# Applications of isogenies between abelian varieties to elliptic curves

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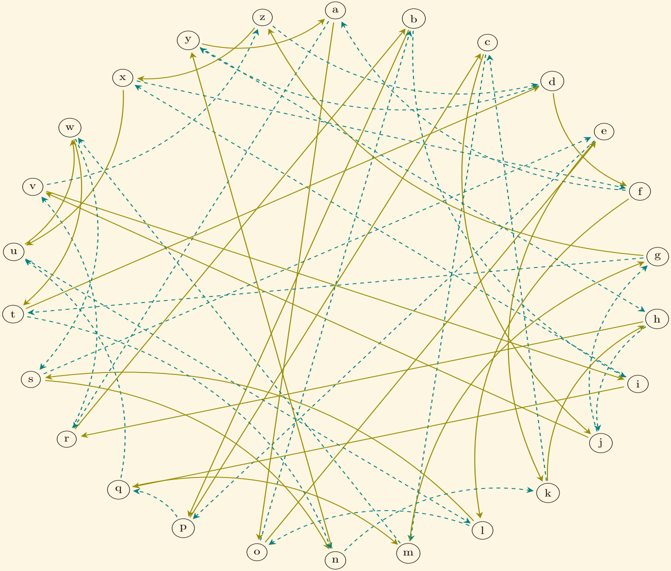
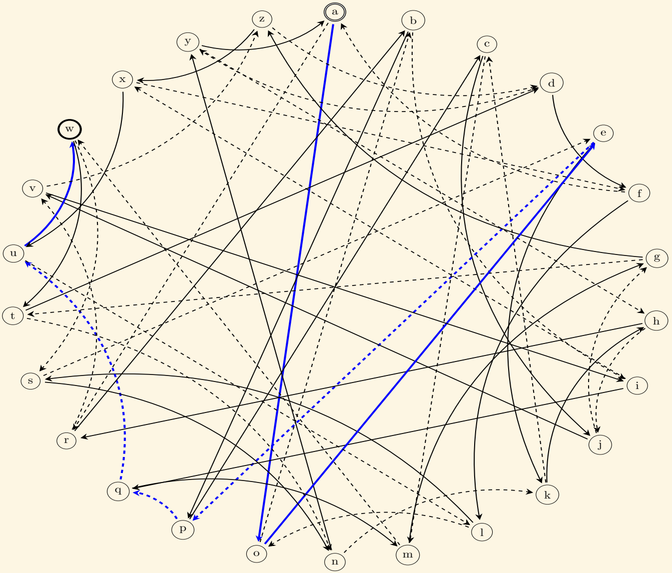
# Key exchange on a (commutative) graph



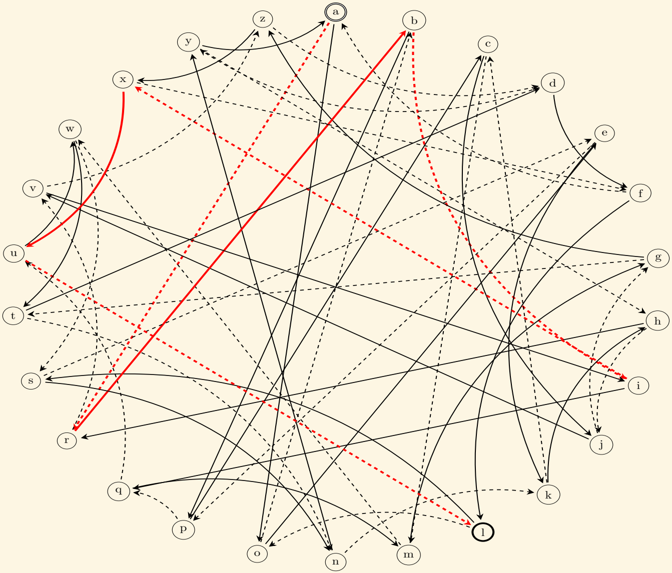
# Key exchange on a (commutative) graph

Alice starts from ‘a’, follows the path 001110, and get ‘w’.



