$$j(\Lambda)=1728\frac{g_2^3}{g_2^3-27g_3^2}.$$

Lattices

- $\wp$ is homogeneous of degree -2 and $\wp'$ of degree -3:

$$\wp(\alpha z, \alpha\Lambda) = \alpha^{-3}\wp(z, \Lambda);$$

- Up to normalisation one has $\Lambda = \tau\mathbb{Z} + \mathbb{Z}$ with $\tau \in \mathfrak{H}_g$ the upper half plane;
- This gives a parametrisation of lattices Λ by $\tau \in \mathfrak{H}_g$;
- If $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Sl}_2(\mathbb{Z})$ then a new basis of Λ is given by $(a\tau + b, c\tau + d)$;
- We can normalize this basis by multiplying by $(c\tau + d)^{-1}$ to get $\Lambda' = \frac{a\tau+b}{c\tau+d}\mathbb{Z} + \mathbb{Z}$;
- The isomorphism class of elliptic curves is then parametrized by $\mathfrak{H}_g / \mathrm{Sl}_2(\mathbb{Z})$.

Elliptic curves over a field k

Definition

An elliptic curve E/k (k perfect) can be defined as

- A nonsingular projective plane curve E/k of genus 1 together with a rational point $0_E \in E(k)$;
- A nonsingular projective plane curve E/k of degree 3 together with a rational point $0_E \in E(k)$;
- A nonsingular projective plane curve E/k of degree 3 together with a rational point $0_E \in E(k)$ which is a point of inflection;
- A non singular projective curve with equation (the Weierstrass equation)

$$Y^2Z + a_1XYZ + a_3YZ^2 = X^3 + a_2X^2Z + a_4XZ^2 + a_6Z^3$$

(in this case $0_E = (0:1:0)$);

Choice of the base point

Remark

- If E is a nonsingular projective plan curve of degree 3 and $O \in E(k)$, then if O is an inflection point there is a linear change of variable which puts E into Weierstrass form and $O = (0:1:0)$, but otherwise needs a non linear change of variable to transform O into an inflection point;
- If $\operatorname{char} k > 3$ then a linear change of variable on the Weierstrass equation gives the short Weierstrass equation:

$$y^2 = x^3 + ax + b.$$

Class of isomorphisms of elliptic curves

- The Weierstrass equation:

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$$

has discriminant $\Delta_E = -b_2b_8 - 8b_3 - 27b_2 + 9b_2b_4b_6$ so it defines an elliptic curve whenever $\Delta_E \neq 0$.
(Here $b_2 = a_1^2 + 4a_2$, $b_4 = 2a_4 + a_1a_3$, $b_6 = a_3^2 + 4a_6$,
$b_8 = a_1^2a_6 + 4a_2a_6 - a_1a_3a_4 + a_2a_3^2 - a_4^2$).

- The j-invariant of E is

$$j_E = \frac{(b_2^2 - 24b_4)^3}{\Delta_E}$$

- When we have a short Weierstrass equation $y^2 = x^3 + ax + b$, the discriminant is $-16(4a^3 + 27b^2)$ and the j-invariant is

$$j_E = 1728\frac{4a^3}{4a^3 + 27b^2}.$$

Theorem

Two elliptic curves E and E' are isomorphic over $\overline{k}$ if and only if $j_E = j_{E'}$.

Isomorphisms and Twists

- The isomorphisms (over $\overline{k}$) of isomorphisms of elliptic curves in Weierstrass form are given by the maps

$$(x, y) \mapsto (u^2 x + r, u^3 y + u^2 s x + t)$$

for $u, r, s, t \in \overline{k}$, $u \neq 0$.

- If we restrict to elliptic curves of the form $y^2 = x^3 + ax + b$ then $s = t = 0$.
- A twist of an elliptic curve E/k is an elliptic curve E'/k isomorphic to E over $\overline{k}$ but not over k.

Example

- Every elliptic curve $E/\mathbb{F}_q : y^2 = x^3 + ax + b$ has a quadratic twist

$$E' : \delta y^2 = x^3 + ax + b$$

for any non square $\delta \in \mathbb{F}_q$. E and E' are isomorphic over $\mathbb{F}_q^2$.

- If $E/\mathbb{F}_q$ is an ordinary elliptic curve with $j_E \notin \{0, 1728\}$ then the only twist of E is the quadratic twist. If $j_E = 1728$, then E admits 4 twists. If $j_E = 0$, then E admits 6 twists.

The addition law

- Let E be an elliptic curve given by a Weierstrass equation
- Then $(E, 0_E)$ is an abelian variety;
- The addition law is recovered by the chord and tangent law;
- If $k = \mathbb{C}$ this addition law coincides with the one on $\mathbb{C}$ modulo the lattice Λ. (The addition law of an abelian variety is fixed by the base point, and the base point $0 \in \mathbb{C}$ corresponds to the point at infinity of E since $\wp$ and $\wp'$ have a pole at 0).
- For $E : y^2 = x^3 + ax + b$ the addition law is given by

$$P + Q = -R = (x_R, -y_{-R})$$

$$\alpha = \frac{y_Q - y_P}{x_Q - x_P} \quad \text{or } \alpha = \frac{f'(x_P)}{2y_P} \text{ when } P = Q$$

$$x_R = \alpha^2 - x_P - x_Q$$

$$y_{-R} = y_P + \alpha(x_R - x_P)$$

- Indeed write $l_{P,Q} : y = \alpha x + \beta$ the line between P and Q (or the tangent to E at P when $P = Q$). Then $y_{-R} = \alpha x_{-R} + \beta$ and $y_P = \alpha x_P + \beta$ so $y_{-R} = \alpha(x_R - x_P) + y_P$. Furthermore x_R, x_P, x_Q are the three roots of $x^3 + ax + b - (\alpha x + \beta)^2$ so $x_P + x_Q + x_R = \alpha^2$.

Elliptic curves over other fields

- Why look at $\mathbb{C}$? For cryptography we work with elliptic curves over finite fields;
- Everything that is true over $\mathbb{C}$ is true over other fields except when it is not true (non algebraically closed fields, characteristic p...). Example: "there are n^2 points of n-torsion".
- For things that are not true over other fields, change the definition so that it remains true. Examples: "the subscheme $E[n]$ has degree n^2", definition of the Tate module $T_p E$ as a p-divisible group when the characteristic is p...

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Transferring results from $\mathbb{C}$ to other fields

- If $\overline{k}$ is an algebraically closed field of characteristic 0 and of cardinality $2^{\aleph}_0$ then $\overline{k}$ is isomorphic to $\mathbb{C}$;
- If $\overline{k}$ is an algebraically closed field of characteristic 0 it is elementary equivalent to $\mathbb{C}$ so the first order statements valid over $\mathbb{C}$ are valid over $\overline{k}$ too;
- If a first order statement is true over $\mathbb{C}$, it is also true for all algebraically closed field of characteristic $p >> 0$ (by compacity arguments);
- If $E/\mathbb{F}_q$ is an elliptic curve over a finite field, it can be lifted to an elliptic curve over $\mathbb{Q}_q$ (and $\mathbb{Q}_q$ is a subfield of $\mathbb{C}_q$ which is isomorphic to $\mathbb{C}$ by the explanation above);
- If $E/\mathbb{F}_q$ is an ordinary elliptic curve, there is a lift to $\mathbb{Q}_q$ which respects $\mathrm{End}(E)$;
- A polynomial in $\mathbb{Z}[X_1, \ldots, X_n]$ which is 0 on a Zariski dense subset of $\mathbb{C}^n$ is identically null.

Example

If $A \in \mathrm{Mat}_n(R)$ is a matrix, then $\mathrm{adj}\ A.A = A.\mathrm{adj}\ A = \det A.\mathrm{Id}$. Indeed this is true for diagonalisable matrices over $\mathbb{C}$ which form a dense Zariski subset (standard linear algebra), so it is true over any ring because the adjoint matrix is given by universal polynomials in the coefficients of A.

Field of definition

- Let E/k be an elliptic curve, and let k_0 be the base field of k;
- There exist an elliptic curve E_0 over $k_0(j(E))$ which is a twist of E;
- E can then be defined over a finite algebraic extension of $k_0(j(E))$;
- $k_0(j(E))$ is either algebraic over k_0 or of transcendance degree 1.