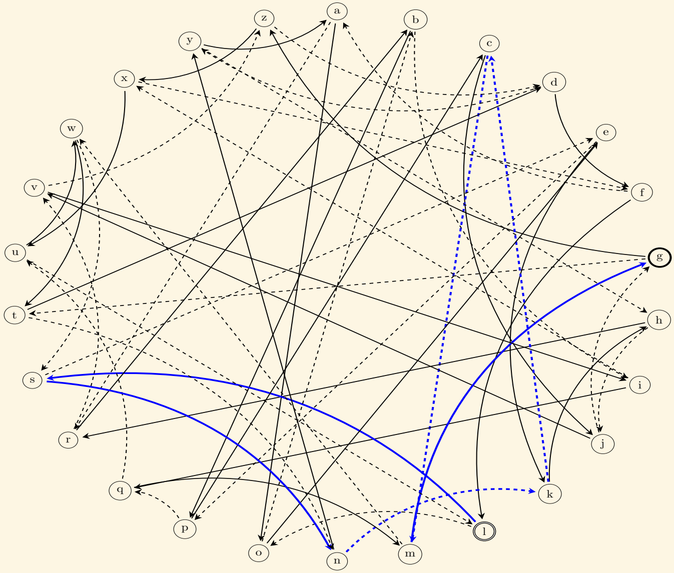
# Key exchange on a (commutative) graph

Bob starts from ‘a’, follows the path 101101, and get ‘l’.



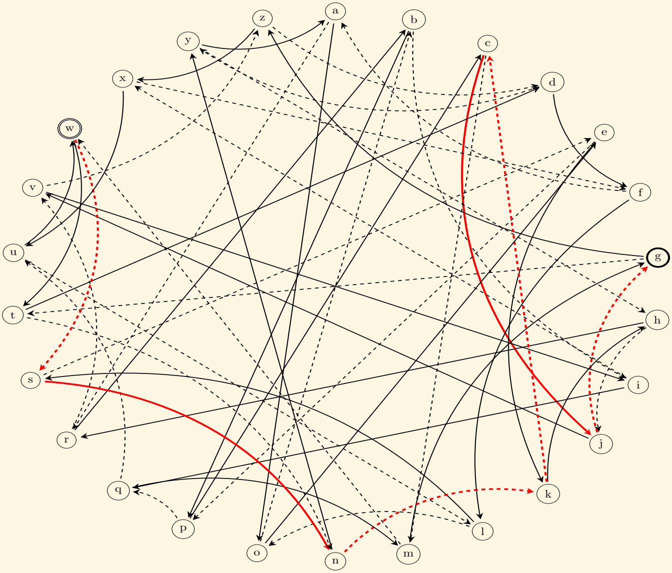
# Key exchange on a (commutative) graph

Alice starts from 'l', follows the path 001110, and get 'g'.



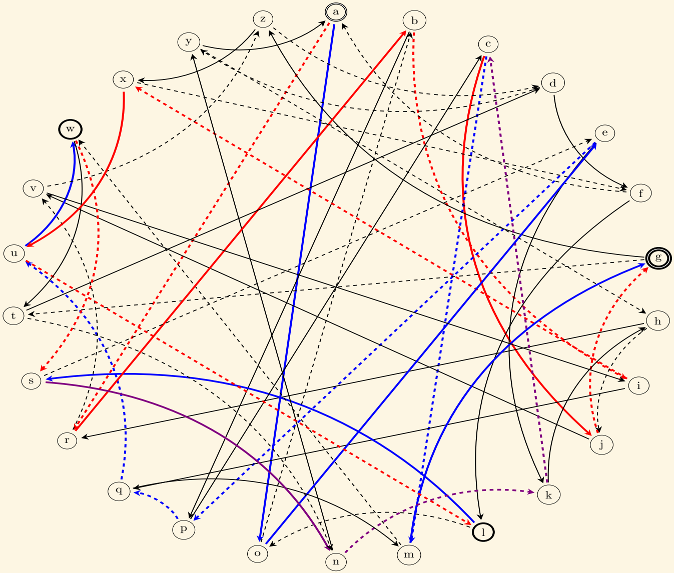
# Key exchange on a (commutative) graph

Bob starts from ‘w’, follows the path 101101, and get ‘g’.



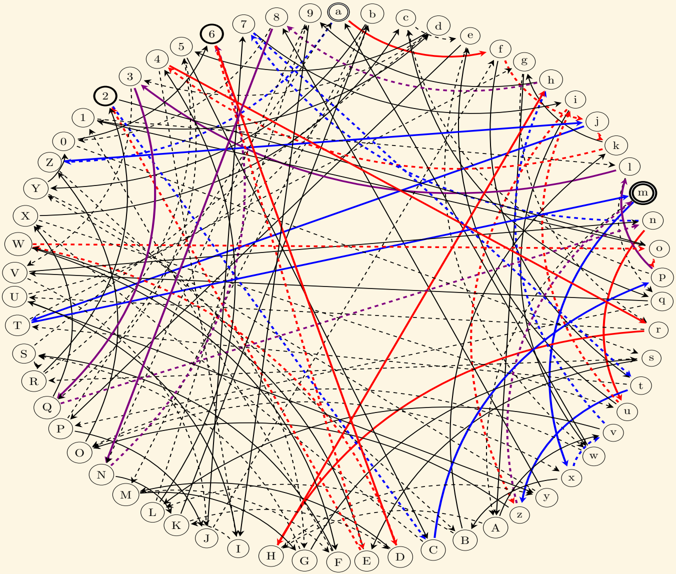
# Key exchange on a (commutative) graph

The full exchange:



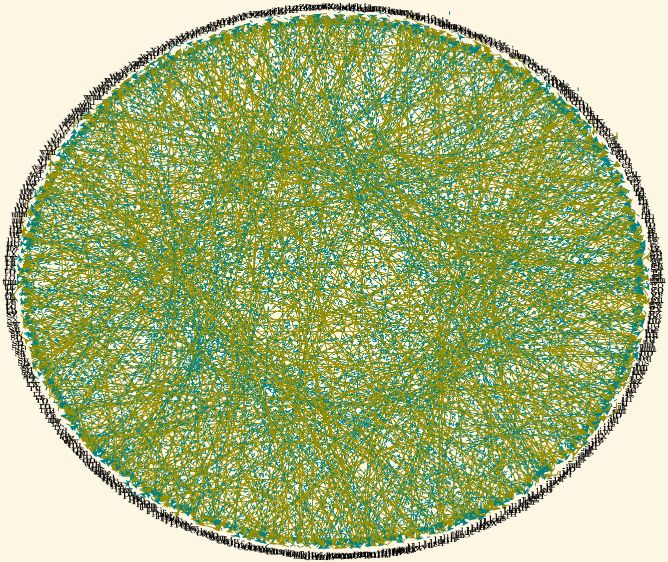
# Key exchange on a (commutative) graph

## Bigger graph (62 nodes)



# Key exchange on a (commutative) graph

Even bigger graph (676 nodes)



# Isogeny graphs for key exchange

- Needs a graph with good mixing properties:
  A path of length $O(\log N)$ gives a uniform node $\Rightarrow$ Ramanujan/expander graph.
- The graph does not fit in memory $\quad (N = 2^{256})$.
- Needs an algorithm taking a node as input and giving the neighbour nodes as output.

- Isogeny graph of ordinary elliptic curves $E/\mathbb{F}_p$ [Couveignes (1997)], [Rostovtsev–Stolbunov (2006)]
- Graph of size $N \approx \sqrt{p}$.
- Torsor (principal homogeneous space) under the class group $\mathrm{Cl}(\mathrm{End}(E_0))$.

☺ Commutative graph!

☹ Hidden shift problem solvable in quantum subexponential $L(1/2)$ time for an abelian group action via Kuperberg's algorithm.

- SIDH: supersingular elliptic curve Diffie-Helmann [De Feo, Jao (2011)],[De Feo, Jao, Plût (2014)]
- Use the isogeny graph of a supersingular elliptic curve $E$ over $\mathbb{F}_{p^2}$ $(N \approx p)$.