Corollary

Every elliptic curve can be defined over a finite extension of $\mathbb{F}_p(T)$ *or* $\mathbb{Q}(T)$*. If* $\operatorname{char} k = 0$*,* E *can be defined over a subfield of* $\mathbb{C}$*.*

n-torsion over $k = \mathbb{C}$

- $E[n] = \{P \in E(k) \mid n.P = 0_E\}$;
- If $E = \mathbb{C}/\Lambda$, $E[n] = \frac{1}{n}\Lambda/\Lambda$;
- $E[n] \simeq (\mathbb{Z}/n\mathbb{Z})^2$.

n-torsion over $k = \overline{k}$

- Let $\overline{k}$ be an algebraically closed field of characteristic p;
- Let $E : y^2 = x^3 + ax + b$ be an elliptic curve (for simplicity we assume $p = 0$ or $p > 3$);
- Since E has dimension one, $E(\overline{k})$ is infinite (Exercice);
- The subscheme $E[n]$ has dimension 0 and degree n^2;

Proof

- Via division polynomials: there exists a unitary polynomial $\varphi_n(x)$ of degree n^2 such that $[n]P = 0_E$ if and only if $\varphi_n(x_P) = 0$ (Exercice: why does φ_n not depend on y?);
- Via dual isogenies: $[n] : E \to E$ is its own dual isogeny, so $[\deg[n]] = [n] \circ \widehat{[n]} = [n^2]$, and $\deg[n] = n^2$;
- Via divisors: if D is a divisor on E, the theorem of the cube shows that $[n]^*D$ is linearly equivalent to $\frac{n^2+n}{2}D + \frac{n^2-n}{2}[-1]^*D$. But $\deg[n]^*D = \deg[n] \deg D$ so $\deg[n] = \frac{n^2+n+n^2-n}{2} = n^2$.

Group structure of the n-torsion

- $d[n]$ is the multiplication by n map on the tangent space $T_{0_E}E$, so $[n]$ is étale whenever $p \nmid n$;
- In this case $\#E[n](\overline{k}) = n^2$ so $E[n] \simeq (\mathbb{Z}/n\mathbb{Z})^2$ (Exercice);
- Either $\#E[p](\overline{k}) = p$ (in which case E is an ordinary elliptic curve), or $\#E[p](\overline{k}) = 0$ (and E is a supersingular elliptic curve);
- If E is ordinary, $E[p^e] = \mathbb{Z}/p^e\mathbb{Z} \oplus \mu_{p^e}$ where $\mu_p = \operatorname{Spec} \mathbb{Z}[T]/(T^{p^e} - 1)$;
- If E is supersingular, $E[p^e] = \alpha_{p^e}^2$ where $\alpha_{p^e} = \operatorname{Spec} \mathbb{Z}[T]/T^{p^e}$ is connected.

Proof

- Let π be the (small) Frobenius, $\widehat{\pi}$ be the Verschiebung, then π is purely inseparable, and $\pi \circ \widehat{\pi} = [p]$, $\widehat{\pi} \circ \pi = [p]$, $\deg \pi = \deg \widehat{\pi} = p$;
- The Weil pairing e_n shows that $E[n]$ (and in particular $E[p]$) is self-dual;
- If $\widehat{\pi}$ is separable, then $\mathbb{Z}/p\mathbb{Z}$ is a subscheme of $E[p]$ and so is its dual μ_p. Taking degrees yield $E[p] = \mathrm{Ker}\, \widehat{\pi} \oplus \mathrm{Ker}\, \pi = \mathbb{Z}/p\mathbb{Z} \oplus \mu_p$.
- Otherwise $\widehat{\pi}$ is not separable, so $\mathrm{Ker}\, \pi$ cannot be μ_p (because its dual $\mathbb{Z}/p\mathbb{Z}$ would be a subscheme of $E[p]$) which implies that $\mathrm{Ker}\, \pi = \alpha_p$ (α_p is self-dual).

Tate modules

- The ℓ-adic numbers $\mathbb{Z}_\ell = \varprojlim \mathbb{Z}/\ell^n\mathbb{Z}$ are a way to handle all the residue rings $\mathbb{Z}/\ell^n\mathbb{Z}$ at once, $\widehat{\mathbb{Z}} = \varprojlim \mathbb{Z}/n\mathbb{Z} = \prod_\ell \mathbb{Z}_\ell$.
- Likewise the Tate modules are a way to encode the (ℓ-primary) torsion subgroup:

$$T_\ell(E) = \varprojlim E[\ell^n](\overline{k})$$
$$T(E) = \varprojlim E[n](\overline{k})$$

- $E[n](\overline{k}) \simeq T(E)/nT(E)$;
- $T_\ell(E) = \mathbb{Z}_\ell^2$ if $p \nmid \ell$;
- If E is ordinary $T_p(E) = \mathbb{Z}_p$, and $T(E) = \widehat{\mathbb{Z}} \times \widehat{\mathbb{Z}}'$ (where $\widehat{\mathbb{Z}}' = \varprojlim_{p \nmid n} \mathbb{Z}/n\mathbb{Z}$) and $E(\overline{k})_{\mathrm{tors}} = \mathbb{Q}/\mathbb{Z} \oplus \mathbb{Z}_{(p)}/\mathbb{Z}$;
- If E is supersingular $T_p(E) = 0$ and $T(E) = \widehat{\mathbb{Z}}' \times \widehat{\mathbb{Z}}'$ and $E(\overline{k})_{\mathrm{tors}} = \mathbb{Z}_{(p)}/\mathbb{Z} \oplus \mathbb{Z}_{(p)}/\mathbb{Z}$.

The group of rational points over a finite field

- If $k = \mathbb{F}_q$ then $E(k)$ is finite;
- In fact (Exercice):

$$E(k) = \mathbb{Z}/n_1\mathbb{Z} \oplus \mathbb{Z}/n_2\mathbb{Z} \quad \text{with } n_1 \mid n_2.$$

- We will study how n_1, and n_2 vary under isogenies and fields extensions.

The Weil pairing over $\mathbb{C}$

- $E = \mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$;
- The function

$$\begin{array}{rcl} e_n \colon E[n]\times E[n] & \longrightarrow & \mu_n \\ (P,Q) & \longmapsto & e^{2\pi i n\left(x_P y_Q - x_Q y_P\right)} \end{array}$$

  where $P = x_P + \tau y_P$ and $Q = x_Q + \tau y_Q$ is bilinear and non degenerate;
- The value does not depend on the choice of basis for the lattice $\Lambda = \mathbb{Z}+\tau\mathbb{Z}$: let $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, then if $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Sl}_2(\mathbb{Z})$,

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x_P \\ y_P \end{pmatrix}\right)^T J \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x_Q \\ y_Q \end{pmatrix}\right) = \begin{pmatrix} x_P \\ y_P \end{pmatrix}^T \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}^t J \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)\begin{pmatrix} x_Q \\ y_Q \end{pmatrix} =$$

$$\begin{pmatrix} x_P \\ y_P \end{pmatrix}^T J \begin{pmatrix} x_Q \\ y_Q \end{pmatrix} = x_P y_Q - x_Q y_P$$

Divisors

- Let C be a projective smooth and geometrically connected curve;
- A divisor D is a formal finite sum of points on C:
  $D = n_1[P_1] + n_2[P_2] + \cdots n_e[P_e]$. The degree $\deg D = \sum n_i$.
- If $f \in k(C)$ is a rational function, then

$$\operatorname{Div} f = \sum_P \operatorname{ord}_P(f)[P]$$

($(O_C)_P$ the stalk of functions defined around P is a discrete valuation ring since C is smooth and $\operatorname{ord}_P(f)$ is the corresponding valuation of f at P).

Example

If $C = \mathbb{P}^1_k$ then $\operatorname{Div} \frac{\prod (X-\alpha_i^{e_i})}{\prod (X-\beta_i^{f_i})} = \sum e_i[\alpha_i] - \sum f_i[\beta_i] + (\sum \beta_i - \sum \alpha_i)\infty$. In particular $\deg \operatorname{Div} f = 0$ and conversely any degree 0 divisor comes from a rational function.