# Isogeny graphs for key exchange

# SIDH in practice

- $p = 2^a 3^b - 1$. $N_A = 2^a, N_B = 3^b$, $N_A$ prime to $N_B$.
- $E_0 : y^2 = x^3 + x$ (supersingular when $a \geq 2$)
- $E_0[N_A] = \langle P_A, Q_A \rangle, E_0[N_B] = \langle P_B, Q_B \rangle$.
- Alice's secret isogeny: $\phi_A$ of kernel $\langle P_A + s_A Q_A \rangle$.
- Bob's secret isogeny: $\phi_B$ of kernel $\langle P_B + s_B Q_B \rangle$.
- Key exchange:

$$\begin{array}{ccc} E_0 & \xrightarrow{\phi_B} & E_B \\ \downarrow \phi_A & & \downarrow \phi'_A \\ E_A & \xrightarrow{\phi'_B} & E_{AB} \end{array}$$

- $E_{AB}$ is the shared secret.
- $\phi'_A \circ \phi_B = \phi'_B \circ \phi_A : E_0 \to E_{AB}$ has kernel $\operatorname{Ker} \phi_A + \operatorname{Ker} \phi_B$.
- $\phi'_A$ has kernel $\langle \phi_B(P_A + s_A Q_A) \rangle$, $\phi'_B$ has kernel $\langle \phi_A(P_B + s_B Q_B) \rangle$.
- Alice publishes: $P'_B = \phi_A(P_B), Q'_B = \phi_A(Q_B)$.
  Bob publishes: $P'_A = \phi_B(P_A), Q'_A = \phi_B(Q_A)$. ("Torsion points".)
- $\operatorname{Ker} \phi'_A = \langle P'_A + s_A Q'_A \rangle$, $\operatorname{Ker} \phi'_B = \langle P'_B + s_B Q'_B \rangle$.
- Key exchange in $\widetilde{O}(\log N_A \ell_A^{1/2} + \log N_B \ell_B^{1/2})$
  (Via fast smooth isogeny computation [De Feo, Jao, Plût (2014)] and Velusqrt [Bernstein, De Feo, Leroux, Smith (2020)]).

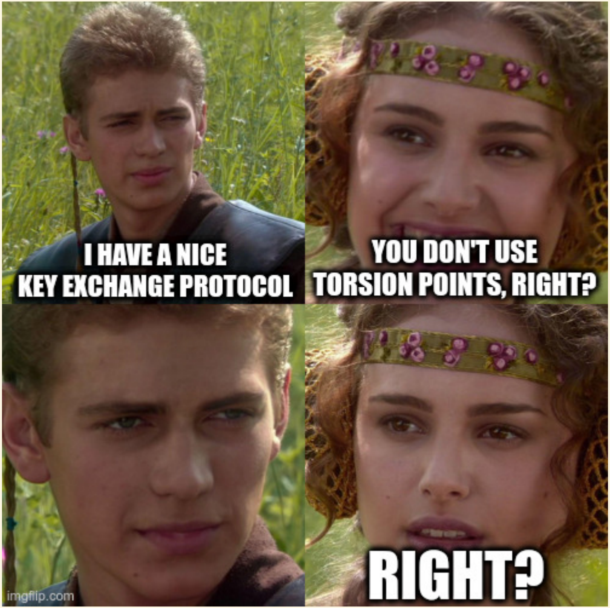
# Isogeny evaluation and interpolation

- Evaluation: given an $N$-isogeny $f$ and a point $Q \in E(\mathbb{F}_q)$, evaluate $f(Q)$.
- $N$-evaluation problem: $f$ is an $N$-isogeny = $\operatorname{Ker} f$ is of degree $N$.

- Interpolation: given a tuple $(P, f(P))$, recover $f$.
- $(N, N')$-interpolation problem: given $f$ an $N$-isogeny and $P$ a point of $N'$-torsion, from $(P, f(P))$ and $Q \in E(\mathbb{F}_q)$, evaluate $f(Q)$ ($N' \geq N$).
- Weak interpolation: we are given $(P_1, f(P_1))$, $(P_2, f(P_2))$ for $(P_1, P_2)$ a basis of $E[N']$.

- SIDH: the key exchange uses the $N_A$ and $N_B$ evaluation problems
- If we can solve the weak interpolation problem when $N = N_A, N' = N_B$ are smooth in polylogarithmic time, we can break SIDH.

# Isogeny evaluation and interpolation



# Evaluation

- $f : E_1 \to E_2$ an $N$-isogeny
- $f(x,y) = \left(\frac{g(x)}{h(x)}, cy\left(\frac{g(x)}{h(x)}\right)'\right), \deg g, \deg h \leq N$
- [Vélu 1971]: given $h(x)$ representing the kernel $\operatorname{Ker} f : \{P \in E \mid h(x(P)) = 0\}$, evaluate $f(Q)$ in $O(N)$ operations in $\mathbb{F}_q$.
- Velusqrt: special case $\operatorname{Ker} f = \langle T \rangle, T \in \mathbb{F}_q$, evaluate $f(Q)$ in $\widetilde{O}(\sqrt{N})$ operations in $\mathbb{F}_q$.
- Linear time.