Linear equivalence class of divisors

- For a general curve, if $f \in k(C)$, $\mathrm{Div}(f)$ is of degree 0 but not any degree 0 divisor D comes from a function f;
- A divisor which comes from a rational function is called a principal divisor. Two divisors D_1 and D_2 are said to be linearly equivalent if they differ by a principal divisor: $D_1 = D_2 + \mathrm{Div}(f)$.
- $\mathrm{Pic}\, C = \mathrm{Div}^0\, C / \text{Principal Divisors}$
- A principal divisor D determines f such that $D = \mathrm{Div}\, f$ up to a multiplicative constant (since the only globally regular functions are the constants).

Divisors on elliptic curves

Theorem

Let $D = \sum n_i[P_i]$ be a divisor of degree 0 on an elliptic curve E. Then D is the divisor of a function $f \in \overline{k}(E)$ (ie D is a principal divisor) if and only if $\sum n_i P_i = 0_E \in E(\overline{k})$ (where the last sum is not formal but comes from the addition on the elliptic curve).
In particular $P \in E(\overline{k}) \to [P] - [0_E] \in \mathrm{Jac}(E)$ is a group isomorphism between the points in E and the linear equivalence classes of divisors;

Proof.

- We will give an algorithm (Miller's algorithm) which starts from a divisor $D = \sum n_i[P_i]$ of degree 0 and constructs a rational function f such that D is linearly equivalent to $[\sum n_i P_i] - [0_E]$. If $\sum n_i P_i = 0_E$ then D is principal.
- Conversely we have to show that if $P = \sum n_i P_i \neq 0_E$ then $[P] - [0_E]$ is not principal. But if we had a function f such that $\mathrm{Div}(f) = [P] - [0_E]$, then the morphism $E \to \mathbb{P}^1_{\overline{k}} : x \mapsto (1 : f(x))$ associated to f would be birational. But this is absurd: E is an elliptic curve so it has genus 1, it cannot have genus 0.

$\square$

Rational divisors

- A divisor D over a perfect field is rational if it is stable under the Galois action;
- If $f \in k(E)$ then $\operatorname{Div} f$ is a rational divisor, conversely if $f \in \overline{k}(E)$ and $\operatorname{Div} f$ is rational then there exists $\alpha \in \overline{k}^*$ such that $\alpha f \in k(E)$;
- A linear equivalence class of divisors $[D]$ is rational if it is stable under the Galois action: $\sigma D \sim D \ \forall \sigma \in \operatorname{Gal}(\overline{k}/k)$;
- Over an elliptic curve E, if $D \simeq [P]-[0_E]$ then D is rational if and only if P is rational;
- Over a curve C with $C(k) \neq 0$ then a rational equivalence class of divisors has a representative given by a rational divisor;
- In particular the map $P \mapsto [P]-[0_E]$ is Galois-equivariant.

Miller's functions

- Let $\mu_{P,Q}$ be a function with divisor $[P]+[Q]-[P+Q]-[0_E]$;
- Using the geometric interpretation of the addition law on E one can construct $\mu_{P,Q}$ explicitly:
- if $P=-Q$ then $\mu_{P,Q}=x-x_P$;
- Otherwise let $l_{P,Q}$ be the line going through P and Q (if $P=Q$ then we take $l_{P,Q}$ to be the tangent to the elliptic curve at P). Then $\mathrm{Div}(l_{P,Q})=[P]+[Q]+[-P-Q]-3[0_E]$.
- Let $v_{P,Q}$ be the vertical line going through $P+Q$ and $-P-Q$; $\mathrm{Div}(v_{P,Q})=[P+Q]+[-P-Q]-2[0_E]$;
- $\mu_{P,Q}=\frac{l_{P,Q}}{v_{P,Q}}$;
- Explicitly if $E: y^2=x^3+ax+b$ is given by a short Weierstrass equation,

$$\mu_{P,Q}=\frac{y-\alpha(x-x_P)-y_P}{x+(x_P+x_Q)-\alpha^2} \tag{1}$$

with $\alpha=\frac{y_P-y_Q}{x_P-x_Q}$ when $P\neq Q$ and $\alpha=\frac{f'(x_P)}{2y_P}$ when $P=Q$.

Miller's algorithm: reducing divisors

- Let $D = [P] + [Q] + D_0$ be a divisor of degree 0;
- Using $\mu_{P,Q}$ we get that $D = \mathrm{Div}(\mu_{P,Q}) + [P+Q] + D_0 + [0_E]$;
- We can iterate the reduction until there is only one non zero point in the support: $D = \mathrm{Div}(g) + [R] - [0_E]$;
- D is principal if and only if $R = 0_E$, in which case g is a function (explicitly written in terms of the $\mu_{P,Q}$) with divisor D (and normalised at 0_E).

Miller's algorithm: double and add

- If $D = n[P] - n[0_E]$ one can combine the reduction above with a double and add algorithm;
- let $\lambda \in \mathbb{N}$ and $P \in E(k)$; we define $f_{\lambda,P} \in k(E)$ to be the function normalized at 0_E thus that:

$$\mathrm{Div}(f_{\lambda,P}) = \lambda[P] - [\lambda P] - (\lambda - 1)[0_E].$$

- In particular $D = \mathrm{Div}\, f_{n,P} + [nP] - [0_E]$;
- If $\lambda, \nu \in \mathbb{N}$, we have

$$f_{\lambda+\nu,P} = f_{\lambda,P} f_{\nu,P} \mathbf{f}_{\lambda,\nu,P}$$

  where $\mathbf{f}_{\lambda,\nu,P} := \mu_{\lambda P,\nu P}$ is the function associated to the divisor $[(\lambda+\nu)P] - [(\lambda)P] - [(\nu)P] + [0_E]$ and normalized at 0_E;

Miller's algorithm: example

- Let D be a general divisor of degree 0. How to apply a double and add algorithm to reduce D?
- Write $D = D_1 + 2D_2 + 4D_4 + \dots$.
- Example: $D = 5[P] + 7[Q] - 12[0_E]$;
- Reduce: $[P] + [Q] - 2[0_E] \sim [P+Q] - [0_E]$;
- Double: $2[P+Q] - 2[0_E] \sim [2P+2Q] - [0_E]$;
- Add: $[2P+2Q] + [Q] - 2[0_E] \sim [2P+3Q] - [0_E]$;
- Double: $2[2P+3Q] - 2[0_E] \sim [4P+6Q] - [0_E]$;
- Add: $[4P+6Q] + [P+Q] - 2[0_E] \sim [5P+7Q] - [0_E]$;

Evaluating functions on divisors

- If f is a function with support disjoint from a divisor $D = \sum n_i[P_i]$, then one can define
$$f(D) = \prod f(P_i)^{n_i}$$
- If D is of degree 0, then $f(D)$ depends only on $\operatorname{Div} f$;
- Miller's algorithm allows, given $\operatorname{Div} f$ to compute $f(D)$ efficiently, the data $\operatorname{Div} f$ can then be seen as a compact way to represent the function f.
- Technicality: during the execution of Miller's algorithm we introduce temporary points in the support of the divisors we evaluate, so we may get a zero or a pole during the evaluation even through f has support disjoint to D;
- One way to proceed is to extend the definition of $f(P)$ when $\operatorname{ord}_P(f) = n$ by fixing a uniformiser u_P (a function with simple zero at P), and defining $f(P)$ to be $(f/u_P^{\operatorname{ord}_P(f)})(P)$. Since C is smooth, $\widehat{O_p} = k[[u_P]]$, $f \in k((u_P))$ and $f(P)$ is then the first coefficient in the Laurent expansion of f along u_P.
- For an elliptic curve a standard uniformiser at 0_E is $u = x/y$; a function f is said to be normalised at 0_E if $f(0_E) = 1$. This fixes uniquely f in its equivalence class $\operatorname{Div} f$.

Evaluating functions on divisors: example

Algorithm (Evaluating $f_{r,P}$ on Q)

Input: $r \in \mathbb{N}$, $P = (x_P, y_P) \in E[r](\mathbb{F}_q)$, $Q = (x_Q, y_Q) \in E(\mathbb{F}_{q^d})$.

Output: $f_{r,P}(Q)$ *where* $\mathrm{Div}\, f_{r,P} = r[P] - r[0_E]$.

1. *Compute the binary decomposition:* $r := \sum_{i=0}^{I} b_i 2^i$. *Let* $T = P, f_1 = 1, f_2 = 1$.
2. *For* i *in* $[I..0]$ *compute*
   1. α, *the slope of the tangent of* E *at* T.
   2. $f_1 = f_1^2(y_Q - \alpha(x_Q - x_T) - y_T)$, $f_2 = f_2^2(x_Q + 2x_T - \alpha^2)$.
   3. $T = 2T$.
   4. *If* $b_i = 1$, *then compute*
      1. α, *the slope of the line going through* P *and* T.
      2. $f_1 = f_1^2(y_Q - \alpha(x_Q - x_T) - y_T)$, $f_2 = f_2(x_Q + x_P + x_T - \alpha^2)$.
      3. $T = T + P$.

Return

$$\frac{f_1}{f_2}.$$

The Weil pairing over algebraically closed fields

Theorem

Let E be an elliptic curve, r a number and P and Q two points of r-torsion on E. Let D_P be a divisor linearly equivalent to $[P]-[0_E]$ and D_Q be a divisor linearly equivalent to $[Q]-[0_E]$. Then

$$e_{W,r}(P,Q) = \varepsilon(D_P, D_Q)^r \frac{(rD_P)\cdot(D_Q)}{(rD_Q)\cdot(D_P)} \tag{2}$$

is well defined.
Furthermore the application $E[r]\times E[r] \to \mu_r : (P,Q) \mapsto e_{W,r}(P,Q)$ is a pairing, called the Weil pairing. The pairing $e_{W,r}$ is an alternate pairing, which means that $e_{W,r}(P,Q) = e_{W,r}(Q,P)^{-1}$.

Proof.

An essential ingredient of the proof is Weil's reciprocity theorem: if $f, g \in K(E)$, then

$$f(\mathrm{Div}(g)) = \varepsilon(\mathrm{Div}\, f, \mathrm{Div}\, g) g(\mathrm{Div}(f)).$$

(Note: $\varepsilon(\mathrm{Div}\, f, \mathrm{Div}\, g) = 1$ if the two divisors have disjoint support.) □

Weil's pairing in practice

- Recall that $f_{r,P}$ is the function with divisor $r[P] - r[0_E]$ (and normalised at 0_E) constructed via Miller's algorithm;
- Similarly $f_{r,Q}$ has divisor $r[Q] - r[0_E]$;
- $e_{W,r}(P,Q) = (-1)^r \frac{f_{r,P}(Q)}{f_{r,Q}(P)}$;
- If during the execution of Miller's algorithm to evaluate $f_{r,P}(Q)$ we find a pole or a zero, then we know that Q is a multiple of P and that $e_{W,r}(P,Q) = 1$.

Embedding degree

- If $\mathbb{F}_q$ is a finite field, the embedding degree e is the smallest integer such that $\mathbb{F}_{q^e} = \mathbb{F}_q(\mu_r)$;
- Alternatively, if $r = \ell$ is prime, it is the smallest integer such that $r \mid q^e - 1$.
- If $\sigma \in \mathrm{Gal}(\overline{k}/k)$, $e_r(\sigma P, \sigma Q) = \sigma(e(P,Q))$ (by unraveling the definition), so if $P, Q \in k$ then $e(P,Q) \in k$;
- In particular if $E[\ell] \subset E(\mathbb{F}_q)$ and ℓ is prime, then $\ell \mid q - 1$.
- More generally if $E[r] \subset E(\mathbb{F}_q)$ then $\mu_r \subset \mathbb{F}_q$.

Application of the Weil pairing

- Extremely useful for cryptography (MOV attack, pairing-based cryptography);
- For cryptography rather use optimised pairings derived from the Tate pairing;
- Application for the group structure: $P, Q \in E[\ell]$ form a basis of the ℓ-torsion if and only if $e_{W,\ell}(P,Q) \neq 1$ (Exercice: compare the complexity with the naive method);
- More generally: $P, Q \in E[r]$ form a basis of the r-torsion if and only if $e_{W,r}(P,Q)$ is a primitive r-root of unity (Exercice: what is the complexity to check this?);

Remark

If $P, Q \in E[n]$, $e_{W,nm}(P,Q) = e_{W,n}(P,Q)^m$ so the Weil pairings glue together to give a symplectic structure on the Tate module $T(E)$.

The Tate pairing over a finite field

Theorem

Let E be an elliptic curve, r a prime number, $P \in E[r](\mathbb{F}_{q^e})$ a point of r-torsion defined over $\mathbb{F}_{q^e}$ and $Q \in E(\mathbb{F}_{q^e})$ a point of the elliptic curve defined over $\mathbb{F}_{q^e}$. Let D_P be a divisor linearly equivalent of $[P]-[0_E]$ and D_Q be a divisor linearly equivalent of $[Q]-[0_E]$. Then

$$e_{T,r}(P,Q) = \left((r D_P)\cdot(D_Q)\right)^{\frac{q^e-1}{r}} \tag{3}$$

is well defined and does not depend on the choice of D_P and D_Q.
Furthermore the application $E[r](\mathbb{F}_{q^e}) \times E(\mathbb{F}_{q^e})/rE(\mathbb{F}_{q^e}) \to \mu_r : (P,Q) \mapsto e_{T,r}(P,Q)$ is a pairing, called the Tate pairing.

Tate's pairing in practice

- Recall that $f_{r,P}$ is the function with divisor $r[P]-r[0_E]$ (and normalised at 0_E) constructed via Miller's algorithm;
- $e_{T,r}(P,Q)=f_{r,P}(Q)^{\frac{q^e-1}{r}}$;
- If during the execution of Tate's algorithm to evaluate $f_{r,P}(Q)$ we find a pole or a zero, then we use $D_Q=[Q+R]-[R]$ instead (for R a random point in $E(\mathbb{F}_{q^e})$) and evaluate

$$e_{T,r}(P,Q)=\left(\frac{f_{r,P}(Q+R)}{f_{r,P}(R)}\right)^{\frac{q^e-1}{r}};$$

- If $R\in E(\mathbb{F}_q)$ and $e>1$ we have

$$e_{T,r}(P,Q)=f_{r,P}(Q+R)^{\frac{q^e-1}{r}}.$$

Tate pairing and the Frobenius

- The Weil pairing, Tate pairing and the Frobenius are related;
- Let $P \in E[r](\mathbb{F}_{q^e})$ and $Q \in E(\mathbb{F}_{q^e})$. Let $Q_0 \in E[r](\overline{k})$ be any point such that $rQ_0 = Q$;
- $\pi^e Q_0 - Q_0 \in E[r]$ (Exercice)
- $$e_{T,r}(P,Q) = e_{W,r}(P,(\pi^e - 1)Q_0)$$
- If $Q' = Q + rR$ with $R \in E(\mathbb{F}_{q^e})$ then one can choose $Q_0' = Q_0 + R$ so that $(\pi^e - 1)(Q_0) = (\pi^e - 1)(Q_0')$;
- So the value of $e_{T,r}(P,Q)$ depends only on the class of $Q \in E(\mathbb{F}_{q^e})/rE(\mathbb{F}_{q^e})$.