- If $N$ is smooth, $f$ can be decomposed into a product of small isogenies.
- Evaluation in $O(\log N \ell_N)$ or $\widetilde{O}(\log N \sqrt{\ell_N})$.
- Logarithmic time.

- The decomposition cost is quasi-logarithmic if $\operatorname{Ker} f = \langle T \rangle$ with $T \in \mathbb{F}_q$; polylogarithmic if $N$ is powersmooth; but linear if $T$ lives in a large extension.

# Interpolation

- Given $(P, f(P))$, $P$ a point of order $N' \geq 2N$, recover the rational function $\frac{g(x)}{h(x)}$ in $\widetilde{O}(N)$ by interpolating the points $(x(mP), x(mf(P)))$, $m = 1, \ldots, N'-1$.
- Can evaluate on $Q$ directly.
- Special case : $P \in T_{0_E}(E)$ a "fat point" of order $p$ ⇒solve a differential equation [Elkies 1992] ($P \neq 0, p > 2N$).
- Quasi-linear time.

- Faster algorithm when $N'$ is smooth?
- Yes if $f(P) = 0$. Then $N = N'$ and $\mathrm{Ker} f = \langle P \rangle$.
- If $N = N'$, the weak interpolation problem reduces via the DLP to the $N'$-evaluation problem.
- This is why the SIDH key exchange is fast: Bob uses the torsion point information published by Alice to find the kernel of his pushforward isogeny.
- No reason to expect a fast algorithm when $N'$ is prime to $N$.

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# Revisiting isogeny evaluation

- Can an $N$-isogeny be evaluated faster than linear time when $N$ has a large prime factor?
- If $f = [\ell]$ (so $N = \ell^2$): double and add in $O(\log \ell)$ to evaluate $\ell Q$.

- $F : E^2 \to E^2, (P_1, P_2) \mapsto (P_1 + P_2, P_1 - P_2)$ is a 2-isogeny in dimension 2.
- Double: $F(T, T) = (2T, 0)$.
- Add: $F(T, Q) = (T + Q, T - Q)$.

- We can evaluate $\ell Q$ as a composition of $O(\log \ell)$ evaluations of $F$, projections $E^2 \to E$ and embeddings $E \to E^2$.
- Double and add on $E$ = 2-isogenies in dimension 2

## Polarisations and isogenies on an abelian variety

- Polarisation on $A$ = a (symmetric) isogeny $\lambda_A : A \to \widehat{A}$
- Principal polarisation: $\lambda_A$ is an isomorphism.
- Warning: $A$ may have several non equivalent principal polarisations if $g > 1$.

- $f : (A, \lambda_A) \to (B, \lambda_B)$ $N$-isogeny between ppav: $f^* \lambda_B = N \lambda_A$.

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[d, "\lambda_B"] \\ \widehat{A} \arrow[u, "\lambda_A^{-1}"] & \hat{B} \arrow[l, "\hat{f}"] \end{tikzcd}$$

- Dual isogeny: $\hat{f} : \hat{B} \to \widehat{A}$
- Contragredient isogeny: $\tilde{f} = \lambda_A^{-1} \hat{f} \lambda_B : B \to A$
- $f$ $N$-isogeny $\Leftrightarrow \tilde{f} f = N \Leftrightarrow f \tilde{f} = N$.

- $\operatorname{Ker} f = \operatorname{Im} \left( \tilde{f} \mid B[N] \right)$.

# Algorithms for $N$-isogenies in higher dimension

- [Cosset-R. (2014), Lubicz-R. (2012–2022)]: An $N$-isogeny in dimension $g$ can be evaluated in linear time $O(N^g)$ arithmetic operations in the theta model given generators of its kernel.
- Warning: exponential dependency $2^g$ or $4^g$ in the dimension $g$.

- [Couveignes-Ezome (2015)]: Algorithm in $O(N^g)$ in the Jacobian model.
- Not hard to extend to product of Jacobians.
- Restricted to $g \leq 3$.