Proof

- The link between the Weil and Tate pairing comes from Weil's reciprocity;
- If $E[r] \subset E(\mathbb{F}_{q^e})$, then $(\pi^e - 1)E[r] = 0$ so $\frac{\pi^e - 1}{r}$ is an endomorphism;
- Since the Weil pairing is non degenerate, to show that the Tate pairing is non degenerate we just need to show that $\frac{\pi^k - 1}{r} : E(\mathbb{F}_{q^e}) \rightarrow E[r]$ is surjective;
- The kernel of $\frac{\pi^k - 1}{r}$ restricted to $E(\mathbb{F}_{q^e})$ is $rE(\mathbb{F}_{q^e})$, so the image is isomorphic to $E(\mathbb{F}_{q^e})/rE(\mathbb{F}_{q^e})$;
- $E(\mathbb{F}_{q^e}) = \mathbb{Z}/a\mathbb{Z} \oplus \mathbb{Z}/b\mathbb{Z}$ with $a \mid b$, and since $E(\mathbb{F}_{q^e}) \supset E[r]$, we know that $r \mid a$ and $r \mid b$;
- We deduce that $E(\mathbb{F}_{q^e})/rE(\mathbb{F}_{q^e})$ is isomorphic to $\mathbb{Z}/r\mathbb{Z} \oplus \mathbb{Z}/r\mathbb{Z}$, in particular it has cardinal r^2 so the application is indeed surjective;
- The general case comes from Galois cohomology applied to the exact sequence $0 \rightarrow E[r] \rightarrow E(\overline{k}) \rightarrow E(\overline{k}) -> 0$.

Field of definition of the r-roots of unity

- By the CRT, we may assume that $r = \ell^n$;
- μ_{ℓ^n} lives in $\mathbb{F}_{q^e}$ whenever $v_\ell(q^e - 1) \geqslant n$;
- If $\mu_\ell \notin \mathbb{F}_q$ then $\mathbb{F}_q(\mu_\ell) = \mathbb{F}_{q^e}$ with $e \mid \ell - 1$;
- If $\mu_\ell \in \mathbb{F}_q$, then $v_\ell(q^e - 1) = v_\ell(q-1)$ unless $\ell \mid e$;
- If $\mu_\ell \in \mathbb{F}_q$, $v_\ell(q^\ell - 1) = v_\ell(q-1) + 1$ (except possibly when $\ell = 2$ and $v_\ell(q-1) = 1$ where $v_\ell(q^\ell - 1)$ can increase by more than 1);
- (Hint: write $q^e - 1 = (q-1)(1 + q + q^2 + \cdots + q^{k-1})) = (q-1)(q - 1 + q^2 - 1 + \cdots + q^{e-1} - 1 + e))$.

Endomorphisms and isogenies

- An isogeny is a non constant rational application $\varphi : E_1 \to E_2$ between two elliptic curves E_1 and E_2 that commutes with the addition law;
- A rational application φ is an isogeny if and only if $\varphi(0_{E_1}) = 0_{E_2}$ (and $\varphi \neq 0$);
- An isogeny is surjective on the $\overline{k}$-points and has finite kernel;
- The degree of φ is $[k(E_2) : \varphi^* k(E_1)]$;
- An isogeny $\varphi : E_1 \to E_2$ admits a dual $\widehat{\varphi} : E_2 \to E_1$ such that $\varphi \circ \widehat{\varphi} = [\deg \varphi]$ and $\widehat{\varphi} \circ \varphi = [\deg \varphi]$;
- We write $E_1[\varphi] = \mathrm{Ker}\, \varphi$; $\deg \varphi = \deg E_1[\varphi]$ (as a scheme), $\mathrm{Ker}\, \varphi$ determines φ (up to automorphisms);
- If φ is separable (for instance if $p \nmid \deg \varphi$) then $E_1[\varphi] = \{P \in E_1(\overline{k}) \mid \varphi P = 0_{E_2}\}$ so $\deg \varphi = \# E_1[\varphi](\overline{k})$;
- Conversely a finite subscheme group K determines an isogeny $E \to E/K$ of degree $\deg K$;
- Over an elliptic curve, every isogeny is (up to isomorphisms) the composition of a separable isogeny and a power of the small Frobenius π_p.
- An endomorphism $\varphi \in \mathrm{End}(E)$ is an isogeny from E to E.

Endomorphism and isogenies over $\mathbb{C}$

- Let $E_1 = \mathbb{C}/\Lambda_1$ and $E_2 = \mathbb{C}/\Lambda_2$;
- An isogeny comes from a linear map $z \mapsto \alpha z$ where $\alpha\Lambda_1 \subset \Lambda_2$;
- The kernel is $\alpha^{-1}\Lambda_2/\Lambda_1$;
- If $E = \mathbb{C}/\Lambda$ an endomorphism comes from a linear map $z \mapsto \alpha z$ where $\alpha\Lambda \subset \Lambda$;
- Write $\Lambda = \mathbb{Z} \oplus \tau\mathbb{Z}$, we get that if $\alpha \notin \mathbb{Z}$ then τ satisfy a quadratic equation and $\alpha \in \mathbb{Z}[\tau]$;
- $\mathbb{Q}(\tau)$ is then a quadratic imaginary field and $\mathrm{End}(E)$ an order (because it stabilizes a lattice).

Field of definition of endomorphisms

- Let E/k be an elliptic curve (k perfect);
- It may happen that endomorphisms of E are defined over a larger field than k (Exercice: but there are always defined over a finite extension of k);
- We let $\mathrm{End}(E) = \mathrm{End}_{\overline{k}}(E)$ and $\mathrm{End}_k(E)$ the subring of rational endomorphisms;
- $\varphi \in \mathrm{End}(E)$ is defined over k if and only if it is stable under $\mathrm{Gal}(\overline{k}/k)$;
- In particular if $k = \mathbb{F}_q$ and π is the Frobenius, then $\mathrm{End}_k(E)$ is the commutant of π in $\mathrm{End}(E)$.
- If l/k is an extension of field, then $\mathrm{End}_l(E)/\mathrm{End}_k(E)$ is torsion free (Exercice: if $m\varphi$ is rational, then so is φ).

Remark

If k is not perfect and l/k is a purely inseparable extension of k then $\mathrm{End}_l(E) = \mathrm{End}_k(E)$.

Characteristic polynomial

Let $\varphi \in \mathrm{End}_k(E)$, the characteristic polynomial $\chi_\varphi \in \mathbb{Z}[X]$ is defined as

- The characteristic polynomial of φ on $T_\ell(E)$ ($\ell \neq p$);
- The only polynomial such that $\deg(\varphi - n\,\mathrm{Id}) = \chi_\varphi(n) \quad \forall n \in \mathbb{Z}$;
- If $\mathrm{End}_k(E)$ is quadratic, as the characteristic polynomial of φ in $\mathrm{End}(E)$;
- If $\varphi \notin \mathbb{Z}$, as the characteristic polynomial of φ in $\mathbb{Q}(\varphi)$;
- If $\varphi \in \mathbb{Z}$, as $X^2 - 2\varphi X + \varphi^2$;
- Let $\mathrm{Tr}(\varphi) = \varphi + \hat{\varphi} \in \mathbb{Z}$ and $N(\varphi) = \varphi\hat{\varphi} = \deg\varphi \in \mathbb{Z}$;

$$\chi_\varphi = X^2 - \mathrm{Tr}(\varphi)X + N(\varphi);$$

Corollary

If $p \nmid n$, the characteristic polynomial of φ acting on $E[n]$ is $\chi_\varphi \mod n$.

Remark

If $\varphi \in \mathrm{End}_k(E)$, $\widehat{\varphi} = \overline{\varphi}$.

Characteristic polynomial of the Frobenius ($k = \mathbb{F}_q$)

- $\chi_\pi = X^2 - tX + q$;
- The roots of χ_π in $\mathbb{C}$ have absolute value $|\sqrt{q}|$ so $|t| \leqslant 2\sqrt{q}$ (Hasse);
- $\#E(\mathbb{F}_q) = \deg(\pi - 1) = \chi_\pi(1)$;
- $$\zeta_E = \exp\left(\sum_{n=1}^{\infty} \#E(\mathbb{F}_{q^n}) \frac{T^n}{n}\right) = \frac{1 - tT + qT^2}{(1-qT)(1-T)};$$
- $\chi_{\pi^n} = \mathrm{Res}_X(\chi_\pi(Y), Y^n - X)$;

Theorem (Tate)

Two elliptic curves over $\mathbb{F}_q$ are isogenous if and only if they have the same cardinal, if and only if they have the same characteristic polynomial of the Frobenius.

Action of the Frobenius on $E[\ell]$

- Let $\Delta_\pi = t^2 - 4q$;
- If $\Delta_\pi = 0 \mod \ell$ then either $\pi = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$ on $E[\ell]$ (and all ℓ-isogenies are rational) or $\pi = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$ (and there is one rational ℓ-isogeny);
- If $\left(\frac{\Delta_\pi}{\ell}\right) = 1$ then $\pi = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$ on $E[\ell]$ with $\lambda \neq \nu \in \mathbb{F}_\ell$, $\lambda\mu = q$ (and there are two rational ℓ-isogenies);
- If $\left(\frac{\Delta_\pi}{\ell}\right) = -1$ then $\pi = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$ on $E[\ell]$ with $\lambda \neq \nu \in \mathbb{F}_{\ell^2}$, $\lambda\mu = q$ (and there are no rational ℓ-isogenies).

Corollary

If $\ell \,||\, \#E(\mathbb{F}_q)$ then

- *If the embedding degree $e > 1$ then $\pi = \begin{pmatrix} 1 & 0 \\ 0 & q \end{pmatrix}$ and $E[\ell] \subset E(\mathbb{F}_{q^e})$;*
- *Otherwise $\pi = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $E[\ell] \subset E(\mathbb{F}_{q^\ell})$.*

Isogenies and Tate modules

- Let $\ell \neq p$ then $\mathrm{Hom}(E_1, E_2) \otimes \mathbb{Z}_\ell \to \mathrm{Hom}(T_\ell E_1, T_\ell E_2)$ is injective [Sil86][Theorem III.7.4] (Exercice: show that $\mathrm{Hom}(E_1, E_2) \to \mathrm{Hom}(T_\ell E_1, T_\ell E_2)$ is injective);
- In particular $\mathrm{End}(E)$ has rank at most 4;

Theorem (Tate,Faltings)

If k is a finite field or a number field, then

$$\mathrm{Hom}_k(E_1, E_2) \otimes \mathbb{Z}_\ell \simeq \mathrm{Hom}_{\mathbb{Z}_\ell(\mathrm{Gal}(\overline{k}/k))}(T_\ell E_1, T_\ell E_2)$$

Remark

Tate's theorem remain valid for $\ell = p$ when considering the Tate module coming from the duality of p-divisible group schemes.

Endomorphism rings and endomorphism fields

$\mathrm{End}_k(E)$ is either

- $\mathbb{Z}$;
- An order in a quadratic imaginary field;
- A maximal order in the definite quaternion algebra ramified at p and ∞.

Remark

If E is an elliptic curve over a finite field $\mathbb{F}_q$, then

- If E is ordinary then $\mathrm{End}(E)$ is an order in a quadratic imaginary field;
- If E is supersingular then $\mathrm{End}(E)$ is a maximal order in the definite quaternion algebra ramified at p and ∞.

Exercice

- In characteristic 0, $\mathrm{End}_k(E)$ is commutative;
- In characteristic p, $\mathrm{End}_k(E) = \mathbb{Z}$ if and only if $j(E)$ is transcendental.

$\mathrm{End}_k^0(E)$

We follow https://rigtriv.wordpress.com/2009/05/14/endomorphisms-of-elliptic-curves-and-the-tate-module/

Lemma

$\mathrm{Hom}(E_1, E_2)$ is torsion free.

Proof.

The degree is multiplicative, so if $[m] \circ f = 0$ then $m = 0$ or $f = 0$. □

Lemma

$\mathrm{End}_k(E)$ has no zero divisors, so $\mathrm{End}_k^0(E) = \mathrm{End}_k(E) \otimes_{\mathbb{Z}} \mathbb{Q}$ is a division algebra

Proof

(We assume here that $p>2$)

- If $\mathrm{End}_k(E)$ has rank 1 then it is $\mathbb{Z}$ (the maximal order of $\mathbb{Q}$);
- Let $\varphi \in \mathrm{End}_k(E) \setminus \mathbb{Z}$, by translating by an integer we can assume that $\mathrm{Tr}\,\varphi = 0$, and since $N(\varphi) = \deg \varphi > 0$ we get that $\mathbb{Z}+\mathbb{Z}\varphi$ is an order in a quadratic imaginary field. If the rank of $\mathrm{End}_k(E) = 2$ then $\mathrm{End}_k(E)$ is an order containing $\mathbb{Z}+\mathbb{Z}\varphi$.
- Otherwise $\psi \mapsto \varphi\psi\varphi^{-1}$ is a linear map of order 2. If ψ is in the -1-eigenspace (Exercice: why does such a ψ exists?) then $(1,\varphi,\psi,\varphi\psi)$ forms a basis of $\mathrm{End}_k(E)$. Thus $\mathrm{End}_k^0(E)$ is a quaternion algebra and $\mathrm{End}_k(E)$ an order in the quaternion algebra.
- Over $\ell \neq p$ we get that $\mathrm{End}_k E \otimes \mathbb{Z}_\ell \subset \mathrm{End}(T_\ell E) = M_2(\mathbb{Z}_\ell)$ so $\mathrm{End}_k^0 E$ is split at ℓ;
- So either $\mathrm{End}_k^0 E = M_2(\mathbb{Q})$ or the definite quaternion algebra ramified at p and ∞. But $M_2(\mathbb{Q})$ has zero divisors so it cannot be $\mathrm{End}_k(E)$.

Endomorphism rings over $\mathbb{F}_q$

- Let $E/\mathbb{F}_q$ be an elliptic curve, π the Frobenius and $\chi_\pi = X^2 - tX + q$;
- E is supersingular if and only if t is not prime to p, if and only if a power of π is an integer, if and only if $\mathrm{End}^0(E)$ is a quaternion algebra if and only if the isogeny class (up to isomorphism) over $\overline{k}$ is finite.
- Either χ_π is irreducible or $\chi_\pi = X^2 - 2 \pm \sqrt{q}X + q = (X \mp \sqrt{q})^2$ and $\pi = \pm\sqrt{q} \in \mathbb{Z}$. If χ_π is irreducible then $\mathrm{End}_k^0 = \mathbb{Q}(\pi) = \mathbb{Q}(\sqrt{t^2 - 4q})$ is quadratic imaginary, otherwise End_k^0 is the definite quaternion algebra ramified at p and ∞;
- If E is ordinary over $\mathbb{F}_q$, then $\mathrm{End}_k(E) = \mathrm{End}(E)$ is an order in $\mathbb{Q}(\pi)$ containing $\mathbb{Z}[\pi]$, $\mathbb{Z}[\pi]$ is maximal at p and p splits.
- If E is supersingular, then $\mathrm{End}_k^0(E)$ is a quaternion algebra if and only if $\pi \in \mathbb{Z}$, and $\mathrm{End}_k(E) = \mathrm{End}(E)$ is then a maximal order. Otherwise $\mathrm{End}_k(E)$ is a quadratic order in $\mathbb{Q}(\pi)$ and is maximal at p (even though $\mathbb{Z}[\pi]$ may not be maximal at p).

Proof (partial)

- If E is supersingular then $\pi_p^2 E \simeq E$. In particular $j_E \in \mathbb{F}_{p^2}$ and $\pi_p^2 = [p] \circ \zeta$ where ζ is an automorphism. ζ is then a root of unity in $\mathrm{End}(E)$ so a power of π is an integer. Reciprocally if $\pi^n \in \mathbb{Z}$ then $p \mid \pi^n$ is inseparable so E is supersingular.
- t is not prime to $p \iff$ a power of π is an integer (Not trivial exercice, see [Wat69][Chapter 4]);
- $\pi^n \in \mathbb{Z} \iff \mathrm{End}^0_{\mathbb{F}_{q^n}}(E)$ is a quaternion algebra (by Tate's theorem);
- If $\mathrm{End}^0(E) = \mathbb{Q}(\pi)$ is a quadratic field, then the isogeny class is infinite (Exercice: look at isogenies $E \to E_i$ of degree a prime ℓ_i inert in O_K and prove that the E_i are non isomorphic). Conversely all supersingular elliptic curves are defined over $\mathbb{F}_{p^2}$ so the isogeny class is finite.