# Kani's lemma [Kani 1997] ($g = 1$), [R. 2022] ($g > 1$)

- $\alpha : A \to B$ a $a$-isogeny, $\beta : A \to C$ a $b$-isogeny.
- $\alpha' : C \to D$ a $a$-isogeny, $\beta' : C \to D$ a $b$-isogeny with $\beta'\alpha = \alpha'\beta$:

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "\beta"] & B \arrow[d, "\beta'"] \\ C \arrow[r, "\alpha'"] & D \end{tikzcd}$$

- If $a$ prime to $b$, the pushforward $\alpha', \beta'$ of $\alpha$ by $\beta$ satisfy these conditions.

- $F = \begin{pmatrix} \alpha & \widetilde{\beta'} \\ -\beta & \widetilde{\alpha'} \end{pmatrix} : A \times D \to B \times C.$
- $\tilde{F} = \begin{pmatrix} \tilde{\alpha} & -\tilde{\beta} \\ \beta' & \alpha' \end{pmatrix} : B \times C \to A \times D, \quad \tilde{F}F = a + b.$
- $F$ is an $a + b$-isogeny with respect to the product polarisations.

- $\operatorname{Ker} F = \{\tilde{\alpha}(P), \beta'(P) \mid P \in B[a+b]\}$ (if $a$ is prime to $b$)

# Revisiting the interpolation

$$\begin{tikzcd} E_1 \arrow[r, "f"] \arrow[d, "\alpha"] & E_2 \arrow[d, "\alpha'"] \\ E_1' \arrow[r, "f'"] & E_2' \end{tikzcd}$$

- $f : E_1 \to E_2$ an $N$-isogeny.
- Goal: replace $f$ by $F$ an $N'$-isogeny.
- Find $\alpha : E_1 \to E_1'$ an $m$-isogeny, with $N' = N + m$.
- Kani's lemma: $F : E_1 \times E_2' \to E_1' \times E_2$ is an $N'$-isogeny.
- Since we know $f(E[N'])$, and we can evaluate $\alpha$ on $E[N']$, we recover Ker $F$ (or Ker $\tilde{F}$)
- Evaluate $F$, hence $f$ at any point: $F(P, 0) = (\alpha(P), -f(P))$.
- This evaluation is fast if $N'$ is (power) smooth.

**Examples:**

- $m$ smooth [Castryck–Decru 2022; Maino–Martindale 2022]
- $m = \ell^2$: take $\alpha = [\ell]$
- $\mathrm{End}(E)$ has an efficient endomorphism $\alpha$ of norm $m$ [Castryck–Decru].

## The general case

- $\alpha = \begin{pmatrix} a_1 & a_2 \\ -a_2 & a_1 \end{pmatrix}$ is always an endomorphism of norm $m = a_1^2 + a_2^2$ on $E^2$ (Gaussian integers $\mathbb{Z}[i]$)
- $\alpha = \begin{pmatrix} a_1 & -a_2 & -a_3 & -a_4 \\ a_2 & a_1 & a_4 & -a_3 \\ a_3 & -a_4 & a_1 & a_2 \\ a_4 & a_3 & -a_2 & a_1 \end{pmatrix}$ is always an endomorphism of norm $m = a_1^2 + a_2^2 + a_3^2 + a_4^2$ on $E^4$ (Hamilton's quaternion algebra)
- Evaluating $\alpha$ costs $O(\log m)$ arithmetic operations
- Every integer is a sum of four squares [Διόφαντος ὁ Ἀλεξανδρεύς, Lagrange].

$$\begin{tikzcd} E_1^4 \arrow[r, "f"] \arrow[d, "\alpha"] & E_2^4 \arrow[d, "\alpha"] \\ E_1^4 \arrow[r, "f"] & E_2^4 \end{tikzcd}$$

- $F : E_1^4 \times E_2^4 \to E_1^4 \times E_2^4$ is an $N'$-isogeny.

# The embedding lemma [R. 2022]

- A $N$-isogeny $f : A \to B$ in dimension $g$ can always be efficiently embedded into a $N'$ isogeny $F : A' \to B'$ in dimension $8g$ (and sometimes $4g, 2g$) for any $N' \geq N$.

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow & & \uparrow \\ A' & \xrightarrow{F} & B' \end{array}$$

- Considerable flexibility (at the cost of going up in dimension).

- Breaks SIDH ([Castryck–Decru], [Maino–Martindale] in dimension $2$, [R.] in dimension $4$ or $8$)
- Reduces the $(N, N')$-weak interpolation problem to the $N'$-evaluation problem in higher dimension
- Only needs ${N'}^2 \geq N$ (uses the dual isogeny)

$\Rightarrow$ Solves the weak interpolation problem when $N'$ is (power) smooth

- Amazing fact: does not requires $\operatorname{Ker} f$, works even if $N$ is prime

- Open question: case $N'$ prime? Need a fast $N'$-evaluation algorithm!

# Efficient representation of isogenies [R. 2022]

- For the $N$-evaluation problem, once we have evaluated $f$ on a basis of the $N'$-torsion this reduces to the $N'$-weak interpolation problem which reduces to the $N'$-evaluation problem (in higher dimension).

$\Rightarrow$ Can always embed an $N$-isogeny $f$ into a $N'$-isogeny $F$ with $N'$ powersmooth

- Then decompose $F$ as a product of small isogenies.

- Polylogarithmic space $O(\log^3 N)$
- Evaluation in polylogarithmic time $O(\log^7 N)$ arithmetic operations.
- Previously: no representation giving better than linear time for a generic isogeny.

- Representation: $(P_i, Q_i, f(P_i), f(Q_i))$ for $(P_i, Q_i)$ basis of $E[\ell_i]$, small torsion points $(\ell_i \mid N')$
- We need to evaluate $f$ on the $N'$-torsion: given the kernel, the decomposition step is quasi-linear.

# Examples

$f$ an $N$-isogeny with an efficient representation.

- Efficient division: evaluate $f/D$ on any point.
- Contragredient isogeny: evaluate $\tilde{f}$ on any point.

$\Rightarrow$ Efficient evaluation of the Verschiebung $\hat{\pi}_p$.

- Efficient lifting of isogenies: embed $f$ into $F$ at precision $m = 1$, then lift $F$ to precision $m > 1$.

## Applications [R. 2022]

- $E/\mathbb{F}_q$ ordinary elliptic curve, $K = \mathrm{End}(E) \otimes_{\mathbb{Z}} \mathbb{Q}$. Given the factorisation of $[O_K : \mathbb{Z}[\pi]]$, compute $\mathrm{End}(E)$ in polynomial time.
  Factorisation: quantum polynomial time, classical subexponential time
- Previously: no quantum polynomial time algorithm known.
  Classical algorithm in $L(1/2)$ under GRH [Bisson–Sutherland 2009].

- Compute the canonical lift $\hat{E}/\mathbb{Z}_q$ in polynomial time.
- Previously: $L(1/2)$ under GRH [Couveignes-i-Henocq 2002]

- Compute the modular polynomial $\Phi_\ell$ in quasi-linear time $O(\ell^3 \log^3 \ell \log\log \ell)$ (no heuristics!).
- Compute $\Phi_\ell \mod p$ in quasi-linear time $\widetilde{O}(\ell^2 \log p)$.
- If $E/K$ elliptic curve of height $H$ over a number field, compute $\Phi_\ell(j(E), Y)$ in quasi-linear time $\widetilde{O}(H\ell^2)$.

- Generalisations to abelian varieties.
- Previously: no algorithm known to compute $\Phi_\ell$ in quasi-linear time when $g > 2$.

# Point counting and canonical lifts

$E/\mathbb{F}_q, q = p^n$.

- [Schoof 1985]: $\widetilde{O}(n^5 \log^5 p)$ (Étale cohomology)
- [SEA 1992]: $\widetilde{O}(n^4 \log^4 p)$ (Heuristic)

- [Kedlaya 2001]: $\widetilde{O}(n^3 p)$ (Rigid cohomology)
- [Harvey 2007]: $\widetilde{O}(n^{3.5} p^{1/2} + n^5 \log p)$

- [Satoh 2000] (canonical lifts of ordinary curves): $\widetilde{O}(n^2 p^2)$ (Crystalline cohomology)
- [Maiga – R. 2021]: $\widetilde{O}(n^2 p)$
- [R. 2022]: $\widetilde{O}(n^2 \log^8 p + n \log^{11} p)$