Reduction and lifting

- Let O be an order in a imaginary quadratic field K. Then there are h_O (the class number of O) elliptic curves over $\overline{\mathbb{Q}}$ with endomorphism ring O. They are defined over the ray class field H_O of O.
- If $p \nmid \Delta_O$, p is a prime of good reduction. Let $\mathfrak{p}$ be a prime above p in H_O. If p is inert in K, $E_\mathfrak{p}$ is supersingular. If p splits, $E_\mathfrak{p}$ is ordinary, and its endomorphism ring is the minimal order containing O of index prime to p.
- Reciprocally, if $E/\mathbb{F}_q$ is an ordinary elliptic curve, the couple $(E, \mathrm{End}(E))$ can be lifted over $\mathbb{Q}_q$.

Corollary

- *If $E/\mathbb{F}_q$ is an ordinary elliptic curve, then $\mathrm{End}(E)$ is an order in $K = \mathbb{Q}(\pi)$ of conductor prime to p. For every order O of K such that $\mathbb{Z}[\pi] \subset O$, there exist an isogenous curve whose endomorphism ring is O.*
- *Reciprocally, for every order O of discriminant a non zero square modulo p, let n be the order of one of the prime above p in the class group of O. Then there exist an (ordinary) elliptic curve E' over $\mathbb{F}_{q^n}$ with $\mathrm{End}(E') = O$.*

Automorphisms and twist

- The automorphisms of E are the inversible elements in $O = \mathrm{End}_k E$.
- All inversible elements are roots of unity.
- We usually have $O^* = \{\pm 1\}$ except in the following exceptions:
  1. $j_E = 1728$ ($p \neq 2,3$), in this case O is the maximal order in $\mathbb{Q}(i)$ and $\#O^* = 4$;
  2. $j_E = 0$ ($p \neq 2,3$), in this case O is the maximal order in $\mathbb{Q}(i\sqrt{3})$ and $\#O^* = 6$;
  3. $j_E = 0$ ($p = 3$), in this case E is supersingular and $\#O^* = 12$;
  4. $j_E = 0$ ($p = 2$), in this case E is supersingular and $\#O^* = 24$.
- The Frobenius $\pi \in K$ characterizes the isogeny class of E (Tate). A twisted isogeny class will correspond to a Frobenius $\pi' \neq \pi$, where there exist n with $\pi^n = \pi'^n$. This give a bijection between the twisted isogeny class and the roots of unity in K.
- More generally, there is a bijection between O^* and the twists of E.

Remark

If E_1 is isogeneous to E_2 over k and $k \subset l$, $\mathrm{Hom}_k(E_1, E_2) = \mathrm{Hom}_l(E_1, E_2)$ when $\mathrm{End}_k(E_1) = \mathrm{End}_l(E_2)$. In particular a twist to E is never isogenous to E over k if E is ordinary.

Isogeny class of elliptic curves over $\mathbb{F}_q$

Let $q = p^n$. The isogeny classes of elliptic curves are given by the value of the trace t by Tate's theorem. The possible value of t are:

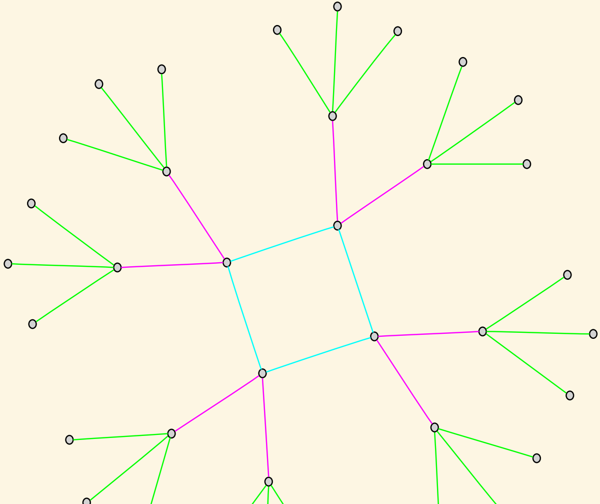
- t prime to p, in this case the isogeny class is ordinary.
- The other cases give supersingular elliptic curves. The endomorphism fraction ring $\mathrm{End}_k^0(\mathscr{E})$ of the isogeny class is either a quaternion algebra of rank 4, or an imaginary quadratic field. In the latter case, it will become maximal after an extension of degree d, with:
  1. If n is even:
     - $t = \pm 2\sqrt{q}$, this is the only case where $\mathrm{End}_k^0(\mathscr{E})$ is a quaternion algebra.
     - $t = \pm\sqrt{q}$ when $p \not\equiv 1 \mod 3$, here $d = 3$.
     - $t = 0$ when $p \not\equiv 1 \mod 4$, here $d = 2$.
  2. If n is odd:
     - $t = 0$, here $d = 2$.
     - $t = \pm\sqrt{2q}$ when $p = 2$, here $d = 4$.
     - $t = \pm\sqrt{3q}$ when $p = 3$, here $d = 6$.

Remark

Any two supersingular elliptic curves become isogenous after a quadratic extension of degree $2d$ (with d the degree where their endomorphism ring become maximal). But a new maximal class and up to 3 commutative classes appear in this extension.

Isogeny graph and endomorphisms of ordinary elliptic curves

The ℓ-isogeny graph looks like a volcano [Koh96; FM02]:
Let f_E be the conductor of $\mathrm{End}(E) \subset O_K$. At each level $v_\ell(f_E)$ increase by one.
At the crater $v_\ell(f_E) = 0$ and at the bottom $v_\ell(f_E) = v_\ell(f) = v_\pi$ where f is the conductor of $\mathbb{Z}[\pi] \subset O_K$.



The α-torsion as an $\mathrm{End}_k(E)$ module

Theorem ([Len96])

- *If* $\mathrm{End}_k(E)$ *is commutative, let* $\alpha \in \mathrm{End}_k(E)$ *be a separable endomorphism. We have an isomorphisme of* $\mathrm{End}_k(E)$*-modules:*

$$E[\alpha] \simeq \mathrm{End}_k(E)/\alpha\,\mathrm{End}_k(E).$$

- *If* $\mathrm{End}_k(E)$ *is non commutative (ie* $\pi \in \mathbb{Z}$*), let* $n \in \mathbb{Z}$*. We have an isomorphism of* $\mathrm{End}_k(E)$*-modules:*

$$E[n] \oplus E[n] \simeq \mathrm{End}_k(E)/n\,\mathrm{End}_k(E).$$

Outline of the proof in the commutative case.

$\mathrm{End}_k(E)$ is a quadratic order so it is a Gorenstein ring. $E[\alpha]$ is faithful over $\mathrm{End}_k(E)/\alpha\,\mathrm{End}_k(E)$, which is a finite Gorenstein ring. So $E[\alpha]$ contains a free $\mathrm{End}_k(E)/\alpha\,\mathrm{End}_k(E)$ module of rank 1, but $\#E[\alpha] = \#\mathrm{End}_k(E)/\alpha\,\mathrm{End}_k(E) = \deg\alpha$ so $E[\alpha]$ is free of rank 1 over $\mathrm{End}_k(E)/\alpha\,\mathrm{End}_k(E)$. □

The structure of the rational points

Theorem (Lenstra)

Let $E/\mathbb{F}_q$ be an ordinary elliptic curve (or suppose that $\pi \notin \mathbb{Z}$). We have as $\mathrm{End}_{\mathbb{F}_q}(E)$-modules:

$$E(\mathbb{F}_{q^n}) \simeq \frac{\mathrm{End}_{\mathbb{F}_q}(E)}{\pi^n - 1}$$

- Let $\Delta_\pi = t^2 - 4q$ and Δ the discriminant of $\mathbb{Q}(\sqrt{\Delta_\pi})$. We have $\Delta_\pi = \Delta f^2$ where f is the conductor of $\mathbb{Z}[\pi] \subset O_K$.
- In practice if $\Delta_\pi = d f_0^2$, then $\Delta = d, f = f_0$ if $d \equiv 1 \mod 4$ or $\Delta = 4d, f = f_0/2$ otherwise;
- Let $\omega = \frac{1+\sqrt{d}}{2}$ if $d \equiv 1 \mod 4$ and $\omega = \sqrt{d}$ otherwise.
- $O_K = \mathbb{Z} \oplus \mathbb{Z}\omega = \mathbb{Z}[\frac{\Delta+\sqrt{\Delta}}{2}]$;
- $\pi = a + f\omega$ with $a = \frac{t-f}{2}$ if $d \equiv 1 \mod 4$ and $a = \frac{t}{2}$ otherwise;
- Let f_E be the conductor of $\mathrm{End}(E) \subset O_K$, $f_E \mid f$ since $\mathbb{Z}[\pi] \subset \mathrm{End}(E)$, $f = f_E\gamma$ where $\gamma_E = [\mathrm{End}(E) : \mathbb{Z}[\pi]]$;
- $E(\mathbb{F}_q) = \mathbb{Z}/n_1\mathbb{Z} \oplus \mathbb{Z}/n_2\mathbb{Z}$ where $n_1 \mid n_2$, $n_1 = \gcd(a-1, \gamma_E)$ and $N = n_1 n_2 = \#E(\mathbb{F}_q)$.

Torsion and conductor of the order

Lemma ([MMS+06])

Let $N = n_1 n_2 = \#E(\mathbb{F}_q)$, $\pi = a + f\omega$, $n_1 = \gcd(a-1, \gamma_E)$.

$$v_\ell(a-1) \geqslant \min(v_\ell(f), v_\ell(N)/2).$$

Proof.

$N = \chi_\pi(1) = (1-\pi)(1-\widehat{\pi})$.
If $d \not\equiv 1 \mod 4$, from $\pi = a + f\omega$ we get

$$N = (a-1)^2 - d f^2$$

so $2v_\ell(a-1) \geq \min(2v_\ell(f), v_\ell(N))$.
If $d \equiv 1 \mod 4$, then $(t-2)^2 = f^2 + 4N$ so $4(a-1)^2 = 4N + f^2(d-1) - 4f(a-1)$, and taking valuations yield the Lemma too. $\square$

Corollary

- *If* $v_\ell(n_1) < v_\ell(N)/2$ *then* $v_\ell(\gamma_E) = v_\ell(n_1)$;
- *If* $v_\ell(n_1) = v_\ell(N)/2$ *then* $v_\ell(\gamma_E) \geqslant v_\ell(N)/2$.

The structure of the ℓ^∞-torsion in the volcano

- If E is on the floor, $E[\ell^\infty](\mathbb{F}_q)$ is cyclic: $E[\ell^\infty](\mathbb{F}_q) = \mathbb{Z}/\ell^m\mathbb{Z}$, with $m = v_\ell(N)$ (possibly $m = 0$).
- If E is on level $\alpha < m/2$ above the floor, then $E[\ell^\infty](\mathbb{F}_q) = \mathbb{Z}/\ell^\alpha \oplus \mathbb{Z}/\ell^{m-\alpha}$.
- If $v \geq m/2$ then m is even and when E is on level $\alpha \geq m/2$, $E[\ell^\infty](\mathbb{F}_q) = \mathbb{Z}/\ell^{m/2} \oplus \mathbb{Z}/\ell^{m/2}$.

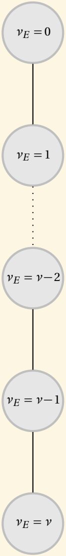
Corollary

When $E[\ell^\infty](\mathbb{F}_q) = \mathbb{Z}/\ell^\alpha \oplus \mathbb{Z}/\ell^{m-\alpha}$ with $\alpha \neq m/2$ we can read the ℓ-valuation of the conductor of $\mathrm{End}_k(E)$ directly from the rational points!

Example

If $\ell \,||\, \#E(\mathbb{F}_q)$ then $\mathrm{End}_k(E)$ is maximal at ℓ and the volcano has height 1.

The structure of the ℓ^∞-torsion in the volcano



$$E[\ell^\infty](\mathbb{F}_q) = \mathbb{Z}/\ell^{m/2}\mathbb{Z} \oplus \mathbb{Z}/\ell^{m/2}\mathbb{Z}$$

$$E[\ell^\infty](\mathbb{F}_q) = \mathbb{Z}/\ell^{m/2}\mathbb{Z} \oplus \mathbb{Z}/\ell^{m/2}\mathbb{Z}$$

$$E[\ell^\infty](\mathbb{F}_q) = \mathbb{Z}/\ell^{2}\mathbb{Z} \oplus \mathbb{Z}/\ell^{m-2}\mathbb{Z}$$

$$E[\ell^\infty](\mathbb{F}_q) = \mathbb{Z}/\ell\mathbb{Z} \oplus \mathbb{Z}/\ell^{m-1}\mathbb{Z}$$

$$E[\ell^\infty](\mathbb{F}_q) = \mathbb{Z}/\ell^{m}\mathbb{Z}$$

Torsion and extensions

- $v_\ell(f_{\pi^e}) = v_\ell(f_\pi)$ when $\ell \nmid e$;
- $v_\ell(f_{\pi^\ell}) = v_\ell(f_\pi) + 1$, except when $\ell = 2$ and $v_\ell(f_{\pi)} = 1$ when the height can increase by more than one [Fou01];
- If $E[\ell^\infty](\mathbb{F}_q) = \mathbb{Z}/\ell^{n_1} \oplus \mathbb{Z}/\ell^{n_2}$ $(n_1 \leqslant n_2)$ with $n_1 > 0$ and $n_2 > 0$ then $E[\ell^\infty](\mathbb{F}_{q^e}) = E[\ell^\infty](\mathbb{F}_q)$ when $\ell \nmid e$;
- With the hypothesis above, if $\ell > 2$, $E[\ell^\infty](\mathbb{F}_q^\ell) = \mathbb{Z}/\ell^{n_1+1} \oplus \mathbb{Z}/\ell^{n_2+1}$;
- If $\ell = 2$, n_1 and n_2 can increase by more than one (but when $v_\ell(f_\pi) > 1$ then n_1 only increase by 1) [IJ13].

Number fields

- If K is a number field, $E(K)$ is finitely generated (Mordell);
- $E(\mathbb{Q})_{\text{tors}} \in \{\mathbb{Z}/n\mathbb{Z} \quad 1 \leqslant n \leqslant 10 \text{ or } n = 12\} \cup \{\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}, \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}, \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/8\mathbb{Z}\}$ (Mazur).

$E(\overline{k})$ [Len96]

- $E(\overline{k}) = E(\overline{k})_{\mathrm{tors}} \oplus E(\overline{k})/E(\overline{k})_{\mathrm{tors}}$;
- $E(\overline{k})/E(\overline{k})_{\mathrm{tors}}$ is equal to 0 if $\overline{k}$ is the algebraic closure of a finite field, otherwise it is isomorphic as en $\mathrm{End}(E)$ module to $\mathrm{End}^0(E)^{\# k}$;
- Let $\mathfrak{p}$ denotes the endomorphisms acting trivially on the tangeant space $T_0(E)$;
- If E is ordinary ($\operatorname{rank}\mathrm{End}(E) = 2$), $E(\overline{k})_{\mathrm{tors}} = \mathrm{End}(E)_{\mathfrak{p}}/\mathrm{End}(E)$;
- Otherwise ($\operatorname{rank}\mathrm{End}(E) = 4$) $E(\overline{k})_{\mathrm{tors}} \oplus E(\overline{k})_{\mathrm{tors}} = \mathrm{End}(E)_{\mathfrak{p}}/\mathrm{End}(E)$.

Corollary

$E(\overline{k}) = E(\overline{k})_{\mathrm{tors}}$ *if and only if* $\overline{k}$ *is algebraic over a finite field.*

Proof.

If $\overline{k}$ is algebraic over a finite field and $P \in E(\overline{k})$, the coordinates of P are defined over a finite field, so P is of torsion.
Conversely we may assume that $\overline{k}$ is algebraic over $\mathbb{F}_p(T)$ or $\mathbb{Q}$ or $\mathbb{Q}(T)$. If $E(\overline{k}) = E(\overline{k})_{\mathrm{tors}}$ the Jordan-Hölder factors of the absolute Galois group would be of the form $\mathrm{PSL}_2(\mathbb{F}_q)$ (up to a finite number of exceptions). But $\mathbb{F}_p(T)$, $\mathbb{Q}$ and $\mathbb{Q}(T)$ all have Galois extension with the symmetric groups S_n for all n. $\square$

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